

# On the connection between almost periodic functions and Blazhko light curves

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Accepted 2017 September 5. Received 2017 September 4; in original form 2017 July 10

## ABSTRACT

In this paper it is shown that the mathematical form which most precisely describes the Blazhko RR Lyrae light curves is connected to almost periodic functions and not to the mathematics of modulation. That is, the Blazhko effect is more than a simple external modulation of the pulsation signal. The mathematical framework of almost periodic functions predicts a new observable effect: a shift of the Fourier harmonics of the main pulsation frequency from the exact harmonic position. This phenomenon is called as harmonic detuning effect (HDE). The published deviations of the harmonics of V445 Lyr are explained with this effect. HDE is also found for V2178 Cyg, another Blazhko star observed by the *Kepler* space telescope. HDE is detectable only if the phase variation part of the Blazhko effect is of large amplitude and non-periodic enough, additionally, the time span of the observed light curve is sufficiently long for obtaining precise frequencies. These three conditions restrict the number of stars showing detectable HDE and explain why this effect has not been discovered up to now.

**Key words:** stars: oscillations – stars: variables: RR Lyrae – methods: analytic – methods: data analysis – space vehicles

## 1 INTRODUCTION

The recent photometric space missions *Kepler* (Borucki et al. 2010) and CoRoT (Baglin et al. 2006) served us long and precise time series for huge amount of variable stars, including RR Lyrae stars showing Blazhko effect. Even the definition of the effect has been refined using space photometry (Benkő et al. 2010; Szabó et al. 2014; Benkő et al. 2014, 2016), namely, that the Blazhko effect means a simultaneous amplitude and frequency variation of the RR Lyrae light curves with the same period(s).

From a mathematical point of view the Blazhko RR Lyrae light curves are described either as modulated signals (Benkő et al. 2011 and references therein) or signals of a beating phenomenon (Kolenberg et al. 2006 and its references). As is shown in Benkő et al. (2011) the modulation framework can explain many but not all observed light curve and Fourier spectrum features. The formulae of the modulation picture were applied to the *Kepler* observations of V445 Lyr, a Blazhko star with a strong effect, and give only a moderately good fit (Guggenberger et al. 2012). The large fitting residual was explained by the influence of the low-amplitude frequencies and the irregular nature of the star's

Blazhko effect. However, Szeidl et al. (2012) demonstrated that the modulation formulae result in an inappropriate fit for CM UMa as well, although the star shows only a simple sinusoidal amplitude variation. Then they suggested a complex formula which fits the light curve properly.

This paper investigates the problem a bit further. As we will see, the solution is to reject the simple modulation framework and to apply a more complex description using almost periodic functions. Beyond the mathematical significance of this new framework it predicts a new observable effect and some physical consequences, as well. The potential usability of almost periodic functions has already been raised in two previous conference presentations (Benkő & Paparó 2013; Benkő & Szabó 2016). Present article discusses their actual use in detail.

## 2 MOTIVATION

The light curves of the RR Lyrae stars are comprehended as non-sinusoidal periodic signals represented by their Fourier sums. Following Benkő et al. (2011) a generally non-sinusoidally modulated RR Lyrae light curve can be de-

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scribed as:

$$m(t) = \left[ a_0^A + g^A(t) \right] \left\{ a_0 + \sum_{i=1}^n a_i \sin \left[ 2\pi i f_0 t + \varphi_i + i g^F(t) \right] \right\}, \quad (1)$$

where the modulation functions are

$$g^M(t) = \sum_{j=1}^{l^M} a_j^M \sin(2\pi j f_m t + \varphi_j^M), \quad M = A, \text{ or } F. \quad (2)$$

Here  $f_0$  and  $f_m$  mean the main pulsation and the modulation frequencies,  $i, j$  are integer running indices,  $a$  and  $\varphi$  coefficients are the Fourier amplitudes and phases, respectively. The  $a_0^A$ ,  $a_0$  are the zero point constants;  $n$  and  $l^M$  integers show the number of terms in the finite Fourier sums. The upper index A indicates the amplitude modulation (AM) and index F means the frequency modulation (FM). From now on the independent variable  $t$  is referred as ‘time’, because this paper deals with time dependent functions only. The formula (1) can be rewritten as:

$$m(t) = a_0^A a_0 + a_0 g^A(t) + \sum_{i=1}^n \left[ a_0^A a_i + a_i g^A(t) \right] \sin \left[ 2\pi i f_0 t + \varphi_i + i g^F(t) \right]. \quad (3)$$

If we do not assume anything about the light curves, we can start with the observed Fourier spectra as well. This was exactly what [Szeidl et al. \(2012\)](#) did. They searched for a closed form which fits the best the observed light curves. They received complicated  $g^M(t)$  modulation functions which depend on the Fourier orders  $i$  of the carrier wave:

$$g_i^M(t) = \sum_{j=1}^{l^M} a_{ij}^M \sin(2\pi j f_m t + \varphi_{ij}^M), \quad M = A \text{ or } F. \quad (4)$$

Their entire light curve fitting formula is

$$m^*(t) = m_0 + \sum_{k=1}^l b_k \sin(2\pi k f_m t + \varphi_k^b) + \sum_{i=1}^n \left[ a_i + g_i^A(t) \right] \sin \left[ 2\pi i f_0 t + \varphi_i + g_i^F(t) \right]. \quad (5)$$

Comparing this formula with Eq. (3) we see an additional difference beyond the different modulation functions: the second term in Eq. (5) describes an independent zero point (a.k.a average brightness) variation, while in the formula (3) this variation directly depends on the amplitude modulation function  $g^A(t)$ .

