

SELF-DUALITY AND SCATTERING MAP FOR THE HYPERBOLIC VAN DIEJEN SYSTEMS WITH TWO COUPLING PARAMETERS (WITH AN APPENDIX BY S. RUIJSENAARS)

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ABSTRACT. In this paper, we construct global action-angle variables for a certain two-parameter family of hyperbolic van Diejen systems. Following Ruijsenaars' ideas on the translation invariant models, the proposed action-angle variables come from a thorough analysis of the commutation relation obeyed by the Lax matrix, whereas the proof of their canonicity is based on the study of the scattering theory. As a consequence, we show that the van Diejen system of our interest is self-dual with a factorized scattering map. Also, in an appendix by S. Ruijsenaars, a novel proof of the spectral asymptotics of certain exponential type matrix flows is presented. This result is of crucial importance in our scattering-theoretical analysis.

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1. INTRODUCTION

The action-angle duality, or Ruijsenaars duality, is one of the most attractive features of the Calogero–Moser–Sutherland (CMS) [1, 2, 3, 4, 5, 6] and the Ruijsenaars–Schneider–van Diejen (RSvD) [7, 8, 9, 10, 11] integrable many-particle systems. At the classical level, in the context of the translation invariant rational and hyperbolic models associated with the A -type root systems, this fascinating phenomenon was discovered by Ruijsenaars in the seminal paper [12]. Due to the importance of this observation, using various advanced techniques ranging from the methods of gauge theory to

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the machinery of symplectic reductions, by now the duality properties have been reinterpreted, and also exhibited for a much wider class of A -type models (see e.g. [13, 14, 15, 16, 17, 18, 19, 20, 21, 22]).

Although the translation invariant members of the RSvD family have drawn considerable attention, the theory of the classical multi-parametric non- A -type integrable deformations introduced by van Diejen [10, 11] is far less developed. For brevity, in the rest of the paper the non- A -type members of the RSvD family are simply referred to as the van Diejen systems. Now, the transparent asymmetry between the maturation of the theories of the classical A -type and non- A -type models is probably best elucidated by the somewhat surprising fact that no Lax representation is known for the most general van Diejen dynamics. Of course, it was observed already in the early stage of the developments that by folding the A -type root systems one can construct Lax matrices of type C and BC , but only with a single coupling parameter [12]. As concerns the D -type models, only partial results are available for small values of the number of particles [23]. Nevertheless, to close the gap, the last couple of years have witnessed the emergence of some new ideas in the literature to cope with the intricacies posed by the classical van Diejen models. Indeed, by working out Lax matrices for the most general rational variants of the RSvD family associated with the BC -type root systems, the duality properties of these special non- A -type van Diejen models are also settled completely (see [24, 25, 26]). Prior to our present work, at the level of the classical hyperbolic systems, non-trivial results could be found only in [27], where the 1-particle BC_1 model is studied by direct techniques. However, as the most recent progress in this research area, in a joint work with Görbe [28] we constructed Lax pairs for certain two-parameter family of *hyperbolic* van Diejen systems, too. As a natural step forward, in this paper we wish to uncover the self-duality property of these special hyperbolic systems.

In order to describe the van Diejen models of our interest, it proves convenient to start with the shorthand notation

$$(1.1) \quad \mathbb{N}_m = \{1, \dots, m\} \subset \mathbb{N} \quad (m \in \mathbb{N}).$$

Furthermore, take an arbitrary $n \in \mathbb{N}$, let $N = 2n$, and consider the open subset

$$(1.2) \quad P = \{p = (\boldsymbol{\xi}, \boldsymbol{\eta}) = (\xi_1, \dots, \xi_n, \eta_1, \dots, \eta_n) \in \mathbb{R}^N \mid \xi_1 > \dots > \xi_n > 0\},$$

that we endow with the smooth manifold structure inherited from the ambient space \mathbb{R}^N . Note that the functions $\lambda_1, \dots, \lambda_n, \theta_1, \dots, \theta_n \in C^\infty(P)$ defined by the formulae

$$(1.3) \quad \lambda_a(p) = \xi_a \quad \text{and} \quad \theta_a(p) = \eta_a \quad (a \in \mathbb{N}_n, p \in P)$$

provide a global coordinate system on P . For consistency with the terminology of the Ruijsenaars–Schneider models, the coordinates λ_a and θ_a are usually called the particle positions and the particle rapidities, respectively. For brevity, we also introduce the notations

$$(1.4) \quad x_a = \lambda_a \quad \text{and} \quad x_{n+a} = \theta_a \quad (a \in \mathbb{N}_n).$$

Note that the even dimensional manifold P can be equipped with the symplectic form

$$(1.5) \quad \omega = \sum_{a=1}^n d\lambda_a \wedge d\theta_a,$$

which is natural in the sense that thereby we can think of (P, ω) as a model of the cotangent bundle of the configuration space

$$(1.6) \quad Q = \{\boldsymbol{\xi} = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n \mid \xi_1 > \dots > \xi_n > 0\}.$$

Upon introducing the $N \times N$ matrix

$$(1.7) \quad \Omega = [\Omega_{k,l}]_{1 \leq k, l \leq N} = \begin{bmatrix} \mathbf{0}_n & \mathbf{1}_n \\ -\mathbf{1}_n & \mathbf{0}_n \end{bmatrix},$$

observe that the Poisson bracket associated with ω (1.5) takes the form

$$(1.8) \quad \{f, g\} = \sum_{k,l=1}^N \Omega_{k,l} \frac{\partial f}{\partial x_k} \frac{\partial g}{\partial x_l} \quad (f, g \in C^\infty(P)).$$

In particular, the distinguished coordinates (1.3) are canonical, i.e.,

$$(1.9) \quad \{\lambda_a, \lambda_b\} = 0, \quad \{\theta_a, \theta_b\} = 0, \quad \{\lambda_a, \theta_b\} = \delta_{a,b} \quad (a, b \in \mathbb{N}_n).$$

To proceed further, let $g = (\mu, \nu) \in \mathbb{R}^2$ be an arbitrary point satisfying

$$(1.10) \quad \sin(\mu) \neq 0 \neq \sin(\nu),$$

and for each $a \in \mathbb{N}_n$ define the function

$$(1.11) \quad u_a^g = \left(1 + \frac{\sin(\nu)^2}{\sinh(2\lambda_a)^2}\right)^{\frac{1}{2}} \prod_{\substack{c=1 \\ (c \neq a)}}^n \left(1 + \frac{\sin(\mu)^2}{\sinh(\lambda_a - \lambda_c)^2}\right)^{\frac{1}{2}} \left(1 + \frac{\sin(\mu)^2}{\sinh(\lambda_a + \lambda_c)^2}\right)^{\frac{1}{2}} \in C^\infty(P).$$

In passing we mention that in the above expression the values of the parameters μ and ν matter only modulo π . Now, the n -particle van Diejen model of our interest is the classical Hamiltonian system (P, ω, H^g) characterized by the smooth Hamiltonian

$$(1.12) \quad H^g = \sum_{a=1}^n \cosh(\theta_a) u_a^g.$$

Remembering the integrable many-particle models introduced in [10, 11], it is clear that this Hamiltonian system does belong to the hyperbolic RSvD family with two independent coupling parameters μ and ν .

Having defined the key player, now we wish to summarize the content of the rest of the paper. In the next section we gather the necessary background material underlying the study of the van Diejen system (1.12). Since our present work can be seen as a natural continuation of [28], the reader may find it convenient to have a copy of [28] on hand for the proofs of the results outlined in Section 2.

Turning to the study of the commutation relation (2.37) obeyed by the Lax matrix L (2.33) of the van Diejen system (1.12), in Section 3 we construct a diffeomorphism Ψ^g (3.102) from the spectral data associated with the Lax matrix. Since its components $\hat{\theta}_a^g$ (3.1) and $\hat{\lambda}_a^g$ (3.100) are defined with the aid of the diagonalization of L , the construction of Ψ^g is of purely algebraic nature. We wish to emphasize that this construction is a direct generalization of Ruijsenaars' approach on the translation invariant models [12], so it is natural to expect that the globally defined smooth functions $\hat{\theta}_a^g$ and $\hat{\lambda}_a^g$ ($a \in \mathbb{N}_n$) are actually action-angle coordinates for the Hamiltonian system (1.12).

To prove that they do form a Darboux system, in Section 4 we turn our attention to the scattering properties of (1.12). Of course, this idea was the other cornerstone of the developments presented in [12], but our implementation is quite different. Indeed, rather than using techniques from the theory of functions of several complex variables, in Section 4 we apply straightforward dynamical system techniques, and a bit real analysis, to prove the canonicity of the proposed action-angle variables. The main result of the paper is formulated in Theorem 24. This theorem allows us to complete the study of the scattering theory, too. Namely, as formulated in Theorem 25, we show that the Møller wave transformations are symplectomorphisms. Moreover, from the explicit formula (4.197) we see that the scattering map S has a factorized form. In Section 5 we briefly discuss the consequences of our results. Also, we pose some open problems related to the theory of the van Diejen type particle systems. We conclude the paper with an appendix by S. Ruijsenaars, on the spectral asymptotics of certain exponential type matrix flows.

2. PRELIMINARIES

In this short section we wish to summarize the fundamental algebraic properties of the Lax matrix of the van Diejen system (1.12) we constructed in [28]. To make the presentation essentially self-contained, and also to fix the notations, it is expedient to start with a brief overview on the underlying Lie theoretical objects. As a rule, the manifolds appearing in this paper are real and smooth.

2.1. Background material from Lie theory. One of the most important observations we made in [28] is that many properties of the van Diejen systems (1.12) can be understood in a geometric setup based on the non-compact reductive matrix Lie group

$$(2.1) \quad G = U(n, n) = \{y \in GL(N, \mathbb{C}) \mid y^* C y = C\},$$

where

$$(2.2) \quad C = \begin{bmatrix} \mathbf{0}_n & \mathbf{1}_n \\ \mathbf{1}_n & \mathbf{0}_n \end{bmatrix} \in GL(N, \mathbb{C}).$$

It is clear that the Lie algebra of G (2.1) can be identified with

$$(2.3) \quad \mathfrak{g} = \mathfrak{u}(u, n) = \{Y \in \mathfrak{gl}(N, \mathbb{C}) \mid Y^* C + C Y = 0\}.$$

Notice also that the set of unitary elements

$$(2.4) \quad K = \{y \in G \mid y^* y = \mathbf{1}_N\} \cong U(n) \times U(n)$$

forms a maximal compact subgroup in G (2.1), and the corresponding Lie subalgebra takes the form

$$(2.5) \quad \mathfrak{k} = \{Y \in \mathfrak{g} \mid Y^* + Y = 0\} \cong \mathfrak{u}(n) \oplus \mathfrak{u}(n).$$

By taking the complementary subspace

$$(2.6) \quad \mathfrak{p} = \{X \in \mathfrak{g} \mid X^* = X\},$$

we end up with the \mathbb{Z}_2 -gradation

$$(2.7) \quad \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p},$$

which is actually orthogonal with respect to the trace pairing defined on the matrix Lie algebra \mathfrak{g} (2.3). Note that at the Lie group level the natural analogue of this decomposition is the diffeomorphism

$$(2.8) \quad \mathcal{C}: \mathfrak{p} \times K \rightarrow G, \quad (X, k) \mapsto e^X k.$$

As a further ingredient, recall that the restriction of the exponential map onto the subspace \mathfrak{p} (2.6) is injective. Moreover, by taking the image of \mathfrak{p} under the exponential map, in G we obtain the closed embedded submanifold

$$(2.9) \quad \mathcal{P} = \exp(\mathfrak{p}) = \{e^X \in G \mid X \in \mathfrak{p}\}.$$

As a matter of fact, it coincides with the set of positive definite elements of G ; that is,

$$(2.10) \quad \mathcal{P} = \{y \in U(n, n) \mid y > 0\}.$$

We mention in passing that, due to the global Cartan decomposition (2.8), \mathcal{P} can be identified with the non-compact symmetric space associated with (G, K) , i.e., $\mathcal{P} \cong G/K$.

Besides the above basic objects, in the following we shall also need some finer elements from the structure theory of G (2.1). As the first step toward this goal, in \mathfrak{p} (2.6) we introduce the maximal Abelian subspace

$$(2.11) \quad \mathfrak{a} = \{X = \text{diag}(x_1, \dots, x_n, -x_1, \dots, -x_n) \mid x_1, \dots, x_n \in \mathbb{R}\}.$$

Moreover, for the subset of the off-diagonal elements of \mathfrak{p} we introduce the notation \mathfrak{a}^\perp . Clearly the centralizer of \mathfrak{a} in K (2.4) is the Abelian Lie group

$$(2.12) \quad M = \{\text{diag}(e^{i\chi_1}, \dots, e^{i\chi_n}, e^{i\chi_1}, \dots, e^{i\chi_n}) \mid \chi_1, \dots, \chi_n \in \mathbb{R}\}$$

with Lie algebra

$$(2.13) \quad \mathfrak{m} = \{\text{diag}(i\chi_1, \dots, i\chi_n, i\chi_1, \dots, i\chi_n) \mid \chi_1, \dots, \chi_n \in \mathbb{R}\}.$$

Now, if \mathfrak{m}^\perp denotes the set of the off-diagonal elements of the subalgebra \mathfrak{k} (2.5), then by (2.7) we can write the refined orthogonal decomposition

$$(2.14) \quad \mathfrak{g} = \mathfrak{m} \oplus \mathfrak{m}^\perp \oplus \mathfrak{a} \oplus \mathfrak{a}^\perp.$$

We proceed by noting that for all

$$(2.15) \quad X = \text{diag}(x_1, \dots, x_n, -x_1, \dots, -x_n) \in \mathfrak{a} \quad (x_1, \dots, x_n \in \mathbb{R})$$

the subspace $\mathfrak{m}^\perp \oplus \mathfrak{a}^\perp$ consisting of the off-diagonal elements of \mathfrak{g} is invariant under the action of the linear operator

$$(2.16) \quad \text{ad}_X: \mathfrak{g} \rightarrow \mathfrak{g}, \quad Y \mapsto [X, Y].$$

Thus, the restriction

$$(2.17) \quad \widetilde{\text{ad}}_X = \text{ad}_X|_{\mathfrak{m}^\perp \oplus \mathfrak{a}^\perp} \in \mathfrak{gl}(\mathfrak{m}^\perp \oplus \mathfrak{a}^\perp)$$

is well-defined, and for its spectrum we have

$$(2.18) \quad \text{Spec}(\widetilde{\text{ad}}_X) = \{x_a - x_b, \pm(x_a + x_b), \pm 2x_c \mid a, b, c \in \mathbb{N}_n, a \neq b\}.$$

Recall that in the study of the reductive Lie groups, the regular part of \mathfrak{a} (2.11) is usually defined by the open subset

$$(2.19) \quad \mathfrak{a}_{\text{reg}} = \{X \in \mathfrak{a} \mid \det(\widetilde{\text{ad}}_X) \neq 0\}.$$

From (2.11), (2.18) and (2.19) we see that

$$(2.20) \quad \mathfrak{c} = \{X = \text{diag}(x_1, \dots, x_n, -x_1, \dots, -x_n) \mid x_1 > \dots > x_n > 0\}$$

is a connected component of $\mathfrak{a}_{\text{reg}}$. So, giving a glance at (1.6), it is clear the configuration space Q of the van Diejen systems (1.12) can be naturally identified with the standard Weyl chamber \mathfrak{c} (2.20), i.e., $Q \cong \mathfrak{c}$.

Due to its importance in the study of the Lax matrix of the van Diejen system (1.12), we conclude this subsection with some important facts related to the notion of regularity. Our discussion is based on the smooth surjective map

$$(2.21) \quad \mathfrak{a} \times K \ni (X, k) \mapsto kXk^{-1} \in \mathfrak{p},$$

that allows us to introduce the regular part of \mathfrak{p} (2.6) by

$$(2.22) \quad \mathfrak{p}_{\text{reg}} = \{kXk^{-1} \mid k \in K \text{ and } X \in \mathfrak{c}\}.$$

Recall that $\mathfrak{p}_{\text{reg}}$ is an open and dense subset of \mathfrak{p} , and the map

$$(2.23) \quad \mathfrak{c} \times (K/M) \ni (X, kM) \mapsto kXk^{-1} \in \mathfrak{p}_{\text{reg}}$$

is a diffeomorphism. Next, by taking the image of $\mathfrak{p}_{\text{reg}}$ under the exponential map, we define

$$(2.24) \quad \mathcal{P}_{\text{reg}} = \exp(\mathfrak{p}_{\text{reg}}) = \{e^X \in G \mid X \in \mathfrak{p}_{\text{reg}}\} \subset \mathcal{P}.$$

Since the exponential map is a diffeomorphism from \mathfrak{p} (2.6) onto \mathcal{P} (2.9), the subset \mathcal{P}_{reg} is open and dense in \mathcal{P} . Therefore, \mathcal{P}_{reg} can be also seen as an embedded submanifold of G . Moreover, the map

$$(2.25) \quad \Upsilon: \mathfrak{c} \times (K/M) \rightarrow \mathcal{P}_{\text{reg}}, \quad (X, kM) \mapsto ke^Xk^{-1}$$

is a diffeomorphism from $\mathfrak{c} \times (K/M)$ onto \mathcal{P}_{reg} . As a closing remark, for the proofs of the Lie theoretic facts appearing in this subsection we recommend [29].

