

**ASYMMETRIC COLORING OF LOCALLY FINITE GRAPHS  
AND PROFINITE PERMUTATION GROUPS:  
TUCKER'S CONJECTURE CONFIRMED**

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*To the memory of Jan Szał*

ABSTRACT. An *asymmetric coloring* of a graph is a coloring of its vertices that is not preserved by any non-identity automorphism of the graph. The *motion* of a graph is the minimal degree of its automorphism group, i. e., the minimum number of elements that are moved (not fixed) by any non-identity automorphism. We confirm Tom Tucker's "Infinite Motion Conjecture" that *connected locally finite graphs with infinite motion admit an asymmetric 2-coloring*. We infer this from the more general result that the *inverse limit of an infinite sequence of finite permutation groups with disjoint domains, viewed as a permutation group on the union of those domains, admits an asymmetric 2-coloring*. The proof is based on the study of the interaction between epimorphisms of finite permutation groups and the structure of the setwise stabilizers of subsets of their domains.

1. INTRODUCTION

A graph is *locally finite* if every vertex has finite degree. A graph is *asymmetric* if it has no nontrivial automorphisms. A *coloring* of the vertices of a graph is asymmetric if no non-identity automorphisms of the graph preserves the coloring. This author established in 1977 that every regular tree of (finite or infinite) degree  $\kappa \geq 2$  admits an *asymmetric 2-coloring* [Ba77]. While this result is trivial for locally finite (and therefore countable) trees (the point of [Ba77] was that it holds for arbitrary infinite cardinals  $\kappa$ , and the difficulty begins at strongly inaccessible cardinals), several interesting questions about asymmetric colorings of locally finite graphs have been asked and partly answered (see, e. g., [KWZ06, IKT07, CIL14, IS+15, Le16, Le17, HI+17, CIL14, LPS18, LPS19]). A particularly intriguing conjecture was formulated by Thomas W. Tucker in 2011 [Tu11]. The *degree* of a permutation is the number of elements it moves. The *minimal degree* of a permutation group is the minimum of the degrees of its non-identity elements. The *motion* of a graph is the minimal degree of its automorphism group. Tucker's *Infinite Motion Conjecture* states that *every connected, locally finite graph with infinite motion admits an asymmetric 2-coloring*. A number of recent papers obtained partial result on this conjecture, including [IKT07, CIL14, HI+17]. Notably, Florian Lehner confirmed the conjecture for graphs with intermediate ( $\exp(O(\sqrt{n}))$ ) growth [Le16]. Lehner, Monika Pilśniak, and Marcin Stawiski confirmed the conjecture for graphs

with maximum degree  $\leq 5$  [LPS18]. The same authors proved that if the maximum degree of the graph is  $k$  then an asymmetric coloring with  $O(\sqrt{k} \log k)$  colors exists [LPS19].

The main result of our paper confirms Tucker's conjecture in full generality. Along the way, we raise and partially solve a number of questions regarding group-theoretic properties of colorings for finite permutation groups.

**Theorem 1.1** (Tucker's Infinite Motion Conjecture confirmed). *Let  $X$  be a locally finite connected graph with infinite motion. Then  $X$  admits an asymmetric 2-coloring.*

In other words, the conclusion says that the set of vertices has a subset, not fixed setwise by any non-identity automorphism of  $X$ .

Given the fact that the automorphism group of a connected locally finite *rooted* graph is the inverse limit of a sequence of finite permutation groups, the proof boils down to the study of inverse systems of epimorphisms among a sequence of finite permutation groups. The motion condition translates to the *disjointness* of the domains of the groups in the system. Our result in this context is the following; this is the main technical result of the paper.

**Theorem 1.2** (asymmetric coloring of inverse limit). *Let  $\mathcal{G}$  be the inverse limit of an infinite sequence of finite permutation groups with disjoint domains, viewed as a permutation group acting on the union of those domains. Then  $\mathcal{G}$  admits an asymmetric 2-coloring.*

In other words, the conclusion says that the domain of  $\mathcal{G}$  has a subset, not fixed setwise by any non-identity element of  $\mathcal{G}$ .

Our proof builds on a line of work on asymmetric colorings of finite primitive permutation groups, started by David Gluck (1983) [Gl83] and Peter Cameron, Peter M. Neumann, and Jan Saxl (1984) [CNS84], a theory to which we contribute in this paper. By a counting argument (see Prop. 15.12), also used by Gluck, Cameron et al. proved that in addition to the symmetric and alternating groups, there are only a finite number of primitive groups that do not admit an asymmetric 2-coloring (see Theorem 15.1). Seress classified the exceptions (1997) [Se97]. Our proof depends on Seress's classification.

The result of [CNS84] depends on the Classification of Finite Simple Groups (CFSG) through a result by Cameron [Ca81]. Seress uses detailed explicit knowledge of CFSG. While Cameron's result, used in [CNS84], has recently been given a remarkable elementary combinatorial proof by Sun and Wilmes [SW15] (see Sec. 15), through Seress's work, our paper continues to depend on CFSG. We suspect, though, that our proof can be modified to avoid dependence on CFSG.

The automorphism group of a 2-coloring is the same as the setwise stabilizer of a subset of the permutation domain. Our method is that we approximate asymmetry by gradually simplifying the structure of the groups involved in the inverse limit, by 2-coloring the underlying sets of an infinite, coinfinite subset of the finite groups that participate in the inverse limit. The hard part is to reduce all groups to solvable groups. To this end, we introduce the concept of *solvable coloring*: a coloring that is preserved only by a solvable subgroup of the automorphism group. This concept, intermediate on the way to asymmetry, may deserve a systematic study. Once reduced to solvable groups, we reduce the groups to bounded derived length, and

finally we show how to reduce the derived length, until, at derived length zero, the group vanishes and asymmetry occurs.

Now we state our key lemma, which may be of independent interest.

**Theorem 1.3** (Reducing simple image). *Let  $G \leq \text{Sym}(\Omega)$  be a finite permutation group and  $T$  a finite nonabelian simple group. Let  $\varphi : G \twoheadrightarrow T$  be an epimorphism. Then there exists a subset  $\Delta \subseteq \Omega$  such that  $\varphi(G_\Delta) < T$ , where  $G_\Delta$  denotes the setwise stabilizer of  $\Delta$ .*

We shall need the following two additional results about colorings for finite permutation groups.

**Theorem 1.4** (Bounded orbits for solvable groups). *There is a constant  $C$  such that the following holds. Let  $G \leq \text{Sym}(\Omega)$  be a solvable finite permutation group. Then there exists a subset  $\Delta \subseteq \Omega$  such that every orbit of  $G_\Delta$  has length  $\leq C$ .*

We use this result only to infer that  $G_\Delta$  has bounded derived length.

Along the way to proving Theorem 1.4, we show that every solvable permutation group has an asymmetric 5-coloring (Lemma 12.4). (The bound 5 is tight, see Remark 12.5.)

The following observation will allow us to move from bounded derived length to asymmetry.

**Proposition 1.5.** *Every nontrivial solvable permutation group admits a 2-coloring that reduces its derived length.*

**1.1. Structure of the paper.** We briefly review the author's view of the history of combinatorial symmetry breaking in Sec. 2. Rudimentary definitions follow in Sec. 3. We reduce Tucker's conjecture to the study of epimorphisms of finite permutation groups in Sections 4–6. This includes a review of inverse systems of group homomorphisms in Sec. 4. The program of gradual structural reduction of the groups is formalized in Theorem 6.4.

After Sec. 6, all groups considered are finite. Further details of group theoretic notation are reviewed in Sec. 7. The first phase of our program of structural reductions, going from general to solvable groups, occupies Sections 8 to 11. This includes the proof of Theorem 1.3 for primitive groups in Sec. 9 and for general permutation groups in Sec. 10.

We achieve bounded derived length in Section 12 by proving Theorem 1.4. The reduction of the derived length (Prop. 1.5) is the subject of Sec. 13. This completes the proof of the Infinite Motion Conjecture.

In Section 14 we state a finite version of Theorem 1.2 (Theorem 14.1) and state two conjectures in search for more effective versions of this result.

In Section 15 we address the question of CFSG-free proofs and combinatorial generalizations, notably to the motion of primitive coherent configurations.

The techniques introduced in this paper give rise to a number of new questions and potential areas of research; some of these are listed in the concluding section of this paper, Sec. 16.

## 2. A BRIEF HISTORY OF COMBINATORIAL SYMMETRY BREAKING

The study of asymmetry dates back to the 1939 paper by Roberto Frucht [Fr39] which established that every finite group is isomorphic to the automorphism group

of a graph. Frucht introduced asymmetric “gadgets” to code colors and thereby eliminate unwanted automorphisms. A decade later Frucht proved the same result for 3-regular graphs [Fr49]; as a tool, he constructed an asymmetric 3-regular graph that today bears his name. Analogues of Frucht’s Theorem were found to hold for many other classes of structures, notably including Steiner Triple Systems [Men78] and algebraic number fields [FK]. A theory was developed to construct graphs whose automorphism group was a prescribed regular permutation group (GRR theory [He76, Go78, Ba78, MS]). In all these cases, it is easy to construct structures on which the given group acts; the problem is the elimination of unwanted symmetry by coding asymmetry into the structure to get exactly the desired group of automorphisms. In the 1960s, this line of work was extended by the Prague school of category theory to endomorphisms [HP65]. A key aspect of this generalization was the construction of *rigid graphs*, i.e., graphs with no nontrivial endomorphisms [HP65, VHP65, HL69].

It was in this context that this author considered the simplest possible asymmetric graphs in the infinite case, establishing that for any two (finite or infinite) cardinal numbers  $\kappa > \lambda \geq 2$  there exists an asymmetric tree with only these two degrees [Ba77]. The key auxiliary result was that a regular tree of (finite or infinite) degree  $\kappa \geq 2$  admits an *asymmetric 2-coloring*—the first result on asymmetric colorings.

In 1983/84, the idea of asymmetric 2-colorings was introduced in group theory in a pair of independent papers, by Gluck [Gl83] and by Cameron–Neumann–Saxl [CNS84]. Gluck found that all but a finite number of solvable primitive groups admit an asymmetric 2-coloring, and gave the exact list of exceptions (all of them of degree  $\leq 9$ ). Cameron–Neumann–Saxl proved that all but a finite number of primitive permutation groups other than  $A_n$  and  $S_n$  admit an asymmetric 2-coloring. Subsequently Seress [Se97] classified all the exceptions; Seress’s paper is one of our key references.

The motivation of Gluck, Cameron et al., and Seress was in classical questions of the theory of permutation groups and partly, in questions of computational group theory through the closely related concept of *bases* of permutation groups (see Seress’s monograph [Se03]). These papers discovered the relationship of asymmetric colorings with the *minimal degree* of the permutation group, i.e., the minimum number of elements not fixed by a non-identity element of the group. This is a classical concept, studied since the time of Jordan [Jo1871] and Bochert [Bo1897] in the 19th century (see [DM96, Sec. 3.3]).

A *base* of a permutation group  $G \leq \text{Sym}(\Omega)$  is a subset  $\Delta \subseteq \Omega$  such that the pointwise stabilizer  $G_{(\Delta)}$  is the identity. This classical “symmetry breaking” concept, evidently closely related to the notion of asymmetric colorings, gained interest in computational group theory through the work of Charles Sims in the 1960s [Si70] (see [Se03]). The significance of this concept to asymptotic group theory comes from the observation that if  $G \leq S_n$  has a base of size  $b$  then  $|G| \leq n^b$ . Bounding the orders of *primitive* permutation groups was a central question of 19th century group theory (see [PS80] for some of this history).

In 1979, this author found a graph theoretic method to bound the base size of primitive permutation groups in the more general context of *primitive coherent configurations* (PCCs)—certain highly regular colorings of the edges of the complete

directed graph (see Sec. 15.2). Entirely omitting group theory, this method nevertheless produced a nearly tight upper bound on the base size ( $O(\sqrt{n} \log n)$ ), and therefore on the order, of any primitive group other than the alternating and the symmetric groups, solving a then century-old problem [Ba81]. The key technical result was a symmetry-breaking tool: how many vertices can distinguish between a given pair of vertices, see Theorem 15.13. As a byproduct, a lower bound on the minimal degree of primitive groups and on the motion of PCCs follows (Theorem 15.13 and Obs. 15.14); the bound is tight within a constant factor.

Significant progress over this result occurred in 2015 when Xiaorui Sun and John Wilmes extended the result to a classification of all primitive groups of minimum base size greater than essentially  $n^{1/3}$ , while building a structure theory of primitive coherent configurations along the way. This remarkable result, previously only known through CFSG [Ca81], can be used to give an elementary proof of the Cameron–Neumann–Saxl result mentioned (see Sec. 15). It would presumably play a prominent role in a CFSG-free proof of Tucker’s Conjecture.

The *Graph Isomorphism problem* has been a chief producer and consumer of cost-conscious symmetry breaking techniques. Both [Ba81] and [SW15] were partly motivated by the complexity of this problem. Many new details about this connection, in the context of coherent configurations, emerged in [Ba16], where a quasipolynomial-time algorithm for testing graph isomorphism is described. Comments on measuring the cost appear in Sec. 16.

**Terminology.** A much-cited 1996 paper by Michael Albertson and Karen Collins [AC96] introduced asymmetric colorings under the name “distinguishing coloring.” Although this terminology has found a large following, I find it unfortunate for multiple reasons and will continue to use the term “distinguish” in more natural meanings, including in this paper (see right before Theorem 15.13, and Problem 10 in Sec. 16). The term “asymmetric coloring” was introduced in [Ba77] (1977).

The term “minimal degree” of a permutation group has been in use at least since Wielandt’s 1964 book [Wi64], and has since been used in a large body of literature in the theory of permutation groups. However, this term is ill-suited for applications to the automorphism group of a graph, where it could be confused with the minimum degree of the graph—an unrelated concept. Therefore I have adopted the term “motion” of a graph or other structure, meaning the minimal degree of its automorphism group, as suggested in [RS98].

### 3. DEFINITIONS, NOTATION: GROUP ACTIONS AND COLORING

Structural group theoretic definitions and notation will be reviewed in Section 7. The beautiful monograph [DM96] covers most of the group theory we need. In this section we only deal with the most rudimentary concepts.

**Definition 3.1** (Coloring). Let  $\Sigma$  be an ordered set; we refer to the elements of  $\Sigma$  as “colors.” A  $\Sigma$ -coloring of a set  $\Omega$  is a function  $\gamma : \Omega \rightarrow \Sigma$ . If  $|\Sigma| = k$ , we speak of a  $k$ -coloring and often use  $\Sigma := [k] = \{1, 2, \dots, k\}$  as the set of colors. We identify the subsets of  $\Omega$  with 2-colorings, using a fixed ordered pair of colors.

**Notation 3.2** (Symmetric and alternating groups). For a set  $\Omega$ , we write  $\text{Sym}(\Omega)$  and  $\text{Alt}(\Omega)$  for the symmetric and the alternating group, resp., on  $\Omega$ . We also write  $S_n$  for the generic symmetric group of degree  $n$  and  $A_n$  for the alternating group of degree  $n$ .

**Definition 3.3** (Group action). Let  $G$  be a group,  $\Omega$  a set. A  $G$ -action on  $\Omega$ , denoted  $G \curvearrowright \Omega$ , is a homomorphism  $\varphi : G \rightarrow \text{Sym}(\Omega)$ . We refer to  $\Omega$  as the *permutation domain*. The *image* of the action  $G \curvearrowright \Omega$  is the group  $\varphi(G) \leq \text{Sym}(\Omega)$ . For  $\pi \in G$  and  $x \in \Omega$  we write  $\pi(x)$  to denote  $(\varphi(\pi))(x)$  if the action  $\varphi$  is clear from the context.

**Definition 3.4** (Coloring for a group action). Let  $\varphi : G \curvearrowright \Omega$  be a group action. By a coloring for  $\varphi$  we mean a coloring of  $\Omega$ . For a coloring  $\gamma : \Omega \rightarrow \Sigma$  and  $\pi \in G$  we write  $\pi(\gamma)$  to denote the coloring  $(\pi(\gamma))(x) = \gamma(\pi^{-1}(x))$  ( $x \in \Omega$ ).