I emphasize that formula (5) was found on empirical basis: this format fits better the observed light curves than Eq. (3), however, there was no explanation why the modulation functions depend on the harmonic orders of the carrier wave’s Fourier solution. These Fourier harmonics have only mathematical meaning. They represent the non-sinusoidal nature of the light curves but they have no physical meaning. How is it possible to have a modulation in the star which influences different ways for each harmonic?

### 3 ALMOST PERIODIC FUNCTIONS

To answer this question, first, let us reproduce some basic definitions. An  $x(t)$  real function is periodic with the period  $P$  if

$$x(t) = x(t + P), \quad \text{or} \quad |x(t) - x(t + P)| = 0. \quad (6)$$

As it is well-known  $x(t)$  function has a unique Fourier representation which can be written in different forms. One of them is as follows:

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \sin(2\pi n f_0 t + \varphi_n), \quad (7)$$

where  $a_0$  is the zero point,  $f_0 = 1/P$  is the frequency,  $a_n$ , and  $\varphi_n$  are the usual Fourier coefficients,  $n$  is an integer index.

Now I define an extension of the real periodic functions. Harald Bohr was the first who defined almost periodic functions in the 1920s. By following his definitions ([Bohr 1947](#)): ‘Let us take an arbitrary function  $z(t)$  continuous for  $-\infty < t < \infty$ . The real number  $\tilde{P}$  will then be called a translation number of  $z(t)$  corresponding to  $\varepsilon$  whenever

$$|z(t) - z(t + \tilde{P})| \leq \varepsilon \quad (8)$$

(...) A  $z(t)$  function will be called almost periodic when, (...) to every  $\varepsilon > 0$ , a length  $L = L(\varepsilon)$  of some sort exists such that each interval of length  $L(\varepsilon)$  contains at least one translation number  $\tilde{P} = \tilde{P}(\varepsilon)$ .’ Comparing the formula (8) with (6) we see that if  $\varepsilon \rightarrow 0$  almost periodic function  $z(t)$  became fully periodic and the translation number plays the role of period.

Almost periodic functions have many similar properties to the periodic ones: e.g. they form a complete orthogonal system, or they have unique Fourier series ([Bohr 1947](#)). The Fourier representation of a  $z(t)$  function in the most suitable form for us now is

$$z(t) = \frac{a_0(t)}{2} + \sum_{n=1}^{\infty} a_n(t) \sin \left[ 2\pi n \int_0^t f_0(\tau) d\tau + \varphi_n(t) \right]. \quad (9)$$

Here, as opposed to the expression (7), all  $a_0(t)$ ,  $f_0(t)$ ,  $a_n(t)$ , and  $\varphi_n(t)$  quantities are (time-dependent) functions.

If we compare Eq. (3) and (5) with Eq. (9) we see that both expressions (3) and (5) show Fourier representation of almost periodic functions. It is easily seen that the zero point variation function ( $a_0(t)/2$ ) could either be the first two terms of (3) or (5). Similarly, the amplitude variation function  $a_n(t)$  can correspond to the coefficients of the sin functions in the square brackets of the third term. The  $\varphi_n(t)$  function can be equated both for Eq. (3) and (5) by the last two terms in the argument of their sin functions. By examining the frequency variation functions we can see that both the modulation picture (3) and the best-fitting mathematical model (5) assume fixed fundamental frequency since

$$\int_0^t f_0(\tau) d\tau = f_0 t, \quad (10)$$

which is an evident simplification. It follows that we cannot distinguish between the frequency and phase variations in an unknown observed signal (see also [Benkő et al. 2011](#) for the discussion).

The uniqueness of the Fourier representation has an important consequence. On the one hand, [Benkő et al. \(2011\)](#)

showed that the Fourier solution of a general periodically modulated periodic signal can be written in the form of Eq. (1). On the other hand, Fourier solution of almost periodic functions is unique, so Eq. (5) belongs to the same function only if it can be transformed equivalently into Eq. (1). But this is not possible, because Eq. (5) represents a more general function than Eq. (1).

As a conclusion of this section we can state that both externally modulated periodic signals and Blazhko light curves can also be described by almost periodic functions but these functions are generally different. Other words, the Blazhko effect cannot be an external modulation on the pulsation. Now we understand Szeidl et al. (2012) finding: they showed, albeit they did not state directly, that the Blazhko light curves are not modulated signals but signals of a different and more complicated physical effect.