2.2. Fundamental properties of the Lax matrix. Having reviewed the necessary objects from Lie theory, now we wish to describe the Lax matrix we constructed in [28] for the two-parameter family of hyperbolic van Diejen systems (1.12). As a preliminary step, for the space of the admissible model parameters we introduce the notation

$$(2.26) \quad \mathfrak{M} = \{g = (\mu, \nu) \in \mathbb{R}^2 \mid \sin(\mu) \neq 0 \neq \sin(\nu)\}.$$

Next, for each $a \in \mathbb{N}_n$ we define the smooth function

$$(2.27) \quad z_a: P \times \mathfrak{M} \rightarrow \mathbb{C}, \quad (p, g) \mapsto z_a(p, g) = z_a^g(p),$$

where for each $g = (\mu, \nu) \in \mathfrak{M}$ the g -section z_a^g is given by

$$(2.28) \quad z_a^g = -\frac{\sinh(i\nu + 2\lambda_a)}{\sinh(2\lambda_a)} \prod_{\substack{c=1 \\ (c \neq a)}}^n \frac{\sinh(i\mu + \lambda_a - \lambda_c)}{\sinh(\lambda_a - \lambda_c)} \frac{\sinh(i\mu + \lambda_a + \lambda_c)}{\sinh(\lambda_a + \lambda_c)} \in C^\infty(P, \mathbb{C}).$$

By taking the modulus of (2.27), we let

$$(2.29) \quad u_a = |z_a| \in C^\infty(P \times \mathfrak{M}) \quad (a \in \mathbb{N}_n).$$

This is consistent with our earlier notation, since for all $g = (\mu, \nu) \in \mathfrak{M}$ the g -section of u_a does coincide with the function we defined in (1.11). An equally important ingredient in the construction of the Lax matrix is the column vector valued smooth function

$$(2.30) \quad F: P \times \mathfrak{M} \rightarrow \mathbb{C}^{N \times 1}, \quad (p, g) \mapsto F(p, g) = F^g(p)$$

with components

$$(2.31) \quad F_a(p, g) = e^{\frac{\theta_a(p)}{2}} u_a(p, g)^{\frac{1}{2}} \quad \text{and} \quad F_{n+a}(p, g) = e^{-\frac{\theta_a(p)}{2}} \bar{z}_a(p, g) u_a(p, g)^{-\frac{1}{2}} \quad (a \in \mathbb{N}_n).$$

Finally, upon introducing the shorthand notation

$$(2.32) \quad \Lambda_a = \lambda_a \quad \text{and} \quad \Lambda_{n+a} = -\lambda_a \quad (a \in \mathbb{N}_n),$$

let us recall that the Lax matrix we worked out in [28] is the matrix valued smooth function

$$(2.33) \quad L: P \times \mathfrak{M} \rightarrow \mathcal{P}, \quad (p, g) \mapsto L(p, g) = L^g(p),$$

where for each $g = (\mu, \nu) \in \mathfrak{M}$ the g -section L^g is given by the entries

$$(2.34) \quad L_{k,l}^g = \frac{i \sin(\mu) F_k^g \bar{F}_l^g + i \sin(\mu - \nu) C_{k,l}}{\sinh(i\mu + \Lambda_k - \Lambda_l)} \quad (k, l \in \mathbb{N}_N).$$

Concerning the relationship between the Hamiltonian H^g (1.12) and the Lax matrix L (2.33), let us note that

$$(2.35) \quad H^g = \frac{1}{2} \text{tr}(L^g).$$

Remember that the fact that L (2.33) takes values in \mathcal{P} (2.9) is itself not entirely trivial (see the proofs of Proposition 1 and Lemma 2 in [28]). However, for our present purposes it is much more important that L obeys a Ruijsenaars type commutation relation (see equation (2.4) and the surrounding ideas in [12]). Indeed, utilizing the matrix valued smooth function

$$(2.36) \quad \mathbf{\Lambda} = \text{diag}(\Lambda_1, \dots, \Lambda_N) \in C^\infty(P, \mathfrak{c}),$$

one can easily show that $\forall g = (\mu, \nu) \in \mathfrak{M}$ we have

$$(2.37) \quad e^{i\mu} e^{\mathbf{\Lambda}} L^g e^{-\mathbf{\Lambda}} - e^{-i\mu} e^{-\mathbf{\Lambda}} L^g e^{\mathbf{\Lambda}} = 2i \sin(\mu) F^g (F^g)^* + 2i \sin(\mu - \nu) C.$$

Although the proof of this statement is quite straightforward (see Lemma 3 in [28]), this commutation relation proved to be one of the cornerstones of the developments presented in [28]. As it turns out, this relation can be seen as the starting point of the present work, too. However, compared to [28], in this paper we take a completely different route by turning our attention to the ‘dual systems’.

Before presenting the new results, let us not forget about an important regularity property of L . Under the slight technical assumption that the coupling parameter belongs to

$$(2.38) \quad \tilde{\mathfrak{M}} = \{g = (\mu, \nu) \in \mathfrak{M} \mid \sin(2\mu - \nu) \neq 0\},$$

in [28] we proved that the spectrum of the Lax matrix (2.33) is simple. More precisely, the statement we made in Lemma 4 of [28] is that $\forall(p, g) \in P \times \tilde{\mathfrak{M}}$ we have $L(p, g) \in \mathcal{P}_{\text{reg}}$ (2.24). During the whole paper we shall assume that the map L (2.33) is regular in this sense.

3. CONSTRUCTION OF THE DUAL OBJECTS

Starting with this section, we wish to present our new results related to the 2-parameter family of classical hyperbolic van Diejen systems (1.12). Following the lead of the paper [12], we shall utilize the commutation relation (2.37) to infer the relevant spectral properties of the Lax matrix L (2.33). As a consequence, our analysis shall naturally give rise to the ‘dual objects’ playing the fundamental role in the construction of action-angle variables for the Hamiltonian system (1.12).

3.1. Diagonalization of the Lax matrix. As we discussed at the end of the previous section, for all $p \in P$ and $g \in \tilde{\mathfrak{M}}$ the self-adjoint Lax matrix $L(p, g)$ (2.34) is regular in the sense that it belongs to \mathcal{P}_{reg} (2.24). Bearing in mind (2.25), we see that the matrix $L(p, g)$ can be conjugated into a unique element of the subset $\exp(\mathfrak{c}) \subset \mathcal{P}_{\text{reg}}$ by certain elements of the compact subgroup K (2.4). In particular, the spectrum of $L(p, g)$ is simple. More precisely, due to the identification $Q \cong \mathfrak{c}$ (see (1.6) and (2.20)), there is a *unique* element

$$(3.1) \quad \hat{\theta}(p, g) = (\hat{\theta}_1(p, g), \dots, \hat{\theta}_n(p, g)) \in Q$$

such that we can write that

$$(3.2) \quad \text{Spec}(L(p, g)) = \{e^{\pm 2\hat{\theta}_a(p, g)} \mid a \in \mathbb{N}_n\}.$$

Utilizing the spectrum of L , let us define the function

$$(3.3) \quad \hat{\theta}: P \times \tilde{\mathfrak{M}} \rightarrow Q, \quad (p, g) \mapsto \hat{\theta}(p, g),$$

together with the matrix valued function

$$(3.4) \quad \hat{\Theta} = \text{diag}(\hat{\Theta}_1, \dots, \hat{\Theta}_N),$$

where the diagonal entries are given by

$$(3.5) \quad \hat{\Theta}_a = \hat{\theta}_a \quad \text{and} \quad \hat{\Theta}_{n+a} = -\hat{\theta}_a \quad (a \in \mathbb{N}_n).$$

Notice that $\hat{\Theta}$ is smooth, i.e.,

$$(3.6) \quad \hat{\Theta} \in C^\infty(P \times \tilde{\mathfrak{M}}, \mathfrak{c}).$$

Indeed, recalling the fact that the smooth function L (2.33) takes values in \mathcal{P}_{reg} , it is evident that with the aid of the diffeomorphism Υ (2.25) we can write

$$(3.7) \quad \hat{\Theta} = \frac{1}{2} \text{pr}_{\mathfrak{c}} \circ \Upsilon^{-1} \circ L,$$

where $\text{pr}_{\mathfrak{c}}: \mathfrak{c} \times (K/M) \rightarrow \mathfrak{c}$ is the (smooth) canonical projection from the product $\mathfrak{c} \times (K/M)$ onto the first factor \mathfrak{c} . As a trivial consequence, the smoothness of $\hat{\theta}$ (3.3) also follows.

As concerns the diagonalization of the matrix $L(p, g)$ at any given point $(p, g) \in P \times \tilde{\mathfrak{M}}$, later on we shall also need information about those elements of K (2.4) that can be used to transform $L(p, g)$ into $\exp(\mathfrak{c})$ by conjugation. Of course, there is a plethora of such diagonalizing elements. Indeed, from (2.25) it is evident that the freedom of choice is completely characterized by the n -dimensional subgroup M (2.12). Thus, aiming for uniqueness, on the diagonalizing matrices we shall impose that the first n components of a certain column vector be strictly positive (see (3.9) below). Namely, keeping in mind the objects defined in (2.31) and (2.36), our observation can be formulated as follows.

Lemma 1. For all $p \in P$ and $g \in \tilde{\mathfrak{M}}$ there is a unique element $\hat{y}(p, g) \in K$ such that

$$(3.8) \quad L(p, g) = \hat{y}(p, g)e^{2\hat{\Theta}(p, g)}\hat{y}(p, g)^{-1},$$

and also $\forall a \in \mathbb{N}_n$ we have

$$(3.9) \quad (e^{-\hat{\Theta}(p, g)}\hat{y}(p, g)^{-1}e^{\Lambda(p)}F(p, g))_a > 0.$$

Moreover, the resulting function

$$(3.10) \quad \hat{y}: P \times \tilde{\mathfrak{M}} \rightarrow K, \quad (p, g) \mapsto \hat{y}(p, g)$$

is smooth, i.e., $\hat{y} \in C^\infty(P \times \tilde{\mathfrak{M}}, K)$.

Proof. We start with the existence part of the Lemma. Let $p \in P$ and $g = (\mu, \nu) \in \tilde{\mathfrak{M}}$ be arbitrary elements and keep them fixed. Recalling the construction of the diagonal matrix (3.4), from (2.25) it follows that there is an element $y \in K$ such that

$$(3.11) \quad L(p, g) = ye^{2\hat{\Theta}(p, g)}y^{-1}.$$

Plugging it into the commutation relation (2.37), we get

$$(3.12) \quad \begin{aligned} e^{i\mu}e^{\Lambda(p)}ye^{2\hat{\Theta}(p, g)}y^{-1}e^{-\Lambda(p)} - e^{-i\nu}e^{-\Lambda(p)}ye^{2\hat{\Theta}(p, g)}y^{-1}e^{\Lambda(p)} \\ = 2i \sin(\mu)F(p, g)F(p, g)^* + 2i \sin(\mu - \nu)C. \end{aligned}$$

By multiplying this equation with the matrices

$$(3.13) \quad e^{-\hat{\Theta}(p, g)}y^{-1}e^{\Lambda(p)} \quad \text{and} \quad e^{\Lambda(p)}ye^{-\hat{\Theta}(p, g)}$$

from the left and the right, respectively, one finds immediately that

$$(3.14) \quad \begin{aligned} e^{i\mu}e^{-\hat{\Theta}(p, g)}y^{-1}e^{2\Lambda(p)}ye^{\hat{\Theta}(p, g)} - e^{-i\nu}e^{\hat{\Theta}(p, g)}y^{-1}e^{2\Lambda(p)}ye^{-\hat{\Theta}(p, g)} \\ = 2i \sin(\mu)(e^{-\hat{\Theta}(p, g)}y^{-1}e^{\Lambda(p)}F(p, g))(e^{-\hat{\Theta}(p, g)}y^{-1}e^{\Lambda(p)}F(p, g))^* + 2i \sin(\mu - \nu)C. \end{aligned}$$

Now, focusing on the diagonal entries on the above matrix equation, it follows that $\forall k \in \mathbb{N}_N$ we have

$$(3.15) \quad (y^{-1}e^{2\Lambda(p)}y)_{k, k} = |(e^{-\hat{\Theta}(p, g)}y^{-1}e^{\Lambda(p)}F(p, g))_k|^2.$$

Notice that the matrix $y^{-1}e^{2\Lambda(p)}y$ is positive definite, whence its diagonal entries are strictly positive. Therefore, we conclude that

$$(3.16) \quad (e^{-\hat{\Theta}(p, g)}y^{-1}e^{\Lambda(p)}F(p, g))_k \neq 0 \quad (k \in \mathbb{N}_N).$$

To proceed further, $\forall a \in \mathbb{N}_n$ we define the complex number

$$(3.17) \quad m_a = \frac{(e^{-\hat{\Theta}(p, g)}y^{-1}e^{\Lambda(p)}F(p, g))_a}{|(e^{-\hat{\Theta}(p, g)}y^{-1}e^{\Lambda(p)}F(p, g))_a|}.$$

Since $|m_a| = 1$, from (2.12) it is clear that

$$(3.18) \quad m = \text{diag}(m_1, \dots, m_n, m_1, \dots, m_n) \in M$$

is well-defined, just as the group element

$$(3.19) \quad \hat{y}(p, g) = ym \in K.$$

Since the subgroup M (2.12) is the centralizer of \mathfrak{a} in K , from (3.11) it is immediate that

$$(3.20) \quad L(p, g) = \hat{y}(p, g)e^{2\hat{\Theta}(p, g)}\hat{y}(p, g)^{-1}.$$

Keeping in mind the definition (3.17), it is also evident that $\forall a \in \mathbb{N}_n$ we can write

$$(3.21) \quad \begin{aligned} (e^{-\hat{\Theta}(p,g)} \hat{y}(p,g)^{-1} e^{\Lambda(p)} F(p,g))_a &= (e^{-\hat{\Theta}(p,g)} m^{-1} y^{-1} e^{\Lambda(p)} F(p,g))_a \\ &= e^{-\hat{\Theta}_a(p,g)} m_a^{-1} (y^{-1} e^{\Lambda(p)} F(p,g))_a = m_a^{-1} (e^{-\hat{\Theta}(p,g)} y^{-1} e^{\Lambda(p)} F(p,g))_a \\ &= |(e^{-\hat{\Theta}(p,g)} y^{-1} e^{\Lambda(p)} F(p,g))_a| > 0, \end{aligned}$$

whence the matrix (3.19) does meet the requirements we imposed in the Lemma.

As concerns the uniqueness part of the Lemma, in the light of the above calculations it is trivial. As a matter of fact, the smoothness of the function \hat{y} (3.10) is also quite straightforward from the above construction. Since \mathcal{Y} (2.25) is a diffeomorphism, and since the smooth map L (2.33) takes values in \mathcal{P}_{reg} , the map

$$(3.22) \quad \mathcal{Y}^{-1} \circ L: P \times \tilde{\mathfrak{M}} \rightarrow \mathfrak{c} \times (K/M)$$

is well-defined and smooth. Now, by composing $\mathcal{Y}^{-1} \circ L$ from the left with appropriate *smooth* local sections of the principal M -bundle

$$(3.23) \quad \mathfrak{c} \times K \ni (X, k) \mapsto (X, kM) \in \mathfrak{c} \times (K/M),$$

it is evident that the diagonalizing matrix y appearing in (3.11) can be chosen smoothly in a small neighborhood of any point $(p, g) \in P \times \tilde{\mathfrak{M}}$. It entails that the complex numbers m_a (3.17) also depend smoothly in the same neighborhood of (p, g) . However, by virtue of the uniqueness of \hat{y} , at each point of this neighborhood it must coincide with the product given in (3.19). As a consequence, the function defined in (3.10) is smooth in a small neighborhood of any point, whence it is smooth everywhere. \square

3.2. The form of the dual Lax matrix. With the aid of the functions L (2.33) and \hat{y} (3.10) let us define the matrix valued function

$$(3.24) \quad \hat{L}: P \times \tilde{\mathfrak{M}} \rightarrow \mathcal{P}_{\text{reg}}, \quad (p, g) \mapsto \hat{L}(p, g) = \hat{y}(p, g)^{-1} e^{2\Lambda(p)} \hat{y}(p, g).$$

Utilizing F (2.30) and Λ (2.36), together with the recently introduced functions $\hat{\Theta}$ (3.4) and \hat{y} (3.10), we also define the column vector valued function

$$(3.25) \quad \hat{F}: P \times \tilde{\mathfrak{M}} \rightarrow \mathbb{C}^{N \times 1}, \quad (p, g) \mapsto \hat{F}(p, g) = e^{-\hat{\Theta}(p,g)} y(p, g)^{-1} e^{\Lambda(p)} F(p, g).$$

By virtue of (3.16) we have $\hat{F}_k(p, g) \neq 0$ for all $k \in \mathbb{N}_N$. Now, from the observations we made in the previous subsection (see (3.6) and Lemma 1) it is evident that

$$(3.26) \quad \hat{L} \in C^\infty(P \times \tilde{\mathfrak{M}}, \mathcal{P}_{\text{reg}}) \quad \text{and} \quad \hat{F} \in C^\infty(P \times \tilde{\mathfrak{M}}, \mathbb{C}^{N \times 1}).$$

Remembering (3.14), we also see that for all $g = (\mu, \nu) \in \tilde{\mathfrak{M}}$ the g -sections of these new objects obey the commutation relation

$$(3.27) \quad e^{i\mu} e^{-\hat{\Theta}^g} \hat{L}^g e^{\hat{\Theta}^g} - e^{-i\mu} e^{\hat{\Theta}^g} \hat{L}^g e^{-\hat{\Theta}^g} = 2i \sin(\mu) \hat{F}^g (\hat{F}^g)^* + 2i \sin(\mu - \nu) C.$$

To proceed further, it is expedient to introduce the shorthand notation

$$(3.28) \quad \hat{g} = (\hat{\mu}, \hat{\nu}) = (-\mu, -\nu).$$

Notice that the map

$$(3.29) \quad \tilde{\mathfrak{M}} \ni g \mapsto \hat{g} \in \tilde{\mathfrak{M}}$$

is a well-defined involution on the space of the admissible parameters (2.38), i.e.,

$$(3.30) \quad \hat{\hat{g}} = g.$$

More importantly, by taking the matrix entries of the equation (3.27), we find immediately that

$$(3.31) \quad \hat{L}_{k,l}^g = \frac{i \sin(\hat{\mu}) \hat{F}_k^g \hat{F}_l^g + i \sin(\hat{\mu} - \hat{\nu}) C_{k,l}}{\sinh(i\hat{\mu} + \hat{\Theta}_k^g - \hat{\Theta}_l^g)} \quad (k, l \in \mathbb{N}_N).$$

Due to its striking similarity with (2.34), it is natural to expect intimate relationships between the matrix valued functions L and \hat{L} . All we need to find is the connection between the column vector valued functions F and \hat{F} .

In order to reveal this missing relationship, we follow the same strategy we applied in our paper [30] to understand the structure of the Lax matrix of the rational C_n RSvD system. As the first step, let us observe that by multiplying both sides of the commutation relation (3.27) with the matrix C (2.2) we get that

$$(3.32) \quad e^{i\mu} e^{\hat{\Theta}^g} (\hat{L}^g)^{-1} e^{-\hat{\Theta}^g} - e^{-i\mu} e^{-\hat{\Theta}^g} (\hat{L}^g)^{-1} e^{\hat{\Theta}^g} = 2i \sin(\mu) (C\hat{F}^g)(C\hat{F}^g)^* + 2i \sin(\mu - \nu) C.$$

Therefore, for the matrix entries of the inverse of \hat{L}^g we obtain at once that

$$(3.33) \quad (\hat{L}^g)_{k,l}^{-1} = \frac{i \sin(\hat{\mu}) (C\hat{F}^g)_k \overline{(C\hat{F}^g)_l} + i \sin(\hat{\mu} - \hat{\nu}) C_{k,l}}{\sinh(i\hat{\mu} - (\hat{\Theta}_k^g - \hat{\Theta}_l^g))} \quad (k, l \in \mathbb{N}_N).$$

Now, the main idea is that from the relationships between certain minors of \hat{L}^g and $(\hat{L}^g)^{-1}$ provided by Jacobi's theorem (see e.g. Theorem 2.5.2 in [31]) we can deduce characterizing equations for the smooth functions

$$(3.34) \quad \hat{z}_c = \hat{F}_c \bar{\hat{F}}_{n+c} \in C^\infty(P \times \tilde{\mathfrak{M}}, \mathbb{C}) \quad (c \in \mathbb{N}_n).$$

In complete analogy with equation (42) of [30], for any $g = (\mu, \nu) \in \tilde{\mathfrak{M}}$ and $c \in \mathbb{N}_n$ it proves handy to introduce the temporary shorthand notations

$$(3.35) \quad \mathcal{D}_c^g = \prod_{\substack{d=1 \\ (d \neq c)}}^n |\hat{F}_{n+d}^g|^2 \prod_{\substack{a,b=1 \\ (c \neq a \neq b \neq c)}}^n \frac{\sinh(\hat{\theta}_a^g - \hat{\theta}_b^g)}{\sinh(i\hat{\mu} + \hat{\theta}_a^g - \hat{\theta}_b^g)} \in C^\infty(P, \mathbb{R} \setminus \{0\}),$$

$$(3.36) \quad \omega_c^g = \prod_{\substack{d=1 \\ (d \neq c)}}^n \frac{\sinh(\hat{\theta}_c^g - \hat{\theta}_d^g) \sinh(\hat{\theta}_c^g + \hat{\theta}_d^g)}{\sinh(i\hat{\mu} + \hat{\theta}_c^g - \hat{\theta}_d^g) \sinh(i\hat{\mu} + \hat{\theta}_c^g + \hat{\theta}_d^g)} \in C^\infty(P, \mathbb{C} \setminus \{0\}).$$

Furthermore, in order to compute the minors of \hat{L}^g and $(\hat{L}^g)^{-1}$, we shall need an appropriate hyperbolic variant of the Cauchy type determinant formulae that we borrow from the paper [32]. Namely, by letting $\lambda \rightarrow \infty$ in equation (B28) of [32], one can easily see that if $\alpha \in \mathbb{R}$ such that $\sin(\alpha) \neq 0$, and if $m \in \mathbb{N}$, $\xi_1, \dots, \xi_m \in \mathbb{R}$ and also $\eta_1, \dots, \eta_m \in \mathbb{R}$, then we can write

$$(3.37) \quad \det \left(\left[\frac{\sinh(i\alpha)}{\sinh(i\alpha + \xi_k - \eta_l)} \right]_{1 \leq k, l \leq m} \right) = \sinh(i\alpha)^m \frac{\prod_{1 \leq k < l \leq m} \sinh(\xi_k - \xi_l) \sinh(\eta_l - \eta_k)}{\prod_{k, l=1}^m \sinh(i\alpha + \xi_k - \eta_l)}.$$

Finally, let us keep in mind that, if $m \in \mathbb{N}$ and $k \in \mathbb{N}_m$, then for any $m \times m$ matrix A its $k \times k$ minor determinant corresponding to the rows $r_1, \dots, r_k \in \mathbb{N}_m$ and the columns $c_1, \dots, c_k \in \mathbb{N}_m$ is given by

$$(3.38) \quad A \begin{pmatrix} r_1 & \dots & r_k \\ c_1 & \dots & c_k \end{pmatrix} = \det([A_{r_a, c_b}]_{1 \leq a, b \leq k}).$$

Now, we are in a position to present two particularly useful relationships for the function \hat{z}_c (3.34). Before proving them, the reader may find it convenient to skim through the formulae appearing in the Appendix of [30].