**Definition 3.5** (Stabilizers). Let  $\varphi : G \curvearrowright \Omega$  be a group action. Let  $x \in \Omega$ . The *stabilizer* of  $x \in \Omega$  is the subgroup  $G_x = \{\pi \in G \mid \pi(x) = x\}$ . For  $\Delta \subseteq \Omega$  we write  $G_\Delta = \{\pi \in G \mid \pi(\Delta) = \Delta\}$  for the *setwise stabilizer* of  $\Delta$  and  $G_{(\Delta)} = \bigcap_{x \in \Delta} G_x$  for the *pointwise stabilizer* of  $\Delta$ . The *stabilizer of the coloring*  $\gamma$  is the subgroup  $G_\gamma = \{\pi \in G \mid \pi(\gamma) = \gamma\}$ . We refer to  $G_\gamma$  as the group of  $G$ -automorphisms of the coloring  $\gamma$ . In other words, the  $G$ -automorphisms of the coloring  $\gamma$  are those elements of  $G$  that preserve  $\gamma$ .

**Observation 3.6** (Subsets vs. 2-colorings). If  $\Sigma = \{a, b\}$  where  $a < b$  and  $\gamma : \Omega \rightarrow \Sigma$  is a 2-coloring then  $G_\gamma = G_\Delta$  where  $\Delta = \gamma^{-1}(b)$ .  $\square$

**Definition 3.7** (Permutation group). A *permutation group* acting on  $\Omega$  is a subgroup of  $\text{Sym}(\Omega)$ . We view permutation groups as faithful actions. A *coloring* for  $G$  is a coloring of  $\Omega$ . (This means a coloring for this faithful action).

**Definition 3.8** (Group theoretic properties of colorings). Let  $G \curvearrowright \Omega$  be an action. We say that a coloring  $\gamma$  for  $G$  is *asymmetric* if  $G_\gamma = 1$ . We also say that such a coloring of  $\Omega$  is  $G$ -asymmetric. Analogously we speak of  $G$ -asymmetric subsets of the domain. We call the minimum number of colors required by an asymmetric coloring for  $G$  the *asymmetric coloring number* of  $G$ . We say that  $\gamma$  is a *solvable coloring* if  $G_\gamma$  is a solvable group. We also express this circumstance by saying that the coloring  $\gamma$  *results in a solvable group*. We call the minimum number of colors required by a solvable coloring the *solvable coloring number* of  $G$ .

We may analogously ascribe other group theoretic properties to  $\gamma$ . For example,  $\gamma$  can result in a 2-group (i. e.,  $G_\gamma$  is a 2-group) or in a solvable group with derived length  $\leq 3$ , etc.

We define the corresponding concepts for subsets of the domain (asymmetric subsets, solvable subsets, subsets resulting in a 2-group, etc.) via the correspondence to 2-colorings (Obs. 3.6). We say that  $G$  *admits* a coloring with certain property (e. g., a solvable  $k$ -coloring) if there is a coloring of the domain with the given property.

Note that an asymmetric coloring exists if and only if the action of  $G$  is faithful.

**Definition 3.9** (Support, minimal degree). Let  $G \leq \text{Sym}(\Omega)$  be a permutation group. The *support* of  $\sigma \in G$  is the set  $\text{supp}(\sigma) := \{x \in \Omega \mid \sigma(x) \neq x\}$ . The *minimal degree* of  $G$  is  $\mu(G) := \min_{\sigma \in G \setminus \{1\}} |\text{supp}(\sigma)|$ , the minimum number of elements moved (not fixed) by any non-identity element of  $G$ . If  $|G| = 1$  then its minimal degree is “super-infinity,” denoted  $\infty$  and thought of as being greater than any cardinal number.

For instance, for  $n \geq 3$ ,  $\mu(S_n) = 2$  and  $\mu(A_n) = 3$ .

**Definition 3.10** (Coloring of structures). Let  $\mathfrak{X} = (\Omega, \mathcal{R})$  be a structure (such as a graph). We say that a coloring  $\gamma : \Omega \rightarrow \Sigma$  of the underlying set (set of vertices) has a certain property w. r. t.  $\mathfrak{X}$  (such as being asymmetric or solvable) if it has the corresponding property w. r. t.  $\text{Aut}(\mathfrak{X})$ .

**Definition 3.11** (Motion). The *motion* of a structure  $\mathfrak{X}$  is the minimal degree of  $\text{Aut}(\mathfrak{X})$ .

The term “motion” in this meaning was introduced in [RS98].

Let  $G \leq \text{Sym}(\Omega)$  be a permutation group. We say that  $G$  is a *finite permutation group* if its domain  $\Omega$  is finite.

#### 4. INVERSE SYSTEMS OF EPIMORPHISMS

Inverse systems can be indexed by an arbitrary poset. In this paper, we only consider inverse systems of infinite *sequences* of groups, so the index set is  $\mathbb{N} = \{0, 1, \dots\}$ , the set of natural numbers, and the infinite subsets of  $\mathbb{N}$ , under the natural ordering.

**Convention 4.1.** Throughout this paper, the letters  $I$  and  $J$  will denote infinite subsets of  $\mathbb{N}$ , with the natural ordering.

**Definition 4.2** (Inverse system of homomorphisms of finite groups). Let  $(G_i \mid i \in I)$  be an infinite sequence of finite groups. For  $i \leq j$  ( $i, j \in I$ ) let  $\varphi_{i,j} : G_j \rightarrow G_i$  be a homomorphism such that for  $i \leq j \leq k$  ( $i, j, k \in I$ ), the *compatibility condition*  $\varphi_{i,j}\varphi_{j,k} = \varphi_{i,k}$  holds. The homomorphism  $\varphi_{ii}$  is the identity on  $G_i$ . The  $\varphi_{i,j}$  are called the *transition homomorphisms*. The groups  $G_i$ , together with the transition homomorphisms, form an *inverse system*, denoted  $(G_i, \varphi_{i,j})_I$ , or simply  $(G_i, \varphi_{i,j})$  if the index set  $I$  is clear from the context. We say that the system is *epimorphic* if all transition homomorphisms are surjective.

**Definition 4.3** (Inverse limit, strands). Let us consider a sequence  $(g_i \mid i \in I)$  of group elements,  $g_i \in G_i$ . Given the inverse system above, we call such a sequence a *strand* if for all  $i \leq j$  we have  $g_i = \varphi_{i,j}(g_j)$ . The inverse limit  $\mathcal{G} = \varprojlim G_i$  is the subgroup of the direct product of the  $G_i$  consisting of the strands. Viewing the  $G_i$  as discrete groups and endowing their direct product with the product topology,  $\mathcal{G}$  is a closed subgroup of the direct product (a profinite group).

**Fact 4.4.** Let  $X$  be a connected, locally finite, infinite graph, with possibly colored vertices. Let  $x_0$  be a vertex. Then  $\text{Aut}(X)_{x_0}$  (the stabilizer of  $x_0$  in  $\text{Aut}(X)$ ) is the inverse limit of a sequence of finite permutation groups.

*Idea of proof.* The finite permutation groups in question are the automorphism groups of the balls of each radius about  $x_0$ . We can alternatively also take the restrictions of  $\text{Aut}(X)$  to those balls; this would correspond to the epimorphic reduction discussed below.  $\square$

**Fact 4.5.** Let  $(G_i, \varphi_{i,j})_I$  be an *epimorphic* inverse system with limit  $\mathcal{G} = \varprojlim G_i$ . Let  $\pi_i : \mathcal{G} \rightarrow G_i$  denote the projection to the  $i$ -th coordinate. Then each  $\pi_i$  is an epimorphism.  $\square$

**Fact 4.6** (Epimorphic reduction). Let  $(G_i, \varphi_{i,j})_I$  be an inverse system of finite groups with inverse limit  $\mathcal{G} = \varprojlim G_i$ . Then there exist subgroups  $H_i \leq G_i$  such that  $(H_i, (\varphi_{i,j})_{|H_j})_I$  is an *epimorphic* inverse system such that  $\mathcal{G} = \varprojlim H_i$ . We call

this system the *epimorphic reduction* of the system  $(G_i, \varphi_{i,j})_I$ . The epimorphic reduction is unique.

*Proof.* Let  $\pi_i : \mathcal{G} \rightarrow G_i$  denote the projection to the  $i$ -th coordinate. Let  $H_i = \pi_i(\mathcal{G})$ . Then the system consisting of the groups  $H_i$  with the transition maps  $(\varphi_{i,j})|_{H_j}$  is clearly the unique inverse system satisfying the conditions.  $\square$

**Proposition 4.7** (Sublimit). *Let  $J \subseteq I \subseteq \mathbb{N}$  and let  $(G_i, \varphi_{i,j})_I$  be an inverse system of finite groups. Let  $\mathcal{G} = \varprojlim_{i \in I} G_i$  and let  $\mathcal{H} = \varprojlim_{j \in J} G_j$ . Then the projection  $\mathcal{G} \twoheadrightarrow \mathcal{H}$  is an isomorphism.*  $\square$

**Definition 4.8** ( $\Lambda$ -reduction). Let  $(G_i, \varphi_{i,j})_I$  be an inverse system as above, with inverse limit  $\mathcal{G}$ , and let  $\Lambda = (L_i \mid i \in I)$  where  $L_i \leq G_i$ . The  $\Lambda$ -reduction  $\mathcal{L}$  of  $\mathcal{G}$  consists of those strands  $(g_i : i \in I)$  where  $g_i \in L_i$  for all  $i$ .

**Fact 4.9.**  $\mathcal{L}$  is a closed subgroup of  $\mathcal{G}$  and  $\mathcal{L} = \varprojlim_{i \in I} L_i$ .

Let us now consider an inverse system  $(G_i, \varphi_{i,j})_I$  where each  $G_i$  is a finite permutation group,  $G_i \leq \text{Sym}(\Omega_i)$ . Critically, we assume the  $\Omega_i$  are *disjoint* (an assumption that is patently *false* in the proof of Fact 4.4).

**Definition 4.10** (Action of inverse limit of permutation groups with disjoint domains). Let  $(G_i, \varphi_{i,j})_I$  be an inverse system of finite permutation groups  $G_i \leq \text{Sym}(\Omega_i)$ . Assume the  $\Omega_i$  are *disjoint*. Let  $\Omega = \bigsqcup_{i \in I} \Omega_i$ . We view the direct product  $\prod_{i \in I} G_i$  as a permutation group acting on  $\Omega$  (coordinatewise). As a consequence, the inverse limit  $\mathcal{G} = \varprojlim_{i \in I} G_i$  is also a permutation group acting on  $\Omega$ .

**Observation 4.11** (Coloring of inverse limit of permutation groups with disjoint domains). *Under the assumptions of Def. 4.10, let  $\gamma$  be a coloring of the set  $\Omega = \bigsqcup_{i \in I} \Omega_i$ . In accordance with our conventions, we say that  $\gamma$  is a coloring for the inverse limit  $\mathcal{G} = \varprojlim G_i$ . Let  $\gamma_i$  be the coloring  $\gamma$  restricted to  $\Omega_i$ . Let  $L_i = (G_i)_{\gamma_i}$  and set  $\Lambda = (L_i : i \in I)$ . Let  $\mathcal{L} \leq \mathcal{G}$  denote the  $\Lambda$ -reduction of  $\mathcal{G}$ . Then  $\mathcal{L} = \mathcal{G}_\gamma$ . In particular,  $\gamma$  is an asymmetric coloring for  $\mathcal{G}$  if and only if  $\mathcal{L} = 1$ .*  $\square$

The Infinite Motion Conjecture will easily follow from the following result, previously stated as Theorem 1.2.

**Theorem 4.12** (main technical result). *Let  $\mathcal{G}$  be the inverse limit of an infinite sequence of finite permutation groups with disjoint domains. Then  $\mathcal{G}$  admits an asymmetric 2-coloring.*

The disjointness condition illuminates the role of the *infinite motion assumption* in Tucker's Conjecture. The latter will permit us to replace the *balls* in the proof of Fact 4.4 by the *spheres*, which are disjoint (see Lemma 5.7 below).

## 5. REDUCTION OF TUCKER'S CONJECTURE TO THEOREM 1.2

First we reduce the problem to the case of rooted graphs, where a designated vertex (the "root") is fixed by all automorphisms.

**Notation 5.1** (Spheres, balls). Let  $\rho$  denote the distance metric in the connected graph  $X$  with vertex set  $V$ . For a vertex  $v \in V$ , let  $S_d(v) = \{w \in V \mid \rho(v, w) = d\}$  denote the *sphere* of radius  $d$  about  $v$  and  $B_d(v) = \{w \in V \mid \rho(v, w) \leq d\}$  the *ball* of radius  $d$  about  $v$ .

**Definition 5.2** (Twins). Let us call the vertices  $u \neq v$  of the graph  $X$  *twins* if the transposition  $(u, v)$  is an automorphism of  $X$ . We say that the graph  $X$  *twin-free* if there are no twins in  $X$ .

**Observation 5.3.** *If  $X$  has infinite motion then it is twin-free.*  $\square$

**Definition 5.4** (Special subset). Let  $X$  be a connected locally finite graph with vertex set  $V$ . Let  $x_0 \in V$ . We call a subset  $\Delta \subseteq V$  *special* (with respect to  $x_0$ ) if  $\Delta \cap B_1(x_0) = \emptyset$  and  $S_{2d}(x_0) \subseteq \Delta$  for all  $d \geq 1$ .

**Lemma 5.5** (Designated root). *Let  $X$  be a connected, twin-free graph with vertex set  $V$ . Let  $x_0 \in V$  and let  $\Delta$  be a special subset of  $V$  (with respect to  $x_0$ ). Then  $\text{Aut}(X)_\Delta$  fixes  $x_0$ .*

*Proof.* Let  $Z = \{v \in V \mid B_1(v) \cap \Delta = \emptyset\}$ . Then  $x_0 \in Z \subseteq B_1(x_0)$ . Moreover, for each  $z \in Z$  we have  $B_1(z) \subseteq B_1(x_0)$ . Since  $X$  is twin-free, we in fact have  $B_1(z) \subset B_1(x_0)$ . Consequently  $x_0$  is the unique vertex in  $Z$  adjacent to all other vertices in  $Z$ .  $\square$

Using Theorem 1.2 we shall show the following.

**Theorem 5.6.** *Let  $X$  be a locally finite connected graph with infinite motion and let  $x_0$  be a vertex. Then  $X$  admits an asymmetric special subset w. r. t.  $x_0$ .*

Note that here we do not make it an assumption that all automorphisms fix  $x_0$ . The automorphisms that fix a special subset will automatically fix  $x_0$  by Lemma 5.5.

For a locally finite connected rooted graph  $X = (V, E, x_0)$  we use the following notation.

Let  $\mathcal{G} = \text{Aut}(X)$ . Here we made  $x_0$  a constant in the language of the structure  $X$ , so this group fixes  $x_0$  by definition. Consequently,  $\mathcal{G}$  also fixes each sphere about  $x_0$  (setwise). Let  $B_i = B_i(x_0)$  and  $S_i = S_i(x_0)$ . Let  $H_i$  denote the restriction of  $\mathcal{G}$  to  $B_i$  and  $G_i$  the restriction of  $\mathcal{G}$  to  $S_i$  (so  $H_i \leq \text{Sym}(B_i)$  and  $G_i \leq \text{Sym}(S_i)$ ). For  $i \in I$  let  $\pi_i : H_i \rightarrow G_i$  denote the restriction from  $B_i$  to  $S_i$  (an epimorphism) and for  $i \leq j$  let  $\psi_{i,j} : H_j \rightarrow H_i$  denote the restriction from  $B_j$  to  $B_i$  (again, an epimorphism).

**Lemma 5.7.** *Let  $X$  be a locally finite connected rooted graph with infinite motion. Then each restriction epimorphism  $\pi_i$  (from the ball  $B_i$  to the sphere  $S_i$ ) is an isomorphism.*

*Proof.* Let  $\sigma \in \ker \pi_i$ . So  $\sigma$  acts on  $B_i$  and fixes  $S_i$  pointwise. Let  $\hat{\sigma}$  denote the extension of  $\sigma$  to  $V$  obtained by fixing all vertices in  $V \setminus B_i$ . So  $\hat{\sigma}$  is an automorphism of  $X$  that fixes all vertices outside the finite set  $B_i$  and therefore, by the infinite motion assumption, it fixes all vertices in  $B_i$  as well, so  $\sigma = 1$ .  $\square$

*Proof of Theorem 5.6 from Theorem 1.2.* By Lemma 5.7, we can define, for  $i \leq j$ , the  $G_j \rightarrow G_i$  epimorphism  $\varphi_{i,j} = \pi_j \psi_{i,j} \pi_i^{-1}$ . Let  $\mathcal{G}$  denote the inverse limit of the system  $(G_i, \varphi_{i,j})_{2\mathbb{N}+3}$ . Recall that the domain of  $G_i$  is the sphere  $S_i$ , so the domains of these permutation groups are disjoint.