#### 4 THE HARMONIC DETUNING EFFECT

As we have seen, the similarities of periodic and almost periodic functions have important consequences. This is true of their differences, as well. Considering the definition of the instantaneous angular frequency  $d\omega(t)/dt = d[2\pi f(t)]/dt$ , and Eq. (9), the instantaneous frequency of the components of an almost periodic signal is:

$$f_n(t) = nf_0(t) + \frac{1}{2\pi} \varphi'_n(t). \quad (11)$$

In general, there is no one-to-one relationship between the instantaneous frequency and the spectral frequency in terms of Fourier decomposition. A detailed review of the topic can be found e.g. in Boashash (1992). Fortunately, the average frequency of the Fourier spectrum equals the time average of the instantaneous frequency. Let us denote the time average of the quantities in Eq. (11) as  $f_n = \langle f_n(t) \rangle$ ,  $f_0 = \langle f_0(t) \rangle$ , and  $\langle \varphi'_n \rangle = \langle \varphi'_n(t) \rangle$ , and

$$f_n = nf_0 + \frac{1}{2\pi} \langle \varphi'_n \rangle. \quad (12)$$

That is, the Fourier spectrum contains not the exact harmonic frequencies  $nf_0$  but frequencies which are shifted by a term containing the time average of the derivative of the phase variation functions  $\varphi_n(t)$ . From now on this phenomenon is referred to as harmonic detuning effect (HDE).

Can this effect be detectable? The generally accepted picture is that the Fourier spectra of the Blazhko light curves contain the main pulsation frequency and its exact harmonics. In the case of V445 Lyr – a *Kepler* star showing extremely complex Blazhko effect –, however, Guggenberger et al. (2012) reported a ‘systematic and significant’ deviation of the detected harmonic frequencies from their exact harmonic position. This finding raises the question whether these detected deviations are due to the HDE or not?

##### 4.1 Observed frequency deviations

###### 4.1.1 The case of V445 Lyr

To decide on this question let us investigate the reported feature in detail. Figure 4 of Guggenberger et al. (2012) showed the detected  $f$  frequencies of V445 Lyr as the function of modulo  $f_0$ , where  $f_0$  means the main pulsation frequency.

(This  $f$  modulo  $f_0$  quantity is denoted here by  $D$ ). If the  $f_n$  components of the Fourier solution were exact harmonics ( $f_n = nf_0$ ), then  $D(f_n) = D_n = 0$  and the points which symbolise the frequencies  $f_n$  would have been on a vertical line. However, we see a slight rightward tilt of the line connecting these frequencies.

The critical point is the significance of these deviations. Since there is no significance test in Guggenberger et al. (2012) paper, here is performed one. If we write the deviation parameter (or shortly deviation) as

$$D_n = \left| \frac{f_n}{f_0} - n \right|, \quad (13)$$

we can see that the  $\sigma(D)$  accuracy depends only on the given frequencies  $f_0$  and  $f_n$  and their accuracies  $\sigma(f_0)$  and  $\sigma(f_n)$ :

$$\sigma(D_n) = \frac{f_n \sigma(f_0) + f_0 \sigma(f_n)}{f_0^2}. \quad (14)$$

So these frequencies are needed with their estimated errors as accurately as possible.

This work used the *Kepler* long cadence observation of V445 Lyr. The photometric data were taken from the paper of Benkő et al. (2014), where all characteristics of this data set were described. It should be mentioned that, as opposed to the case of Guggenberger et al. (2012), now the entire *Kepler* observations are available which resulted here a ~2.5 times longer (1426 days vs. 588 days) data set.

The usual way for obtaining the frequency content of a variable star’s light curve is a consecutive pre-whitening process. This method, however, could sometimes cause serious troubles as well (see Balona 2014). In our particular case the problem is that each pre-whitening step adds to the frequency uncertainty. To avoid this effect, the raw (unwhitened) spectrum was exclusively used. Generally this needs extra caution, because the structure of the lower amplitude frequencies are dominated by the window function. Fortunately, the investigated frequencies are amongst the highest amplitude ones, so this problem does not affect the results.

The accuracy of the frequency determination is often estimated through the  $\chi^2$  error of the non-linear fit of the given harmonic component (e.g. Montgomery & O’Donoghue 1999; Lenz & Breger 2005). However, because the present investigation does not need amplitudes and phases, the involvement of the amplitude and phase errors into the frequency determination could be avoided. A common simple frequency accuracy estimation which does not need any fits is the Rayleigh resolution ( $\sigma_R(f) = 1/\Delta t$ , where  $\Delta t$  is the total time span of the observation). For the *Kepler* data of V445 Lyr it is  $0.0007 \text{ d}^{-1}$ . Kallinger, Reegen & Weiss (2008) showed, however, that the Rayleigh resolution value is a drastic overestimate of the real uncertainty and found the value  $\sigma_K(f) = 1/(\Delta t \sqrt{s_f})$  to be a reliable upper estimate for the frequency determination error. The quantity  $s_f$  here the spectral significance of the frequency defined by Reegen (2007). This error estimation  $\sigma_K(f) = \sigma(f)$  was accepted as a first approximation.