Proposition 2. *For any $c \in \mathbb{N}_n$ and $g = (\mu, \nu) \in \tilde{\mathfrak{M}}$ the g -section of the function \hat{z}_c (3.34) obey the equations*

$$(3.39) \quad \frac{\sin(\hat{\mu})}{\sinh(i\hat{\mu} + 2\hat{\theta}_c^g)} \omega_c^g \hat{z}_c^g + \frac{\sin(\hat{\mu})}{\sinh(i\hat{\mu} - 2\hat{\theta}_c^g)} \bar{\omega}_c^g \bar{\hat{z}}_c^g + \frac{\sin(\hat{\mu} - \hat{\nu})}{\sinh(i\hat{\mu} + 2\hat{\theta}_c^g)} + \frac{\sin(\hat{\mu} - \hat{\nu})}{\sinh(i\hat{\mu} - 2\hat{\theta}_c^g)} = 0,$$

$$(3.40) \quad \sinh(2\hat{\theta}_c^g)^2 |\omega_c^g \hat{z}_c^g|^2 - \sin(\hat{\mu}) \sin(\hat{\mu} - \hat{\nu}) (\omega_c^g \hat{z}_c^g + \bar{\omega}_c^g \bar{\hat{z}}_c^g) = \sin(\hat{\mu})^2 + \sin(\hat{\mu} - \hat{\nu})^2 + \sinh(2\hat{\theta}_c^g)^2.$$

Proof. We can be brief here, since our proof is modeled on the ideas presented in subsection 3.2 of [30]. Fix an arbitrary $g = (\mu, \nu) \in \mathfrak{M}$ and let $c \in \mathbb{N}_n$. From the definition of \hat{L} (3.24) we know that \hat{L}^g takes values in $\exp(\mathfrak{p}_{\text{reg}})$, whence it is clear that $\det(\hat{L}^g) = 1$. Thus, recalling the notation we introduced in (3.38), the application of Jacobi's theorem, as formulated in Theorem A1 of [30], leads to the relationship

$$(3.41) \quad ((\hat{L}^g)^{-1})^\top \begin{pmatrix} 1 & \dots & c & \dots & n \\ 1 & \dots & n+c & \dots & n \end{pmatrix} = -\hat{L}^g \begin{pmatrix} n+1 & \dots & n+c & \dots & 2n \\ n+1 & \dots & c & \dots & 2n \end{pmatrix},$$

where \top is the shorthand for taking transpose. Now, for brevity let $X^{(c)}$ denote the $n \times n$ matrix corresponding to the minor determinant on the left hand side of the above equation. By inspecting the entries of $(\hat{L}^g)^{-1}$ (3.33), one finds immediately that

$$(3.42) \quad X^{(c)} = Y^{(c)} + \frac{i \sin(\hat{\mu} - \hat{\nu})}{\sinh(i\hat{\mu} + 2\hat{\theta}_c^g)} E_{c,c},$$

where $Y^{(c)} = [Y_{a,b}^{(c)}]_{1 \leq a, b \leq n}$ is a Cauchy type matrix with entries

$$(3.43) \quad Y_{a,b}^{(c)} = \begin{cases} \frac{\bar{F}_{n+a}^g \sinh(i\hat{\mu})}{\sinh(i\hat{\mu} + \hat{\Theta}_a^g - \hat{\Theta}_b^g)} \hat{F}_{n+b}^g, & \text{if } b \neq c, \\ \frac{\bar{F}_{n+a}^g \sinh(i\hat{\mu})}{\sinh(i\hat{\mu} + \hat{\Theta}_a^g - \hat{\Theta}_{n+c}^g)} \hat{F}_c^g, & \text{if } b = c, \end{cases}$$

whereas $E_{c,c}$ is the $n \times n$ elementary matrix $E_{c,c} = [\delta_{c,a} \delta_{c,b}]_{1 \leq a, b \leq n}$. Keeping in mind the definitions (3.35) and (3.36), the application of the determinant formula (3.37) immediately yields

$$(3.44) \quad \det(Y^{(c)}) = \frac{\sinh(i\hat{\mu})}{\sinh(i\hat{\mu} + 2\hat{\theta}_c^g)} \mathcal{D}_c^g \omega_c^g \hat{z}_c^g.$$

Next, giving a glance at (3.42), we see that matrix $X^{(c)}$ is actually a rank one perturbation of $Y^{(c)}$. As is known (see e.g. equation (A.6) in [30]), in such cases we can write that

$$(3.45) \quad \det(X^{(c)}) = \det(Y^{(c)}) + \frac{i \sin(\hat{\mu} - \hat{\nu})}{\sinh(i\hat{\mu} + 2\hat{\theta}_c^g)} \mathcal{C}_{c,c}^{(c)},$$

where $\mathcal{C}_{c,c}^{(c)}$ is the *cofactor* of $Y^{(c)}$ associated with the entry $Y_{c,c}^{(c)}$. In other words, $\mathcal{C}_{c,c}^{(c)}$ can be computed by taking $(-1)^{c+c} = 1$ times the determinant of the $(n-1) \times (n-1)$ submatrix obtained by deleting the c -th row and the c -th column of $Y^{(c)}$. Since this submatrix is also of Cauchy type, by applying (3.37) we get immediately that $\mathcal{C}_{c,c}^{(c)} = \mathcal{D}_c^g$. Thus, by putting the above formulae together, we end up with the expression

$$(3.46) \quad \det(X^{(c)}) = \frac{i \sin(\hat{\mu})}{\sinh(i\hat{\mu} + 2\hat{\theta}_c^g)} \mathcal{D}_c^g \omega_c^g \hat{z}_c^g + \frac{i \sin(\hat{\mu} - \hat{\nu})}{\sinh(i\hat{\mu} + 2\hat{\theta}_c^g)} \mathcal{D}_c^g.$$

To proceed further, we turn to the study of the right hand side of (3.41). Along the same lines as above, one obtains that

$$(3.47) \quad \hat{L}^g \begin{pmatrix} n+1 & \dots & n+c & \dots & 2n \\ n+1 & \dots & c & \dots & 2n \end{pmatrix} = \overline{\det(X^{(c)})},$$

and so the relationship (3.41) can be rewritten as

$$(3.48) \quad \det(X^{(c)}) + \overline{\det(X^{(c)})} = 0.$$

Since \mathcal{D}_c^g (3.35) is non-zero and real at each point of the phase space, the relationship displayed in (3.39) emerges at once as an immediate consequence of the equations (3.46) and (3.48).

In order to get an independent relationship for \hat{z}_c (3.34), notice that by Jacobi's theorem we can also write that

$$(3.49) \quad ((\hat{L}^g)^{-1})^\top \begin{pmatrix} 1 & \dots & n & n+c \\ 1 & \dots & n & n+c \end{pmatrix} = \hat{L}^g \begin{pmatrix} n+1 & \dots & \widehat{n+c} & \dots & 2n \\ n+1 & \dots & \widehat{n+c} & \dots & 2n \end{pmatrix},$$

where $\widehat{n+c}$ on the right means that the indicated row and column are omitted. To make it practical, let $Z^{(c)}$ denote the $(n+1) \times (n+1)$ matrix corresponding to the minor determinant on the left hand side of (3.49). Upon introducing the temporary shorthand notation

$$(3.50) \quad \xi_k = \begin{cases} \hat{\Theta}_k^g, & \text{if } 1 \leq k \leq n, \\ \hat{\Theta}_{n+c}^g, & \text{if } k = n+1, \end{cases}$$

together with

$$(3.51) \quad f_k = \begin{cases} \hat{F}_{n+k}^g, & \text{if } 1 \leq k \leq n, \\ \hat{F}_c^g, & \text{if } k = n+1, \end{cases}$$

one finds immediately that

$$(3.52) \quad Z^{(c)} = S^{(c)} + \frac{i \sin(\hat{\mu} - \hat{\nu})}{\sinh(i\hat{\mu} + 2\hat{\theta}_c^g)} E_{c,n+1} + \frac{i \sin(\hat{\mu} - \hat{\nu})}{\sinh(i\hat{\mu} - 2\hat{\theta}_c^g)} E_{n+1,c},$$

where $S^{(c)}$ is a Cauchy type $(n+1) \times (n+1)$ matrix with entries

$$(3.53) \quad S_{k,l}^{(c)} = \bar{f}_k \frac{\sinh(i\hat{\mu})}{\sinh(i\hat{\mu} + \xi_k - \xi_l)} f_l \quad (k, l \in \mathbb{N}_{n+1}),$$

whereas

$$(3.54) \quad E_{c,n+1} = [\delta_{c,k} \delta_{n+1,l}]_{1 \leq k, l \leq n+1} \quad \text{and} \quad E_{n+1,c} = [\delta_{n+1,k} \delta_{c,l}]_{1 \leq k, l \leq n+1}.$$

Now, by applying (3.37), it is straightforward to verify that

$$(3.55) \quad \det(S^{(c)}) = \frac{\sinh(2\hat{\theta}_c^g)^2}{|\sinh(i\hat{\mu} + 2\hat{\theta}_c^g)|^2} \mathcal{D}_c^g |\omega_c^g \hat{z}_c^g|^2.$$

Of course, the computation of the determinant of $Z^{(c)}$ is a bit more subtle. Nevertheless, the relevant determinant formula for rank two perturbations of invertible Hermitian matrices can be also found in the Appendix of [30]. Indeed, equation (A.8) in [30] tells us that, upon introducing the complex valued function

$$(3.56) \quad \alpha_c = \frac{i \sin(\hat{\mu} - \hat{\nu})}{\sinh(i\hat{\mu} + 2\hat{\theta}_c^g)},$$

for the determinant of $Z^{(c)}$ (3.52) we can write

$$(3.57) \quad \det(Z^{(c)}) = \det(S^{(c)}) + \alpha_c \mathcal{C}_{c,n+1}^{(c)} + \bar{\alpha}_c \bar{\mathcal{C}}_{c,n+1}^{(c)} + |\alpha_c|^2 \frac{|\mathcal{C}_{c,n+1}^{(c)}|^2 - \mathcal{C}_{c,c}^{(c)} \mathcal{C}_{n+1,n+1}^{(c)}}{\det(S^{(c)})},$$

where $\mathcal{C}_{k,l}^{(c)}$ now stands for the cofactor of $S^{(c)}$ associated with the entry $S_{k,l}^{(c)}$ ($k, l \in \mathbb{N}_{n+1}$). Invoking (3.37) again, it is a routine exercise to verify that

$$(3.58) \quad \mathcal{C}_{n+1,c}^{(c)} = \bar{\mathcal{C}}_{c,n+1}^{(c)} = -\frac{i \sin(\hat{\mu})}{\sinh(i\hat{\mu} + 2\hat{\theta}_c^g)} \mathcal{D}_c^g \omega_c^g \hat{z}_c^g,$$

whilst the other two relevant cofactors are given by

$$(3.59) \quad \mathcal{C}_{c,c}^{(c)} = \mathcal{D}_c^g |\hat{F}_c^g|^2 \prod_{\substack{d=1 \\ (d \neq c)}}^n \left| \frac{\sinh(\hat{\theta}_c^g + \hat{\theta}_d^g)}{\sinh(i\hat{\mu} + \hat{\theta}_c^g + \hat{\theta}_d^g)} \right|^2,$$

$$(3.60) \quad \mathcal{C}_{n+1,n+1}^{(c)} = \mathcal{D}_c^g |\hat{F}_{n+c}^g|^2 \prod_{\substack{d=1 \\ (d \neq c)}}^n \left| \frac{\sinh(\hat{\theta}_c^g - \hat{\theta}_d^g)}{\sinh(i\hat{\mu} + \hat{\theta}_c^g - \hat{\theta}_d^g)} \right|^2.$$

Inserting the above formulae into (3.57), for the left hand side of (3.49) we obtain

$$(3.61) \quad \det(Z^{(c)}) = \mathcal{D}_c^g \frac{\sinh(2\hat{\theta}_c^g)^2 |\omega_c^g \hat{z}_c^g|^2 - \sin(\hat{\mu}) \sin(\hat{\mu} - \hat{\nu}) (\omega_c^g \hat{z}_c^g + \bar{\omega}_c^g \bar{\hat{z}}_c^g) - \sin(\hat{\mu} - \hat{\nu})^2}{|\sinh(i\hat{\mu} + 2\hat{\theta}_c^g)|^2}.$$

As concerns the right hand side of (3.49), from (3.31) it is clear that the corresponding $(n-1) \times (n-1)$ matrix is of Cauchy type, whence by invoking (3.37) we obtain at once that

$$(3.62) \quad \hat{I}^g \begin{pmatrix} n+1 & \dots & \widehat{n+c} & \dots & 2n \\ n+1 & \dots & \widehat{n+c} & \dots & 2n \end{pmatrix} = \mathcal{D}_c^g.$$

Now, simply by plugging the formulae (3.61) and (3.62) into (3.49), we end up with the quadratic relationship (3.40). \square

Our next goal is to solve the system of equations given in Proposition 2 for the g -section of \hat{z}_c (3.34). By taking the real and the imaginary parts of the smooth function $\omega_c^g \hat{z}_c^g$, we introduce the shorthand notation

$$(3.63) \quad R_c^g = \operatorname{Re}(\omega_c^g \hat{z}_c^g) \in C^\infty(P) \quad \text{and} \quad I_c^g = \operatorname{Im}(\omega_c^g \hat{z}_c^g) \in C^\infty(P) \quad (c \in \mathbb{N}_n).$$

Notice that (3.39) can be cast into the form

$$(3.64) \quad \sin(\hat{\mu}) \cosh(2\hat{\theta}_c^g) R_c^g - \cos(\hat{\mu}) \sinh(2\hat{\theta}_c^g) I_c^g + \sin(\hat{\mu} - \hat{\nu}) \cosh(2\hat{\theta}_c^g) = 0,$$

and so we get

$$(3.65) \quad R_c^g = \cot(\hat{\mu}) \tanh(2\hat{\theta}_c^g) I_c^g - \frac{\sin(\hat{\mu} - \hat{\nu})}{\sin(\hat{\mu})}.$$

On the other hand, (3.40) can be rewritten as

$$(3.66) \quad \sinh(2\hat{\theta}_c^g)^2 ((R_c^g)^2 + (I_c^g)^2) - 2 \sin(\hat{\mu}) \sin(\hat{\mu} - \hat{\nu}) R_c^g - \sin(\hat{\mu})^2 - \sin(\hat{\mu} - \hat{\nu})^2 - \sinh(2\hat{\theta}_c^g)^2 = 0,$$

thus by exploiting (3.65), for the smooth function I_c^g we end up with

$$(3.67) \quad (I_c^g)^2 - 2 \cos(\hat{\mu}) \sin(\hat{\mu} - \hat{\nu}) \coth(2\hat{\theta}_c^g) I_c^g + (\sin(\hat{\mu} - \hat{\nu})^2 - \sin(\hat{\mu})^2) \coth(2\hat{\theta}_c^g) = 0.$$

Solving this quadratic equation for I_c^g , it is clear that there is a function

$$(3.68) \quad S_c: P \times \tilde{\mathfrak{M}} \rightarrow \mathbb{R}, \quad (p, g) \mapsto S_c(p, g) = S_c^g(p),$$

such that $|S_c| = 1$ and

$$(3.69) \quad I_c^g = (\cos(\hat{\mu}) \sin(\hat{\mu} - \hat{\nu}) + S_c^g \sin(\hat{\mu}) \cos(\hat{\mu} - \hat{\nu})) \coth(2\hat{\theta}_c^g).$$

Note that, a priori, the ‘sign’ S_c^g may change from point to point in an uncontrolled manner. To cure the problem, in $\tilde{\mathfrak{M}}$ (2.38) we define the open subset

$$(3.70) \quad \tilde{\mathfrak{M}}_0 = \{g = (\mu, \nu) \in \tilde{\mathfrak{M}} \mid \cos(\mu - \nu) \neq 0\},$$

and for the restriction of the function S_c (3.68) to $P \times \tilde{\mathfrak{M}}_0$ we introduce the notation \tilde{S}_c . The point is that, if $g = (\mu, \nu) \in \tilde{\mathfrak{M}}_0$, then from (3.69) we get

$$(3.71) \quad \tilde{S}_c^g = \frac{\tanh(2\hat{\theta}_c^g)I_c^g - \cos(\hat{\mu}) \sin(\hat{\mu} - \hat{\nu})}{\sin(\hat{\mu}) \cos(\hat{\mu} - \hat{\nu})}.$$

From this formula it is also clear that the ‘sign’ function \tilde{S}_c is *smooth*, i.e.,

$$(3.72) \quad \tilde{S}_c \in C^\infty(P \times \tilde{\mathfrak{M}}_0).$$

As a consequence, the function \tilde{S}_c is constant on each connected component of the product manifold $P \times \tilde{\mathfrak{M}}_0$. Thus, keeping in mind (3.69), (3.65), (3.63) and (3.36), one finds immediately that on any connected component of $P \times \tilde{\mathfrak{M}}_0$ where $\tilde{S}_c = 1$ we have

$$(3.73) \quad \hat{z}_c(p, g) = \frac{\sinh(i(2\hat{\mu} - \hat{\nu}) + 2\hat{\theta}_c^g(p))}{\sinh(2\hat{\theta}_c^g(p))} \prod_{\substack{d=1 \\ (d \neq c)}}^n \frac{\sinh(i\hat{\mu} + \hat{\theta}_c^g(p) - \hat{\theta}_d^g(p)) \sinh(i\hat{\mu} + \hat{\theta}_c^g(p) + \hat{\theta}_d^g(p))}{\sinh(\hat{\theta}_c^g(p) - \hat{\theta}_d^g(p)) \sinh(\hat{\theta}_c^g(p) + \hat{\theta}_d^g(p))}.$$

On the other hand, on a connected component of $P \times \tilde{\mathfrak{M}}_0$ where $\tilde{S}_c = -1$, we can write that

$$(3.74) \quad \hat{z}_c(p, g) = -\frac{\sinh(i\hat{\nu} + 2\hat{\theta}_c^g(p))}{\sinh(2\hat{\theta}_c^g(p))} \prod_{\substack{d=1 \\ (d \neq c)}}^n \frac{\sinh(i\hat{\mu} + \hat{\theta}_c^g(p) - \hat{\theta}_d^g(p)) \sinh(i\hat{\mu} + \hat{\theta}_c^g(p) + \hat{\theta}_d^g(p))}{\sinh(\hat{\theta}_c^g(p) - \hat{\theta}_d^g(p)) \sinh(\hat{\theta}_c^g(p) + \hat{\theta}_d^g(p))}.$$

To proceed further, the following trivial observation proves to be instrumental in selecting the correct form of \hat{z}_c .