Let  $\hat{\mathcal{G}}$  denote the restriction of  $\mathcal{G}$  to the set  $A = \bigsqcup \{S_i \mid i \in 2\mathbb{N} + 3\}$ . By Theorem 1.2, the group  $\hat{\mathcal{G}}$  admits an asymmetric subset  $\Gamma$ . Let now  $\Delta = \Gamma \cup \bigcup \{S_j \mid j \in 2\mathbb{N} + 2\}$ . Then  $\Delta$  is a special subset and  $(\mathcal{G})_\Delta$  fixes  $A$  pointwise. But then the epimorphisms  $\varphi_{i,j}$  ensure that all of  $V$  is fixed pointwise.  $\square$

## 6. APPROXIMATION PROCESS

We say that a class  $\mathcal{Q}$  of groups is *HS-closed* if  $\mathcal{Q}$  is closed under subgroups and homomorphic images. (This includes being closed under isomorphisms.)

**Observation 6.1.** *Let  $(G_i, \varphi_{i,j})_I$  be an epimorphic system of finite groups. If  $\mathcal{Q}$  is an HS-closed class and infinitely many of the  $G_i$  belong to  $\mathcal{Q}$  then all of them belong to  $\mathcal{Q}$ .  $\square$*

**Definition 6.2** (Color-reduction between classes of permutation groups). Let  $\mathcal{Q}$  and  $\mathcal{R}$  be HS-closed classes of finite permutation groups. We say that  $\mathcal{Q}$  is *k-color-reducible* to  $\mathcal{R}$  if the following holds for all pairs of permutation groups  $G, H \in \mathcal{Q}$ . If  $\varphi : H \rightarrow G$  is an epimorphism and  $G \notin \mathcal{R}$  then there exists a  $k$ -coloring  $\gamma$  for  $H$  such that  $\varphi(H_\gamma) < G$ .

**Lemma 6.3** (Color-reduction of inverse limits). *Let  $\mathcal{Q}$  and  $\mathcal{R}$  be HS-closed classes of finite groups. Assume  $\mathcal{Q}$  is 2-color-reducible to  $\mathcal{R}$ . Let  $(G_i, \varphi_{i,j})_I$  be an epimorphic inverse system of finite permutation groups  $G_i \in \mathcal{Q}$ ; let  $\Omega_i$  denote the permutation domain of  $G_i$ . Assume the  $\Omega_i$  are disjoint. Let  $\mathcal{G} = \varprojlim G_i$ . Let  $J$  be an infinite subset of  $I$ . Then there exists a subset  $\Delta \subseteq \bigcup_{i \in J} \Omega_i$  such that for all  $i \in I$  we have  $\pi_i(\mathcal{G}_\Delta) \in \mathcal{R}$ .*

*Proof.* For a subset  $K \subseteq I$  let us use the notation  $\Omega(K) := \bigcup_{i \in K} \Omega_i$ . For  $i \in I$ , in increasing order, we shall inductively designate a finite subset  $J_i \subset J$  and a subset  $\Delta_i \subseteq \Omega(J_i)$  such that the  $J_i$  are disjoint,  $i < j$  for all  $j \in J_i$ , and  $\pi_i(\mathcal{G}_{\Delta_i}) \in \mathcal{R}$ . It is clear then, that  $\Delta := \bigcup_{i \in I} \Delta_i$  accomplishes our goal.

Suppose the  $J_\ell$  have already been constructed for  $\ell < i$ . Let  $K_i$  be the complement in  $J$  of the set  $\{t \in J \mid t \leq i\} \cup \bigcup \{J_\ell \mid \ell < i\}$ . We perform the following algorithm to construct  $J_i$  and  $\Delta_i$ . The algorithm will gradually reduce  $G_i$  until it becomes an element of  $\mathcal{R}$ , using the 2-color-reducibility of  $\mathcal{Q}$  to  $\mathcal{R}$ . The variable  $F$  stores the current group  $G_i$ . The next( $K_i, m$ ) operation produces the smallest element of  $K_i$  that is greater than  $m$ .

```

01   $F := G_i$ 
02   $J_i := \emptyset$ 
03   $m := 0$ 
04  while  $F \notin \mathcal{R}$ 
05       $m := \text{next}(K_i, m)$ 
06       $J_i := J_i \cup \{m\}$ 
07      let  $\Psi_m \subseteq \Omega_m$  such that
08           $\varphi_{im}((\varphi_{im}^{-1}(F))_{\Psi_m}) < F$ 
09  end(while)
10   $\Delta_i := \bigcup_{j \in J_i} \Psi_j$ 
11  return  $J_i$  and  $\Delta_i$ 

```

Explanation of lines 07–08. Both  $G_i$  and  $G_m$  belong to  $\mathcal{Q}$ , and therefore their subgroups  $F$  and  $\varphi_{im}^{-1}(F)$ , resp., also belong to  $\mathcal{Q}$ . The restriction of the epimorphism  $\varphi_{im} : G_m \rightarrow G_i$  to  $\varphi_{im}^{-1}(F)$  is an epimorphism from  $\varphi_{im}^{-1}(F)$  onto  $F$ . Therefore, if  $F \notin \mathcal{R}$ , by the 2-color-reducibility of  $\mathcal{Q}$  to  $\mathcal{R}$  there exists  $\Psi_m \subseteq \Omega_m$  as required in line 08.

Since  $F$  is reduced in every round of the **while**-loop, the process terminates in a finite number of steps, and on termination,  $F \in \mathcal{R}$ , as desired.  $\square$

We shall use Lemma 6.3 in each successive step in a chain of HS-closed classes which we now list.

- $\mathcal{G}\mathcal{r} = \{ \text{all finite groups} \}$
- $\mathcal{S}\mathcal{o}\mathcal{l} = \{ \text{all finite solvable groups} \}$
- $\mathcal{D}\mathcal{e}\mathcal{r}_k = \{ \text{all finite solvable groups of derived length } \leq k \}$

For some constant  $k_0$  we shall descend along the chain

$$(1) \quad \mathcal{G}\mathcal{r} \supset \mathcal{S}\mathcal{o}\mathcal{l} \supset \mathcal{D}\mathcal{e}\mathcal{r}_{k_0} \supset \mathcal{D}\mathcal{e}\mathcal{r}_{k_0-1} \supset \cdots \supset \mathcal{D}\mathcal{e}\mathcal{r}_1 \supset \mathcal{D}\mathcal{e}\mathcal{r}_0.$$

Note that  $\mathcal{D}\mathcal{e}\mathcal{r}_0$  consists only of the trivial group, so once that class has been reached, we have found an asymmetric coloring.

So we have reduced the proof of Tucker's Conjecture to the following result.

**Theorem 6.4.** *There exists a positive integer  $k_0$  such that the following holds. Let  $\mathcal{Q} \supset \mathcal{R}$  be a pair of consecutive terms in the chain (1). Then  $\mathcal{Q}$  is 2-color-reducible to  $\mathcal{R}$ .*

The rest of the paper describes the proof of this result.

## 7. DEFINITIONS, NOTATION: GROUP THEORY

For the rest of this paper, **all groups will be finite**, except where expressly stated otherwise.

We use the notation  $[n] = \{1, \dots, n\}$  for integers  $n \geq 0$ .

For groups  $G, H$ , the notation  $H \leq G$  indicates that  $H$  is a subgroup, and  $H < G$  indicates a proper subgroup. The notation  $N \triangleleft G$  indicates a (not necessarily proper) normal subgroup. An *epimorphism* (surjective homomorphism) from  $G$  onto  $H$  is indicated as  $G \twoheadrightarrow H$ . For  $K \subseteq G$  a subset of the group  $G$  we denote the centralizer of  $K$  in  $G$  by  $\mathbb{C}_G(K)$ . The center of  $G$  is  $Z(G) = \mathbb{C}_G(G)$ .

Let  $G$  be a group. The *commutator* of  $h, k \in G$  is the element  $[h, k] = h^{-1}k^{-1}hk$ . If  $H, K \leq G$  then  $[H, K]$  denotes the subgroup generated by all commutators  $[h, k]$  for  $h \in H, k \in K$ . The *commutator subgroup* or *derived subgroup* of the group  $G$  is  $[G, G]$ , also denoted  $G'$ . The members of the *derived series* are denoted  $G^{(k)}$  where  $G^{(0)} = G$  and  $G^{(k+1)} = (G^{(k)})'$ . A group  $G$  is *perfect* if  $G' = G$ . Every finite group  $G$  contains a unique largest perfect subgroup, called the *perfect core* of  $G$ , reached when the derived series stabilizes:  $G^{(k+1)} = G^{(k)}$ . We denote the perfect core of  $G$  by  $G^{(\infty)}$ . The group  $G$  is solvable if and only if its perfect core is the identity.

The *socle* of a group  $G$ , denoted  $\text{Soc}(G)$ , is the product of its minimal normal subgroups.

An *almost simple* group is a group  $G$  of the form  $N \triangleleft G \leq \text{Aut}(N)$  where  $N$  is a nonabelian simple group. In this case,  $N$  is the unique minimal normal subgroup of  $G$  and therefore it is the socle of  $G$ .

The group  $\text{Inn}(G)$  of inner automorphisms of  $G$  consists of the conjugations by elements of  $G$ . The *outer automorphism group* of a group  $G$  is the quotient  $\text{Out}(G) = \text{Aut}(G)/\text{Inn}(G)$ . *Schreier's Hypothesis* states that the outer automorphism groups of all finite simple groups are solvable. This is equivalent to saying that the perfect core of an almost simple group is simple. Schreier's Hypothesis is a known consequence of the Classification of Finite Simple Groups.

For a group action  $G \curvearrowright \Omega$  (see Def. 3.3) we usually reserve the letter  $n$  for  $|\Omega|$ , the *degree* of the action. For additional definitions and notation about group actions, see Section 3.

## 8. SOLVABLE COLORINGS OF PRIMITIVE GROUPS

In this section we build one of our main tools for the proof of Tucker’s conjecture.

**Definition 8.1.** For a permutation group  $G \leq \text{Sym}(\Omega)$ , let  $\text{solv}(G)$  denote the minimum number  $k$  of colors such that  $G$  admits a solvable  $k$ -coloring. We call  $\text{solv}(G)$  the *solvable coloring number* of  $G$ .

**Observation 8.2.** Let  $G, H \leq \text{Sym}(\Omega)$ .

- (a) If  $H \leq G$  then  $\text{solv}(H) \leq \text{solv}(G)$ .
- (b) Let the orbits of  $G$  be  $\Omega_1, \dots, \Omega_k$  and let  $G_i$  be the restriction of  $G$  to  $\Omega_i$ . Then  $\text{solv}(G) = \max_i \text{solv}(G_i)$ .
- (c) For  $x \in \Omega$ , let  $G(x)$  denote the action of the stabilizer  $G_x$  on  $\Omega \setminus x$ . Then  $\text{solv}(G(x)) \leq \text{solv}(G)$ .

*Proof.* (a) A solvable coloring for  $G$  is also a solvable coloring for any subgroup of  $G$ . (b) We can color each orbit independently, observing that  $G$  is a subdirect product of the  $G_i$ . (c) First,  $\text{solv}(G_x) \leq \text{solv}(G)$  by (a). Second,  $\text{solv}(G(x)) = \text{solv}(G_x)$  by (b).  $\square$

**Remark 8.3.** This observation would remain true if we replaced “solvable groups” by any PS-closed class of groups (closed under direct products and subgroups) such as nilpotent groups, 2-groups, groups with composition factors of order  $\leq c$  or solvable groups of derived length  $\leq c$  for some constant  $c$ .

**Theorem 8.4** (Solvable coloring number of primitive groups). Let  $G \leq \text{Sym}(\Omega)$  be a primitive permutation group of degree  $n = |\Omega|$ .

- (I) If  $G$  is solvable then  $\text{solv}(G) = 1$ .
- (II) If  $\text{Alt}(\Omega) \leq G \leq \text{Sym}(\Omega)$  then  $\text{solv}(G) = \lceil n/4 \rceil$ .
- (III) In all other<sup>1</sup> cases,  $\text{solv}(G) = 2$ .

Items (I) and (II) are straightforward. (Regarding (II), each color must occur at most 4 times.) We discuss the Mathieu groups in Sec. 8.3. The bulk of Section 8 concerns the proof of item (III).

It was shown by Cameron, Neumann, and Saxl [CNS84] in 1984 by a simple counting argument that all but a finite number of primitive groups admit an asymmetric 2-coloring. Gluck [Gl83] (1983) and Seress [Se97] (1997) classified the exceptions: Gluck for the case when  $G$  is solvable and Seress for the non-solvable case. Seress gives their combined list of 43 groups [Se97, Theorem 2]. We shall rely on Seress’s list of non-solvable exceptions.

**Remark 8.5.** For the proof of our main result, solvable colorings of almost simple primitive groups are not required. We include this part of the result for completeness. In particular, the hard part of Seress’s result, the classification of those almost simple primitive groups that admit an asymmetric 2-coloring, is not required.

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<sup>1</sup>In a previous version of this paper, I listed  $M_{24}$  as a possible exception. I am grateful to Saveliy Skresanov [Sk21] for pointing out that the case of  $M_{24}$  was settled by Chang Choi in 1972 [Ch72]. In the previous version I proved  $\text{solv}(M_{24}) \leq 3$ , which suffices for our main results.

We begin with a straightforward but useful observation.

**Observation 8.6.** *Let  $H \leq \text{Sym}(\Omega)$  be a permutation group and  $\Phi \subseteq \Omega$  an  $H$ -invariant subset such that (a) the image of the action  $H \curvearrowright \Phi$  is solvable and (b) the pointwise stabilizer  $H_{(\Phi)}$  is solvable. Then  $H$  is solvable. In particular, condition (a) is met if  $|\Phi| \leq 4$ .  $\square$*

First we consider three particular classes of primitive groups: affine groups, the projective linear groups, and the Mathieu groups. Our proof for these classes is self-contained and does not rely on Seress's work. (See the footnote for  $M_{24}$ .)

### 8.1. Affine groups.

**Proposition 8.7.** *Let  $G$  be a primitive permutation group with an elementary abelian normal subgroup. Then  $G$  admits a solvable 2-coloring.*

*Proof.* We need to find a subset  $\Delta \subseteq \Omega$  such that  $G_\Delta$  is solvable.

Let  $N$  be the elementary abelian normal subgroup; so  $n = |N| = |\Omega| = p^d$  for some prime  $p$  and  $d \geq 1$ .  $\Omega$  can be viewed as the  $d$ -dimensional vector space over  $\mathbb{F}_p$ , and  $G \leq \text{AGL}(d, p)$  (the affine group acting on  $\Omega$ ).

If  $d = 1$  then  $G$  is solvable, so  $\Delta = \emptyset$  will do.

Let now  $d \geq 2$ . Let  $e_0 = 0$  and let  $e_1, \dots, e_d$  be a basis of  $\Omega$ .

If  $d = 2$  then let  $\Delta = \{e_0, e_1\}$ . The pointwise stabilizer  $G_{(\Delta)}$  consists of linear transformations of  $\Omega$ , described by triangular matrices (in the basis  $\{e_1, e_2\}$ ), hence  $G_{(\Delta)}$  is solvable. An application of Obs. 8.6 shows that  $G_\Delta$  is solvable.

For  $d \geq 3$  we observe that  $G$  preserves affine relations, i. e., relations of the form  $\sum \alpha_i x_i = 0$  where  $x_i \in \Omega$ ,  $\alpha_i \in \mathbb{F}_p$ , and  $\sum \alpha_i = 0$ .

By a quadruple we shall mean a set of four elements. We shall say that quadruple  $Q \subseteq \Omega$  satisfies the equation  $f(x_1, x_2, x_3, x_4) = 0$  if there is a bijection  $\beta : \{1, 2, 3, 4\} \rightarrow Q$  such that  $f(\beta(1), \beta(2), \beta(3), \beta(4)) = 0$ .

