

On the symmetries of the ^{12}C nucleus

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Abstract

The consequences of some symmetries of the three-alpha system are discussed. In particular, the recent description of the low-energy spectrum of the ^{12}C nucleus in terms of the Algebraic Cluster Model (ACM) is compared to that of the Semimicroscopic Algebraic Cluster Model (SACM). The previous one applies interactions of a D_{3h} geometric symmetry [1], while the latter one has a $U(3)$ multichannel dynamical symmetry, that connects the shell and cluster pictures. The available data is in line with both descriptions.

Keywords: alpha-clustering, geometrical, dynamical, and permutational symmetries

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The ^{12}C nucleus plays a crucial role in the evolution of stars, it is an important test ground of different models of atomic nuclei, furthermore, it is a rich laboratory of symmetries. Therefore, it is in the focus of both the experimental and the theoretical research. New states or transitions detected in this nucleus can choose between competitive models.

The recent observation of the 5^- state by the Birmingham group [1] was interpreted as an evidence for the D_{3h} symmetry in nuclear structure. The algebraic cluster model [2, 3] describes the detailed rotational-vibrational spectrum of the three-alpha system, and the application of its D_{3h} limit results in an energy spectrum that is in good agreement with the experimental observation [1].

Alpha-cluster models have been applied to the ^{12}C very extensively [4], from the beginning of nuclear structure studies [5] until very recently [6]. It seems to be a general agreement of these cluster studies that the ground state of the ^{12}C nucleus has a triangular shape at the vertices with the three alpha-particles. The novel feature of the algebraic cluster model interpretation [1, 2, 3] is twofold. First: in this model not only the ground state, or the ground band is considered to have the D_{3h} symmetry, but the excitation spectrum, too, including the Hoyle-band, and several others. Second: the algebraic cluster model describes in detail the spectrum of the three-alpha system with the triangular symmetry, as opposed to e.g. the Bloch–Brink alpha-cluster model, which gives this shape for the ground state, but it does not provide us with detailed spectrum [7].

In Table 1 we show the symmetries of some alpha-cluster models applied to the ^{12}C nucleus, including the present work. The interactions applied in the calculations are also listed. The microscopic cluster model (MCM) in the last line actually indicates a family of models (GCM, RGM,

OCM...). From the viewpoint of the applied symmetries they are similar to each other. For their detailed discussion and comparison we refer to [8].

A new chapter of the structure studies of ^{12}C is the application of those microscopic calculations, which describe each nucleonic degrees of freedom without supposing any cluster structure [9, 10, 11], and in some cases realistic nucleon-nucleon forces are applied, i.e. real ab initio calculations are carried out [12]. Obviously, for the understanding of the structure, these approaches are the most promising.

In this paper we investigate the three-alpha cluster system from a different angle. In particular, we compare the consequences of different symmetries of the system. The D_{3h} point symmetry, mentioned beforehand, is a geometrical one. In addition to this, the S_3 permutational symmetry of the three identical alpha-particles is also essential. It is involved both in the algebraic and in the other cluster models. A further basic symmetry is the antisymmetry of the 12 nucleons building up the ^{12}C , resulting in the Pauli-exclusion principle. This is taken into account in the microscopic (and semimicroscopic) cluster models, but it is not involved in the phenomenologic ones, like the algebraic cluster model. Here we make a comparison between the performance of the algebraic cluster model with D_{3h} symmetry and that of a $U(3)$ multichannel dynamical symmetry [13, 14]. This latter one is the connecting symmetry of the cluster and shell (quartet) models, which is formulated in the semimicroscopic algebraic approach [15, 16].

In what follows, first we introduce the semimicroscopic algebraic quartet (SAQM) and cluster (SACM) models and their connecting multichannel dynamical symmetry (MUSY). Then we present the $U(3)$ MUSY spectrum in comparison with the experimental one and with the

Model	Interaction	S_3	D_{3h}	Pp	Spectrum
ACM	D_{3h}	+	+	−	+
SACM	U(3) MUSY	+	−	+	+
Bl-Br	<i>Volkov, Brink</i>	+	−	+	−
MCM	<i>various</i>	+	−	+	+

Table 1: The symmetries of some 3-alpha-models of the ^{12}C nucleus. ACM: algebraic cluster model, SACM: semimicroscopic algebraic cluster model, Bl-Br: Bloch–Brink alpha-cluster model, MCM: microscopic cluster model. S_3 stands for the permutational symmetry of the three identical particles, D_{3h} is the geometrical symmetry of the equilateral triangle, while Pp means Pauli-principle. The last column indicates if detailed spectrum is provided by the model.

spectrum of the D_{3h} model. Finally, some conclusions are drawn.