Proposition 3. *The functions z_c (2.27) and \hat{z}_c (3.34) obey the equation*

$$(3.75) \quad \sum_{c=1}^n \operatorname{Re}(\hat{z}_c) = \sum_{c=1}^n \operatorname{Re}(z_c).$$

Proof. Looking back to (3.25), we see that $\hat{F} = e^{-\hat{\Theta}} \hat{y}^{-1} e^{\Lambda} F$, and so

$$(3.76) \quad (C\hat{F})^* = (e^{\hat{\Theta}} \hat{y}^{-1} e^{-\Lambda} CF)^* = (CF)^* e^{-\Lambda} \hat{y} e^{\hat{\Theta}}.$$

As a consequence, we obtain

$$(3.77) \quad \operatorname{tr}(\hat{F}(C\hat{F})^*) = \operatorname{tr}(e^{-\hat{\Theta}} \hat{y}^{-1} e^{\Lambda} F (CF)^* e^{-\Lambda} \hat{y} e^{\hat{\Theta}}) = \operatorname{tr}(F(CF)^*).$$

However, keeping in mind the definition (2.31), we can write

$$(3.78) \quad \begin{aligned} \operatorname{tr}(F(CF)^*) &= \sum_{j=1}^N F_j \overline{(CF)_j} = \sum_{c=1}^n \left(F_c \overline{(CF)_c} + F_{n+c} \overline{(CF)_{n+c}} \right) \\ &= \sum_{c=1}^n (F_c \bar{F}_{n+c} + F_{n+c} \bar{F}_c) = \sum_{c=1}^n (z_c + \bar{z}_c) = 2 \sum_{c=1}^n \operatorname{Re}(z_c). \end{aligned}$$

Along the same lines, from the definition (3.34) it is also evident that

$$(3.79) \quad \operatorname{tr}(\hat{F}(C\hat{F})^*) = 2 \sum_{c=1}^n \operatorname{Re}(\hat{z}_c),$$

and so by (3.77) the proof is complete. \square

Lemma 4. *For all $c \in \mathbb{N}_n$ and $g = (\mu, \nu) \in \tilde{\mathfrak{M}}$ the g -section of the function \hat{z}_c (3.34) has the form*

$$(3.80) \quad \hat{z}_c^g = -\frac{\sinh(i\hat{\nu} + 2\hat{\theta}_c^g)}{\sinh(2\hat{\theta}_c^g)} \prod_{\substack{d=1 \\ (d \neq c)}}^n \frac{\sinh(i\hat{\mu} + \hat{\theta}_c^g - \hat{\theta}_d^g) \sinh(i\hat{\mu} + \hat{\theta}_c^g + \hat{\theta}_d^g)}{\sinh(\hat{\theta}_c^g - \hat{\theta}_d^g) \sinh(\hat{\theta}_c^g + \hat{\theta}_d^g)}.$$

Proof. Recalling (3.70), it is clear that the open set $P \times \tilde{\mathfrak{M}}_0$ is *dense* in $P \times \tilde{\mathfrak{M}}$. Thus, by continuity, it is enough to prove that $\forall c \in \mathbb{N}_n$ and $\forall g \in \tilde{\mathfrak{M}}_0$ the g -section of the smooth function \hat{z}_c is given by (3.80). So, recalling our discussion surrounding the derivation of the formulae (3.73) and (3.74), it is sufficient to show that on each connected component of $P \times \tilde{\mathfrak{M}}_0$, for all $c \in \mathbb{N}_n$ we have $\tilde{S}_c = -1$.

Arguing by contradiction, let us *suppose* that there is an index $c_0 \in \mathbb{N}_n$ and a connected component \mathcal{O}_0 of $P \times \tilde{\mathfrak{M}}_0$ such that on \mathcal{O}_0 we have $\tilde{S}_{c_0} = 1$. Since the phase space P (1.2) is connected, we have $\mathcal{O}_0 = P \times \mathfrak{N}_0$, where \mathfrak{N}_0 is a connected component of $\tilde{\mathfrak{M}}_0$ (3.70). Now, let $x_0 \in \mathcal{O}_0$ be arbitrary and define

$$(3.81) \quad \mathcal{N}_0 = \{c \in \mathbb{N}_n \mid \tilde{S}_c(x_0) = 1\}.$$

Since \mathcal{O}_0 is connected, \mathcal{N}_0 is *independent* of the choice of the point x_0 . Moreover, it is a non-empty subset of \mathbb{N}_n , since $c_0 \in \mathcal{N}_0$.

Equipped with the above objects, let $g = (\mu, \nu) \in \mathfrak{N}_0$, and keep it fixed. Furthermore, from the configuration space Q (1.6) take an arbitrary $\xi = (\xi_1, \dots, \xi_n)$, and in the phase space (1.2) for all $r, s \in \mathbb{N}$ define a point

$$(3.82) \quad p_{r,s} = (r\xi, s\xi) \in P.$$

Looking back to the definition (2.28), it is clear that $\forall c \in \mathbb{N}_n$ and $\forall s \in \mathbb{N}$ we have the limit relation

$$(3.83) \quad z_c^g(p_{r,s}) \rightarrow -\exp(i\nu + (N - 2c)i\mu) \quad (r \rightarrow \infty).$$

Thus, it is straightforward to verify that

$$(3.84) \quad \sum_{c=1}^n \operatorname{Re}(z_c^g(p_{r,s})) \rightarrow -\cos(\nu + (n-1)\mu) \frac{\sin(n\mu)}{\sin(\mu)} \quad (r \rightarrow \infty).$$

Turning to the study of \hat{z}_c^g (3.34), from the definition of u_c^g (1.11) it is obvious that $\forall c \in \mathbb{N}_n$ and $\forall s \in \mathbb{N}$ we have $u_c^g(p_{r,s}) \rightarrow 1$ as $r \rightarrow \infty$, and so recalling (2.31) we obtain the limit relations

$$(3.85) \quad F_c^g(p_{r,s}) \rightarrow e^{s\xi_c/2} \quad \text{and} \quad F_{n+c}^g(p_{r,s}) \rightarrow -e^{-s\xi_c/2} e^{-i\nu - (N-2c)i\mu} \quad (r \rightarrow \infty).$$

Looking back to the entries of the Lax matrix L (2.33), it is clear that

$$(3.86) \quad \mathcal{P}_{\text{reg}} \ni L(p_{r,s}, g) \rightarrow \operatorname{diag}(e^{s\xi_1}, \dots, e^{s\xi_n}, e^{-s\xi_1}, \dots, e^{-s\xi_n}) \in \mathcal{P}_{\text{reg}} \quad (r \rightarrow \infty).$$

Now, let us observe that the diffeomorphism Υ (2.25) allows us to write that

$$(3.87) \quad \operatorname{diag}(e^{s\xi_1}, \dots, e^{s\xi_n}, e^{-s\xi_1}, \dots, e^{-s\xi_n}) = \Upsilon(\operatorname{diag}(s\xi_1, \dots, s\xi_n, -s\xi_1, \dots, -s\xi_n), \mathbf{1}_N M).$$

Moreover, due to Lemma 1, we can also write that

$$(3.88) \quad L(p_{r,s}, g) = \hat{y}(p_{r,s}, g) e^{2\hat{\Theta}(p_{r,s}, g)} \hat{y}(p_{r,s}, g)^{-1} = \Upsilon(2\hat{\Theta}(p_{r,s}, g), \hat{y}(p_{r,s}, g)M).$$

Thus, from (3.86) it follows that

$$(3.89) \quad (2\hat{\Theta}(p_{r,s}, g), \hat{y}(p_{r,s}, g)M) \rightarrow (\operatorname{diag}(s\xi_1, \dots, s\xi_n, -s\xi_1, \dots, -s\xi_n), \mathbf{1}_N M) \quad (r \rightarrow \infty).$$

In particular, $\forall c \in \mathbb{N}_n$ and $\forall s \in \mathbb{N}$ we obtain

$$(3.90) \quad \hat{\theta}_c(p_{r,s}, g) \rightarrow \frac{1}{2} s\xi_c \quad (r \rightarrow \infty).$$

Now, from the above observations the behavior of $\hat{z}_c^g(p_{r,s})$ (3.34) for $r \rightarrow \infty$ comes almost effortlessly. Indeed, recalling (3.81), from (3.73) it follows immediately that $\forall c \in \mathcal{N}_0$ and $\forall s \in \mathbb{N}$ we have

$$(3.91) \quad \hat{z}_c^g(p_{r,s}) \rightarrow \frac{\sinh(i(2\hat{\mu} - \hat{\nu}) + s\xi_c)}{\sinh(s\xi_c)} \prod_{\substack{d=1 \\ (d \neq c)}}^n \frac{\sinh(i\hat{\mu} + s(\xi_c - \xi_d)/2) \sinh(i\hat{\mu} + s(\xi_c + \xi_d)/2)}{\sinh(s(\xi_c - \xi_d)/2) \sinh(s(\xi_c + \xi_d)/2)},$$

whereas equation (3.74) entails that $\forall c \in \mathbb{N}_n \setminus \mathcal{N}_0$ and $\forall s \in \mathbb{N}$ we can write

$$(3.92) \quad \hat{z}_c^g(p_{r,s}) \rightarrow -\frac{\sinh(i\hat{\nu} + s\xi_c)}{\sinh(s\xi_c)} \prod_{\substack{d=1 \\ (d \neq c)}}^n \frac{\sinh(i\hat{\mu} + s(\xi_c - \xi_d)/2) \sinh(i\hat{\mu} + s(\xi_c + \xi_d)/2)}{\sinh(s(\xi_c - \xi_d)/2) \sinh(s(\xi_c + \xi_d)/2)}.$$

Keeping in mind the limit relation (3.84), notice that $\forall s \in \mathbb{N}$ from Proposition 3 we can infer that

$$(3.93) \quad -\cos(\nu + (n-1)\mu) \frac{\sin(n\mu)}{\sin(\mu)} = \sum_{c \in \mathcal{N}_0} \operatorname{Re} \left(\lim_{r \rightarrow \infty} \hat{z}_c^g(p_{r,s}) \right) + \sum_{c \in \mathbb{N}_n \setminus \mathcal{N}_0} \operatorname{Re} \left(\lim_{r \rightarrow \infty} \hat{z}_c^g(p_{r,s}) \right),$$

where the limits on the right can be read off from (3.91) and (3.92).

To proceed further, in the following we shall study the behavior of equation (3.93) for large values of s . From (3.91) it follows immediately that $\forall c \in \mathcal{N}_0$ we have

$$(3.94) \quad \lim_{s \rightarrow \infty} \lim_{r \rightarrow \infty} \hat{z}_c^g(p_{r,s}) = \exp(i(2\hat{\mu} - \hat{\nu}) + (N-2c)i\hat{\mu}),$$

whilst from (3.92) we see that $\forall c \in \mathbb{N}_n \setminus \mathcal{N}_0$ we can write

$$(3.95) \quad \lim_{s \rightarrow \infty} \lim_{r \rightarrow \infty} \hat{z}_c^g(p_{r,s}) = -\exp(i\hat{\nu} + (N-2c)i\hat{\mu}).$$

Therefore, as $s \rightarrow \infty$, the above relationship (3.93) yields

$$(3.96) \quad -\cos(\nu + (n-1)\mu) \frac{\sin(n\mu)}{\sin(\mu)} = \sum_{c \in \mathcal{N}_0} \operatorname{Re}(e^{i(2\hat{\mu} - \hat{\nu}) + (N-2c)i\hat{\mu}}) - \sum_{c \in \mathbb{N}_n \setminus \mathcal{N}_0} \operatorname{Re}(e^{i\hat{\nu} + (N-2c)i\hat{\mu}}).$$

It is straightforward to see that this equation is equivalent to

$$(3.97) \quad \cos(\nu - \mu) \sum_{c \in \mathcal{N}_0} \cos((2c - N - 1)\mu) = 0.$$

However, since $g = (\mu, \nu) \in \mathfrak{N}_0 \subset \tilde{\mathfrak{M}}_0$ (3.70), we have $\cos(\mu - \nu) \neq 0$, thus we can also write that

$$(3.98) \quad \sum_{c \in \mathcal{N}_0} (e^{i(2c-N-1)\mu} + e^{-i(2c-N-1)\mu}) = 0.$$

Now, notice that in the above sum the powers of $e^{i\mu}$ are all distinct. Recall also that our argument is valid for any $g = (\mu, \nu) \in \mathfrak{N}_0$, which is an open and non-empty condition on μ as well. However, if μ varies in an open and non-empty subset of \mathbb{R} , then the family of functions $\{\mu \mapsto e^{i\ell\mu}\}_{\ell \in \mathbb{Z}}$ is clearly linearly independent, thus (3.98) expresses a contradiction. So, necessarily, $\mathcal{N}_0 = \emptyset$, and the proof is complete. \square

To proceed further, for each $c \in \mathbb{N}_n$ we define $\hat{u}_c = |\hat{z}_c|$. Due to Lemma 4, it is obvious that for all $g = (\mu, \nu) \in \tilde{\mathfrak{M}}$ we can write

$$(3.99) \quad \hat{u}_c^g = \left(1 + \frac{\sin(\hat{\nu})^2}{\sinh(2\hat{\theta}_c^g)^2} \right)^{\frac{1}{2}} \prod_{\substack{d=1 \\ (d \neq c)}}^n \left(1 + \frac{\sin(\hat{\mu})^2}{\sinh(\hat{\theta}_c^g - \hat{\theta}_d^g)^2} \right)^{\frac{1}{2}} \left(1 + \frac{\sin(\hat{\mu})^2}{\sinh(\hat{\theta}_c^g + \hat{\theta}_d^g)^2} \right)^{\frac{1}{2}} \in C^\infty(P).$$

Let us observe that $\hat{u}_c > 1$. Recalling Lemma 1 and the definition (3.25), it is also clear that $\hat{F}_c > 0$, whence the function

$$(3.100) \quad \hat{\lambda}_c: P \times \tilde{\mathfrak{M}} \rightarrow \mathbb{R}, \quad (p, g) \mapsto \hat{\lambda}_c(p, g) = 2 \ln(\hat{F}_c(p, g)) - \ln(\hat{u}_c(p, g))$$

is well-defined and smooth. Keeping in mind (3.34), it is obvious that

$$(3.101) \quad \hat{F}_c = e^{\frac{\hat{\lambda}_c}{2}} \hat{u}_c^{\frac{1}{2}} \quad \text{and} \quad \hat{F}_{n+c} = \frac{1}{\hat{F}_c} \bar{\hat{z}}_c = e^{-\frac{\hat{\lambda}_c}{2}} \bar{\hat{z}}_c \hat{u}_c^{-\frac{1}{2}}.$$

Now, with the aid of the functions $\hat{\theta}_c$ (3.1) and $\hat{\lambda}_c$ (3.100), for each $g \in \tilde{\mathfrak{M}}$ we introduce the *smooth* map

$$(3.102) \quad \Psi^g: P \rightarrow P, \quad p \mapsto \Psi^g(p) = (\hat{\theta}_1^g(p), \dots, \hat{\theta}_n^g(p), \hat{\lambda}_1^g(p), \dots, \hat{\lambda}_n^g(p)).$$

Comparing the formulae (2.28) and (3.80), it is clear that the components of F (2.31) and \hat{F} (3.101) look alike. As a consequence, now we can uncover the precise connection between the matrix entries of L (2.34) and \hat{L} (3.31), too. Indeed, making use of Ψ^g (3.102), our observations can be formulated as follows.