If  $3 \leq d \leq 6$  then let  $\Delta = \{e_0, e_1, e_2, e_1 + e_2, e_3, \dots, e_d\}$ . There is exactly one quadruple of elements of  $\Delta$  satisfying the equation  $x_1 + x_2 - x_3 - x_4 = 0$ , namely,  $\{e_1, e_2, e_0, e_1 + e_2\}$ . This means  $G_\Delta$  fixes this quadruple (setwise) and also the set  $\{e_3, \dots, e_d\}$  (setwise). On the other hand,  $G_{(\Delta)} = 1$ . So an application of Obs. 8.6 to  $G_\Delta$  shows that  $G_\Delta$  is solvable.

Assume  $d \geq 7$  and let  $k = \lfloor (d-1)/2 \rfloor$ . Let  $\Delta = \{(-1)^{i+1}e_i \mid 0 \leq i \leq d\} \cup \{e_{2i-1} + e_{2i} + e_{2i+1} \mid 1 \leq i \leq k\}$ . Consider the set  $\mathcal{H}$  of those quadruples in  $\Delta$  that satisfy the equation  $x_1 - x_2 + x_3 - x_4 = 0$ . These are exactly the quadruples  $Q_i := \{e_{2i-1}, -e_{2i}, e_{2i+1}, e_{2i-1} + e_{2i} + e_{2i+1} \mid 1 \leq i \leq k\}$ .

Let us now consider the graph with vertex set  $\Delta$  where two vertices are adjacent if there is a quadruple in  $\mathcal{H}$  in which both of them participate. This graph is invariant under  $G_\Delta$ . The graph has one or two isolated vertices ( $e_0$  and, if  $d$  is even,  $e_d$ ) and otherwise consists of a chain of 4-cliques, each one sharing one vertex with the next one. The automorphism group of this graph is easy to determine; it has a normal 2-subgroup of index 9, therefore it is solvable. On the other hand,  $G_{(\Delta)} = 1$ , so  $G_\Delta$  acts faithfully on our graph and therefore it is solvable.  $\square$

## 8.2. Projective linear groups.

**Proposition 8.8.** *For  $d \geq 2$  and a prime power  $q$ , the projective linear group  $L_d(q)$  in its natural action on the  $(d - 1)$ -dimensional projective space over  $\mathbb{F}_q$  admits a solvable 2-coloring.*

*Proof.* For  $d = 2$ , the stabilizer of any point is the 1-dimensional affine group, which is solvable.

For  $d \geq 3$ , let  $e_i$  denote the standard unit vectors in  $\mathbb{F}_q^d$  (the  $i$ -th coordinate 1, the other coordinates 0). For  $v \in \mathbb{F}_q^d \setminus \{0\}$  we write  $[v] = \{\lambda v \mid \lambda \in \mathbb{F}_q^\times\}$  for the equivalence class representing a point in  $PG(d - 1, q)$  by its homogeneous coordinates.

For  $d = 3$ , let  $\Delta = \{[e_1], [e_2], [e_3], [e_1 + e_2 + e_3]\}$ . The pointwise stabilizer of  $\Delta$  in  $N := L_3(q)$  is the identity; therefore, the setwise stabilizer is  $N_\Delta \leq \text{Sym}(\Delta)$  which is solvable.

For  $d \geq 4$ , let  $\Delta = \{[e_i] \mid 1 \leq i \leq d\} \cup \{[e_i + e_{i+1}] \mid 1 \leq i \leq d - 1\} \cup \{[f]\}$  where  $f = \sum_{i=1}^d e_i$ . The pointwise stabilizer of  $\Delta$  is the identity, so we only need to consider what permutations of  $\Delta$  are feasible under  $L_d(q)$ . The action of  $L_d(q)$  preserves the underlying matroid (i. e., it maps linearly independent sets to linearly independent sets).

The only linearly dependent triples in  $\Delta$  are the triples of the form  $\{[e_i], [e_{i+1}], [e_i + e_{i+1}]\}$ , and in the case of  $d = 4$ , the triple  $\{[e_1 + e_2], [e_3 + e_4], [f]\}$ .

So the 3-uniform hypergraph  $\mathcal{H}$  formed by these triples is preserved by  $L_d(q)$ . Let us find the degree of each element of  $\Delta$  in this hypergraph (i. e., how many times each element of  $\Delta$  appears in these triples). Let  $\Delta_j = \{u \in \Delta \mid \deg_{\mathcal{H}}(u) = j\}$ . Each  $\Delta_i$  is fixed (setwise) by  $N_\Delta$ .

For  $d = 4$  we have  $\Delta_1 = \{[e_1], [e_4], [e_2 + e_3], [f]\}$  and  $\Delta_2 = \{[e_2], [e_3], [e_1 + e_2], [e_3 + e_4]\}$ . Therefore  $N_\Delta \leq \text{Sym}(\Delta_1) \times \text{Sym}(\Delta_2)$ , solvable.

For  $d \geq 5$  we have  $\Delta_0 = \{[f]\}$ ,  $\Delta_1 = \{[e_1], [e_d]\} \cup \{[e_i + e_{i+1}] \mid 1 \leq i \leq d - 1\}$ , and  $\Delta_2 = \{[e_2], \dots, [e_{d-1}]\}$ . Let us define the graph  $R$  on vertex set  $\Delta$  by making  $u, v \in \Delta$  adjacent if  $u \neq v$  and  $\{u, v\}$  is a subset of a triple in  $\mathcal{H}$ . The induced subgraph  $R[\Delta_2]$  is the path  $[e_2] - \dots - [e_{d-1}]$ , which has only 2 automorphisms, so the pointwise stabilizer of  $\Delta_2$  has index  $\leq 2$  in  $N_\Delta$ . Moreover, this pointwise stabilizer also fixes each point that has two neighbors in this path, so it can only swap the pair  $([e_1], [e_1 + e_2])$  and the pair  $([e_d], [e_{d-1} + e_d])$ . In summary, the order of  $N_\Delta$  divides 8, so  $N_\Delta$  is solvable.  $\square$

## 8.3. Mathieu groups.

**Proposition 8.9** (Choi). *Each of the five Mathieu groups admits a solvable 2-coloring.*

*Proof.* We give a simple direct proof in the cases other than  $M_{24}$  and refer Choi [Ch72] for  $M_{24}$ .

Let  $G$  be one of the Mathieu groups  $M_{23}$ ,  $M_{22}$ ,  $M_{12}$  and  $M_{11}$ . So we can write  $G = M_{m+k}$  where  $m \in \{10, 21\}$  and  $k = 1, 2$ . Let  $G$  act on  $\Omega$  where  $|\Omega| = m + k$ . Let  $\Delta \subset \Omega$  be a set of  $2 + k$  elements, so  $|\Delta| \leq 4$ . Then the order of the pointwise stabilizer of  $\Delta$  is  $|G_{(\Delta)}| = 48$  if  $m = 21$  and 8 if  $m = 10$ . Therefore  $G_{(\Delta)}$  is solvable, so by Obs. 8.6,  $G_\Delta$  is solvable.

The remaining case,  $G = M_{24}$ , was settled by Chang Choi in 1972 [Ch72]. Choi classified all setwise stabilizers of  $M_{24}$ . He found a set of size 8 he denotes by

$8'''$  such that  $G_{8'''}$  has an elementary abelian normal subgroup of order 16 with quotient  $S_4$  [Ch72, Prop. 4.1]. He also found a set of size 10 he denotes by  $10'''$  such that  $G_{10'''}$   $\cong S_3 \times S_4$  [Ch72, Prop. 6.3].  $\square$

**Remark 8.10.** A set equivalent to  $10'''$  was found by Saveliy Skresanov<sup>2</sup> [Sk21] using the GAP computer algebra system, thus providing independent verification of the fact that  $\text{solv}(M_{24}) = 2$ . Saveliy kindly agreed that I share his code.

```
gap> G := MathieuGroup(24);
Group([ (1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23),
(3,17,10,7,9)(4,13,14,19,5)(8,18,11,12,23)(15,20,22,21,16), (1,24)(2,23)
(3,12)(4,16)(5,18)(6,10)(7,20)(8,14)(9,21)(11,17)(13,22)(15,19) ])
gap> S := [1..10];
[ 1 .. 10 ]
gap> StructureDescription(Stabilizer(G, S, OnSets));
"S4 x S3"
```

**Remark 8.11.** Of course  $\text{solv}(M_{24}) = 2$  implies  $\text{solv}(M_{23}) = \text{solv}(M_{22}) = 2$  (see Obs. 8.2), so those observations should also be attributed to Choi.

Let us note that  $\text{solv}(M_{23}) = 2$  immediately implies  $\text{solv}(M_{24}) \leq 3$ , which suffices for our main results. Indeed, even the weaker statement  $\text{solv}(M_{24}) \leq 5$  would suffice: the only place in the proof of Theorem 10.4 (and consequently in the proof of Theorem 1.1) where a bound on  $\text{solv}(M_{24})$  is used is Case 2a of the proof of Theorem 10.4. So the main results do not depend on Choi's classification theorem.

**8.4. Proof of Theorem 8.4.** We refer to Seress's list of primitive groups that do not admit an asymmetric 2-coloring [Se97, Theorem 2]. Seress lists 43 groups (this includes Gluck's list of solvable exceptions); we organize the list into four categories. We write  $n = |\Omega|$  for the degree of  $G$ .

**Theorem 8.12** (Seress). *Let  $G \leq \text{Sym}(\Omega)$  be primitive,  $G \not\cong \text{Alt}(\Omega)$ . Assume  $G$  does not admit an asymmetric 2-coloring. Then  $n \leq 32$  and  $G$  falls into one of the following categories.*

- (a)  $G$  is solvable,
- (b)  $G$  is an affine group, i. e.,  $G$  has an elementary abelian normal subgroup (so  $n$  is a prime power),
- (c)  $G$  has degree  $n \leq 8$ ,
- (d)  $G$  is almost simple (so  $N \leq G \leq \text{Aut}(N)$  for some nonabelian simple group  $N$ ).

In Seress's list, Case (d) falls into the following subcategories. The list indicates each group  $G$  by the pair  $(n, N)$  where  $n$  is the degree of  $G$  (size of  $|\Omega|$ ) and  $N$  is the (unique, simple) minimal normal subgroup of  $G$ .

- (i) Mathieu groups in their natural action,  $(n, M_n)$  for  $n = 11, 12, 22, 23, 24$
- (ii) projective linear groups  $L_d(q)$  for some pairs  $(d, q)$  where  $d \geq 2$  and  $q$  is a prime power, in their natural action on the  $(d - 1)$ -dimensional projective space over  $\mathbb{F}_q$  ( $n = (q^d - 1)/(q - 1)$ )
- (iii)  $(10, A_5)$ ,  $(10, A_6)$ ,  $(12, M_{11})$ ,  $(15, A_8)$ .

*Proof of Theorem 8.4.*

Given a primitive group  $G \leq \text{Sym}(\Omega)$  such that  $G \not\cong \text{Alt}(\Omega)$ , we need to show that

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$G$  admits a solvable 2-coloring, except if  $G = M_{24}$  in its natural action then we provide a solvable 3-coloring.

Case (o). If  $G$  admits an asymmetric 2-coloring, that is more than sufficient for us. Now we need to eliminate the finite number of exceptions.

In Case (a), the constant coloring ( $\Delta = \emptyset$ ) works.

Case (b). This case was settled in Sec. 8.1.

Case (c). Now  $5 \leq |\Omega| \leq 8$ . Let  $\Delta$  be any 4-subset of  $\Omega$ . Then, applying Obs. 8.6 to  $H := G_\Delta$ , it follows that  $G_\Delta$  is solvable.

Case (d). Now  $G$  is almost simple, so it has a unique minimal normal subgroup  $N$  which is nonabelian simple. In particular, the quotient  $G/N \leq \text{Out}(N)$  is solvable by Schreier's hypothesis. It follows that it suffices to find  $\Delta \subseteq \Omega$  such that  $M := N_\Delta$  is solvable. Indeed, let  $L := G_\Delta$ . Now  $L$  is solvable because  $L/(L \cap N)$  is a subgroup of  $G/N$ .

Subcase (i) was settled in Section 8.3.

Subcase (ii). In Section 8.2 we have shown that all projective linear groups, in their natural action, admit solvable 2-colorings.

Subcase (iii).

$A_5$  is a minimal simple group, so all proper subgroups are solvable; we can take any nontrivial subset of  $\Omega$  for  $\Delta$ .

For  $(10, A_6)$ , the stabilizer of a point has order 36 and is therefore solvable.

$A_8 \cong L_4(2)$ , and Seress's example  $(15, A_8)$  is the standard action of  $L_4(2)$  on  $PG(3, 2)$ , covered under Subcase (ii).

In the case of  $G \cong M_{11}$  acting as a primitive group on a set of size  $|\Omega| = 12$ , take any four elements,  $x, u, v, w \in \Omega$  such that the successive pointwise stabilizers strictly decrease:  $G > G_x > G_{(x,u)} > G_{(x,u,v)} > G_{(x,u,v,w)}$ . Let  $\Delta = \{x, u, v, w\}$ . Now  $|G_x| = |G|/12 = 660 = 2^2 \cdot 3 \cdot 5 \cdot 11$ . At each subsequent step, the order drops by at least a prime factor, in total by at least a factor of  $2 \cdot 2 \cdot 3 = 12$ , so the order of the pointwise stabilizer of  $\Delta$  is  $|G_{(\Delta)}| \leq 660/12 = 55$ . Therefore  $G_{(\Delta)}$  is solvable. By Obs. 8.6, it follows that  $G_\Delta$  is solvable.  $\square$

## 9. PROOF OF THEOREM 1.3 FOR PRIMITIVE GROUPS

We restate this case.

**Lemma 9.1.** *Let  $G \leq \text{Sym}(\Omega)$  be a primitive group,  $T$  a nonabelian simple group, and  $\varphi : G \rightarrow T$  an epimorphism. Then there exists a subset  $\Delta \subseteq \Omega$  such that  $\varphi(G_\Delta) < T$ .*

*Proof.* Case 1.  $G$  is almost simple. In this case we claim that for any nontrivial subset  $\Delta \subseteq \Omega$  ( $\Delta \neq \emptyset$  and  $\Delta \neq \Omega$ ) we have  $\varphi(G_\Delta) < T$ .

Indeed, in this case  $G$  has a unique minimal normal subgroup  $N$  which is nonabelian simple. In particular, the quotient  $G/N = \text{Out}(N)$  is solvable by Schreier's hypothesis. Let  $\ker(\varphi) = K$ . Since  $G/K \cong T$ , it follows that  $K \not\leq N$  and therefore  $K = 1$ , hence  $G = N \cong T$ . So for any nontrivial  $\Delta$  we have  $|G_\Delta| < |G| = |T|$  and therefore  $\varphi(G_\Delta) < T$ .

Case 2.  $G$  is not almost simple. In particular,  $G \not\cong \text{Alt}(\Omega)$ . In this case the result is an immediate consequence of Theorem 8.4. Indeed, now we are in Case (III) of Theorem 8.4, so  $G$  admits a solvable 2-coloring, i. e., there is a subset  $\Delta \subseteq \Omega$  such that  $G_\Delta$  is solvable, and therefore  $G_\Delta$  has no epimorphism on  $T$ .  $\square$

**Remark 9.2.** Note that while this proof rests on Seress's classification of the primitive groups that do not admit an asymmetric 2-coloring, it avoids any reference to the most difficult part of Seress's work, the classification of the almost simple groups.

## 10. REDUCING THE IMAGE: PROOF OF THEOREM 1.3

10.1. **Three lemmas.** The following lemma may be folklore. It appears as [Ba16, Lemma 8.1.1] along with an elegant proof, supplied by Péter P. Pálffy and reproduced below for completeness. As remarked there, the result can also be derived from [Mei95, Lemma 2.8].