The steps of the frequency finding process are as follows: the SIGSPEC program (Reegen 2011) was run once for finding the dominant frequency  $f_0$  and its accuracy  $\sigma(f_0)$ . Then it was run again in a small interval around each  $nf_0$  harmonic position ( $nf_0 - 0.05 \leq f \leq nf_0 + 0.05$ ) separately

**Table 1.** An excerpt from the electronic table `dn_all.dat` with the values of V445 Lyrae. The individual columns contain the harmonic order,  $n$ ; the Fourier frequency,  $f_n$ ; its error,  $\sigma(f_n)$ ; the  $D_n$  deviation of  $f_n$  frequency from the exact harmonic; and its error,  $\sigma(D_n)$ .

$n$	$f_n$ ( $d^{-1}$ )	$\sigma(f_n)$ ( $d^{-1}$ )	$D_n$	$\sigma(D_n)$
...				
1	1.9489699	8.75E-06	0.00E+00	8.98E-06
2	3.8979614	3.12E-05	1.10E-05	2.50E-05
3	5.8470079	1.38E-04	5.03E-05	8.43E-05
4	7.7968746	2.23E-04	5.10E-04	1.32E-04
5	9.7457488	4.23E-04	4.61E-04	2.40E-04
...				

for finding the proper position of  $f_n$ . Then the highest amplitude frequency was double checked whether it is really a harmonic and not a Blazhko side frequency which has higher amplitude than its central peak. In the latter case the best harmonic candidate was selected from the SIGSPEC result file and run the program on a narrower interval around this candidate frequency. With this process it was achieved that all frequencies are determined in the raw (unwhitened) spectrum. The use of the raw spectrum provides better control on the errors but at the same times it reduces the number of significant frequencies. For V445 Lyr only five harmonics have been found to be significant ( $s_f > 5$ ).

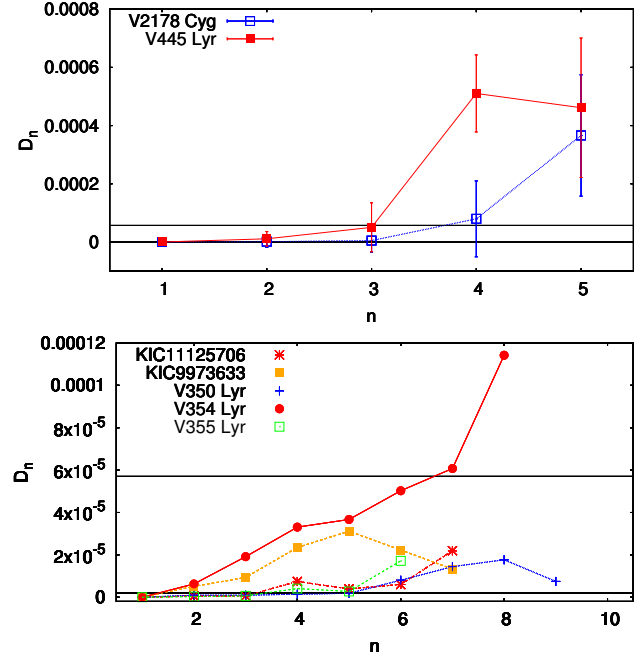
The obtained frequencies,  $f_n$ , their calculated formal errors,  $\sigma(f_n)$ , the deviation values,  $D_n$ , and their errors,  $\sigma(D_n)$  are given in the electronic table attached to this paper from which the values of V445 Lyr are shown as an excerpt in Table 1.

Plotting the deviation parameter,  $D_n$ , with its uncertainty,  $\sigma(D_n)$ , in the function of the harmonic order,  $n$ , we receive the result in the upper panel of Fig. 1. The red filled squares show the values of V445 Lyr (for better visibility of the trend the points are connected). As we can see, with the increasing harmonic order the deviation also tends to increase, and this deviation is significant above the 4th order. With these statements the findings of Guggenberger et al. (2012) about the systematic and significant deviation trend are validated.

#### 4.1.2 The Kepler sample

After the cited Guggenberger et al. (2012) paper no similar deviation trends have been reported, what is most probably, the position of Blazhko star's harmonic frequencies have not been investigated at all.