Theorem 5. *For all $g \in \tilde{\mathfrak{M}}$, $c \in \mathbb{N}_n$, and $j \in \mathbb{N}_N$ we can write*

$$(3.103) \quad \hat{z}_c^g = z_c^{\hat{g}} \circ \Psi^g \quad \text{and} \quad \hat{F}_j^g = F_j^{\hat{g}} \circ \Psi^g,$$

whilst the matrix valued functions L (2.33) and \hat{L} (3.24) are related by

$$(3.104) \quad \hat{L}^g = L^{\hat{g}} \circ \Psi^g.$$

Following the terminology of [12], at this point we may call the matrix valued function \hat{L} (3.24) the *dual Lax matrix*, whereas the name *duality map* seems adequate for Ψ^g (3.102). Of course, to fully justify these names, the map Ψ^g (3.102) must meet some further conditions, that we wish to examine in the rest of the paper. Heading toward this goal, take an arbitrary $g \in \tilde{\mathfrak{M}}$ and keep it fixed. Recalling (2.25), (2.36), (3.4) and (3.102), it is evident that $\hat{\Theta}^g = \Lambda \circ \Psi^g$, whence (3.8) can be rewritten as

$$(3.105) \quad L^g = \hat{y}^g e^{2\Lambda \circ \Psi^g} (\hat{y}^g)^{-1} = \Upsilon(2\Lambda \circ \Psi^g, \hat{y}^g M).$$

Thus, by composing the above functions with $\Psi^{\hat{g}}$, we obtain

$$(3.106) \quad L^g \circ \Psi^{\hat{g}} = \Upsilon(2\Lambda \circ \Psi^g \circ \Psi^{\hat{g}}, (\hat{y}^g \circ \Psi^{\hat{g}}) M).$$

On the other hand, keeping in mind the involution (3.29) and the definition (3.24), the observation we made in (3.104) entails that

$$(3.107) \quad L^g \circ \Psi^{\hat{g}} = L^{\hat{g}} \circ \Psi^{\hat{g}} = \hat{L}^{\hat{g}} = (\hat{y}^{\hat{g}})^{-1} e^{2\Lambda} \hat{y}^{\hat{g}} = \Upsilon(2\Lambda, (\hat{y}^{\hat{g}})^{-1} M).$$

Since Υ (2.25) is a diffeomorphism, the above two equations imply that

$$(3.108) \quad \Lambda \circ \Psi^g \circ \Psi^{\hat{g}} = \Lambda.$$

Moreover, there is a unique smooth function $m^g \in C^\infty(P, M)$ such that

$$(3.109) \quad (\hat{y}^{\hat{g}})^{-1} = (\hat{y}^g \circ \Psi^{\hat{g}}) m^g.$$

By applying the involution (3.29) on the last two equations, we get

$$(3.110) \quad \Lambda \circ \Psi^{\hat{g}} \circ \Psi^g = \Lambda \circ \Psi^{\hat{g}} \circ \Psi^{\hat{g}} = \Lambda \quad \text{and} \quad (\hat{y}^g)^{-1} = (\hat{y}^{\hat{g}})^{-1} = (\hat{y}^{\hat{g}} \circ \Psi^{\hat{g}}) m^{\hat{g}} = (\hat{y}^{\hat{g}} \circ \Psi^g) m^{\hat{g}}.$$

Thus, recalling (3.107) and (3.106), it also follows that

$$(3.111) \quad \begin{aligned} L^g \circ \Psi^{\hat{g}} \circ \Psi^g &= (L^g \circ \Psi^{\hat{g}}) \circ \Psi^g = ((\hat{y}^{\hat{g}})^{-1} \circ \Psi^g) e^{2\Lambda \circ \Psi^g} (\hat{y}^{\hat{g}} \circ \Psi^g) \\ &= m^{\hat{g}} \hat{y}^g e^{2\Lambda \circ \Psi^g} (\hat{y}^g)^{-1} (m^{\hat{g}})^{-1} = m^{\hat{g}} L^g (m^{\hat{g}})^{-1}. \end{aligned}$$

To proceed further, take an arbitrary point $p = (\boldsymbol{\xi}, \boldsymbol{\eta}) \in P$ and let

$$(3.112) \quad p' = (\boldsymbol{\xi}', \boldsymbol{\eta}') = \Psi^{\hat{g}} \circ \Psi^g(p) \in P.$$

Applying (3.110), we can clearly write that

$$(3.113) \quad \Lambda(p') = \Lambda(p),$$

so the definition (2.36) leads to the relation $\boldsymbol{\xi}' = \boldsymbol{\xi}$. Next, let us recall that the function $m^{\hat{g}}$ takes values in the subgroup M (2.12), whence it is *diagonal*. Therefore, due to the relationship (3.111), for all $c \in \mathbb{N}_n$ we can write that

$$(3.114) \quad L_{c,c}^g(p') = L_{c,c}^g(p).$$

Recalling (2.34) and (2.31), it is clear that

$$(3.115) \quad L_{c,c}^g(p) = |F_c^g(p)|^2 = e^{\eta_c} u_c^g(p) \quad \text{and} \quad L_{c,c}^g(p') = |F_c^g(p')|^2 = e^{\eta'_c} u_c^g(p').$$

Since $\boldsymbol{\xi}' = \boldsymbol{\xi}$, from the definition (1.11) it is evident that $u_c^g(p') = u_c^g(p)$. Consequently, from (3.114) we infer that

$$(3.116) \quad \eta'_c = \eta_c \quad (c \in \mathbb{N}_n).$$

Now, putting the above observations together, we see that $p' = p$; that is,

$$(3.117) \quad \Psi^{\hat{g}} \circ \Psi^g(p) = p \quad (p \in P).$$

In other words, $\Psi^{\hat{g}} \circ \Psi^g = \text{Id}_P$. Applying the involution (3.29) once more, we also obtain that

$$(3.118) \quad \Psi^g \circ \Psi^{\hat{g}} = \Psi^{\hat{g}} \circ \Psi^g = \text{Id}_P.$$

Theorem 6. *For all $g \in \tilde{\mathfrak{M}}$ the map Ψ^g (3.102) is a diffeomorphism with inverse $(\Psi^g)^{-1} = \Psi^{\hat{g}}$.*

To sum up, using essentially algebraic techniques, in this section we proved that the proposed duality map Ψ^g (3.102) is a diffeomorphism. To complete the study of Ψ^g , we still have to investigate its relationship with the symplectic form ω (1.5). Since our approach is built upon the scattering theory of the van Diejen system (1.12), using mainly analytical techniques, it is favorable to relegate this material to a separate section.

4. SCATTERING THEORY AND DUALITY

In this section we provide a rigorous treatment on the scattering theory of the particle system governed by the Hamiltonian function H^g (1.12). Our analysis hinges on Ruijsenaars' theorem about the asymptotic properties of certain exponential type matrix flows (see Theorem A2 in [12]). Merging this result with our projection method presented in [28], we shall work out the asymptotic properties of the particle trajectories. By pushing forward this approach, the wave and the scattering maps also become accessible. As an added bonus, in this section we shall complete the study of the self-duality property of the van Diejen type systems (1.12), too, by proving that the duality map Ψ^g (3.102) is an anti-symplectomorphism.

4.1. Recapitulation of Ruijsenaars' theorem. To make our presentation self-contained, in this subsection we briefly recapitulate Ruijsenaars' aforementioned theorem. Given a quadratic matrix

$$(4.1) \quad M = [M_{k,l}]_{1 \leq k, l \leq N} \in \mathbb{C}^{N \times N},$$

for its leading principal minors we introduce the temporary notation

$$(4.2) \quad \pi_j(M) = \det([M_{k,l}]_{1 \leq k, l \leq j}) \in \mathbb{C} \quad (j \in \mathbb{N}_N).$$

Clearly the subset

$$(4.3) \quad \mathcal{M} = \{M \in \mathbb{C}^{N \times N} \mid \pi_j(M) \neq 0 \quad (j \in \mathbb{N}_N)\}$$

is open in $\mathbb{C}^{N \times N}$, and the functions $m_j: \mathcal{M} \rightarrow \mathbb{C}$, $M \mapsto m_j(M)$ defined by the formulae

$$(4.4) \quad m_1(M) = M_{1,1}, \quad m_j(M) = \frac{\pi_j(M)}{\pi_{j-1}(M)} \quad (2 \leq j \leq N)$$

are smooth. We also need the special family of diagonal matrices

$$(4.5) \quad \mathcal{D} = \{D = \text{diag}(d_1, \dots, d_N) \in \mathbb{C}^{N \times N} \mid \text{Re}(d_1) > \dots > \text{Re}(d_N)\},$$

and with each pair $X = (M, D) \in \mathcal{M} \times \mathcal{D}$ we associate the exponential type matrix flow

$$(4.6) \quad E_X: \mathbb{R} \rightarrow \mathbb{C}^{N \times N}, \quad s \mapsto E_X(s) = Me^{sD}.$$

Counting with multiplicities, let $\lambda_1^X(s), \dots, \lambda_N^X(s)$ denote the not necessarily distinct eigenvalues of $E_X(s)$. For convenience, we shall assume that

$$(4.7) \quad |\lambda_1^X(s)| \geq \dots \geq |\lambda_N^X(s)|.$$

Equipped with the above objects, we are now in a position to state Ruijsenaars' theorem in the form that is the most convenient in the later developments.

Theorem 7. *For each pair $X_0 = (M_0, D_0) \in \mathcal{M} \times \mathcal{D}$ there exist positive constants $T_0, C_0, R_0 > 0$ and a compact subset $\mathcal{K}_0 \subset \mathcal{M} \times \mathcal{D}$ containing X_0 in its topological interior such that $\forall s \in [T_0, \infty)$ and $\forall X \in \mathcal{K}_0$ we have*

$$(4.8) \quad |\lambda_1^X(s)| > \dots > |\lambda_N^X(s)|.$$

Moreover, $\forall s \in [T_0, \infty)$, $\forall X = (M, D) \in \mathcal{K}_0$ and $\forall j \in \mathbb{N}_N$ we can write

$$(4.9) \quad \lambda_j^X(s) = m_j(M)e^{sD_{j,j}} (1 + \rho_j^X(s)),$$

whereas for its derivative with respect to s we have

$$(4.10) \quad \dot{\lambda}_j^X(s) = m_j(M)e^{sD_{j,j}} (D_{j,j} + D_{j,j}\rho_j^X(s) + \dot{\rho}_j^X(s)),$$

where

$$(4.11) \quad |\rho_j^X(s)| \leq C_0 e^{-sR_0} \leq \frac{1}{2} \quad \text{and} \quad |\dot{\rho}_j^X(s)| \leq C_0 e^{-sR_0} \leq \frac{1}{2}.$$

We mention in passing that the theorem in its original form is sharper in the sense that in [12] we find finer estimates on the decay (4.11) for each individual eigenvalue. For even sharper estimates, the reader is kindly advised to study Theorem A.1 in Appendix A.

4.2. Application of Ruijsenaars' theorem. Starting with this subsection, we wish to analyze the dynamics generated by the Hamiltonian function H^g (1.12). Notationwise, in the rest of the whole section we take an arbitrary $g = (\mu, \nu) \in \tilde{\mathfrak{M}}$ and keep it fixed. Also, for the g -section of any function defined on $P \times \tilde{\mathfrak{M}}$ we shall omit the superscript g , i.e., we shall write $H \equiv H^g$, $\Psi \equiv \Psi^g$, $L \equiv L^g$, $\hat{L} \equiv \hat{L}^g$, etc.

Now, recall that the Hamiltonian vector field $\mathbf{X}_H \in \mathfrak{X}(P)$ corresponding to the Hamiltonian H (1.12) is defined by

$$(4.12) \quad \mathbf{X}_H[f] = \{f, H\} \quad (f \in C^\infty(P)).$$

In our paper [28] we proved that the vector field \mathbf{X}_H is complete. In other words, if $p \in P$, $I \subseteq \mathbb{R}$ is an open interval containing 0, and

$$(4.13) \quad I \ni s \mapsto \gamma_p(s) \in P$$

is the maximally defined integral curve of \mathbf{X}_H satisfying the initial condition $\gamma_p(0) = p$, then we have $I = \mathbb{R}$. This fact allows us to introduce the flow

$$(4.14) \quad \Phi: \mathbb{R} \times P \rightarrow P, \quad (s, p) \mapsto \Phi(s, p) = \gamma_p(s),$$

that is a smooth map (see e.g. Theorem 9.12 in [33]). Conforming with the standard convention, for any $s \in \mathbb{R}$ and $p \in P$ we define the s -section $\Phi_s: P \rightarrow P$ and the p -section $\Phi^p: \mathbb{R} \rightarrow P$ by

$$(4.15) \quad \Phi_s(p) = \Phi^p(s) = \Phi(s, p).$$

In the following we shall apply this notation for each function defined on $\mathbb{R} \times P$.

Next, for the natural extensions of the coordinates (1.4) onto $\mathbb{R} \times P$ we introduce the notations

$$(4.16) \quad \tilde{x}_j(s, p) = x_j(p) \quad (j \in \mathbb{N}_N, (s, p) \in \mathbb{R} \times P).$$

Upon defining

$$(4.17) \quad t: \mathbb{R} \times P \rightarrow \mathbb{R}, \quad (s, p) \mapsto s,$$

that we call the time variable, it is evident that the family of smooth functions $t, \tilde{x}_1, \dots, \tilde{x}_N$ provides a global coordinate system on the product manifold $\mathbb{R} \times P$. Notice that the defining property of Φ (4.14) can be rephrased as

$$(4.18) \quad \frac{\partial(f \circ \Phi)}{\partial t} = \{f, H\} \circ \Phi \quad (f \in C^\infty(P))$$

and $\Phi_0 = \text{Id}_P$. Occasionally, in the rest of the paper the partial differentiation with respect to t will be denoted by a dot.

Equipped with the above objects, now we are in a position to make a closer inspection of the dynamics. Due to the completeness of \mathbf{X}_H (4.12), it is natural to inquire about the properties of the flow (4.14) for large values of s . In this respect our key observation is that the trajectories (4.13) can be recovered from the projection method we worked out in [28]. More precisely, due to Theorem 12 in [28], we have the spectral identification

$$(4.19) \quad \{e^{\pm 2\lambda_a \circ \Phi(s,p)} \mid a \in \mathbb{N}_n\} = \text{Spec} \left(e^{2\Lambda(p)} e^{s(L(p)-L(p)^{-1})} \right) \quad (s \in \mathbb{R}, p \in P).$$

To put it simple, finding $\lambda_a \circ \Phi$ amounts to determining the eigenvalues of the matrix flow

$$(4.20) \quad \mathbb{R} \ni s \mapsto e^{2\Lambda(p)} e^{s(L(p)-L(p)^{-1})} \in GL(N, \mathbb{C}).$$

As concerns the θ_a coordinate of Φ , notice that the equation of motion (4.18) entails

$$(4.21) \quad \frac{\partial(\lambda_a \circ \Phi)}{\partial t} = \{\lambda_a, H\} \circ \Phi = \frac{\partial H}{\partial \theta_a} \circ \Phi = \sinh(\theta_a(\Phi)) u_a(\Phi).$$

Therefore, since the function u_a (1.11) is independent of the particle rapidities, from the knowledge of $\lambda_a \circ \Phi$ it comes effortlessly that

$$(4.22) \quad \theta_a \circ \Phi = \text{arcsinh} \left(\frac{1}{u_a(\Phi)} \frac{\partial(\lambda_a \circ \Phi)}{\partial t} \right).$$

In the light of the above observations our plan is quite straightforward. Taking into account Ruijsenaars' theorem, we can get close control over the asymptotics of the eigenvalues of the matrix flow (4.20). Therefore, by exploiting (4.19) and (4.22), we can squeeze information about the asymptotic properties of the flow Φ (4.14) as well. The rest is technique.

In order to meet the requirements of Theorem 7, we still have to diagonalize the exponent of the matrix flow (4.20). We can be brief here, since the same idea already appeared in Subsection 4.5 our paper [28]. Recalling Lemma 1 and definition (3.24), we can write that

$$(4.23) \quad L = \hat{y} e^{2\hat{\Theta}} \hat{y}^{-1} \quad \text{and} \quad \hat{L} = \hat{y}^{-1} e^{2\Lambda} \hat{y},$$

thus

$$(4.24) \quad L - L^{-1} = 2\hat{y} \sinh(2\hat{\Theta}) \hat{y}^{-1} \quad \text{and} \quad e^{2\Lambda} = \hat{y} \hat{L} \hat{y}^{-1}.$$

It follows that $\forall s \in \mathbb{R}$ and $\forall p \in P$ we have

$$(4.25) \quad \text{Spec} \left(e^{2\Lambda(p)} e^{s(L(p)-L(p)^{-1})} \right) = \text{Spec} \left(\hat{L}(p) e^{2s \sinh(2\hat{\Theta}(p))} \right).$$

To proceed further, for each $m \in \mathbb{N}$ we introduce the $m \times m$ Hermitian matrix \mathcal{R}_m with entries

$$(4.26) \quad (\mathcal{R}_m)_{k,l} = \delta_{k+l, m+1} \quad (1 \leq k, l \leq m),$$

and the $N \times N$ Hermitian matrix

$$(4.27) \quad W = \begin{bmatrix} \mathbf{1}_n & \mathbf{0}_n \\ \mathbf{0}_n & \mathcal{R}_n \end{bmatrix}.$$

Since $\mathcal{R}_m^2 = \mathbf{1}_m$, both \mathcal{R}_m and W are invertible with inverses

$$(4.28) \quad \mathcal{R}_m^{-1} = \mathcal{R}_m \quad \text{and} \quad W^{-1} = W.$$

Next, upon introducing the matrix valued functions

$$(4.29) \quad \Theta^+ = 2W\hat{\Theta}W^{-1} \quad \text{and} \quad \tilde{L} = W\hat{L}W^{-1},$$

from (4.19) and (4.25) we conclude that

$$(4.30) \quad \{e^{\pm 2\lambda_a \circ \Phi(s,p)} \mid a \in \mathbb{N}_n\} = \text{Spec} \left(\tilde{L}(p)e^{2s \sinh(\Theta^+(p))} \right) \quad (s \in \mathbb{R}, p \in P).$$

Now, two observations are in order. First, notice that by construction the matrix $\tilde{L}(p)$ is positive definite, whence for its leading principal minors we have $\pi_j(\tilde{L}(p)) > 0$ ($j \in \mathbb{N}_N$). Recalling (4.4), it follows that $m_j(\tilde{L}(p)) > 0$. As a consequence, $\tilde{L}(p) \in \mathcal{M}$ (4.3), and for each $a \in \mathbb{N}_n$ the function

$$(4.31) \quad \lambda_a^+ : P \rightarrow \mathbb{R}, \quad p \mapsto \frac{1}{2} \ln(m_a(\tilde{L}(p)))$$

is well-defined and smooth. Second, from (4.29) it is also clear that

$$(4.32) \quad \Theta^+ = \text{diag}(\theta_1^+, \dots, \theta_n^+, -\theta_n^+, \dots, -\theta_1^+),$$

where

$$(4.33) \quad \theta_a^+ = 2\hat{\theta}_a \in C^\infty(P) \quad (a \in \mathbb{N}_n).$$

As a consequence, $\forall p \in P$ we have $2 \sinh(\Theta^+(p)) \in \mathcal{D}$ (4.5). The upshot of the above discussion is that, for each $p \in P$, Theorem 7 is directly applicable on the matrix flow

$$(4.34) \quad \mathbb{R} \ni s \mapsto \tilde{L}(p)e^{2s \sinh(\Theta^+(p))} \in GL(N, \mathbb{C}),$$

and so by (4.30) we can obtain information about the asymptotic properties of the flow Φ (4.14). To formulate our first result in this direction, for each $a \in \mathbb{N}_n$ we introduce the smooth functions

$$(4.35) \quad \mathcal{E}_a : \mathbb{R} \times P \rightarrow \mathbb{R}, \quad (s, p) \mapsto \lambda_a(\Phi(s, p)) - s \sinh(\theta_a^+(p)) - \lambda_a^+(p),$$

$$(4.36) \quad \mathcal{F}_a : \mathbb{R} \times P \rightarrow \mathbb{R}, \quad (s, p) \mapsto \sinh(\theta_a(\Phi(s, p)))u_a(\Phi(s, p)) - \sinh(\theta_a^+(p)).$$

Lemma 8. *For each point $p_0 \in P$ there are strictly positive constants $T_0, C_0, R_0 > 0$ and a compact subset $K_0 \subset P$ containing p_0 in its interior such that $\forall s \in [T_0, \infty)$, $\forall p \in K_0$, and $\forall a \in \mathbb{N}_n$ we have*

$$(4.37) \quad |\mathcal{E}_a(s, p)| \leq C_0 e^{-sR_0} \leq \frac{1}{2} \quad \text{and} \quad |\mathcal{F}_a(s, p)| \leq C_0 e^{-sR_0} \leq \frac{1}{2}.$$

Proof. From our earlier discussion it is clear that the map

$$(4.38) \quad \Gamma : P \rightarrow \mathcal{M} \times \mathcal{D}, \quad p \mapsto (\tilde{L}(p), 2 \sinh(\Theta^+(p)))$$

is well-defined and smooth. Now, take an arbitrary point $p_0 \in P$. Associated with $X_0 = \Gamma(p_0)$ we also choose the non-negative constants $T_0, C_0, R_0 > 0$ and the compact subset $\mathcal{K}_0 \subset \mathcal{M} \times \mathcal{D}$ containing X_0 in its interior, whose existence is guaranteed by Theorem 7. Note that by the continuity of Γ we can find a compact subset $K_0 \subset P$ such that

$$(4.39) \quad p_0 \in \text{int}(K_0) \subset K_0 \subset \Gamma^{-1}(\text{int}(\mathcal{K}_0)).$$