The semimicroscopic algebraic quartet model (SAQM) [16] is a symmetry-governed truncation of the no-core shell model [17], that describes the quartet excitations in a nucleus. A quartet is formed by two protons and two neutrons, which interact with each other very strongly, as a consequence of the short-range attractive forces between the nucleons inside a nucleus [18]. The interaction between the different quartets is weaker. In this approach the L-S coupling is applied, the model space has a spin-isospin sector characterized by Wigner’s $U^{ST}(4)$ group [19], and a space part described by Elliott’s $U(3)$ group [20]. Four nucleons form a quartet [21] when their spin-isospin symmetry is $\{1,1,1,1\}$, and their permutational symmetry is $\{4\}$. This definition allows two protons and two neutrons to form a quartet even if they sit in different shells. As a consequence, the quartet model space incorporates 0, 1, 2, 3, 4, ... major shell excitations (in the language of the shell model), contrary to the original interpretation of [18], when the four nucleons had to occupy the same single-particle orbital, therefore, only 0, 4, 8, ... major shell excitations could be described.

The model is fully algebraic, therefore, group theoretical methods can be applied in calculating the matrix elements. The operators contain parameters to fit to the experimental data, that is why the model is called semimicroscopic: phenomenologic operators are combined with microscopic model spaces. Due to the quartet symmetry, only a single $\{1,1,1,1\}$ $U^{ST}(4)$ sector plays a role in the calculation of the physical quantities, thus the $U(3)$ space-group and its subgroups are sufficient for characterizing the situation:

$$U(3) \supset SU(3) \supset SO(3) \supset SO(2) \\ |[n_1, n_2, n_3], (\lambda, \mu), K, L, M]. \quad (1)$$

In Eq. (1) we have indicated also the representation labels of the groups, which serve as quantum numbers of the basis states. Here $n = n_1 + n_2 + n_3$ is the number of the oscillator quanta, and $\lambda = n_1 - n_2$, $\mu = n_2 - n_3$. The angular momentum content of a (λ, μ) representation is as follows [20]: $L = K, K + 1, \dots, K + \max(\lambda, \mu)$, $K = \min(\lambda, \mu), \min(\lambda, \mu) - 2, \dots, 1$ or 0, with the exception of

$K_L = 0$, for which $L = \max(\lambda, \mu), \max(\lambda, \mu) - 2, \dots, 1$ or 0. In the limiting case of the dynamical symmetry, when the Hamiltonian is expressed in terms of the invariant operators of this group-chain, an analytical solution is available for the energy-eigenvalue problem (an example is shown below).

The SAQM can be considered as an effective model in the sense of [22]: the bands of different quadrupole shapes are described by their lowest-grade $U(3)$ irreducible representations (irreps) without taking into account the giant-resonance excitations, built upon them, and the model parameters are renormalized for the subspace of the lowest $U(3)$ irreps.

The semimicroscopic algebraic cluster model (SACM) [15], just like the other cluster models, classifies the relevant degrees of freedom of the nucleus into two categories: they belong either to the internal structure of the clusters, or to their relative motion. In other words, the description is based on a molecule-like picture. The internal structure of the clusters is handled in terms of Elliott’s shell model [20] with $U^{ST}(4) \otimes U(3)$ group structure (as discussed beforehand). The relative motion is taken care of by algebraic models with a $U(3)$ basis. In particular, for a binary configuration it is the (modified) vibron model of $U(4)$ dynamical algebra [23], which is a group-theoretical model of the dipole motion. (The modification means a truncation of the basis due to the Pauli-principle [15].) For a ternary configuration the two independent Jacobi-coordinates are described by the $U(7)$ model [2, 3, 24, 25]. For a three-cluster configuration this model has a group-structure of $U_{C_1}^{ST}(4) \otimes U_{C_1}(3) \otimes U_{C_2}^{ST}(4) \otimes U_{C_2}(3) \otimes U_{C_3}^{ST}(4) \otimes U_{C_3}(3) \otimes U_R(7)$. The exclusion of the Pauli-forbidden states requires the truncation of the basis of the $U(7)$ model, that determines the lowest allowed major shell, and in addition one needs to distinguish between the Pauli-allowed and forbidden states within a major shell, too. Different methods can be applied to this purpose; e.g. one can make an intersection with the $U(3)$ shell model basis of the nucleus, which is constructed to be free from the forbidden states.