That is why the light curves of the fundamental mode RR Lyrae (RRab) stars of the original *Kepler* field were checked. The measurements of this sample are the best ever observed in the sense of accuracy and continuous sampling. The quarterly stitched and tailor-made aperture photometric light curves were used from the papers Benkő et al. (2014); Benkő & Szabó (2015). The sample contains 17 Blazhko RRab stars (see their names in column 1 of Table 2) which – apart from RRLyr, the prototype – is the complete set of the published data. RRLyr was omitted because its



**Figure 1.** Deviation parameter  $D_n$  as the function of the harmonic order  $n$ . (top) The found two significantly detuned stars are V445 Lyr (filled red squares) and V2178 Cyg (open blue squares). The black horizontal lines symbolise the range of  $D_n$  parameters of non-Blazhko stars. The formal calculated errors are shown with the error bars. (bottom) The next five Blazhko stars' deviation function are shown with different symbols. The black horizontal lines are the same as the top panel. None of the plotted function show significant deviation, but V354 Lyr (filled red circles) shows higher deviations than the non-Blazhko stars. For the sake of clarity the error bars are omitted here and the points are connected with continuous lines in both panels.

overexposed measurements would have needed special handling.

As an additional check for the accuracy of the frequency position the *Kepler* non-Blazhko RRab sample were also processed. Up to now 19 non-Blazhko stars<sup>1</sup> were published in the original *Kepler* field (Nemec et al. 2011, 2013). The names of the used stars are given in Table 2. Stitched and tailor-made aperture photometric data of non-Blazhko stars, however, have not been available still now. For uniform handling this data set was prepared. The *Kepler* long-cadence Q0-Q17 data were processed the same way as it was described in Benkő et al. (2014) in details. The resulted light curves can be downloaded from the same site as the Blazhko data<sup>2</sup>.

The  $f_n$ ,  $\sigma(f_n)$ ,  $D_n$ , and  $\sigma(D_n)$  parameters were determined for all stars. The applied frequency searching method was identical with the above described one for the case of V445 Lyrae. The result is given in the same electronic table

<sup>1</sup> The present non-Blazhko sample is the same as listed by Nemec et al. (2013) apart from those two stars (V350 Lyr and KIC 7021124) which turned out to be showing a slight Blazhko effect (Benkő & Szabó 2015).

<sup>2</sup> [http://www.konkoly.hu/KIK/data\\_en.html](http://www.konkoly.hu/KIK/data_en.html)

in the same form as it is shown for V445 Lyr in Table 1. Based on the results the following statements can be made:

(1) Beyond the case of V445 Lyr, significant deviation values were found for V2178 Cyg as well. Its  $D_n$  vs.  $n$  function is shown in the top panel of Fig. 1 (blue open squares). The systematic increase of the deviation with the increasing harmonic order is clearly visible. The deviation of the 5th component has become significant.

(2) No further stars of the sample show significant deviation. The deviation function of the five stars showing the next highest deviation parameters are plotted in the bottom panel of Fig. 1. Due to the short range shown in the vertical axis, indicating the errors would make the figure obscure, the error bars are omitted.

(3) If one selects the highest  $D_n$  values for each non-Blazhko star ( $D_{\max}$ ) separately one found that these numbers spread in a very limited interval between  $2 \cdot 10^{-6}$  and  $5.7 \cdot 10^{-5}$  (see column 4 in Table 2). These two limit values are represented by horizontal lines in the panels of Fig. 1. As we see the Blazhko stars' values are within this range except for V354 Lyr.

(4) If we compare the measured  $D_{\max}$  values in column 4 of Table 2 with the formal errors  $\sigma(D_{\max})$  (column 3) calculated from the  $\sigma_K$  values (in column 2), one finds that this calculated formal error might be a 5-10 times overestimation of the measured deviations. In the light of this, V354 Lyr is a possible candidate star for showing HDE.

#### 4.1.3 Alternate explanation for the deviations

In the previous subsections significant harmonic detuning was detected for at least two *Kepler* Blazhko RR Lyrae stars. Since the light curve description using almost periodic functions directly predicts such an effect (the HDE) for the detected deviations HDE seems to be the most plausible explanation.

The paper of Guggenberger et al. (2012) gives two possible explanations for the detected deviations of V445 Lyr: (1) 'result of period changes', or (2) 'close unresolved peaks' in the Fourier spectrum. The first case is closely related to the HDE explanation since the period change means phase variation and the phase variation functions play the key role in HDE (see Eq. 12). The second option implicitly assumes the existence of a very long period (maybe secular) or a nearly pulsation period change that would imply an unresolved close frequency  $f_c$ . The other 'unresolved' frequencies would be the linear combinations of  $f_c$  with the harmonics of the main pulsation frequency. Because of their identical frequency differences the observed different  $D_n$  values would be difficult to interpret.