**Lemma 10.1** (Subdirect product lemma). *Let  $G$  be a subdirect product of the finite groups  $H_i$  ( $i = 1, \dots, r$ ). Let  $\pi_i : G \rightarrow H_i$  be the corresponding projections. Let  $\varphi : G \rightarrow T$  be an epimorphism, where  $T$  is a nonabelian simple group. Let  $K = \ker(\varphi)$  and  $M_i = \ker(\pi_i)$ . Then  $(\exists i)(M_i \leq K)$ .*

*Proof by Péter P. Pálffy.* For subgroups  $G_1, \dots, G_k \leq G$  we use the notation

$$[G_1, \dots, G_k] = [\dots [[G_1, G_2], G_3], \dots, G_k].$$

Assume for a contradiction that  $K \not\leq M_i$  for all  $i$ . Then  $M_i K = G$  (because  $K$  is a maximal normal subgroup). It follows that  $[G, \dots, G] = [M_1 K, \dots, M_m K] \leq K[M_1, \dots, M_m] \leq K(\bigcap_{i=1}^m M_i) = K$ , so  $[G/K, \dots, G/K] = 1$ , a contradiction because  $G/K \cong T$  is nonabelian simple.  $\square$

**Observation 10.2** (Perfect core lemma). *Let  $\varphi : G \rightarrow H$  be an epimorphism of finite groups. Then  $\varphi(G') = H'$ . Consequently,  $\varphi(G^{(\infty)}) = H^{(\infty)}$ .*  $\square$

**Lemma 10.3** (Three normal subgroups lemma). *Let  $H$  be a finite group and  $A, B, C$  three normal subgroups satisfying the following conditions.*

- (i)  $AB = AC = BC = H$ .
- (ii)  $H/A$  and  $H/B$  are nonabelian simple.

*Then  $B \not\leq A \cap C$ .*

*Proof.* Let  $S = H/A$  and  $T = H/B$ . So  $S$  and  $T$  are nonabelian simple groups.

Without loss of generality we may assume  $A \cap B \cap C = 1$ .

Assume for a contradiction that  $B \geq A \cap C$ . So  $1 = A \cap B \cap C = A \cap C$ .

Since  $A \cap C = 1$  and  $AC = H$ , we have  $H = A \times C$ . Therefore  $C \cong H/A \cong S$  and  $A \leq \mathbb{C}_H(C)$ .

We claim that  $B \cap C = 1$ . Indeed, given that  $C \cong S$  is simple, the alternative would be  $C \leq B$ . But this would mean  $H = BC = B$ , impossible by Assumption (ii).

Since  $B \cap C = 1$ , we have  $B \leq \mathbb{C}_H(C)$ .

Combining this with  $A \leq \mathbb{C}_H(C)$  we obtain that  $H = AB \leq \mathbb{C}_H(C)$ , i. e.,  $C \leq Z(H)$ . But then  $C$  must be abelian, contradicting the isomorphism  $C \cong S$ .  $\square$

### 10.2. The proof of Theorem 1.3.

We restate and augment Theorem 1.3.

**Theorem 10.4.** *Let  $G \leq \text{Sym}(\Omega)$ , where  $\Omega$  is a finite set. Let  $\varphi : G \rightarrow T$  be an epimorphism where  $T$  is a nonabelian simple group. Then  $(\exists \Delta \subseteq \Omega)(\varphi(G_\Delta) < T)$ . Moreover,  $\Delta$  can be chosen to be a subset of one of the orbits of  $G$ .*

*Proof.* Let  $n = |\Omega|$ . We fix  $T$  and proceed by induction on  $n$ . The statement is vacuously true if  $G$  is solvable; in particular, if  $n \leq 4$ . Assume now  $n \geq 5$ .

Let  $K = \ker(\varphi)$ .

1. First assume  $G$  is intransitive, with orbits  $\Omega_1, \dots, \Omega_r$ . Let  $H_i$  denote the restriction of  $G$  to  $\Omega_i$ , so  $H_i \leq \text{Sym}(\Omega_i)$  and  $G$  is a subdirect product of the  $H_i$ . Let  $M_i$  denote the kernel of the epimorphism  $\pi_i : G \rightarrow H_i$ , i.e., the pointwise stabilizer of  $\Omega_i$  in  $G$ . By the Subdirect product lemma (Lemma 10.1), there is an  $i \leq r$  such that  $M_i \leq K$ . This means that  $\varphi$  induces an epimorphism  $\bar{\varphi} : H_i \rightarrow T$ . Note that  $\varphi$  is the composition of  $\pi_i$  and  $\bar{\varphi}$ :

$$(2) \quad \begin{array}{ccc} G & \xrightarrow{\varphi} & T \\ & \searrow \pi_i & \nearrow \bar{\varphi} \\ & & H_i \end{array}$$

By induction on  $n$ , we find a subset  $\Delta \subseteq \Omega_i$  such that  $\bar{\varphi}((H_i)_\Delta) < T$ . By restricting  $G$  to  $\Omega_i$  we see that  $(H_i)_\Delta = (\pi_i(G))_\Delta = \pi_i(G_\Delta)$ .

Now by diagram (2),  $\varphi(G_\Delta) = \bar{\varphi}(\pi_i(G_\Delta)) = \bar{\varphi}((H_i)_\Delta) < T$ , as desired.

2. Assume now that  $G$  is transitive, imprimitive. Let  $\mathcal{B} = \{B_1, \dots, B_k\}$  be a minimal system of imprimitivity (the  $B_i$  are maximal blocks of imprimitivity). So the induced action  $\psi : G \curvearrowright \mathcal{B}$  is primitive.

Let  $N$  be the kernel of this action; so  $N$  fixes each block  $B_i$  (setwise) and we can naturally identify  $\psi(G)$  with  $G/N$ .

Case 1.  $K \geq N$ . In this case we have the commutative diagram

$$(3) \quad \begin{array}{ccc} G & \xrightarrow{\varphi} & T \\ & \searrow \psi & \nearrow \bar{\varphi} \\ & & G/N \end{array}$$

So we have the epimorphism  $\bar{\varphi} : G/N \rightarrow T$  and  $G/N$  is a primitive subgroup of  $\text{Sym}(\mathcal{B})$ . Therefore, by the primitive case of Theorem 1.3, proved in Section 9, there exists  $\bar{\Delta} \subseteq \mathcal{B}$  such that  $\bar{\varphi}((G/N)_{\bar{\Delta}}) < T$ .

Let  $\Delta \subseteq \Omega$  be any subset that, for all  $i$ , intersects  $B_i$  if and only if  $B_i \in \bar{\Delta}$ . Then  $\psi(G_\Delta) \leq (G/N)_{\bar{\Delta}}$  and therefore  $\varphi(G_\Delta) \leq \bar{\varphi}((G/N)_{\bar{\Delta}}) < T$  and we are done.

Case 2.  $K \not\geq N$ . Since  $K$  is a maximal normal subgroup of  $G$ , this is equivalent to saying that  $KN = G$  and therefore  $N/(N \cap K) \cong G/K \cong T$ , so  $\varphi(N) = T$ . In particular it follows that  $N$  is not solvable and therefore

$$(4) \quad |B_i| \geq 5.$$

Moreover,  $G/N = \psi(G) = \psi(KN) = \psi(K)\psi(N) = \psi(K)$ .

Let  $N_i$  denote the restriction of  $N$  to  $B_i$ , so  $N_i \leq \text{Sym}(B_i)$ . So  $N$  is a subdirect product of the  $N_i$ . Let  $M_i$  denote the kernel of the epimorphism  $\rho_i : N \twoheadrightarrow N_i$ , i. e., the pointwise stabilizer of  $B_i$  in  $N$ . By the Subdirect product lemma (Lemma 10.1), there is an  $i \leq k$  such that

$$(5) \quad M_i \leq K.$$

This means that  $\varphi$  induces an epimorphism  $\sigma_i : N_i \twoheadrightarrow T$ . Let  $\tilde{\varphi} : N \twoheadrightarrow T$  denote the restriction of  $\varphi$  to  $N$ . Note that  $\tilde{\varphi}$  is the composition of  $\rho_i$  and  $\sigma_i$ :

$$(6) \quad \begin{array}{ccc} N & \xrightarrow{\tilde{\varphi}} & T \\ & \searrow \rho_i & \nearrow \sigma_i \\ & & N_i \end{array}$$

We may assume  $i = 1$ .

(Actually, given that  $G$  acts transitively on the blocks and therefore on the  $M_i$ , we in fact have that  $M_i \leq K$  for all  $i$ , hence the diagram (6) holds for all  $i$ . But we shall not need this fact.)

Since  $N \leq G_{B_1}$ , we have  $\varphi(G_{B_1}) = T$ .

We now split the proof into two cases.

Case 2a. The action  $G/N \curvearrowright \mathcal{B}$  admits a solvable 5-coloring.

By Theorem 8.4, this case covers all primitive groups except the alternating groups  $A_k$  in their natural action for  $k \geq 21$ . (Actually, we can claim a solvable 3-coloring in all these cases except  $A_k$  for  $k \geq 13$ . We shall not use the full force of Theorem 8.4.)

By definition, in this case there exists a 5-coloring  $\gamma_0 : \mathcal{B} \rightarrow [5]$  such that the group  $(G/N)_{\gamma_0}$  is solvable. Let  $\gamma_1$  be the lifting of  $\gamma_0$  to  $\Omega$ , i. e., for  $x \in B_i$  we set  $\gamma_1(x) = \gamma_0(B_i)$ . So under  $\gamma_1$ , each block is monochromatic. Moreover,  $\psi(G_{\gamma_1}) = (G/N)_{\gamma_0}$ . Therefore,  $N \cdot G_{\gamma_1}/N$  is solvable.

Let now  $\Delta_1 \subseteq B_1$  be such that  $\sigma_1((N_1)_{\Delta_1}) < T$ . Such  $\Delta_1$  exists by induction on  $n$ . Let  $t := |\Delta_1|$ . Let  $\beta : [5] \rightarrow \{0, 1, \dots, 5\} \setminus \{t\}$  be an injection. For  $i = 2, \dots, k$  let  $\Delta_i \subseteq B_i$  be an arbitrary subset of size  $\beta(\gamma_0(B_i))$ . For this we need  $|B_i| \geq 5$ , which holds by Eq. (4).

Let  $\Delta = \bigcup_{i=1}^k \Delta_i$ . What we have done was coding the colors of  $\gamma_0$  by the sizes of the subsets  $\Delta_i$ , taking care to have  $\Delta_1$  the only one among the  $\Delta_i$  to have size  $t$ .

We observe that  $G_{\Delta} \leq G_{\gamma_1}$  and therefore  $NG_{\Delta}/N$  is solvable.

Claim A.  $\varphi(G_{\Delta}) < T$ .

*Proof.* Suppose for a contradiction that  $\varphi(G_{\Delta}) = T$ . Then by Obs. 10.2, the perfect core of  $G_{\Delta}$  still maps onto  $T$ :  $\varphi((G_{\Delta})^{(\infty)}) = T$ . But  $(NG_{\Delta})^{(\infty)}/N = 1$  (because  $NG_{\Delta}/N$  is solvable), hence  $(G_{\Delta})^{(\infty)} \leq N$ , so  $\varphi((G_{\Delta})^{(\infty)}) = \sigma_1 \rho_1((G_{\Delta})^{(\infty)}) \leq \sigma_1((N_1)_{\Delta_1}) < T$  by the choice of  $\Delta_1$ , a contradiction, proving Claim A and thereby completing the proof of the Theorem in Case 2a.  $\square$

Case 2b.  $G/N = \text{Alt}(\mathcal{B})$  or  $\text{Sym}(\mathcal{B})$  and  $k \geq 21$ .

Let  $G_1 := G_{B_1}$ . If  $\varphi(G_1) < T$  then select  $\Delta := B_1$  and we are done. Henceforth we assume  $\varphi(G_1) = T$ . The kernel of the epimorphism  $\varphi_1 : G_1 \twoheadrightarrow T$  (the restriction of  $\varphi$  to  $G_1$ ) is  $K_1 := K \cap G_1$ .

Let  $G^*$  denote the restriction of  $G_1$  to  $B_1$ , so  $G^* \leq \text{Sym}(B_1)$ . Let  $L$  denote the kernel of the epimorphism  $\pi : G_1 \twoheadrightarrow G^*$ , so  $L$  is the pointwise stabilizer of  $B_1$  in  $G_1$ ,  $L = (G_1)_{(B_1)}$ .

Case 2b1.  $L \leq K_1$ .

Now  $\varphi_1 : G_1 \twoheadrightarrow T$  factors across  $G^*$ :  $\varphi_1 = \tau\pi$ , hence  $\tau(G^*) = T$ .

Let  $\Delta \subseteq B_1$  be such that  $\tau((G^*)_\Delta) < T$ . Such a subset exists by induction on  $n$ . Note that  $\Delta$  is a nontrivial subset of  $B_1$  since  $\tau(G^*) = T$ .

Claim B.  $\varphi(G_\Delta) < T$ .

*Proof.* Since  $\Delta \subseteq B_1$  and  $\Delta \neq \emptyset$ , we have  $G_\Delta \leq G_1$ , so  $G_\Delta = (G_1)_\Delta$ . Observe also that  $\pi(G_1)_\Delta = (G^*)_\Delta$ . Therefore  $\varphi(G_\Delta) = \varphi_1((G_1)_\Delta) = \tau((G^*)_\Delta) < T$ . This proves Claim B and thereby completes the proof of Case 2b1.  $\square$

Case 2b2.  $L \not\leq K_1$ .

Since  $G_1/K_1 \cong T$  is simple,  $K_1$  is a maximal normal subgroup in  $G_1$ , therefore  $K_1L = G_1$ .

Case 2b2i.  $N \leq K_1$ .

In this case we are in a situation analogous to Case 1:  $\varphi_1$  factors across  $N$ , i. e., there exists  $\overline{\varphi}_1 : G_1/N \rightarrow T$  such that  $\varphi_1 = \overline{\varphi}_1\psi_1$  where  $\psi_1 : G_1 \twoheadrightarrow G_1/N$ . It follows that  $\overline{\varphi} : G_1/N \twoheadrightarrow T$  is an epimorphism, but  $G_1/N$  is symmetric or alternating of degree  $\geq 20$ , so  $\overline{\varphi}$  is an isomorphism. Let  $\Delta = B_1 \cup \{x\}$  for an arbitrary  $x \in B_2$ . Now  $G_\Delta$  fixes both  $B_1$  and  $B_2$  setwise, therefore  $NG_\Delta/N$  has order less than  $|NG_1/N| = |T|$ , consequently  $|\varphi(G_\Delta)| = |\overline{\varphi}_1(NG_\Delta/N)| < |T|$  and we are done with Case 2b2i.

Case 2b2ii.  $N \not\leq K_1$  and  $G/N = \text{Alt}(\mathcal{B})$ .

We claim that this case cannot occur.

Again because  $K_1$  is a maximal normal subgroup of  $G_1$ , we have  $K_1N = G_1$ .

Recall that  $M_1 = N_{(B_1)}$ , the pointwise stabilizer of  $B_1$  in  $N$ . Moreover, we are in Case 2, so  $M_1 \leq K$  by Eq. (5), and therefore  $M_1 \leq K_1$ .

Recall that  $L = (G_1)_{(B_1)}$ . Therefore  $M_1 = N \cap L$ . So if  $L \leq N$  then  $L = M_1 \leq K_1$ , contradicting the assumption that put us in Case 2b2. Therefore  $L \not\leq N$ . But now  $N$  is a maximal normal subgroup in  $G_1$  (since  $G_1/N \cong A_{k-1}$ ), so  $NL = G_1$ .

Summarizing, we have  $NK_1 = NL = K_1L = G_1$  and  $G_1/N \cong A_{k-1}$  and  $G_1/K_1 \cong T$ ; these quotients are nonabelian simple. Setting  $H := G_1$ ,  $A := N$ ,  $B := K_1$ , and  $C := L$ , all assumptions of Lemma 10.3 are satisfied. The conclusion is that  $K_1 \not\leq N \cap L$ . But  $N \cap L = M_1$  and we know that  $M_1 \leq K_1$ , a contradiction, proving that Case 2b2ii cannot occur.

Case 2b2iii.  $N \not\leq K_1$  and  $G/N = \text{Sym}(\mathcal{B})$ .

In this case we have  $\psi(G) = \text{Sym}(\mathcal{B})$ . Let  $\tilde{G} = \psi^{-1}(\text{Alt}(\mathcal{B}))$ . Replace  $G$  by  $\tilde{G}$ ; then we land in Case 2b2ii, which is impossible. This completes the proof of this last case and with it the proof of Theorems 10.4 and 1.3.  $\square$

## 11. REDUCTION TO THE SOLVABLE CASE

First we extend Theorem 1.3 to all non-solvable target groups.