The SACM is fully algebraic, and semimicroscopic in the sense discussed above.

When we are interested only in spin-isospin zero states of the nucleus (a typical problem in cluster studies, and being our case here, too), then only the space symmetries are relevant (apart from the construction of the model space). For a ternary cluster configuration the corresponding group-chain is

$$U_{C_1}(3) \otimes U_{C_2}(3) \otimes U_{C_3}(3) \otimes U_R(7) \supset \\ U_C(3) \otimes \{U_R(6) \supset U_R(3)\} \supset \\ U(3) \supset SU(3) \supset SO(3) \supset SO(2). \quad (2)$$

The basis defined by this chain is especially useful for treating the exclusion principle, since the $U(3)$ generators commute with those of the permutation group, therefore, all the basis states of an irrep are either Pauli-allowed, or for-

bidden [26]. By applying basis (2) we can pick up the allowed cluster states from the U(3) shell model basis (1).

A Hamiltonian corresponding to the dynamical symmetry of group-chain (2) reads as

$$\begin{aligned}\hat{H} &= \hat{H}_{C_1} + \hat{H}_{C_2} + \hat{H}_{C_3} + \hat{H}_{U_R(7)} \\ &+ \hat{H}_{U_C(3)} + \hat{H}_{U_R(6)} + \hat{H}_{U_R(3)} \\ &+ \hat{H}_{U(3)} + \hat{H}_{SU(3)} + \hat{H}_{SO(3)}.\end{aligned}\quad (3)$$

We note here that the first part

$$\begin{aligned}\hat{H}_{CM} &= \hat{H}_{C_1} + \hat{H}_{C_2} + \hat{H}_{C_3} + \hat{H}_{U_R(7)} \\ &+ \hat{H}_{U_C(3)} + \hat{H}_{U_R(6)} + \hat{H}_{U_R(3)}\end{aligned}\quad (4)$$

is an operator that corresponds to the pure cluster picture, while the second part

$$\hat{H}_{SM} = \hat{H}_{U(3)} + \hat{H}_{SU(3)} + \hat{H}_{SO(3)}\quad (5)$$

is a shell model Hamiltonian (of the united nucleus).

The *multichannel dynamical symmetry* (MUSY) [13, 14] connects different cluster configurations (including the shell model limit) in a nucleus. Here the word channel refers to the reaction channel that defines the cluster configuration.

The simplest case is a two-channel symmetry connecting two different clusterizations. It holds when both cluster configurations can be described by a U(3) dynamical symmetry, and in addition a further symmetry connects them to each other. This latter symmetry acts in the pseudo space of the particle indices [27, 14]. The \hat{H}_{SM} Hamiltonian of Eq. (5) is symmetric with respect to these transformations, therefore, it is invariant under the changes from one clusterization to the other. The cluster part of the Hamiltonian, \hat{H}_{CM} is affected by the transformation from one configuration to the other, of course. Nevertheless, it may remain invariant, which is the case for simple operators, like the harmonic oscillator Hamiltonian, or the quadrupole operator [14]. Due to this symmetry of the quadrupole operator, the $E2$ transitions of different clusterizations also coincide, when the MUSY holds, just like the energy eigenvalues of the symmetric Hamiltonians [14].

The MUSY is a composite symmetry of a composite system. Its logical structure is somewhat similar to that of the dynamical supersymmetry (SUSY) of nuclear spectroscopy [28]. In the SUSY case the system has two components, a bosonic and a fermionic one, each of them showing a dynamical symmetry, and a further symmetry connects them to each other. The connecting symmetry is that of the supertransformations which change bosons into fermions, or vice versa. In the MUSY case the system has two (or more) different clusterizations, each of them having dynamical symmetries which are connected to each other by the symmetry of the pseudo space of the particle indices that change from one configuration to the other.

When the multichannel dynamical symmetry holds then the spectra of different clusterizations are related to each