The case of V354 Lyr raises a third possibility. The  $D_n$  values of V354 Lyr fit well to a linear trend up to  $n = 7$  with the slope of  $1.03 \cdot 10^{-5}$ . This can be explained as follows. The frequencies are always inaccurately known:  $f_0^\delta = f_0 + \delta_0$ ,  $f_n^\delta = f_n + \delta_n$ . (Here the measured inaccurate frequencies are upper indexed by a  $\delta$ , their errors are denoted by  $\delta_0$  and  $\delta_n$ , the exact quantities are  $f_0$  and  $f_n$ .) If  $\delta_n = n(\delta_0 + \delta')$  ( $\delta' \neq 0$ ), the  $D_n$  value increases linearly with the harmonic order,  $n$ , even if the harmonic frequencies are in the exact harmonic position ( $f_n = n f_0$ ). This argument assumes that the error of the harmonic frequencies is systematically different from

**Table 2.** The used *Kepler* RR Lyrae sample. The columns show the star's name; the maximal value of the estimated frequency error of the star's harmonic components; the highest calculated deviation error; the maximum value of the deviation.

Name/KIC	$\sigma_{K,\max}$ ( $\times 10^{-4} \text{d}^{-1}$ )	$\sigma(D)_{\max}$ ( $\times 10^{-4}$ )	$D_{\max}$ ( $\times 10^{-4}$ )
Blazhko stars			
V783 Cyg	1.89	1.49	0.04
V808 Cyg	4.90	3.01	0.12
V838 Cyg	2.83	1.71	0.16
V1104 Cyg	2.59	1.42	0.03
V2178 Cyg	3.88	2.09	3.66
V350 Lyr	2.42	1.82	0.18
V353 Lyr	2.15	1.48	0.12
V354 Lyr	4.02	2.63	1.14
V355 Lyr	1.90	1.11	0.17
V360 Lyr	2.40	1.57	0.03
V366 Lyr	2.27	1.46	0.08
V445 Lyr	4.23	2.40	5.10
V450 Lyr	1.88	1.16	0.07
7021124	2.56	2.00	0.03
7257008	4.99	3.29	0.07
9973633	6.14	4.16	0.31
11125706	2.93	2.15	0.22
non-Blazhko stars			
AW Dra	2.92	2.44	0.31
V715 Cyg	2.70	1.61	0.13
V782 Cyg	2.95	1.80	0.09
V784 Cyg	2.49	1.55	0.02
V839 Cyg	4.15	2.38	0.34
V894 Cyg	2.87	2.08	0.22
V1107 Cyg	2.20	1.52	0.04
V1510 Cyg	2.95	2.13	0.15
V2470 Cyg	2.44	1.60	0.33
FN Lyr	2.14	1.57	0.46
NQ Lyr	2.61	1.87	0.26
NR Lyr	2.42	2.09	0.57
V346 Lyr	2.74	1.95	0.22
V349 Lyr	1.98	1.29	0.12
V368 Lyr	2.63	1.49	0.06
6100702	3.04	1.72	0.06
7030715	2.87	2.34	0.39
9658012	5.52	3.78	0.33
9717032	6.02	4.22	0.28

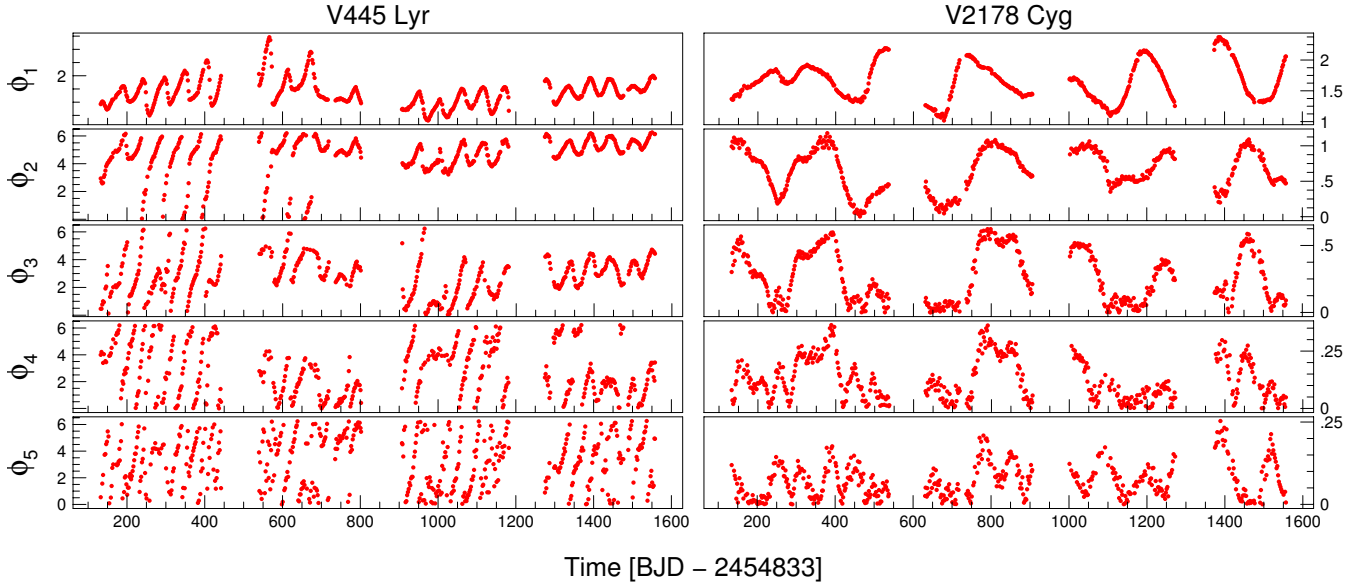
the error of the main frequency ( $\delta_n \neq n\delta_0$ ). This cannot be ruled out completely even though we have done everything to avoid it.