To proceed further, let $s \in [T_0, \infty)$, $p \in K_0$ and $a \in \mathbb{N}_n$ be arbitrary elements. Since by (4.39) we have $X = \Gamma(p) \in \mathcal{K}_0$, from (4.9) we infer that

$$(4.40) \quad e^{2\lambda_a(\Phi(s,p))} = m_a(\tilde{L}(p))e^{2s \sinh(\theta_a^+(p))} (1 + \rho_a^X(s)),$$

whereas (4.10) leads to

$$(4.41) \quad \frac{\partial e^{2\lambda_a \circ \Phi}}{\partial t}(s, p) = m_a(\tilde{L}(p))e^{2s \sinh(\theta_a^+(p))} (2 \sinh(\theta_a^+(p))(1 + \rho_a^X(s)) + \dot{\rho}_a^X(s)).$$

From (4.40) and (4.41) it is clear that both $\rho_a^X(s)$ and $\dot{\rho}_a^X(s)$ are real numbers. Moreover, due to the estimates displayed in (4.11), they are of magnitude less than or equal to $1/2$. So, by taking the logarithm of (4.40), from the definitions (4.31) and (4.35) it is straightforward to verify that

$$(4.42) \quad \mathcal{E}_a(s, p) = \frac{1}{2} \ln(1 + \rho_a^X(s)),$$

and so we may write

$$(4.43) \quad |\mathcal{E}_a(s, p)| = \frac{1}{2} |\ln(1 + \rho_a^X(s))| \leq |\rho_a^X(s)| \leq C_0 e^{-sR_0} \leq \frac{1}{2}.$$

Next, from the equation of motion (4.21) and the relationship (4.40) we see that

$$(4.44) \quad \begin{aligned} \frac{\partial e^{2\lambda_a \circ \Phi}}{\partial t}(s, p) &= 2 \frac{\partial(\lambda_a \circ \Phi)}{\partial t}(s, p) e^{2\lambda_a(\Phi(s, p))} = 2 \sinh(\theta_a(\Phi(s, p))) u_a(\Phi(s, p)) e^{2\lambda_a(\Phi(s, p))} \\ &= 2 \sinh(\theta_a(\Phi(s, p))) u_a(\Phi(s, p)) m_a(\tilde{L}(p)) e^{2s \sinh(\theta_a^+(p))} (1 + \rho_a^X(s)). \end{aligned}$$

Notice that by plugging this formula into the left hand side of (4.41) we obtain that

$$(4.45) \quad \sinh(\theta_a(\Phi(s, p))) u_a(\Phi(s, p)) = \sinh(\theta_a^+(p)) + \frac{1}{2} \frac{\dot{\rho}_a^X(s)}{1 + \rho_a^X(s)}.$$

Thus, recalling the definition (4.36), we end up with the estimate

$$(4.46) \quad |\mathcal{F}_a(s, p)| = \frac{1}{2} \frac{|\dot{\rho}_a^X(s)|}{1 + \rho_a^X(s)} \leq |\dot{\rho}_a^X(s)| \leq C_0 e^{-sR_0} \leq \frac{1}{2},$$

and so the proof is complete. \square

4.3. Asymptotics along the trajectories. Somewhat informally, Lemma 8 tells us that the time evolution of the particle positions is asymptotically linear. However, looking back to the definition (4.36), the asymptotic characterization of the particle rapidities given by Lemma 8 is quite implicit. To make it explicit, we need control over the value of the function u_a (1.11) along the trajectories for large values of time. *As a rule for this subsection, we find it convenient to fix an arbitrary point $p_0 \in P$, together with the corresponding non-negative constants T_0, C_0, R_0 and the compact subset K_0 given by Lemma 8.* Furthermore, we introduce

$$(4.47) \quad l_0 = \max\{|\lambda_a^+(p)| \mid a \in \mathbb{N}_n, p \in K_0\} \geq 0,$$

and

$$(4.48) \quad \Delta_0 = \min\{\min\{\theta_1^+(p) - \theta_2^+(p), \dots, \theta_{n-1}^+(p) - \theta_n^+(p), 2\theta_n^+(p)\} \mid p \in K_0\} > 0.$$

Since K_0 is compact, by the extreme value theorem both constants are well-defined.

Proposition 9. *There is a constant $T_1 \geq T_0$ such that if $s \in [T_1, \infty)$, $p \in K_0$, $a, b, c \in \mathbb{N}_n$ and $a \neq b$, then*

$$(4.49) \quad |\lambda_a(\Phi(s, p)) - \lambda_b(\Phi(s, p))| \geq s \frac{\Delta_0}{2} \geq \ln(\sqrt{2}) \quad \text{and} \quad 2\lambda_c(\Phi(s, p)) \geq s \frac{\Delta_0}{2} \geq \ln(\sqrt{2}).$$

Proof. Let $p \in K_0$, $a, b, c \in \mathbb{N}_n$ and suppose that $a \neq b$. Recalling (4.48), it is obvious that

$$(4.50) \quad |\theta_a^+(p) - \theta_b^+(p)| \geq \Delta_0 \quad \text{and} \quad 2\theta_c^+(p) \geq \Delta_0.$$

Thus, remembering the definitions (4.35) and (4.47), from Lemma 8 it is clear $\forall s \in [T_0, \infty)$ we can write that

$$\begin{aligned}
(4.51) \quad & |\lambda_a(\Phi(s, p)) - \lambda_b(\Phi(s, p))| \\
& = |s(\sinh(\theta_a^+(p)) - \sinh(\theta_b^+(p))) + \lambda_a^+(p) - \lambda_b^+(p) + \mathcal{E}_a(s, p) - \mathcal{E}_b(s, p)| \\
& \geq s|\sinh(\theta_a^+(p)) - \sinh(\theta_b^+(p))| - |\lambda_a^+(p)| - |\lambda_b^+(p)| - |\mathcal{E}_a(s, p)| - |\mathcal{E}_b(s, p)| \\
& \geq s\Delta_0 - 2l_0 - 1 = s\frac{\Delta_0}{2} + \frac{\Delta_0}{2} \left(s - \frac{4l_0 + 2}{\Delta_0} \right).
\end{aligned}$$

Along the same lines, one can easily infer that

$$(4.52) \quad 2\lambda_c(\Phi(s, p)) \geq s\frac{\Delta_0}{2} + \frac{\Delta_0}{2} \left(s - \frac{4l_0 + 2}{\Delta_0} \right).$$

So, with the constant

$$(4.53) \quad T_1 = \max\{T_0, (4l_0 + 2)/\Delta_0, \ln(2)/\Delta_0\}$$

the Proposition follows. \square

As an important remark, note that *in the rest of this subsection* T_1 shall always stand for the constant provided by Proposition 9.

Before going into the study of the asymptotic properties of u_a (1.11), notice that $\forall y \in \mathbb{R}$ satisfying $|y| \geq \ln(\sqrt{2})$ we have the trivial inequalities

$$(4.54) \quad \frac{1}{|\sinh(y)|} \leq 4e^{-|y|} \leq 2\sqrt{2}, \quad \frac{1}{\sinh(y)^2} \leq 16e^{-2|y|} \leq 8, \quad |\coth(y)| \leq 3.$$

Proposition 10. *There is a constant $C_1 \geq 0$ such that $\forall s \in [T_1, \infty)$, $\forall p \in K_0$, $\forall a \in \mathbb{N}_n$ we have*

$$(4.55) \quad u_a(\Phi(s, p)) \leq 1 + C_1 e^{-s\Delta_0}.$$

Proof. Since $\forall y \geq 0$ we have $\ln(1 + y) \leq y$, by taking the logarithm of u_a ($a \in \mathbb{N}_n$) we can write that

$$(4.56) \quad 0 < \ln(u_a) \leq \frac{1}{2} \frac{1}{\sinh(2\lambda_a)^2} + \frac{1}{2} \sum_{\substack{c=1 \\ (c \neq a)}}^n \left(\frac{1}{\sinh(\lambda_a - \lambda_c)^2} + \frac{1}{\sinh(\lambda_a + \lambda_c)^2} \right).$$

Thus, if $s \geq T_1$ and $p \in K_0$, then from Proposition 9 and the inequalities appearing in (4.54) we can easily infer that

$$(4.57) \quad \ln(u_a(\Phi(s, p))) \leq \frac{1}{2} \frac{1 + 2(n-1)}{\sinh\left(s\frac{\Delta_0}{2}\right)^2} \leq 8(N-1)e^{-s\Delta_0} \leq 4(N-1).$$

Since the exponential function is convex on the interval $[0, 4(N-1)]$, with the non-negative constant

$$(4.58) \quad C_1 = 2(e^{4(N-1)} - 1)$$

we can write that

$$(4.59) \quad 1 < u_a(\Phi(s, p)) \leq \exp(8(N-1)e^{-s\Delta_0}) \leq 1 + C_1 e^{-s\Delta_0},$$

whence the proof is complete. \square

During the later developments we shall also need control over the partial derivatives of the smooth functions u_a (1.11) along the trajectories. For brevity, we introduce the notations

$$(4.60) \quad \mathcal{L}_b^a = \frac{\partial \ln(u_a)}{\partial \lambda_b} \in C^\infty(P) \quad \text{and} \quad \mathcal{L}_{b,c}^a = \frac{\partial \mathcal{L}_b^a}{\partial \lambda_c} = \frac{\partial^2 \ln(u_a)}{\partial \lambda_b \partial \lambda_c} \in C^\infty(P) \quad (a, b, c \in \mathbb{N}_n).$$

Utilizing the auxiliary function

$$(4.61) \quad \Xi(y, \alpha) = \frac{\sin(\alpha)^2 \coth(y)}{\sin(\alpha)^2 + \sinh(y)^2} \quad (y \in \mathbb{R} \setminus \{0\}, \alpha \in \mathbb{R} \setminus \mathbb{Z}\pi),$$

it is straightforward to verify that

$$(4.62) \quad \mathcal{L}_a^a = -2\Xi(2\lambda_a, \nu) - \sum_{\substack{d=1 \\ (d \neq a)}}^n (\Xi(\lambda_a - \lambda_d, \mu) + \Xi(\lambda_a + \lambda_d, \mu)) \quad (a \in \mathbb{N}_n),$$

while

$$(4.63) \quad \mathcal{L}_b^a = -\Xi(\lambda_b - \lambda_a, \mu) - \Xi(\lambda_b + \lambda_a, \mu) \quad (a, b \in \mathbb{N}_n, a \neq b).$$

As an immediate consequence of (4.54), let us also observe that

$$(4.64) \quad |\Xi(y, \alpha)| \leq \frac{|\coth(y)|}{\sinh(y)^2} \leq 48e^{-2|y|} \quad (|y| \geq \ln(\sqrt{2})).$$

Thus, by Proposition 9 the following result is immediate.

Proposition 11. *There is a constant $C_2 \geq 0$ such that $\forall s \in [T_1, \infty)$, $\forall p \in K_0$, $\forall a, b \in \mathbb{N}_n$ we have*

$$(4.65) \quad |\mathcal{L}_b^a(\Phi(s, p))| \leq C_2 e^{-s\Delta_0}.$$

Proof. Let $s \in [T_1, \infty)$ and $p \in K_0$ be arbitrary elements, and also introduce the temporary shorthand notation

$$(4.66) \quad p' = \Phi(s, p) \in P.$$

Keeping in mind Proposition 9 and the estimate (4.64), from (4.62) it is clear that $\forall a \in \mathbb{N}_n$ we can write that

$$(4.67) \quad \begin{aligned} |\mathcal{L}_a^a(\Phi(s, p))| &\leq 2|\Xi(2\lambda_a(p'), \nu)| + \sum_{\substack{d=1 \\ (d \neq a)}}^n (|\Xi(\lambda_a(p') - \lambda_d(p'), \mu)| + |\Xi(\lambda_a(p') + \lambda_d(p'), \mu)|) \\ &\leq 96e^{-2|2\lambda_a(p')|} + 48 \sum_{\substack{d=1 \\ (d \neq a)}}^n \left(e^{-2|\lambda_a(p') - \lambda_d(p')|} + e^{-2|\lambda_a(p') + \lambda_d(p')|} \right) \leq 48Ne^{-s\Delta_0}. \end{aligned}$$

On the other hand, if $a, b \in \mathbb{N}_n$ and $a \neq b$, then (4.63) yields

$$(4.68) \quad \begin{aligned} |\mathcal{L}_b^a(\Phi(s, p))| &\leq |\Xi(\lambda_b(p') - \lambda_a(p'), \mu)| + |\Xi(\lambda_b(p') + \lambda_a(p'), \mu)| \\ &\leq 48e^{-2|\lambda_b(p') - \lambda_a(p')|} + 48e^{-2|\lambda_b(p') + \lambda_a(p')|} \leq 96e^{-s\Delta_0}. \end{aligned}$$

Therefore, with $C_2 = 48N$ the Proposition follows. \square

Turning to the second order partial derivatives of $\ln(u_a)$, bear in mind that the partial derivative of the function Ξ (4.61) with respect to its first variable y is given by

$$(4.69) \quad \Xi'(y, \alpha) = -\frac{2\sin(\alpha)^2 \cosh(y)^2}{(\sin(\alpha)^2 + \sinh(y)^2)^2} - \frac{\sin(\alpha)^2}{\sin(\alpha)^2 + \sinh(y)^2} \frac{1}{\sinh(y)^2}.$$

Making use of this derivative, from (4.62) and (4.63) we find immediately that

$$(4.70) \quad \mathcal{L}_{a,a}^a = -4\Xi'(2\lambda_a, \nu) - \sum_{\substack{d=1 \\ (d \neq a)}}^n (\Xi'(\lambda_a - \lambda_d, \mu) + \Xi'(\lambda_a + \lambda_d, \mu)) \quad (a \in \mathbb{N}_n).$$

Notice that the remaining second order partial derivatives take much simpler forms. Indeed, for all $a, b \in \mathbb{N}_n$ satisfying $a \neq b$ we have

$$(4.71) \quad \mathcal{L}_{b,b}^a = -\Xi'(\lambda_b - \lambda_a, \mu) - \Xi'(\lambda_b + \lambda_a, \mu),$$

$$(4.72) \quad \mathcal{L}_{b,a}^a = \mathcal{L}_{b,a}^a = \Xi'(\lambda_b - \lambda_a, \mu) - \Xi'(\lambda_b + \lambda_a, \mu),$$

whereas for all $a, b, c \in \mathbb{N}_n$ satisfying $c \neq a \neq b \neq c$ we can write that

$$(4.73) \quad \mathcal{L}_{b,c}^a = \mathcal{L}_{c,b}^a = 0.$$

Now, in order to cook up convenient estimates for the above objects, let us observe that by applying (4.54) on (4.69) we obtain

$$(4.74) \quad |\Xi'(y, \alpha)| \leq \frac{2 \cosh(y)^2}{\sinh(y)^4} + \frac{1}{\sinh(y)^2} = \frac{2 \coth(y)^2 + 1}{\sinh(y)^2} \leq 304e^{-2|y|} \quad (|y| \geq \ln(\sqrt{2})).$$

Therefore, without going into the details, the same ideas we presented in the proof of Proposition 11 lead to the following result immediately.

Proposition 12. *There is a constant $C_3 \geq 0$ such that $\forall s \in [T_1, \infty)$, $\forall p \in K_0$, $\forall a, b, c \in \mathbb{N}_n$ we have*

$$(4.75) \quad |\mathcal{L}_{b,c}^a(\Phi(s, p))| \leq C_3 e^{-s\Delta_0}.$$

Picking an arbitrary $a \in \mathbb{N}_n$, at this point we are in a position to measure effectively the deviation of the rapidity $\theta_a \circ \Phi$ from θ_a^+ (4.33). So, for convenience, we define the smooth function

$$(4.76) \quad \mathcal{G}_a: \mathbb{R} \times P \rightarrow \mathbb{R}, \quad (s, p) \mapsto \theta_a(\Phi(s, p)) - \theta_a^+(p).$$

Furthermore, with the aid of the Hamiltonian H (1.12), we introduce the strictly positive constant

$$(4.77) \quad \mathcal{H}_0 = \max\{H(p) \mid p \in K_0\} > 0,$$

which is well-defined by the compactness of K_0 . Since H is a first integral, let us note that $\forall s \in \mathbb{R}$, $\forall p \in K_0$, $\forall a \in \mathbb{N}_n$ we can write that

$$(4.78) \quad \begin{aligned} |\theta_a(\Phi(s, p))| &\leq \sinh(|\theta_a(\Phi(s, p))|) \leq \cosh(\theta_a(\Phi(s, p))) \\ &\leq H(\Phi(s, p)) = H(\Phi(0, p)) = H(p) \leq \mathcal{H}_0. \end{aligned}$$

Before formulating our explicit result about the asymptotic properties of the rapidities, it proves convenient to bring in the new constant

$$(4.79) \quad \delta_0 = \min\{R_0, \Delta_0\} > 0.$$

Lemma 13. *There is a constant $C_4 \geq 0$ such that $\forall s \in [T_1, \infty)$, $\forall p \in K_0$, $\forall a \in \mathbb{N}_n$ we can write*

$$(4.80) \quad |\mathcal{G}_a(s, p)| \leq C_4 e^{-s\delta_0}.$$

Proof. Let $s \in [T_1, \infty)$, $p \in K_0$, $a \in \mathbb{N}_n$ be arbitrary elements. Keeping in mind (4.76) and (4.36), it is clear that

$$(4.81) \quad \begin{aligned} |\mathcal{G}_a(s, p)| &\leq |\sinh(\theta_a(\Phi(s, p))) - \sinh(\theta_a^+(p))| \\ &\leq |\mathcal{F}_a(s, p)| + |\sinh(\theta_a(\Phi(s, p)))(1 - u_a(\Phi(s, p)))|. \end{aligned}$$

However, recalling (4.79), by Lemma 8 we can write

$$(4.82) \quad |\mathcal{F}_a(s, p)| \leq C_0 e^{-sR_0} \leq C_0 e^{-s\delta_0},$$

whereas from Proposition 10 and the observation we made in (4.78) it is immediate that

$$(4.83) \quad \begin{aligned} |\sinh(\theta_a(\Phi(s, p)))(1 - u_a(\Phi(s, p)))| &= \sinh(|\theta_a(\Phi(s, p))|)(u_a(\Phi(s, p)) - 1) \\ &\leq \mathcal{H}_0 C_1 e^{-s\Delta_0} \leq \mathcal{H}_0 C_1 e^{-s\delta_0}. \end{aligned}$$

Thus, with the constant $C_4 = C_0 + \mathcal{H}_0 C_1$ the Lemma follows. \square

To conclude this subsection, notice that by Lemmas 8 and 13 we may call the functions λ_a^+ (4.31) the *asymptotic positions*, and the name *asymptotic rapidities* is also justified for θ_a^+ (4.33).

4.4. Derivation of the equations of variation. From Lemma 8 the importance of the functions defined in (4.35) and (4.36) is evident. Not surprisingly, the whole scattering theory of the van Diejen systems (1.12) can be built upon the careful study of these objects. The partial differential equations we set up in this subsection form the basis of their finer analysis.