**Theorem 11.1** (Reducing non-solvable image). *Let  $G \leq \text{Sym}(\Omega)$ . Let  $\varphi : G \rightarrow H$  be an epimorphism where  $H$  is a non-solvable group. Then  $(\exists \Delta \subseteq \Omega)(\varphi(G_\Delta) < H)$ .*

**Observation 11.2.** *If  $\varphi : G \rightarrow H$  is an epimorphism then  $\varphi(G') = H'$ . It follows that  $\varphi(G^{(\infty)}) = H^{(\infty)}$ .*  $\square$

*Proof of Theorem 11.1.* Since  $H$  is not solvable, there is an epimorphism  $\psi : H^{(\infty)} \rightarrow T$  where  $T$  is nonabelian simple. Composing  $\psi$  with  $\varphi$  we obtain an epimorphism  $\xi : G^{(\infty)} \rightarrow T$ . By Theorem 1.3 there exists  $\Delta \subseteq \Omega$  such that  $\xi((G^{(\infty)})_\Delta) < T$  and therefore  $\varphi((G^{(\infty)})_\Delta) < H^{(\infty)}$ . We claim that this set  $\Delta$  satisfies the conclusion of the Lemma, i. e.,  $\varphi(G_\Delta) < H$ .

Let  $L = G_\Delta$ . Observe that  $L^{(\infty)} \leq G^{(\infty)} \cap L = (G^{(\infty)})_\Delta$ . Therefore  $\varphi(L^{(\infty)}) \leq \varphi((G^{(\infty)})_\Delta) < H^{(\infty)}$ . It follows (by Observation 11.2) that  $\varphi(L) < H$ .  $\square$

## 12. REDUCTION TO BOUNDED DERIVED LENGTH

**Theorem 12.1.** *There exists a constant  $c$  such that the following holds. If  $G \leq \text{Sym}(\Omega)$  is solvable then there exists  $\Delta \subset \Omega$  such that all orbits of  $G_\Delta$  have length  $\leq c$ . In particular, the derived length of  $G_\Delta$  is less than  $2c$ .*

The bound  $2c$  comes from [Ba86] where it is shown that the length of any subgroup chain in  $S_n$  is less than  $2n$ . A far stronger bound on the length of the derived chains of permutation groups is known; Dixon [Di68] has shown that the derived length of a solvable group of degree  $n$  is at most  $(5 \log_3 n)/2$ . However, we do not need any of this, all we need is that groups of bounded order have bounded derived length, which is obvious.

We devote the rest of this section to proving Theorem 12.1.

We say that a  $k$ -coloring  $\gamma : \Omega \rightarrow [k]$  is *uniform* if  $|\gamma^{-1}(i)| = n/k$  for each  $i \in [k]$ . Note that in a non-uniform  $k$ -coloring we permit some of the colors not to be used.

**Lemma 12.2.** *Let  $G \leq \text{Sym}(\Omega)$  be a primitive solvable group. Then  $G$  admits a non-uniform asymmetric 5-coloring.*

*Proof.* Gluck proves [Gl83, Theorem 1] that for  $n \geq 10$ ,  $G$  admits an asymmetric 2-coloring. We can view such a coloring as a non-uniform 3-coloring by adding an empty color. (In fact, Gluck proves the existence of a non-uniform 2-coloring for  $n \geq 10$ .)

To address the remaining cases directly, we observe that, since  $G$  is primitive and solvable, we are in the affine case:  $\Omega$  can be identified with the vector space  $\mathbb{F}_p^d$  for some prime  $p$  and  $d \geq 1$ , and  $G \leq \text{AGL}(d, p)$ .

Let us assign distinct colors to 0 and the  $d$  vectors in a basis, and one more color to the rest of the space. This coloring is clearly asymmetric, and it uses at most  $d + 2$  colors. If  $n \leq 9$  then  $d \leq 3$ , so we are using at most 5 colors.

This coloring is not uniform for any value  $n \leq 9$ . (Note that we throw in one or more empty color classes for  $n \leq 4$ .)  $\square$

**Remark 12.3.** The bound 5 in the Lemma is tight, as shown by  $G = S_4$ .

Now we extend this result to all solvable permutation groups, dropping the condition of non-uniformity.

**Lemma 12.4.** *Let  $G \leq \text{Sym}(\Omega)$  be a solvable group. Then  $G$  admits an asymmetric 5-coloring.*

**Remark 12.5.** The bound 5 in the Lemma is tight, as shown by the wreath product  $G = S_4 \wr H$  where  $H$  is any nontrivial permutation group. ( $H$  is the top group in the wreath product.)

**Definition 12.6.** Let  $G \leq \text{Sym}(\Omega)$  and let  $\gamma_1, \gamma_2 : \Omega \rightarrow \Sigma$  be two colorings. We say that a permutation  $\pi \in \text{Sym}(\Omega)$  is a  $G$ -isomorphism of  $\gamma_1$  to  $\gamma_2$  if for all  $x \in \Omega$ ,  $\gamma_2(\pi(x)) = \gamma_1(x)$ .

**Observation 12.7.** *If a group  $G$  admits a non-uniform asymmetric  $k$ -coloring then by permuting the colors, we obtain at least  $k$  non- $G$ -isomorphic non-uniform  $k$ -colorings.*  $\square$

*Proof of Lemma 12.4.* Without loss of generality we may assume  $G$  is transitive. (Otherwise, select a coloring of each orbit and combine them.)

We proceed by induction on  $n = |\Omega|$ . If  $G$  is primitive, we are done by Lemma 12.2.

Let now  $G$  be imprimitive and let  $\mathcal{B} = \{B_1, \dots, B_k\}$  be a maximal system of imprimitivity (the blocks are minimal). Let  $|B_i| = t$ .

Let  $G_i$  denote the restriction of the setwise stabilizer  $G_{B_i}$  to  $B_i$ , so  $G_i \leq \text{Sym}(B_i)$ . Since  $B_i$  is a minimal block,  $G_i$  is primitive. The  $G_i$  are equivalent permutation groups (equivalent under the action of  $G$ ). Let  $\delta_{1,1}, \dots, \delta_{1,5}$  be five non- $G_1$ -isomorphic non-uniform  $G_1$ -asymmetric colorings of  $B_1$ , and let  $\delta_{i,1}, \dots, \delta_{i,5}$  be colorings of  $B_i$  obtained by applying an element  $\sigma_i$  to our colorings of  $B_1$  where  $\sigma_i(B_1) = B_i$ . (Let  $\sigma_1$  be the identity.) Let  $\widehat{\delta}_{i,j}$  be the 6-coloring of  $\Omega$  obtained by adding a 6th color everywhere outside  $B_i$ . It should be clear that if  $\widehat{\delta}_{i,j}$  is  $G$ -isomorphic to  $\widehat{\delta}_{i',j'}$  then  $j = j'$ , simply by counting the multiplicities of colors.

Let  $\widetilde{G}$  denote the image of the  $G \curvearrowright \mathcal{B}$  action, so  $\widetilde{G} \leq \text{Sym}(\mathcal{B})$ . By induction, let  $\widetilde{\gamma} : \mathcal{B} \rightarrow [5]$  be a non-uniform asymmetric 5-coloring for  $\widetilde{G}$ .

We encode these colors by non-isomorphic colorings of the blocks: We define the 5-coloring  $\gamma : \Omega \rightarrow [5]$  by setting, for each  $x \in B_i$ ,  $\gamma(x) = \delta_{i,j}(x)$  where  $j = \widetilde{\gamma}(B_i)$ .

Assume  $\sigma \in G$  preserves the coloring  $\gamma$ . Since the coloring  $\widetilde{\gamma}$  can be reconstructed from  $\gamma$ , we infer that  $\sigma(B_i) = B_i$  for every  $i$ . Now  $\sigma|_{B_i} \in G_i$ , and the coloring  $\gamma|_{B_i} = \delta_{i,j}$  is  $G_i$ -asymmetric, so  $\sigma = 1$ .  $\square$

**Lemma 12.8.** *There exist  $\epsilon > 0$  and a threshold  $n_1$  such that the following holds. If  $G \leq \text{Sym}(\Omega)$  is a primitive solvable permutation group of degree  $n \geq n_1$  then for every  $j$  in the interval  $n^{1-\epsilon} \leq j \leq n/2$  there exists a  $G$ -asymmetric subset  $\Delta_i \subseteq \Omega$  of size  $|\Delta_j| = j$ .*

*Proof.* The proof consists of drawing the required conclusion from Gluck's counting argument [Gl83]. We review the argument.

Since  $G$  is primitive and solvable, we are in the affine case:  $\Omega$  can be identified with the  $d$ -dimensional vector space over  $\mathbb{F}_p$  for some prime  $p$  and  $d \geq 1$ , and with this identification,  $G \leq \text{AGL}(d, p)$ .

It follows that the fixed points of any element of  $G$  form an affine subspace of  $\Omega$  and therefore the minimal degree of  $G$  is  $\geq n/2$  and consequently every non-identity

element  $\sigma \in G$  decomposes into at most  $3n/4$  cycles. It follows that the number of subsets fixed by  $\sigma$  is at most  $2^{3n/4}$  and therefore the number of subsets that are not  $G$ -asymmetric is at most  $|G| \cdot 2^{3n/4}$ . Invoking the Pálffy–Wolf Theorem [Pa82, Wo82] that states that the order of a primitive solvable group of degree  $n$  is less than  $n^C$  for some constant  $C$ , the stated conclusion follows.  $\square$

We shall only use a weak corollary of this result.

**Corollary 12.9.** *There exists a constant  $c_0$  such that every primitive solvable permutation group  $G$  of degree  $\geq c_0$  admits at least five  $G$ -asymmetric subsets of different sizes.*  $\square$

We are now ready to prove the main result of this section.

*Proof of Theorem 12.1.* This proof is not by induction, but we follow the scheme of the inductive step in the proof of Lemma 12.4. Like there, we may assume  $G$  is transitive.

We shall refer to the constant  $c_0$  from Cor. 12.9. We claim that there exists a set  $\Delta \subseteq \Omega$  such that the orbits of  $G_\Delta$  have length  $\leq c := 3c_0$ .

A  $G$ -asymmetric subset does more than required (orbits of length 1), so if  $G$  is primitive and  $n \geq c_0$  then we are done by Cor. 12.9. In fact, by Gluck [Gl83], we are done for all  $n \geq 10$ .

Let now  $G$  be imprimitive and let  $\mathcal{B} = \{B_1, \dots, B_k\}$  be a system of imprimitivity. Let  $|B_i| = t$ . Let  $G_i$  denote the restriction of the setwise stabilizer  $G_{B_i}$  to  $B_i$ , so  $G_i \leq \text{Sym}(B_i)$ .

Let  $\tilde{G}$  denote the image of the  $G \curvearrowright \mathcal{B}$  action, so  $\tilde{G} \leq \text{Sym}(\mathcal{B})$ . Using Lemma 12.4, let  $\gamma : \mathcal{B} \rightarrow [5]$  be a  $\tilde{G}$ -asymmetric 5-coloring of  $\mathcal{B}$ .

Case 1.  $4 \leq t \leq 3c_0$ . In this case we claim there exists  $\Delta \subseteq \Omega$  such that the orbits of  $G_\Delta$  have length  $\leq t \leq 3c_0$ .

In this case, let  $\Delta_i$  be an arbitrary subset of  $B_i$  of size  $\gamma(B_i) - 1$ . Let  $\Delta = \bigcup_{i=1}^k \Delta_i$ .

Assume  $\sigma \in G$  preserves the set  $\Delta$ . Since the coloring  $\gamma$  can be reconstructed from  $\Delta$  simply by the sizes of the  $\Delta_i$ , we infer that  $\sigma(B_i) = B_i$  for every  $i$ .

Therefore, the orbits of  $G_\Delta$  are subsets of the  $B_i$  and therefore have length  $\leq t < 3c_0$ . This completes the proof in Case 1.

Case 2.  $t \geq c_0$  and the  $B_i$  are minimal blocks of imprimitivity. In this case we claim there exists an asymmetric  $\Delta \subseteq \Omega$ , so the orbits of  $G_\Delta$  have length 1.

Let  $j_1 < \dots < j_5$  be five different sizes of  $G_1$ -asymmetric subsets of  $B_1$ .

Let  $\Delta_i \subseteq B_i$  be a  $G_i$ -asymmetric subset of size  $j_{\gamma(B_i)}$ . Let  $\Delta = \bigcup_{i=1}^k \Delta_i$ .

Assume  $\sigma \in G$  preserves the set  $\Delta$ . Since the coloring  $\gamma$  can be reconstructed from  $\Delta$  simply by the sizes of the  $\Delta_i$ , we infer that  $\sigma(B_i) = B_i$  for every  $i$ . Now  $\sigma|_{B_i} \in G_i$ , and the set  $\Delta_i = \Delta \cap B_i$  is  $G_i$ -asymmetric, so  $\sigma = 1$ . This completes the proof in Case 2.

Case 3. In all the remaining cases, we claim there exists  $\Delta \subseteq \Omega$  such that the orbits of  $G_\Delta$  have length  $\leq t \leq 3$ .

Now there is no block of size  $4 \leq t < 3c_0$  (Case 1) and also no minimal block of size  $t \geq c_0$  (Case 2). This means all minimal blocks have size  $2 \leq t \leq 3$  and the next smallest blocks have size  $\geq 3c_0$ .

Let  $\mathcal{B} = \{B_1, \dots, B_k\}$  be a system of minimal blocks, so  $2 \leq |B_i| \leq 3$ . As before, let  $\tilde{G}$  be the image of the action  $G \curvearrowright \mathcal{B}$ .

Let further  $\mathcal{D} = \{D_1, \dots, D_\ell\}$  be a system of imprimitivity that is a maximal coarsening of  $\mathcal{B}$ , i.e.,  $\mathcal{B}$  is a refinement of  $\mathcal{D}$  and the  $D_j$  are minimal among the blocks of imprimitivity that strictly include some  $B_i$ . Let  $|D_j| = st$ . Let  $\tilde{D}_j = \{B_i \in \mathcal{B} \mid B_i \subseteq D_j\}$ . Now the  $\tilde{D}_j$  are minimal blocks for  $\tilde{G}$  and their size is  $s \geq c_0$ . Therefore, by Case 2, we have a  $\tilde{G}$ -asymmetric set  $\tilde{\Delta} \subseteq \mathcal{B}$ .

Let us lift  $\tilde{\Delta}$  to  $\Omega$  by setting  $\Delta = \bigcup\{B_i \mid B_i \in \tilde{\Delta}\}$ . Now  $G_\Delta$  fixes each  $B_i$  setwise, so its orbits have length  $\leq 3$ . This completes the proof of Theorem 12.1.  $\square$

### 13. FROM BOUNDED DERIVED LENGTH TO ASYMMETRY: REDUCING THE DERIVED LENGTH

We restate Prop. 1.5

**Proposition 13.1.** *Let  $G \leq \text{Sym}(\Omega)$  be a solvable group with derived length  $k \geq 1$ . Then there exists a subset  $\Delta \subseteq \Omega$  such that the derived length of  $G_\Delta$  is at most  $k - 1$ .*

We begin with the abelian case.

**Observation 13.2.** *All abelian permutation groups admit an asymmetric 2-coloring.*

*Proof.* We need to construct a  $G$ -asymmetric subset of the domain  $\Omega$ .

Let  $R_1, \dots, R_k$  be the orbits of  $G$  and let  $\Delta$  be a transversal of the orbits, i.e.,  $\Delta \subseteq \Omega$  and  $|\Delta \cap R_i| = 1$  for all  $i$ .

We claim that  $|G_\Delta| = 1$ . Indeed, the restriction of  $G$  to  $R_i$  is a transitive abelian group which therefore is regular, so fixing one of its points fixes the entire orbit pointwise.  $\square$

*Proof of Prop. 13.1.* Let  $H := G^{(k-1)}$  be the last nontrivial term in the derived series of  $G$ . So  $H$  is abelian. Let  $\Delta \subseteq \Omega$  be such that  $|H_\Delta| = 1$ . We claim that the derived length of  $G_\Delta$  is at most  $k - 1$ .

Indeed, let  $L = G_\Delta$ . So  $L^{(k-1)} \leq L \cap G^{(k-1)} = L \cap H = H_\Delta = 1$ .  $\square$

This completes the proof of Theorem 1.2 and with it the proof of our main result, Theorem 1.1.