If we could get compatible numerical values from the observed material and the prediction of HDE, it would clearly demonstrate that the measured differences were caused by HDE. We will try this in the next section.

## 4.2 Phase variation function

In Sec. 4.1 the deviation parameter,  $D_n$ , was measured with the formula (13). We can express this quantity by an alternate form as well. From Eq. (12) and (13)

$$D_n = \left| \frac{\langle \varphi'_n \rangle}{2\pi f_0} \right|. \quad (15)$$



**Figure 2.** The first five Fourier phase variation function  $\varphi_n(t)$  ( $n = 1, 2, \dots, 5$ ) for the *Kepler* Blazhko stars V445 Lyr and V2178 Cyg.

If we determine the  $D_n$  values from both (13) and (15) and find that these are compatible values then we would justify formula (12) which would make the almost periodic explanation more robust.

Expression (15) includes the main pulsation frequency  $f_0$  as it can be determined from the spectrum and the time average of the derivative of the phase variation function. The latter parameter has never been determined for any stars. The temporal variability of the Fourier phases of the Blazhko stars' harmonic frequencies were discovered and studied already from the best ground-based observations (e.g. Jurcsik et al. 2005, 2006, 2008) but in these cases the phase variations were typified by Fourier phases vs. Blazhko phase functions. The first phase-time functions could only be prepared in the space photometry era (Guggenberger et al. 2011, 2012; Nemeč et al. 2011, 2013).

There are two obvious ways of determining  $\langle \varphi'_n \rangle$  values.

(1) Using the definition of the time average of a function:

$$\langle \varphi'_n \rangle = \frac{1}{\Delta t} \int_{t_0}^{t_1} \varphi'_n(t) dt = \frac{1}{\Delta t} [\varphi_n(t_1) - \varphi_n(t_0)], \quad (16)$$

where  $t_0$ ,  $t_1$ , and  $\Delta t$  mean the starting and ending epochs, and the total time span of the observation, respectively. (2) The alternate possibility is to determine the  $\varphi'_n(t)$  functions (by a numerical derivation of the  $\varphi_n(t)$  functions) then to average them. It is well known that the phases are the least accurately determinable Fourier parameters in any Fourier fits. Unfortunately both methods need the  $\varphi_n(t)$  functions. Moreover, the second method requires the numerical derivation as well, which causes additional numerical inaccuracies, so the first one has been chosen.

The  $\varphi_n(t)$  functions were determined for the two stars which show the HDE (V445 Lyr and V2178 Cyg). The  $\varphi_n(t)$  functions were prepared with the phase variation calculation tool of the PERIOD04 (Lenz & Breger 2005) package. This tool subdivides the total observed time span into short time intervals (bins) then it applies a non-linear fit for each bin

separately. For compatibility, in the case of V445 Lyr, the same Fourier solution ( $f_0$  and its ten harmonics) and bin size (2 days) was used as Guggenberger et al. (2012). As they mentioned, this bin size is optimal because in such a short time-scale the impact of the Blazhko effect is negligible. So this work used the 2-d bin size for V2178 Cyg as well. In the latter case the used Fourier fit contains  $f_0$  and its 12 harmonics. The number of harmonics here is higher than we found to be significant in Sec. 4.1, but now the goal is to calculate the  $\varphi_n(t)$  functions as accurately as possible, and therefore all harmonics from the conventional pre-whitening process should be considered.

The first five  $\varphi_n(t)$  functions ( $n = 1, 2, \dots, 5$ ) of V445 Lyr and V2178 Cyg are shown in Fig. 2. The functions of V445 Lyr for  $n > 1$  show long time segments, where the phase continuously increases exceeding the  $(0, 2\pi)$  interval. The summed length of such time intervals increases with increasing  $n$ . The functions of V2178 Cyg do not show this feature, what is more, their amplitudes tend to decrease with the increasing harmonic order. It must be mentioned that the scatter shown in the figures is, partly, because of an intrinsic effect: the signal of the excited low amplitude modes present in the stars (see Guggenberger et al. 2012; Benkő et al. 2014) disturb the light curves on short time-scale causing higher scatter.

By applying Eq. (15) and (16) we obtain for the deviation parameters  $D_4 = 0.33 \pm 9.3 \cdot 10^{-4}$ ,  $D_5 = 0.93 \pm 3.6 \cdot 10^{-4}$  for V445 Lyr, and  $D_5 = 0.62 \pm 7.5 \cdot 10^{-5}$  for V2178 Cyg. First, we see that estimated errors are quite large, as we predicted. Second, these values are systematically lower than the values obtained in Sec. 4.1 (see in Table 1 and Table 2). If we consider the errors as well, the values become consistent within  $3\sigma$  intervals, but because of the relatively large errors, all we can state is that the calculated deviation values of this section do not contradict the results of Sec. 4.1.