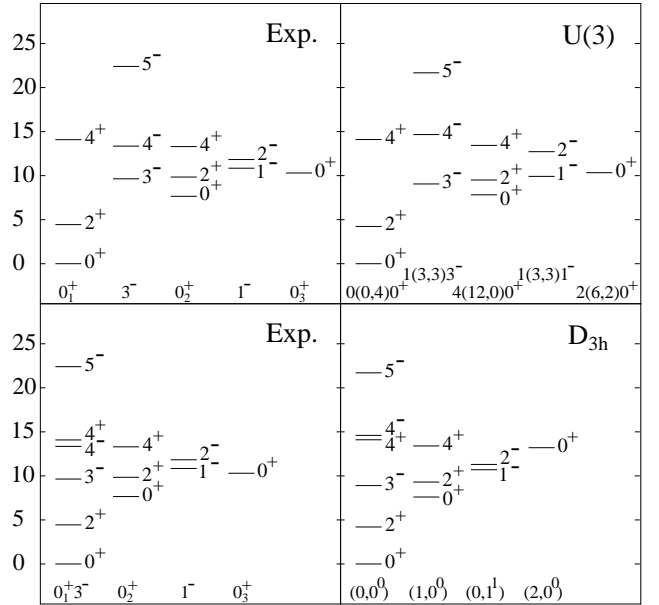


Figure 1: Comparison between the experimental and model spectra of the ^{12}C nucleus. The energy scale is given in MeV, and the bands of the experimental part are labelled by the K^π values. The D_{3h} (as well as the experimental spectrum) is taken from [1], and the model bands are labelled by the rotational-vibrational quantum numbers of the D_{3h} model. In the U(3) MUSY spectrum the bands have the $n(\lambda, \mu)K^\pi$ quantum numbers.

other by very strong constraints. The MUSY provides us with a unified multiplet structure of different cluster configurations, furthermore the corresponding energies and $E2$ transitions coincide exactly. Of course, it can not be decided a priori whether the MUSY holds or not, rather one can suppose the symmetry and compare its consequences with the experimental data.

The energy spectrum of Figure 1 was calculated from the formula

$$\begin{aligned}E &= e n + a (\lambda^2 + \mu^2 + \lambda\mu + 3\lambda + 3\mu) \\ &+ b (\lambda - \mu)(\lambda + 2\mu + 3)(2\lambda + \mu + 3) \\ &+ f K^2 + d \frac{1}{2\theta} L(L + 1).\end{aligned}\quad (6)$$

In the first term n is the number of oscillator quanta. The second term is the expectation value of the second-order invariant operator of the SU(3) algebra, which represents quadrupole-quadrupole interaction. The third one is the eigenvalue of the third-order invariant distinguishing between the prolate and oblate shapes. The K -dependent term splits the bands belonging to the same SU(3) representation. (The corresponding operator is determined by the operators of the integrity basis of the SU(3) algebra, and is very nearly diagonal in the SU(3) basis states [29].) In the last part θ is the moment of inertia calculated classically for the rigid shape determined by the U(3) quantum numbers (for a rotor with axial symmetry) [16]. The parameters were fitted to the experimental data:

$e = 37.8$ MeV, $a = -2.4879$ MeV, $b = 0.045432$ MeV, $f = -0.98366$ MeV, $d = 1.1661$ MeV.

As it is discussed above, the U(3) MUSY connects the cluster and quartet (i.e. shell) descriptions. Therefore, in determining its parameters some shell-model constraints (e.g. systematics) can be, and in some cases has been applied [30]. In the present study, however, we determined the parameters from the experimental spectrum, like in the work [1], in order to treat the two descriptions (based on the D_{3h} and U(3) MUSY) on an equal footing. For comparison we show in the lower part of Figure 1 the result of the ACM, too from [1]. The number of parameters is comparable in the two cases: 5 in our Eq. (6), and 6 in Eq. (2) of [1].

In comparing our semimicroscopic description to that of the no-core symplectic shell model [11], it is worth noting that the lowest-grade SU(3) symmetries we associate with the ground-state band and to the Hoyle-band are the dominating ones in the fully microscopic description, too. In particular, the (0,4) basis has the largest contribution to the ground state, and (12,0) symmetry is the head of the symplectic band, dominating the Hoyle-states.

To sum up: In Figure 1 we compared the model spectra of two algebraic descriptions to the experimental spectrum of the ^{12}C nucleus. The result illustrates both the usefulness of the U(7) dynamical algebra in the treatment of the three-cluster problem, and the fact that it incorporates different models. In particular, the algebraic cluster model with D_{3h} interaction and the semimicroscopic algebraic cluster model with the U(3) multichannel dynamical symmetry give very similar descriptions. Therefore, further experimental details (on the “missing states” and on the transitions) seem to be essential in order to decide which symmetry is realized to a better approximation. (At the same time they can deepen our understanding in terms of the fully microscopic theories.) From the viewpoint of the symmetry studies the combination of the two algebraic methods can also be very informative: the operators with the D_{3h} symmetry of the ACM could be applied on the model space of the SACM, which incorporates the Pauli-principle. This algebraic treatment would include all the symmetries applied only partly so far by different models (Table 1).

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