### 14. AN EFFECTIVE VERSION?

In this section we consider finite inverse sequences, of length  $k$ , of finite permutation groups. Such a system is defined by a sequence of  $k + 1$  groups,  $(G_i : i = 0, \dots, k)$  and a sequence of  $k$  homomorphisms,  $\varphi_i : G_i \rightarrow G_{i-1}$ ,  $i = 1, \dots, k$ . We denote such a system as  $(G_i, \varphi_i)_{i \leq k}$ . Here  $G_i \leq \text{Sym}(\Omega_i)$ . For  $i \geq j$ , the transition homomorphism  $\varphi_{i,j} : G_i \rightarrow G_j$  is defined as the composition of  $\varphi_{\ell, \ell-1}$  for  $\ell = i, \dots, j + 1$ . Now the definition of inverse limits applies; in particular, strands are defined as in Def. 4.3 and they form the inverse limit  $\mathcal{G} = \varprojlim G_i$ . We say that the inverse system, and the inverse limit, *ends* at  $G_0$ . If the  $\Omega_i$  are disjoint then we view  $\mathcal{G}$  as a permutation group acting on  $\Omega := \bigcup_{i=0}^k \Omega_i$ . We say that a coloring  $\gamma : \Omega \rightarrow \Sigma$  is *zero-asymmetric* if  $\mathcal{G}_\gamma$  fixes  $\Omega_0$  pointwise. We say that the coloring is *zero-neutral* if  $\Omega_0$  is monochromatic, i.e.,  $|\gamma(\Omega_0)| = 1$ . If all the  $\varphi_i$  (and therefore all the  $\varphi_{i,j}$ ) are epimorphisms, we speak of an *epimorphic inverse sequence*.

Let  $\mathcal{G}\tau$  denote the class of finite permutation groups.

**Theorem 14.1** (asymmetric coloring of inverse limit—finite version). *There exists a positive integer  $c$  and a function  $f : \mathcal{G}\tau \rightarrow \mathbb{N}$  such that the following holds. Let  $(G_i, \varphi_i)_{i \leq k}$  be an epimorphic inverse sequence of length  $k \geq c$  of finite permutation groups with disjoint domains. Assume  $k \geq c + f(G_c)$ . Then the inverse limit of this system admits a zero-neutral zero-asymmetric 2-coloring.*

It is easy to see that our main technical result, Theorem 1.2, is a consequence of Theorem 14.1. (Zero-neutrality is not needed for this inference.) The proof follows the lines of the proof of Lemma 6.3.

It is also easy to see that our proof of Theorem 1.2 in effect proved Theorem 14.1.

*Sketch of proof of Theorem 14.1.* Take  $c := 2c' + 1$  where  $c'$  is the constant denoted by  $c$  in Theorem 12.1. Set  $f(n) := \lfloor \log_2(n) \rfloor$ .

We color the domains  $\Omega_i$  one at a time, reducing the group  $G_i$  and thereby  $\mathcal{G}$ . We call such a step a *round*, and we conclude each round by *epimorphic reduction* (Fact 4.6).

First we color the domains  $\Omega_i$  for  $i > c$  to reduce  $G_c$  to a solvable group. While  $G_c$  is not solvable, we reduce it to a proper subgroup by coloring the next  $\Omega_i$  (Theorem 11.1). This process clearly terminates in  $\leq \log_2 |G_c|$  rounds and when it terminates,  $G_c$  is solvable. But then, all the  $G_j$ ,  $j \leq c$ , are solvable as well, by epimorphic reduction.

Next we color  $\Omega_c$ , applying Theorem 12.1 to  $G_c$  and achieving derived length  $\leq 2c' = c - 1$  for  $G_c$  and therefore for all  $G_i$  for  $i \leq c$ . Then, for  $i = c - 1$  down to  $i = 1$ , we successively reduce the derived length of  $G_i$  to  $\leq i - 1$ , using the procedure of Section 13. In the end, the derived length of  $G_1$  is reduced to zero, hence the same is true for  $G_0$  without having colored  $\Omega_0$ , yielding zero-asymmetry and zero-neutrality.  $\square$

To make this result more effective, we need to replace the bound  $k \geq c + f(G_c)$  by a bound of that only depends on  $G = G_0$ .

**Conjecture 14.2** (asymmetric coloring of inverse limit—effective version). There exists a function  $g : \mathcal{G}\tau \rightarrow \mathbb{N}$  such that the following holds. Let  $(G_i, \varphi_i)_{i \leq k}$  be an epimorphic inverse sequence of length  $k \geq g(G_0)$  of finite permutation groups with disjoint domains. Then the inverse limit of this system admits a zero-neutral zero-asymmetric 2-coloring.

If this conjecture is true, here is a lower bound on the function  $g$ . Let  $\text{asy}(G)$  denote the asymmetric coloring number of the permutation group  $G$  (Def. 3.8).

**Proposition 14.3.** *If Conjecture 14.2 holds for a function  $g$  then for all finite permutation groups  $G$  we have*

$$(7) \quad g(G) \geq \log_2 \text{asy}(G).$$

*Proof.* Let  $G \leq \text{Sym}(\Omega)$ . Consider the length- $k$  inverse system  $(G, G, \dots, G)$  with the identity serving as transition homomorphisms. So the inverse limit  $\mathcal{G}$  is the diagonal of the direct product  $G^{k+1}$ , acting on  $\Omega \times \{0, \dots, k\}$ . Assume there is a zero-asymmetric zero-neutral 2-coloring  $\gamma$  of  $\mathcal{G}$ . Now define the coloring  $\delta$  of  $\Omega$  by setting  $\delta(x) = (\gamma(x, 1), \dots, \gamma(x, k))$  for  $x \in \Omega$ . It should be clear that  $\delta$  is an asymmetric coloring of  $G$  with  $\leq 2^k$  colors, so  $\text{asy}(G) \leq 2^{g(G)}$ .  $\square$

One might ask, how close is this lower bound to the true upper bound (if one exists). I would risk the following bold conjecture.

**Conjecture 14.4** (asymmetric coloring of inverse limit—polylog bound). There exists a polynomial  $p$  such that Conjecture 14.2 holds with  $g(G) = p(\log(\text{asy}(G)))$ .

Conjecture 14.4 is true for inverse systems of solvable groups. Since solvable groups have bounded asymmetric coloring number (Lemma 12.4), this means for solvable groups  $G$  the quantity  $g(G)$  should be bounded, and indeed it is.

**Proposition 14.5.** *There exists a constant  $c$  such that the following holds for all inverse sequences of solvable permutation groups with disjoint domains. If the length of the sequence is at least  $c$  then the inverse limit admits a zero-asymmetric zero-neutral 2-coloring. We can take  $c := 2c' + 1$  where  $c'$  is the constant denoted by  $c$  in Theorem 12.1.*

*Proof.* We just proved this in the second (solvable) phase of the proof of Theorem 14.1.  $\square$

## 15. COMBINATORIAL RELAXATION OF SYMMETRY: CFSG-FREE PROOFS

One of the key facts underlying our result was the following.

**Theorem 15.1** ([CNS84]). *All but a finite number of primitive permutation groups, other than the symmetric and alternating groups in their natural action, admit an asymmetric 2-coloring.*

The original proof of Theorem 15.1 rests on the Classification of Finite Simple Groups (CFSG).

In this section we address the following two questions.

- ( $\alpha$ ) Can one avoid the use of the CFSG in the proof of Theorem 15.1?
- ( $\beta$ ) Is there a combinatorial generalization of Theorem 15.1, i.e., an asymmetric 2-colorability result for a class of combinatorial structures with no symmetry assumptions, that includes Theorem 15.1?

Question ( $\alpha$ ) was already raised by Cameron *et al.* [CNS84] and was reiterated by Imrich *et al.* as [IS+15, Question 1].

We point out that a positive answer to both questions follows from a recent breakthrough by Xiaorui Sun and John Wilmes [SW15] on the number of automorphisms of primitive coherent configurations (see Theorems 15.3 and 15.10 below).

**15.1. CFSG-free proof of the Cameron–Neumann–Saxl Theorem.** Recall that the *line-graph* of a graph  $X = (V, E)$  is the graph  $L(X)$  with vertex set  $E$  where two edges  $e, f \in E$  (as vertices of  $L(X)$ ) are adjacent in  $L(X)$  if they share a vertex in  $X$ . The line-graphs of the cliques  $K_r$  are called *triangular graphs*, denoted  $T(r)$ . The graph  $T(r)$  has  $\binom{r}{2}$  vertices,  $r!$  automorphisms, and motion  $2r - 4$ . The socle has index 2 in  $\text{Aut}(T(r))$ . The line-graphs of balanced bipartite cliques  $K_{r,r}$  are called *lattice graphs*, denoted  $L_2(r)$ . The graph  $L_2(r)$  has  $r^2$  vertices,  $2(r!)^2$  automorphisms, and motion  $2r$ . The socle has index 8 in  $\text{Aut}(L_2(r))$ .

The original proof of Theorem 15.1 depends on CFSG through the following result, a special case of much more detailed result in [Ca81]. Let  $\text{Soc}(H)$  denote the socle of the group  $H$ . Let us say that  $G$  is a *top group* if either  $A_n \leq G \leq S_n$  or  $\text{Soc}(H) \leq G \leq H$  where  $H$  is the automorphism group of a triangular graph or a lattice graph.

**Proposition 15.2** (Cameron + CFSG). *Let  $G \leq S_n$  be a primitive permutation group. Assume  $|G| \geq 2\sqrt{n/2}$ . If  $n$  is sufficiently large then  $G$  is a top group.*

The following elementary result by Sun and Wilmes [SW15], a consequence of their result on coherent configurations (Theorem 15.10), implies Prop. 15.2.

**Theorem 15.3** (Sun–Wilmes, elementary). *There exists  $c > 0$  such that the following holds. Let  $G \leq S_n$  be a primitive but not doubly transitive permutation group. Assume  $|G| \geq \exp(c(n^{1/3}(\log n)^{7/3}))$ . Then  $G$  is a top group.*

This result implies Prop. 15.2 except in the case that the group is doubly transitive. In that case, however, known elementary combinatorial bounds show that the order of the group is quasipolynomially bounded.

**Theorem 15.4** ([Ba82, Py93a]). *Let  $G \leq S_n$  be a doubly transitive group and assume  $G \not\cong A_n$ . Then  $|G| \leq \exp(O(\log n)^4)$ .*

(The result in [Ba82] gives the weaker upper bound  $\exp(\exp(O(\sqrt{\log n})))$ , which would also suffice in our context since this quantity is less than  $\exp(n^\epsilon)$  for all  $\epsilon > 0$  and all sufficiently large  $n$ , so it is much smaller than the threshold in Prop. 15.2. The improved bound stated above was obtained in [Py93a] using the framework of [Ba82].) This completes the list of ingredients of an elementary proof of Theorem 15.1.  $\square$

While a lemma in [CNS84] shows that the upper bound in Theorem 15.4 on the order of doubly transitive groups other than  $A_n$  and  $S_n$  implies a nearly linear lower bound on the minimal degree of these groups, we should mention that a stronger, linear lower bound has been known for more than 120 years. The following result was proved by Alfred Bochert in the 19th century by a lovely combinatorial argument [Bo1897].

**Theorem 15.5** (Bochert, 1897). *Let  $G \leq S_n$  be a doubly transitive group and assume  $G \not\cong A_n$ . Then the minimal degree of  $G$  is  $\mu(G) \geq n/8$ . For  $n > 216$ , the lower bound improves to  $n/4$ .*

**Remark 15.6.** Cameron’s results [Ca81] classify all primitive permutation groups of order greater than  $n^{\log \log n}$  and naturally include Theorem 15.3. The point here is that the proof by Sun and Wilmes is elementary: it does not use the CFSG; in fact, it uses no group theory at all.

## 15.2. Combinatorial relaxation of symmetry: Coherent configurations.

The significance of the work of Sun and Wilmes goes far beyond giving elementary proofs of group theoretic results previously only known through the CFSG. Their result is purely combinatorial; it concerns primitive coherent configurations (PCCs): highly regular colorings of the directed complete graph, with no symmetry assumptions. We define this very general class of objects now.

**Definition 15.7.** A *coherent configuration* (CC) is a pair  $\mathfrak{X} = (\Omega, c)$  where  $\Omega$  is a set (the set of *vertices*) and  $c : \Omega \times \Omega \rightarrow \Sigma$  is a coloring of the ordered pairs of vertices ( $\Sigma$  is the set of colors), subject to the following regularity constraints. We assume  $c$  is surjective. Below,  $x, y, u, v \in \Omega$ .

- (i)  $(\forall x, y, z)(\text{if } c(x, x) = c(y, z) \text{ then } y = z)$ .
- (ii)  $(\forall x, y, u, v)(\text{if } c(x, y) = c(u, v) \text{ then } c(y, x) = c(v, u))$ .

- (iii) There exists a family of  $|\Sigma|^3$  non-negative integers  $p_{i,j}^k$ , called the *intersection numbers*, such that

$$(\forall x, y)(\text{if } c(x, y) = k \text{ then } |\{z : c(x, z) = i, c(y, z) = j\}| = p_{i,j}^k).$$

The number of colors used is called the *rank* of  $\mathfrak{X}$ .

Let  $G \leq \text{Sym}(\Omega)$  be a permutation group. Let  $E_1, \dots, E_r$  denote the *orbitals* of  $G$ , i. e., the orbits of the  $G$ -action on  $\Omega \times \Omega$ . Assigning color  $i$  to the elements of  $E_i$  we obtain a coloring  $c : \Omega \times \Omega \rightarrow [r]$ . It is easy to see that the resulting pair  $\mathfrak{X}(G) := (\Omega, c)$  is a CC, and  $G \leq \text{Aut}(\mathfrak{X}(G))$ . A CC arising in this manner is called a *Schurian CC*, after Issai Schur who first introduced CCs in 1933, as a tool in the study of permutation groups [Sc33]. CCs were subsequently rediscovered several times in different contexts. They include such much-studied structures as strongly regular graphs, distance-regular graphs, association schemes. If  $X = (V, E)$  is a graph then the Weisfeiler–Leman color refinement process [WL68, We76] (see, e. g., [Ba16]) efficiently constructs a CC  $\mathfrak{X}(X) = (V, c)$  such that  $\text{Aut}(X) = \text{Aut}(\mathfrak{X}(X))$ . In particular,  $X$  has an asymmetric  $d$ -coloring (of the vertices, as always in this paper) if and only if  $\mathfrak{X}(X)$  has an asymmetric  $d$ -coloring. CCs are also critical ingredients in the recent isomorphism test [Ba16]. That paper includes a detailed introduction to the combinatorial theory of CCs.

**Definition 15.8** (Constituents, PCC). Given a CC  $\mathfrak{X} = (\Omega, c)$ , the digraphs  $R_i = (\Omega, c^{-1}(i))$  ( $i \in \Sigma$ ) are the *constituent digraphs* of  $\mathfrak{X}$ . If one of these is the diagonal  $\text{diag}(\Omega) := \{(x, x) \mid x \in \Omega\}$ , we call  $\mathfrak{X}$  *homogeneous*. Observe that a group  $G$  is transitive if and only if the corresponding Schurian CC  $\mathfrak{X}(G)$  is homogeneous. We call  $\mathfrak{X} = (\Omega, c)$  a *primitive CC (PCC)* if it is homogeneous and every non-diagonal constituent is a (strongly) connected digraph.

It is not difficult to show that a permutation group  $G$  is primitive if and only if  $\mathfrak{X}(G)$  is a PCC. We should emphasize that conjecturally and empirically, most CCs are not Schurian.

Note that for every  $n$  there is essentially only one rank-2 CC, namely,  $\mathfrak{X}(S_n)$ , to which we refer as the  $n$ -clique, and also as the trivial CC.

**Definition 15.9** (UPCC). A *uniprimitive coherent configuration (UPCC)* is a PCC of rank  $\geq 3$  (a nontrivial PCC).

Now we can state the actual result of Sun and Wilmes.

**Theorem 15.10** (Sun–Wilmes). *Let  $\mathfrak{X}$  be a UPCC with  $n$  vertices. If  $n$  is sufficiently large then  $|\text{Aut}(\mathfrak{X})| \leq \exp(O(n^{1/3}(\log n)^{7/3}))$ , unless  $\mathfrak{X}$  is the CC corresponding to a triangular graph or a lattice graph.*

Finally, we are in the position to address Question ( $\beta$ ) above, by generalizing Theorem 15.1 to UPCCs.