Lemma 14. *For each $a \in \mathbb{N}_n$ we have $\dot{\mathcal{E}}_a = \mathcal{F}_a$ and $\dot{\mathcal{F}}_a = \varphi_a \circ \Phi$, where*

$$(4.84) \quad \varphi_a = \sinh(\theta_a)u_a \sum_{c=1}^n \sinh(\theta_c)u_c \mathcal{L}_c^a - \cosh(\theta_a)u_a \sum_{c=1}^n \cosh(\theta_c)u_c \mathcal{L}_c^a \in C^\infty(P).$$

Proof. Let $a \in \mathbb{N}_n$ be arbitrary, then from (4.35) and (4.21) it is evident that

$$(4.85) \quad \frac{\partial \mathcal{E}_a}{\partial t} = \frac{\partial (\lambda_a \circ \Phi - t \sinh(\theta_a^+) - \lambda_a^+)}{\partial t} = \sinh(\theta_a(\Phi))u_a(\Phi) - \sinh(\theta_a^+) = \mathcal{F}_a,$$

whereas (4.18) and the definition (4.36) lead to

$$(4.86) \quad \begin{aligned} \frac{\partial \mathcal{F}_a}{\partial t} &= \frac{\partial (\sinh(\theta_a(\Phi))u_a(\Phi) - \sinh(\theta_a^+))}{\partial t} = \cosh(\theta_a(\Phi)) \frac{\partial \theta_a(\Phi)}{\partial t} u_a(\Phi) + \sinh(\theta_a(\Phi)) \frac{\partial u_a(\Phi)}{\partial t} \\ &= (\cosh(\theta_a)\{\theta_a, H\}u_a + \sinh(\theta_a)\{u_a, H\}) \circ \Phi. \end{aligned}$$

Recalling the notation (4.60), notice that for the constituent Poisson brackets we can write that

$$(4.87) \quad \{\theta_a, H\} = -\frac{\partial H}{\partial \lambda_a} = -\sum_{c=1}^n \cosh(\theta_c)u_c \mathcal{L}_c^a,$$

$$(4.88) \quad \{u_a, H\} = \sum_{c=1}^n \frac{\partial u_a}{\partial \lambda_c} \frac{\partial H}{\partial \theta_c} = u_a \sum_{c=1}^n \mathcal{L}_c^a \sinh(\theta_c)u_c,$$

and so the Lemma follows immediately. \square

There is no doubt that any change in the initial condition of the maximally defined integral curve γ_p (4.13) has an inevitable effect on the asymptotic variables (4.31) and (4.33), too. To measure the effect of small changes, we introduce the smooth functions

$$(4.89) \quad \mathcal{V}_a^{(j)} = \frac{\partial \mathcal{E}_a}{\partial \tilde{x}_j} \in C^\infty(\mathbb{R} \times P) \quad \text{and} \quad \mathcal{W}_a^{(j)} = \frac{\partial \mathcal{F}_a}{\partial \tilde{x}_j} \in C^\infty(\mathbb{R} \times P) \quad (a \in \mathbb{N}_n, j \in \mathbb{N}_N),$$

simply by taking the first order partial derivatives of (4.35) and (4.36) with respect to the coordinates (4.16). In order to understand their asymptotic properties, first we focus on their time evolution. Remembering the notations introduced in (4.60), the following result is immediate.

Proposition 15. *For all $a \in \mathbb{N}_n$ and for all $j \in \mathbb{N}_N$ we have*

$$(4.90) \quad \frac{\partial \mathcal{V}_a^{(j)}}{\partial t} = \mathcal{W}_a^{(j)} \quad \text{and} \quad \frac{\partial \mathcal{W}_a^{(j)}}{\partial t} = \sum_{b=1}^n \left(\frac{\partial \varphi_a}{\partial \lambda_b} \circ \Phi \right) \frac{\partial (\lambda_b \circ \Phi)}{\partial \tilde{x}_j} + \sum_{b=1}^n \left(\frac{\partial \varphi_a}{\partial \theta_b} \circ \Phi \right) \frac{\partial (\theta_b \circ \Phi)}{\partial \tilde{x}_j}$$

with the partial derivatives

$$(4.91) \quad \begin{aligned} \frac{\partial \varphi_a}{\partial \lambda_b} &= \sinh(\theta_a)u_a \sum_{c=1}^n \sinh(\theta_c)u_c (\mathcal{L}_b^a \mathcal{L}_c^a + \mathcal{L}_b^c \mathcal{L}_c^a + \mathcal{L}_{c,b}^a) \\ &\quad - \cosh(\theta_a)u_a \sum_{c=1}^n \cosh(\theta_c)u_c (\mathcal{L}_b^a \mathcal{L}_c^c + \mathcal{L}_b^c \mathcal{L}_c^c + \mathcal{L}_{a,b}^c), \end{aligned}$$

$$(4.92) \quad \begin{aligned} \frac{\partial \varphi_a}{\partial \theta_b} &= \sinh(\theta_a)u_a \cosh(\theta_b)u_b \mathcal{L}_b^a - \cosh(\theta_a)u_a \sinh(\theta_b)u_b \mathcal{L}_a^b \\ &\quad + \delta_{a,b} \left(\cosh(\theta_a)u_a \sum_{c=1}^n \sinh(\theta_c)u_c \mathcal{L}_c^a - \sinh(\theta_a)u_a \sum_{c=1}^n \cosh(\theta_c)u_c \mathcal{L}_c^a \right). \end{aligned}$$

Proof. Due to the smoothness of the maps \mathcal{E}_a (4.35) and \mathcal{F}_a (4.36), from Lemma 14 and the definitions displayed in (4.89) it is evident that

$$(4.93) \quad \frac{\partial \mathcal{W}_a^{(j)}}{\partial t} = \frac{\partial}{\partial t} \frac{\partial \mathcal{E}_a}{\partial \tilde{x}_j} = \frac{\partial}{\partial \tilde{x}_j} \frac{\partial \mathcal{E}_a}{\partial t} = \frac{\partial \mathcal{F}_a}{\partial \tilde{x}_j} = \mathcal{W}_a^{(j)},$$

whereas

$$(4.94) \quad \frac{\partial \mathcal{W}_a^{(j)}}{\partial t} = \frac{\partial}{\partial t} \frac{\partial \mathcal{F}_a}{\partial \tilde{x}_j} = \frac{\partial}{\partial \tilde{x}_j} \frac{\partial \mathcal{F}_a}{\partial t} = \frac{\partial(\varphi_a \circ \Phi)}{\partial \tilde{x}_j} = \sum_{k=1}^N \left(\frac{\partial \varphi_a}{\partial x_k} \circ \Phi \right) \frac{\partial(x_k \circ \Phi)}{\partial \tilde{x}_j}.$$

Now, simply by working out the partial derivatives of φ_a (4.84), the Proposition follows. \square

To proceed further, we wish to sharpen the above Proposition by expressing the derivative $\mathcal{W}_a^{(j)}$ in terms of the functions (4.89). Elementary algebraic manipulations lead to the following result.

Lemma 16. *For all $a \in \mathbb{N}_n$ and for all $j \in \mathbb{N}_N$ we have*

$$(4.95) \quad \frac{\partial \mathcal{W}_a^{(j)}}{\partial t} = \sum_{c=1}^n \mathcal{A}_{a,c} \mathcal{V}_c^{(j)} + \sum_{c=1}^n \mathcal{B}_{a,c} \mathcal{W}_c^{(j)} + \mathcal{Z}_a^{(j)},$$

where the coefficients are smooth functions on $\mathbb{R} \times P$ of the form

$$(4.96) \quad \mathcal{A}_{a,c} = \left(\frac{\partial \varphi_a}{\partial \lambda_c} - \sum_{d=1}^n \tanh(\theta_d) \mathcal{L}_c^d \frac{\partial \varphi_a}{\partial \theta_d} \right) \circ \Phi, \quad \mathcal{B}_{a,c} = \left(\frac{1}{\cosh(\theta_c) u_c} \frac{\partial \varphi_a}{\partial \theta_c} \right) \circ \Phi,$$

$$(4.97) \quad \mathcal{Z}_a^{(j)} = \sum_{c=1}^n \mathcal{A}_{a,c} \left(t \frac{\partial \sinh(\theta_c^+)}{\partial x_j} + \frac{\partial \lambda_c^+}{\partial x_j} \right) + \sum_{c=1}^n \mathcal{B}_{a,c} \frac{\partial \sinh(\theta_c^+)}{\partial x_j}.$$

Proof. Let $b \in \mathbb{N}_n$ and $j \in \mathbb{N}_N$ be arbitrary indices. Looking back to the definition (4.35), it is clear that

$$(4.98) \quad \lambda_b \circ \Phi = \mathcal{E}_b + t \sinh(\theta_b^+) + \lambda_b^+,$$

whence by (4.89) we obtain

$$(4.99) \quad \frac{\partial(\lambda_b \circ \Phi)}{\partial \tilde{x}_j} = \mathcal{V}_b^{(j)} + t \frac{\partial \sinh(\theta_b^+)}{\partial x_j} + \frac{\partial \lambda_b^+}{\partial x_j}.$$

Next, from the definitions (4.36) and (4.89) it follows that

$$(4.100) \quad \begin{aligned} \mathcal{W}_b^{(j)} &= \frac{\partial(\sinh(\theta_b(\Phi)) u_b(\Phi) - \sinh(\theta_b^+))}{\partial \tilde{x}_j} \\ &= \cosh(\theta_b(\Phi)) \frac{\partial(\theta_b \circ \Phi)}{\partial \tilde{x}_j} u_b(\Phi) + \sinh(\theta_b(\Phi)) \frac{\partial(u_b \circ \Phi)}{\partial \tilde{x}_j} - \frac{\partial \sinh(\theta_b^+)}{\partial x_j}, \end{aligned}$$

and so we can write that

$$(4.101) \quad \frac{\partial(\theta_b \circ \Phi)}{\partial \tilde{x}_j} = \frac{1}{\cosh(\theta_b(\Phi)) u_b(\Phi)} \left(\mathcal{W}_b^{(j)} + \frac{\partial \sinh(\theta_b^+)}{\partial x_j} - \sinh(\theta_b(\Phi)) \frac{\partial(u_b \circ \Phi)}{\partial \tilde{x}_j} \right).$$

However, keeping in mind (4.99), it is immediate that

$$(4.102) \quad \frac{\partial(u_b \circ \Phi)}{\partial \tilde{x}_j} = \sum_{d=1}^n \left(\frac{\partial u_b}{\partial \lambda_d} \circ \Phi \right) \frac{\partial(\lambda_d \circ \Phi)}{\partial \tilde{x}_j} = u_b(\Phi) \sum_{d=1}^n \mathcal{L}_d^b(\Phi) \left(\mathcal{V}_d^{(j)} + t \frac{\partial \sinh(\theta_d^+)}{\partial x_j} + \frac{\partial \lambda_d^+}{\partial x_j} \right).$$

Thus, by inserting (4.102) into (4.101), we end up with the formula

$$(4.103) \quad \begin{aligned} \frac{\partial(\theta_b \circ \Phi)}{\partial \tilde{x}_j} &= \frac{1}{\cosh(\theta_b(\Phi))u_b(\Phi)} \mathcal{W}_b^{(j)} - \tanh(\theta_b(\Phi)) \sum_{d=1}^n \mathcal{L}_d^b(\Phi) \mathcal{V}_d^{(j)} \\ &+ \frac{1}{\cosh(\theta_b(\Phi))u_b(\Phi)} \frac{\partial \sinh(\theta_b^+)}{\partial x_j} \\ &- \tanh(\theta_b(\Phi)) \sum_{d=1}^n \mathcal{L}_d^b(\Phi) \left(t \frac{\partial \sinh(\theta_d^+)}{\partial x_j} + \frac{\partial \lambda_d^+}{\partial x_j} \right). \end{aligned}$$

Now, by plugging the formulae (4.99) and (4.103) into (4.90), the Lemma follows. \square

Now, let us observe that with the aid of the matrix valued functions

$$(4.104) \quad \mathcal{A} = [\mathcal{A}_{a,c}]_{1 \leq a, c \leq n}, \quad \mathcal{B} = [\mathcal{B}_{a,c}]_{1 \leq a, c \leq n},$$

and the column vector valued functions

$$(4.105) \quad \mathcal{V}^{(j)} = [\mathcal{V}_a^{(j)}]_{1 \leq a \leq n}, \quad \mathcal{W}^{(j)} = [\mathcal{W}_a^{(j)}]_{1 \leq a \leq n}, \quad \mathcal{Z}^{(j)} = [\mathcal{Z}_a^{(j)}]_{1 \leq a \leq n} \quad (j \in \mathbb{N}_N),$$

the differential equations displayed in Proposition 15 and Lemma 16 can be cast into the more concise matrix form

$$(4.106) \quad \dot{\mathcal{V}}^{(j)} = \mathcal{W}^{(j)}, \quad \dot{\mathcal{W}}^{(j)} = \mathcal{A}\mathcal{V}^{(j)} + \mathcal{B}\mathcal{W}^{(j)} + \mathcal{Z}^{(j)}.$$

Following the standard terminology, this inhomogeneous linear system may be called the system of *equations of variation* associated with the non-linear differential equations appearing in Lemma 14. (For background information on the theory of the equations of variation see. e.g. Chapter V in the classic textbook [34].)

4.5. Analyzing the equations of variation. By merging our former asymptotic results with the equations of variation (4.106), in this subsection we wish to extend our asymptotic analysis to the functions (4.89). As a preliminary step, we begin with two remarks on the notations.

First, recall that for any $m \times m'$ matrix

$$(4.107) \quad A = [A_{k,l}]_{\substack{1 \leq k \leq m, \\ 1 \leq l \leq m'}} \in \mathbb{C}^{m \times m'}$$

its *Hilbert–Schmidt norm* is given by

$$(4.108) \quad \|A\| = \sqrt{\sum_{k=1}^m \sum_{l=1}^{m'} |A_{k,l}|^2}.$$

Besides the above simple formula, we shall often exploit the fact that it is submultiplicative. In other words, if $A \in \mathbb{C}^{m \times m'}$ and $B \in \mathbb{C}^{m' \times m''}$, then $\|AB\| \leq \|A\| \|B\|$.

Second, *throughout this subsection we fix an arbitrary point $p_0 \in P$, and the earlier notations for the associated constants introduced in subsections 4.2 and 4.3 are also kept in effect.* In particular, remembering the compact subset $K_0 \subset P$ given in Lemma 8, the constant T_1 provided by Proposition 9, and the shorthand notation (4.79), for the Hilbert–Schmidt norms of the coefficient matrices appearing in the system of the equations of variation (4.106) one can easily establish the following result.

Proposition 17. *There is a constant $c_0 \geq 0$ such that $\forall s \in [T_1, \infty)$ and $\forall p \in K_0$ we have*

$$(4.109) \quad \|\mathcal{A}(s, p)\| \leq c_0 e^{-s\delta_0}, \quad \|\mathcal{B}(s, p)\| \leq c_0 e^{-s\delta_0},$$

and also $\forall j \in \mathbb{N}_N$ we can write that

$$(4.110) \quad \|\mathcal{Z}^{(j)}(s, p)\| \leq c_0(s+1)e^{-s\delta_0}.$$

Proof. Let $a, b, c \in \mathbb{N}_n$, $s \in [T_1, \infty)$ and $p \in K_0$. By combining the estimate (4.78) with Propositions 10, 11 and 12, from the explicit expressions (4.91) and (4.92) it is evident that

$$(4.111) \quad \left| \frac{\partial \varphi_a}{\partial \lambda_b}(\Phi(s, p)) \right| \leq c_1 e^{-s\delta_0} \quad \text{and} \quad \left| \frac{\partial \varphi_a}{\partial \theta_b}(\Phi(s, p)) \right| \leq c_1 e^{-s\delta_0}$$

with some constant $c_1 \geq 0$. Thus, giving a glance at (4.96), it readily follows that there is a constant $c_2 \geq 0$ such that

$$(4.112) \quad |\mathcal{A}_{a,c}(s, p)| \leq c_2 e^{-s\delta_0} \quad \text{and} \quad |\mathcal{B}_{a,c}(s, p)| \leq c_2 e^{-s\delta_0}.$$

Finally, from (4.97) it is also clear that $\forall j \in \mathbb{N}_N$ we can write

$$(4.113) \quad \begin{aligned} |\mathcal{Z}_a^{(j)}(s, p)| &\leq \sum_{c=1}^n |\mathcal{A}_{a,c}(s, p)| \left(s |\cosh(\theta_c^+(p))| \left| \frac{\partial \theta_c^+}{\partial x_j}(p) \right| + \left| \frac{\partial \lambda_c^+}{\partial x_j}(p) \right| \right) \\ &\quad + \sum_{c=1}^n |\mathcal{B}_{a,c}(s, p)| |\cosh(\theta_c^+(p))| \left| \frac{\partial \theta_c^+}{\partial x_j}(p) \right|. \end{aligned}$$

Since the smooth functions λ_c^+ , θ_c^+ , and their partial derivatives are also bounded on the compact subset K_0 , the estimates displayed in (4.112) entail that

$$(4.114) \quad |\mathcal{Z}_a^{(j)}(s, p)| \leq (c_3 s + c_4) e^{-s\delta_0}$$

with some constants $c_3, c_4 \geq 0$. So, recalling (4.108), with $c_0 = \max\{nc_2, \sqrt{nc_3}, \sqrt{nc_4}\}$ the Proposition follows. \square

As can be seen below, Proposition 17 itself allows us to give a very rough estimate on the growth of the functions (4.89).