Although formula (15) proved to be not suitable for calculating deviation parameter, it gives ideas when the HDE

becomes detectable: shortly, if  $\langle \varphi'_n \rangle$  has a relatively large, non-zero value. The  $\varphi(t)$  functions of the Blazhko stars (and consequently their derivatives  $\varphi'_n(t)$  as well) are generally nearly periodic with the Blazhko period(s). Let us assume the phase variation functions to be strictly periodic and represent them with their Fourier sum:

$$\varphi_n(t) = \frac{a_{n0}^\varphi}{2} + \sum_{i=1}^{\infty} a_{ni}^\varphi \sin(2\pi i f_B t + \varphi_{ni}^\varphi), \quad (17)$$

where  $a_{n0}^\varphi, a_{ni}^\varphi, \varphi_{ni}^\varphi$  are the constant Fourier coefficients, and  $f_B$  is the Blazhko frequency. Using (16) and some trigonometric identities we obtain the trivial result:

$$\langle \varphi'_n \rangle = \frac{2}{\Delta t} \sum_{i=0}^{\infty} a_{ni}^\varphi \cos[\pi i f_B (2t_0 + \Delta t) + \varphi_{ni}^\varphi] \times \sin(\pi i f_B \Delta t) = 0. \quad (18)$$

That is, in the case of fully periodic phase variation functions the deviation parameter is zero. As the phase variation functions differ from periodic ones,  $D_n$  became non-zero. If we find a Blazhko star with highly non-periodic Blazhko effect (which causes non-periodic phase variation functions) we expect the HDE to appear.

The O–C diagram well characterises the frequency variation part of the Blazhko effect. It contains the total frequency (or phase) variation (not the individual components), but if we see the O–C diagrams of the *Kepler* sample in Fig. 7 of Benkő et al. (2014) we find that V445 Lyr and V2178 Cyg show the two largest amplitude and most exotic (non-periodic) diagrams. In other words, checking the phase variation or O–C curves could help us to select those stars which might show detectable HDE.

## 5 CONCLUSIONS

In this short paper it is shown that two common assumptions concerning RR Lyrae light curves seems to be invalid.

- (1) It was shown that the mathematical formalism currently best describing the observed Blazhko light curves (Szeidl et al. 2012) is the Fourier representation of general almost periodic functions. It follows that, the widely used phrase ‘Blazhko modulation’ is not precise. Although many features of the Blazhko light curves can be described in the modulation framework, the real light curves are more than a simple modulated signals. That is, the external modulation explanation of the Blazhko effect is deficient, which gives a further support for those theoretical models which produce the effect as an inherent feature of the pulsation (e.g. Buchler & Kolláth 2011; Kolláth 2016).

- (2) Up to now most studies assumed that the Fourier spectra of the light curve of an RR Lyrae star, either mono-periodic or showing the Blazhko effect, is dominated by the main pulsation frequency and its harmonics. The Fourier representation of almost periodic functions, however, does not consist of the harmonics of the main pulsation frequency, but terms with frequencies which are slightly different from the exact harmonics. This was called as harmonic detuning effect (HDE).

By investigating the *Kepler* Blazhko RR Lyrae sample concerning the HDE the following conclusions can be drawn:

- On the one hand, the previously found deviation of the

harmonic frequencies in the Fourier spectrum of V445 Lyr can be explained with HDE. On the other hand, an additional star – V2178 Cyg – was found showing significant HDE. The detected deviation of V354 Lyr is formally non-significant, but unusually high compared to other Blazhko and non-Blazhko stars. That is why V354 Lyr can be considered a candidate HDE showing star.

- As a by-product of this work, stitched tailor-made aperture light curves for the 19 *Kepler* non-Blazhko RRab stars were prepared and made publicly available<sup>3</sup>.

- The HDE can be detected for those Blazhko stars only which have (i) high amplitude phase variation – otherwise the deviations are below the detection limit. (ii) These phase variations should also be non-periodic enough – otherwise the effect is averaged out. There is a third condition for the detectability of HDE: (iii) the Blazhko light curves should be long and precise enough – otherwise the error of frequency determination makes this small effect undetectable. These strong conditions explain why HDE was not previously revealed.

## ACKNOWLEDGEMENTS

The author thanks to R. Szabó for reading and to L. Szabados for language editing the manuscript, likewise thanks to the anonymous referee for her/his valuable comments. This project has been supported by the Hungarian National Research, Development and Innovation Office – NKFIH K-115709.

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<sup>3</sup> [http://www.konkoly.hu/KIK/data\\_en.html](http://www.konkoly.hu/KIK/data_en.html)

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