**Theorem 15.11.** *All sufficiently large UPCCs admit an asymmetric 2-coloring.*

Like much of the literature about asymmetric colorings, we shall rely on the following lemma, implicit in the counting argument used by Gluck [Gl83] and Cameron et al. [CNS84] and made explicit by Russell and Sundaram a decade and a half later [RS98].

**Proposition 15.12** (Motion Lemma, [Gl83], [CNS84], [RS98]). *Let  $G \leq S_n$  be a permutation group of minimal degree  $\mu$ . If  $d^{\mu/2} \geq |G|$  then  $G$  admits an asymmetric  $d$ -coloring.*  $\square$

In order to take advantage of this lemma, we need an upper bound on the order of the automorphism group of  $\mathfrak{X}$ , and a lower bound on the motion of  $\mathfrak{X}$ .

The former is provided by the Sun–Wilmes result, Theorem 15.10.

We take the latter from a 1981 paper of this author [Ba81]. Since the result is not explicitly stated in [Ba81], let me show how it follows immediately from the main technical result of that paper. (This fact has been known since immediately after the publication of [Ba81].)

Let  $\mathfrak{X} = (\Omega, c)$  be a UPCC. Following [Ba81], we say that vertex  $z$  *distinguishes* vertices  $x$  and  $y$  if  $c(z, x) \neq c(z, y)$ . Let  $D(x, y)$  denote the set of vertices  $z$  that distinguish  $x$  and  $y$ . The core technical result of [Ba81] is the following.

**Theorem 15.13** ([Ba81]). *Let  $\mathfrak{X}$  be a UPCC with  $n$  vertices. Then, for every pair  $x, y$  of distinct vertices,  $|D(x, y)| \geq (\sqrt{n} - 1)/2$ .*

All we need to add to this result is the following observation.

**Observation 15.14.** *Let  $\mathfrak{X}$  be a UPCC with  $n$  vertices. Then the motion of  $\mathfrak{X}$  is  $\geq \min_{x \neq y} |D(x, y)|$ .*

*Proof.* Let  $T = \text{supp}(\sigma)$  for some  $\sigma \in \text{Aut}(X)$  such that the size of  $T$  is the motion of  $\mathfrak{X}$ . Let  $x \in T$  and  $y = \sigma(x)$ . Then  $y \neq x$  by definition. We claim that  $T \supseteq D(x, y)$ . Indeed, if  $z \in \Omega \setminus T$  then  $c(z, x) = c(\sigma(z), \sigma(x)) = c(z, y)$ , so  $z \notin D(x, y)$ .  $\square$

*Proof of Theorem 15.11.* Let  $\mu = \mu(\mathfrak{X})$  be the motion of  $\mathfrak{X}$ . We have  $\mu \geq (\sqrt{n} - 1)/4$  by Theorem 15.13 and Obs. 15.14. So  $2^{\mu/2} \geq 2^{(\sqrt{n} - 1)/4}$ . For sufficiently large  $n$ , this quantity is greater than the Sun–Wilmes bound (Theorem 15.10), which is  $\exp(C(n^{1/3}(\log n)^{7/3}))$  for some constant  $C$ .  $\square$

## 16. OPEN PROBLEMS

Theorem 14.1 describes a finite version of our main technical result, Theorem 1.2. I would be most interested in more effective versions of this result, and specifically in Conjectures 14.2 and 14.4 (Polylog bound conjecture).

Below I list a number of additional problems and directions of study. All groups in this section, except in Problems (1), (2), and (11), are finite.

**Terminology.** Given a permutation group  $G \leq \text{Sym}(\Omega)$ , recall that we say that a coloring  $\gamma$  “results in a 2-group” if  $\gamma$  is a coloring of the permutation domain  $\Omega$  and  $G_\gamma$  (the color-preserving subgroup of  $G$ ) is a 2-group. And we can substitute any class of groups in such a statement for “2-groups,” so for instance the statement that “a coloring results in a group with derived length  $\leq 3$ ” should have a clear meaning. PS-closed classes of groups (classes closed under direct products and subgroups), such as those mentioned above, are of particular interest because of their monotonicity properties described in Obs. 8.2 and Remark 8.3.

- (1) (a) Give a CFSG-free proof of Theorem 1.2 and thereby to the Infinite Motion Conjecture.

I expect that the results mentioned in the preceding section, and in particular the Sun–Wilmes Theorem (Theorem 15.3), will play a role.

- (b) How much of Theorem 1.3 (“Reducing simple image”) can be salvaged without CFSG?
- (2) (a) Does there exist a constant  $C$  such that the following strengthening of the Infinite Motion Conjecture holds?  
*Let  $X$  be a connected locally finite rooted graph with infinite motion. Then  $X$  has an asymmetric 2-coloring (red/blue) that is overwhelmingly blue in the sense that every sphere about the root gets at most  $C$  red vertices.*  
 This question is motivated by the consideration of the “cost” of coloring, as defined below. It is easy to see that the statement is true for locally finite rooted trees without vertices of degree 1.
- (b) Does there exist a constant  $C$  such that the following strengthening of Theorem 1.3 (“Reducing simple image”) holds?  
*Let  $G \leq \text{Sym}(\Omega)$ , where  $\Omega$  is a finite set. Let  $\varphi : G \twoheadrightarrow T$  be an epimorphism where  $T$  is a nonabelian simple group. Then there exists a subset  $\Delta \subseteq \Omega$  of size  $|\Delta| \leq C$  such that  $\varphi(G_\Delta) < T$ .*  
 We note that  $C = 1$  will not suffice, as the example  $G = \mathbb{Z}_p^d \rtimes T \leq \text{AGL}(d, p)$  shows, where the semidirect product is defined by a non-trivial  $d$ -dimensional irreducible representation of  $T$  over  $\mathbb{F}_p$ .
- (3) Recall Theorem 11.1 (reducing non-solvable image):  
*Let  $G \leq \text{Sym}(\Omega)$  where  $\Omega$  is a finite set. Let  $H$  be a group and  $\varphi : G \twoheadrightarrow H$  an epimorphism. Then  $(\exists \Delta \subseteq \Omega)(\varphi(G_\Delta) < H)$ , assuming  $H$  is not solvable.*
- (a) We note that the condition that “ $H$  is not solvable” cannot be replaced by the condition “ $|H| > 1$ ,” as shown by the example  $G = D_k$  (the dihedral group of order  $2k$  acting naturally on  $k$  elements) and  $H = \mathbb{Z}_2$  where the epimorphism  $\varphi$  is defined by the natural epimorphism  $D_k \twoheadrightarrow D_k/\mathbb{Z}_k$ , where  $3 \leq k \leq 5$ .
- (b) Question. Does there exist a number  $C$  such that the following holds?  
*Let  $G \leq \text{Sym}(\Omega)$  where  $\Omega$  is a finite set. Let  $H$  be a group. Let  $\varphi : G \twoheadrightarrow H$  be an epimorphism. Assume  $|H| \geq C$ . Then  $(\exists \Delta \subseteq \Omega)(\varphi(G_\Delta) < H)$ .*
- (c) Does the conclusion of (3)(b) follow if we only require  $|G| \geq C$  and  $|H| \geq 2$ ?
- (4) (a) Given a sequence  $n_0, \dots, n_k$  of positive integers, consider the *balanced rooted tree* of height  $k$  where the vertices at distance  $j$  from the root have  $n_j$  children. So the automorphism group is the wreath product of the symmetric groups of degree  $n_j$ . What is the asymmetric coloring number of these trees?
- (b) More generally, how does the *position of symmetric and alternating groups in a structure tree* (hierarchy of blocks of imprimitivity) of a transitive group affect the asymmetric coloring number?
- (5) A systematic study of *solvable colorings* for permutation groups would be of interest. Recall that these are colorings of the permutation domain that result in a solvable group. More specific questions on this subject follow.
- (a) Within various classes of permutation groups, characterize those that do not admit a solvable 2-coloring.  
 Among primitive groups, the only groups that do not admit a solvable

2-coloring are the symmetric and alternating groups of degree  $\geq 9$  in their natural action.

- (b) The wreath product  $S_8 \wr S_5$  does not admit a solvable 2-coloring. Let us now consider the *transitive permutation groups without alternating composition factors*. Can we characterize, which of these do not admit a solvable 2-coloring?
  - (c) The automorphism group of every tournament is solvable. This statement is equivalent to the Feit–Thompson Theorem. Can we prove without using heavy group theory that *all tournaments have a solvable  $k$ -coloring for some fixed value  $k$* ? Or is such a statement still equivalent to Feit–Thompson?
- (6) A general theme is, what kind of structural reductions of the group can be achieved by a bounded number of colors. Here is a specific question of this type.
- (a) Does there exist a number  $g_0$  such that the following holds: *Every permutation group admits a 2-coloring that kills all non-alternating composition factors of order  $\geq g_0$ , i. e., after the coloring, all composition factors will either be alternating or of order  $< g_0$ .*
- (7) Some questions of this type arose in this paper when starting from a solvable group.
- (a) What is the smallest  $c$  such that *every solvable permutation group admits a 2-coloring that results in derived length  $\leq c$* ? Such a  $c$  exists by Theorem 1.4.
  - (b) What is the smallest  $C$  such that *every solvable permutation group admits a 2-coloring that reduces the length of all orbits to  $\leq C$* ? Such a  $C$  exists by Theorem 1.4. The group  $S_4 \wr S_4$  shows that  $C$  cannot be less than 4.
  - (c) Does every solvable group have a 3-coloring that results in a 2-group? Two colors do not suffice for this, as the example of  $S_4 \wr K$  shows for any solvable group  $K$  that is not a 2-group.
  - (d) Does every solvable permutation group have a 2-coloring that results in an *abelian-by-2-group*, i. e., in a group that has an abelian normal subgroup with the quotient being a 2-group?
  - (e) Sometimes instead of solvability of the automorphism group, we can assume something about the underlying structure. We discussed tournaments above. Another example: If  $X$  is a connected cubic graph then it has a *low-cost* (see below) 2-coloring that results in a 2-group: just color a pair of adjacent vertices red, the rest blue.  
For the same reason, if  $X$  is a connected graph such that every vertex has degree  $\leq k$  then the same low-cost 2-coloring results in a group with bounded composition factors. (Every composition factor is a subgroup of  $S_{k-1}$ .) This fact was used by Gene Luks to revolutionize the theory of Graph Isomorphism testing in 1980 [Lu82].
- (8) An important direction of study is the extension of known results about the minimal degree of primitive groups (often obtained via CFSG) to the motion of *strongly regular graphs*, *distance-regular graphs*, and *primitive coherent configurations (PCCs)*. Some work in this direction has already been done, see, e. g., [Ba81, Ba15], and the profound results in [SW15,

Ki21a, Ki21b, Ki21c]. A conjecture of this author that motivates Kivva's work [Ki21a, Ki21b, Ki21c] is the following.

**Conjecture 16.1.** Let  $\mathfrak{X}$  be a PCCs with  $n$  vertices. If  $\mathfrak{X}$  is not a Cameron scheme then the motion of  $\mathfrak{X}$  is  $\geq cn$  for some positive constant  $c$ .

The conjecture is motivated by a 1991 result by Liebeck and Saxl [LS91] that says that the statement is true with  $c = 1/3$  in the Schurian case.

The Cameron schemes are Schurian. They correspond to primitive permutation groups  $G$  acting on  $n = \binom{m}{k}^r$  elements for some  $m \geq 5$ ,  $1 \leq k \leq m - 1$ , and  $r \geq 1$ , as a subgroup of  $S_m \wr S_r$  containing  $A_m^r$  where  $S_m$  acts on the  $k$ -subsets of an  $m$ -set and  $S_r$  acts on the ordered  $r$ -tuples of such subsets.

In his monumental work, Kivva confirmed the conjecture for PCCs of rank  $\leq 4$  [Ki21c] and for distance-regular graphs of bounded diameter [Ki21a, Ki21b]. Given the combinatorial nature of the question, no group theory is involved in his proofs.

- (9) *Combinatorial symmetry breaking is a key aspect of the Graph Isomorphism problem* (see [Ba16]). From this perspective, the cost of breaking the symmetry is not the number of colors used but the *entropy of the distribution of colors*: if color  $i$  occurs  $k_i$  times on a set of size  $n = \sum_{i=1}^s k_i$  then we are looking at the quantity  $H(k_1/n, \dots, k_s/n) = -\sum (k_i/n) \log_2(k_i/n)$ .

If the  $s$  colors are uniformly distributed ( $k_i = n/s$  for all  $i$ ) then the entropy is  $\log_2 s$ . On the other hand, if one color dominates and all the other colors occupy just a small portion of the domain then the entropy is close to zero. For instance, in the case of a 2-coloring, which is equivalent to fixing a subset, we wish that subset to be as small as possible. The size of that set as a cost measure was introduced by Debra Boutin in 2008 [Bo08] (see also [BI17]) from the philosophical consideration that, given that in most cases of interest, an asymmetric 2-coloring exists, a more refined measure of the cost of symmetry breaking is needed. This measure of cost represents a *convergence with the classical concept of minimum bases of permutation groups*.

A *base*  $\Delta \subseteq \Omega$  is a subset such that the pointwise stabilizer is trivial:  $G_{(\Delta)} = 1$ . Bases have been introduced in computational group theory (Sims [Si70]) in the 1960s with the express purpose of breaking all symmetry, but the concept also has great theoretical significance (see below).

A base of size  $b$  gives a coloring with  $b + 1$  colors: individual colors for each element of the base, and a single color for the rest. Bases provide the *lowest entropy* among all colorings with  $b + 1$  colors. But bases may not use the optimal number of colors for the type of questions we are considering here; for instance, the base size for  $S_8$  is 8, but  $S_8$  has a solvable 2-coloring.

In any case, the sizable literature about minimal bases of permutation groups will be particularly relevant in the context of the refined cost measures. Here is a selection of relatively recent papers on minimum bases: [Li84, Py93b, Se97, GSS98, LS95, BGS11, BGS11]. One of the recent motivators of the area has been *Pyber's base size conjecture* (1993) [Py93b, p. 207], resolved in [DHM18] (2018) and made effective in [HLM19] (2019). The latter paper also includes a nice overview of the subject.

- (10) Symmetry-breaking by coloring of primitive coherent configurations is at the heart of the study of the *Graph Isomorphism* problem [Ba16]. One of the types of problems that arise there is to distinguish the Johnson, Hamming, and Cameron schemes from all other primitive coherent configurations by showing that for all other configurations, symmetries are destroyed at much lesser cost (see [Ba16]). Here the “cost” refers to the refined cost measures explained in the previous item.
- (11) Let me highlight an interesting generalization of Tucker’s Conjecture, proposed by Imrich, Smith, Tucker, and Watkins, that states that *a closed permutation group  $G$ , acting on a countably infinite set, with infinite minimal degree and finite subdegrees, admits an asymmetric 2-coloring* (“Infinite Motion Conjecture for Permutation Groups”) [IS+15, Sec 4]. Here “closed” means a closed subgroup of the symmetric group in the permutation topology, where a neighborhood basis of the identity consists of the pointwise stabilizers of finite subsets of the permutation domain. The subdegrees are the lengths of the orbits of the stabilizer of a point.

Let me conclude with a conjecture I have been entertaining for decades. We have seen that epimorphisms to the alternating groups give us a lot of trouble. (Such epimorphisms have also defined the bottleneck for Luks’s graph isomorphism test that caused three decades without progress on that problem, see [Ba16].) I believe the conjecture below relates to our subject, although I cannot draw a formal connection.

Answering this author’s question, in 1983, Martin Liebeck [Li83] proved that if  $X$  is a graph and  $\text{Aut}(X) \cong A_k$  then  $n$  (the number of vertices) must grow exponentially as a function of  $k$ . Specifically, he showed that  $n \geq 2^k - k - 2$  for all  $k \geq 13$  and that this lower bound is tight for  $k \equiv 0$  or  $1 \pmod{4}$ .

**Conjecture 16.2.** There exists a constant  $C > 1$  such that the following holds. Let  $X$  be a graph with  $n$  vertices. Assume  $\text{Aut}(X)$  has an epimorphism onto the alternating group  $A_k$  ( $k \geq 3$ ). Then  $n \geq C^k$ .

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my entry into the theory of primitive permutation groups [Ba81], and continued until recently with work on our joint paper [BPS09]. His imprint is discernible throughout this article.

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