Lemma 18. *There are constants $v_0, w_0 \geq 0$ such that $\forall s \in [T_1, \infty)$, $\forall p \in K_0$, and $\forall j \in \mathbb{N}_N$ we can write that*

$$(4.115) \quad \|\mathbf{V}^{(j)}(s, p)\| \leq v_0 + (s - T_1)w_0 \quad \text{and} \quad \|\mathbf{W}^{(j)}(s, p)\| \leq w_0.$$

Proof. Notice that the dependence of the norms $\|\mathbf{V}^{(j)}(T_1, p)\|$ and $\|\mathbf{W}^{(j)}(T_1, p)\|$ on p is continuous, thus by the compactness of K_0 there are some constants $v_0, c_1 \geq 0$ such that

$$(4.116) \quad \|\mathbf{V}^{(j)}(T_1, p)\| \leq v_0 \quad \text{and} \quad \|\mathbf{W}^{(j)}(T_1, p)\| \leq c_1 \quad (j \in \mathbb{N}_N, p \in K_0).$$

To proceed further, let $s \in [T_1, \infty)$, $p \in K_0$, and $j \in \mathbb{N}_N$ be arbitrary elements and keep them fixed. Upon integrating the equations of variation (4.106) with respect to time t , we obtain

$$(4.117) \quad \mathbf{V}^{(j)}(s, p) - \mathbf{V}^{(j)}(T_1, p) = \int_{T_1}^s \dot{\mathbf{V}}^{(j)}(\tau, p) \, d\tau = \int_{T_1}^s \mathbf{W}^{(j)}(\tau, p) \, d\tau,$$

therefore we can write

$$(4.118) \quad \|\mathbf{V}^{(j)}(s, p)\| \leq \|\mathbf{V}^{(j)}(T_1, p)\| + \int_{T_1}^s \|\mathbf{W}^{(j)}(\tau, p)\| \, d\tau \leq v_0 + \int_{T_1}^s \|\mathbf{W}^{(j)}(\tau, p)\| \, d\tau.$$

Along the same lines, utilizing the integral equation

$$(4.119) \quad \mathbf{W}^{(j)}(s, p) - \mathbf{W}^{(j)}(T_1, p) = \int_{T_1}^s \left(\mathcal{A}(\tau, p) \mathbf{V}^{(j)}(\tau, p) + \mathcal{B}(\tau, p) \mathbf{W}^{(j)}(\tau, p) + \mathcal{Z}^{(j)}(\tau, p) \right) \, d\tau,$$

we infer that

$$(4.120) \quad \begin{aligned} \|\mathbf{W}^{(j)}(s, p)\| &\leq \|\mathbf{W}^{(j)}(T_1, p)\| + \int_{T_1}^s \|\mathcal{A}(\tau, p)\| \|\mathbf{V}^{(j)}(\tau, p)\| \, d\tau \\ &\quad + \int_{T_1}^s \|\mathcal{B}(\tau, p)\| \|\mathbf{W}^{(j)}(\tau, p)\| \, d\tau + \int_{T_1}^s \|\mathcal{Z}^{(j)}(\tau, p)\| \, d\tau. \end{aligned}$$

Now, let us examine each term appearing on the right hand side of the above inequality. Due to the observation we made in (4.116), the first term is under control. Turning to the second term, let us note that by exploiting (4.118) we can write

$$(4.121) \quad \int_{T_1}^s \|\mathcal{A}(\tau, p)\| \|\mathcal{V}^{(j)}(\tau, p)\| d\tau \leq v_0 \int_{T_1}^s \|\mathcal{A}(\tau, p)\| d\tau + \int_{T_1}^s \|\mathcal{A}(\tau, p)\| \left(\int_{T_1}^{\tau} \|\mathcal{W}^{(j)}(u, p)\| du \right) d\tau.$$

However, recalling (4.109), it is obvious that

$$(4.122) \quad \int_{T_1}^s \|\mathcal{A}(\tau, p)\| d\tau \leq c_0 \int_{T_1}^s e^{-\tau\delta_0} d\tau \leq c_0 \int_{T_1}^{\infty} e^{-\tau\delta_0} d\tau = \frac{c_0}{\delta_0} e^{-T_1\delta_0} \leq \frac{c_0}{\delta_0}.$$

As concerns the double integral appearing on the right hand side of (4.121), the Fubini–Tonelli theorem is clearly applicable, and we find that

$$(4.123) \quad \int_{T_1}^s \|\mathcal{A}(\tau, p)\| \left(\int_{T_1}^{\tau} \|\mathcal{W}^{(j)}(u, p)\| du \right) d\tau = \int_{T_1}^s \left(\int_u^s \|\mathcal{A}(\tau, p)\| d\tau \right) \|\mathcal{W}^{(j)}(u, p)\| du.$$

Remembering (4.122), at this point let us notice that on the right hand side of the above equation we can apply the inequality

$$(4.124) \quad \int_u^s \|\mathcal{A}(\tau, p)\| d\tau \leq \frac{c_0}{\delta_0} e^{-u\delta_0}.$$

Thus, keeping in mind (4.122), (4.123) and (4.124), from (4.121) we infer at once that

$$(4.125) \quad \int_{T_1}^s \|\mathcal{A}(\tau, p)\| \|\mathcal{V}^{(j)}(\tau, p)\| d\tau \leq v_0 \frac{c_0}{\delta_0} + \frac{c_0}{\delta_0} \int_{T_1}^s e^{-u\delta_0} \|\mathcal{W}^{(j)}(u, p)\| du.$$

Finally, by inspecting the last two terms on the right hand side of (4.120), the application of the estimates displayed in (4.109) leads to the inequality

$$(4.126) \quad \int_{T_1}^s \|\mathcal{B}(\tau, p)\| \|\mathcal{W}^{(j)}(\tau, p)\| d\tau \leq c_0 \int_{T_1}^s e^{-\tau\delta_0} \|\mathcal{W}^{(j)}(\tau, p)\| d\tau,$$

whereas (4.110) yields

$$(4.127) \quad \int_{T_1}^s \|\mathcal{Z}^{(j)}(\tau, p)\| d\tau \leq c_0 \int_{T_1}^s (\tau + 1) e^{-\tau\delta_0} d\tau \leq c_0 \int_{T_1}^{\infty} (\tau + 1) e^{-\tau\delta_0} d\tau \leq c_2$$

with some constant $c_2 \geq 0$.

To sum up, by applying the estimates displayed in (4.116), (4.125), (4.126) and (4.127), from the inequality (4.120) we can derive that

$$(4.128) \quad \|\mathcal{W}^{(j)}(s, p)\| \leq c_3 + c_4 \int_{T_1}^s e^{-\tau\delta_0} \|\mathcal{W}^{(j)}(\tau, p)\| d\tau,$$

where $c_3 = c_1 + v_0 c_0 \delta_0^{-1} + c_2$ and $c_4 = c_0(1 + \delta_0^{-1})$. As a consequence, by invoking Grönwall's lemma (see e.g. Theorem 1.1 in Chapter III of [34]), we obtain that

$$(4.129) \quad \|\mathcal{W}^{(j)}(s, p)\| \leq c_3 \exp \left(c_4 \int_{T_1}^s e^{-\tau\delta_0} d\tau \right).$$

So, with the non-negative constant

$$(4.130) \quad w_0 = c_3 \exp \left(c_4 \int_{T_1}^{\infty} e^{-\tau\delta_0} d\tau \right) = c_3 \exp \left(\frac{c_4}{\delta_0} e^{-T_1\delta_0} \right)$$

we end up with the desired inequality $\|\mathcal{W}^{(j)}(s, p)\| \leq w_0$. Moreover, utilizing (4.118) and (4.116), it also follows that

$$(4.131) \quad \|\mathcal{V}^{(j)}(s, p)\| \leq v_0 + w_0(s - T_1),$$

whence the proof is complete. \square

Of course, later on we shall need much sharper estimates on the functions defined in (4.89) than the rudimentary inequalities given in the previous Lemma. To make progress, let $a \in \mathbb{N}_n$, $p \in K_0$, and suppose that $s' \geq s \geq T_1$. Recalling the time derivatives appearing in Lemma 14, it is clear that

$$(4.132) \quad \mathcal{E}_a(s', p) - \mathcal{E}_a(s, p) = \int_s^{s'} \mathcal{F}_a(\tau, p) d\tau \quad \text{and} \quad \mathcal{F}_a(s', p) - \mathcal{F}_a(s, p) = \int_s^{s'} \varphi_a(\Phi(\tau, p)) d\tau.$$

Notice that by Lemma 8 we have

$$(4.133) \quad \mathcal{E}_a(s', p) \rightarrow 0 \quad \text{and} \quad \mathcal{F}_a(s', p) \rightarrow 0 \quad (s' \rightarrow \infty).$$

Furthermore, from the decay conditions given in Lemma 8 we also see that the function

$$(4.134) \quad [s, \infty) \ni \tau \mapsto \mathcal{F}_a(\tau, p) \in \mathbb{C}$$

is integrable in Lebesgue's sense over the interval $[s, \infty)$. Combining the estimate given in (4.78) with Propositions 10 and 11, from (4.84) it is also clear that $\varphi_a(\Phi(\tau, p))$ decays exponentially fast for $\tau \rightarrow \infty$, whence the function

$$(4.135) \quad [s, \infty) \ni \tau \mapsto \varphi_a(\Phi(\tau, p)) \in \mathbb{C}$$

also belongs to $L^1[s, \infty)$. Therefore, a trivial application of Lebesgue's dominated convergence theorem yields immediately that for $s' \rightarrow \infty$ we can write

$$(4.136) \quad \int_s^{s'} \mathcal{F}_a(\tau, p) d\tau \rightarrow \int_s^\infty \mathcal{F}_a(\tau, p) d\tau \quad \text{and} \quad \int_s^{s'} \varphi_a(\Phi(\tau, p)) d\tau \rightarrow \int_s^\infty \varphi_a(\Phi(\tau, p)) d\tau.$$

Thus, combining (4.133) and (4.136) with (4.132), we end up with the integral representations

$$(4.137) \quad \mathcal{E}_a(s, p) = - \int_s^\infty \mathcal{F}_a(\tau, p) d\tau \quad \text{and} \quad \mathcal{F}_a(s, p) = - \int_s^\infty \varphi_a(\Phi(\tau, p)) d\tau.$$

Making use of the above observations, now we turn back to the study of the partial derivatives of the functions \mathcal{E}_a and \mathcal{F}_a ($a \in \mathbb{N}_n$). More precisely, we wish to set up integral representations for the first order partial derivatives defined in (4.89). In this respect the only non-trivial question is whether in the relationships displayed in (4.137) the differentiation with respect to \tilde{x}_j ($j \in \mathbb{N}_N$) can be performed under the integral sign.

Starting with the integral representation of \mathcal{F}_a , notice that the integrand $\varphi_a \circ \Phi$ is smooth, and from Lemmas 14 and 16 it is evident that

$$(4.138) \quad \frac{\partial(\varphi_a \circ \Phi)}{\partial \tilde{x}_j} = \frac{\partial}{\partial \tilde{x}_j} \frac{\partial \mathcal{F}_a}{\partial t} = \frac{\partial}{\partial t} \frac{\partial \mathcal{F}_a}{\partial \tilde{x}_j} = \frac{\partial \mathcal{W}_a^{(j)}}{\partial t} = \sum_{c=1}^n \mathcal{A}_{a,c} \mathcal{V}_c^{(j)} + \sum_{c=1}^n \mathcal{B}_{a,c} \mathcal{W}_c^{(j)} + \mathcal{Z}_a^{(j)}.$$

Since still $s \in [T_1, \infty)$ and $p \in K_0$, from Proposition 17 and Lemma 18 we see that for any $\tau \geq s$ the above partial derivative can be estimated from above as

$$(4.139) \quad \left| \frac{\partial(\varphi_a \circ \Phi)}{\partial \tilde{x}_j}(\tau, p) \right| \leq \sum_{c=1}^n |\mathcal{A}_{a,c}(\tau, p)| |\mathcal{V}_c^{(j)}(\tau, p)| + \sum_{c=1}^n |\mathcal{B}_{a,c}(\tau, p)| |\mathcal{W}_c^{(j)}(\tau, p)| + |\mathcal{Z}_a^{(j)}(\tau, p)| \\ \leq c_1(\tau + 1)e^{-\tau\delta_0}$$

with some constant $c_1 \geq 0$. Now, the point is that the majorizing function

$$(4.140) \quad [s, \infty) \ni \tau \mapsto c_1(\tau + 1)e^{-\tau\delta_0} \in \mathbb{C}$$

is integrable on $[s, \infty)$. Moreover, the majorization (4.139) is uniform in the sense that it is independent of $p \in K_0$. So, a trivial application of Lebesgue's dominated convergence theorem yields that if p belongs to the interior of K_0 , then the differentiation can be performed under the integral sign. (This nice fact can be found in any textbook on real analysis, see e.g. Theorem 2.27 in [35].) That is, it is fully justified to write that

$$(4.141) \quad \mathcal{W}_a^{(j)}(s, p) = \frac{\partial \mathcal{F}_a}{\partial \tilde{x}_j}(s, p) = - \int_s^\infty \frac{\partial(\varphi_a \circ \Phi)}{\partial \tilde{x}_j}(\tau, p) d\tau \quad (s \in [T_1, \infty), p \in \text{int}(K_0)).$$

Now, due to (4.139), it follows that $\forall s \in [T_1, \infty)$ and $\forall p \in \text{int}(K_0)$ we can write

$$(4.142) \quad |\mathcal{W}_a^{(j)}(s, p)| \leq \int_s^\infty \left| \frac{\partial(\varphi_a \circ \Phi)}{\partial \tilde{x}_j}(\tau, p) \right| d\tau \leq c_1 \int_s^\infty (\tau + 1)e^{-\tau\delta_0} d\tau \leq c_2(s + 1)e^{-s\delta_0}$$

with some constant $c_2 \geq 0$. Since the closure of $\text{int}(K_0)$ coincides with K_0 , by continuity it follows that

$$(4.143) \quad |\mathcal{W}_a^{(j)}(s, p)| \leq c_2(s + 1)e^{-s\delta_0} \quad (s \in [T_1, \infty), p \in K_0).$$

Turning to the integral representation of \mathcal{E}_a , from (4.137) we see that in this case the integrand in question is given by the smooth function \mathcal{F}_a . By definition (4.89), the derivative of \mathcal{F}_a with respect to \tilde{x}_j is the smooth function $\mathcal{W}_a^{(j)}$, for which we have just derived the estimate (4.143). We proceed by noting that the upper bound in (4.143) is independent of the choice of $p \in K_0$. Moreover, this upper bound is an integrable function of s on $[T_1, \infty)$, thus performing the differentiation under the integral sign is completely legitimate. More precisely, we are entitled to write that

$$(4.144) \quad \mathcal{V}_a^{(j)}(s, p) = \frac{\partial \mathcal{E}_a}{\partial \tilde{x}_j}(s, p) = - \int_s^\infty \mathcal{W}_a^{(j)}(\tau, p) d\tau \quad (s \in [T_1, \infty), p \in \text{int}(K_0)).$$

As a consequence, $\forall s \in [T_1, \infty)$ and $\forall p \in \text{int}(K_0)$ we have

$$(4.145) \quad |\mathcal{V}_a^{(j)}(s, p)| \leq \int_s^\infty |\mathcal{W}_a^{(j)}(\tau, p)| d\tau \leq c_2 \int_s^\infty (\tau + 1)e^{-\tau\delta_0} d\tau \leq c_3(s + 1)e^{-s\delta_0}$$

with some constant $c_3 \geq 0$. Again, by continuity, we conclude that

$$(4.146) \quad |\mathcal{V}_a^{(j)}(s, p)| \leq c_3(s + 1)e^{-s\delta_0} \quad (s \in [T_1, \infty), p \in K_0).$$

At this point we are in a position to formulate the most important technical result of this section.

Theorem 19. *There is a constant $C \geq 0$ such that $\forall a \in \mathbb{N}_n, \forall j \in \mathbb{N}_N, \forall s \in [T_1, \infty)$ and $\forall p \in K_0$ we can write that*

$$(4.147) \quad |\mathcal{V}_a^{(j)}(s, p)| \leq C(s + 1)e^{-s\delta_0} \quad \text{and} \quad |\mathcal{W}_a^{(j)}(s, p)| \leq C(s + 1)e^{-s\delta_0}.$$

Just as earlier in Lemma 8, in the above Theorem we face the problem that the relevant asymptotic information about the particle rapidities is somewhat hidden. Although the above estimate on $\mathcal{W}_a^{(j)}$ proves to be crucial, what we rather need is the control over the partial derivative

$$(4.148) \quad \mathcal{U}_a^{(j)} = \frac{\partial \mathcal{G}_a}{\partial \tilde{x}_j} \in C^\infty(\mathbb{R} \times P) \quad (a \in \mathbb{N}_n, j \in \mathbb{N}_N)$$

with \mathcal{G}_a defined in (4.76). Since by construction we have

$$(4.149) \quad \frac{\partial(\theta_a \circ \Phi)}{\partial \tilde{x}_j} = \mathcal{U}_a^{(j)} + \frac{\partial \theta_a^+}{\partial x_j},$$

notice that the application of the formula (4.103) entails that

$$(4.150) \quad \mathcal{U}_a^{(j)} = \frac{1}{\cosh(\theta_a(\Phi))u_a(\Phi)} \mathcal{W}_a^{(j)} - \frac{\cosh(\theta_a(\Phi))u_a(\Phi) - \cosh(\theta_a^+) \frac{\partial \theta_a^+}{\partial x_j}}{\cosh(\theta_a(\Phi))u_a(\Phi)} \frac{\partial \theta_a^+}{\partial x_j} \\ - \tanh(\theta_a(\Phi)) \sum_{c=1}^n \mathcal{L}_c^a(\Phi) \left(\mathcal{V}_c^{(j)} + t \cosh(\theta_c^+) \frac{\partial \theta_c^+}{\partial x_j} + \frac{\partial \lambda_c^+}{\partial x_j} \right).$$

Thus, it is evident that

$$(4.151) \quad |\mathcal{U}_a^{(j)}| \leq |\mathcal{W}_a^{(j)}| + |\cosh(\theta_a(\Phi))u_a(\Phi) - \cosh(\theta_a^+)| \left| \frac{\partial \theta_a^+}{\partial x_j} \right| \\ + \sum_{c=1}^n |\mathcal{L}_c^a(\Phi)| \left(|\mathcal{V}_c^{(j)}| + |t| \cosh(\theta_c^+) \left| \frac{\partial \theta_c^+}{\partial x_j} \right| + \left| \frac{\partial \lambda_c^+}{\partial x_j} \right| \right).$$

Let $s \in [T_1, \infty)$ and $p \in K_0$ be arbitrary, and examine the right hand side of the above inequality at the point (s, p) . Due to Theorem 19 and Proposition 11, the first term and also the terms in the last sum are under direct control, so only the second term requires attention. However, by exploiting the relationship (4.77) together with Proposition 10 and Lemma 13, it is a routine exercise to verify that

$$(4.152) \quad |\cosh(\theta_a(\Phi(s, p)))u_a(\Phi(s, p)) - \cosh(\theta_a^+(p))| \leq c'_1 e^{-s\delta_0}$$

with some constant $c'_1 \geq 0$. Thus, the following result is immediate.

Proposition 20. *There is a constant $C' \geq 0$ such that $\forall a \in \mathbb{N}_n, \forall j \in \mathbb{N}_N, \forall s \in [T_1, \infty)$ and $\forall p \in K_0$ we can write that*

$$(4.153) \quad |\mathcal{U}_a^{(j)}(s, p)| \leq C'(s+1)e^{-s\delta_0}.$$

As a trivial corollary of Lemma 8, Lemma 13, Theorem 19 and Proposition 20, we conclude this subsection with the following observation.

Proposition 21. *As $s \rightarrow \infty, \forall a \in \mathbb{N}_n, \forall j \in \mathbb{N}_N$ and $\forall m \in \mathbb{N}$ we have*

$$(4.154) \quad s^m \mathcal{E}_a(s, p_0) \rightarrow 0, \quad s^m \mathcal{G}_a(s, p_0) \rightarrow 0, \quad s^m \mathcal{V}_a^{(j)}(s, p_0) \rightarrow 0, \quad s^m \mathcal{U}_a^{(j)}(s, p_0) \rightarrow 0.$$

4.6. Canonicity of the dual variables. Our first goal in this subsection is to compute the Poisson brackets of the asymptotic variables defined in (4.31) and (4.33). Recalling (4.35), (4.76), and the notation for the sections introduced in (4.15), it is clear that $\forall a \in \mathbb{N}_n$ and $\forall s \in \mathbb{R}$ we can write that

$$(4.155) \quad \lambda_a \circ \Phi_s = s \sinh(\theta_a^+) + \lambda_a^+ + (\mathcal{E}_a)_s \quad \text{and} \quad \theta_a \circ \Phi_s = \theta_a^+ + (\mathcal{G}_a)_s.$$

Let us keep in mind that in the previous subsections we established tight control on the ‘error terms’ \mathcal{E}_a and \mathcal{G}_a . Therefore, by exploiting the fundamental Poisson brackets (1.9) and the fact that $\forall s \in \mathbb{R}$ the time- s flow $\Phi_s : P \rightarrow P$ is a *symplectomorphism*, the Poisson brackets of the asymptotic variables also become accessible.

Lemma 22. *For all $a, b \in \mathbb{N}_n$ we have $\{\lambda_a^+, \lambda_b^+\} = 0 = \{\theta_a^+, \theta_b^+\}$ and $\{\lambda_a^+, \theta_b^+\} = \delta_{a,b}$.*

Proof. Throughout the proof let $a, b \in \mathbb{N}_n$ and $p \in P$ be arbitrary, but fixed elements. Starting with the Poisson brackets of the asymptotic rapidities, from (4.155) it is evident that $\forall s \in \mathbb{R}$ we have

$$(4.156) \quad 0 = \{\theta_a, \theta_b\} \circ \Phi_s = \{\theta_a \circ \Phi_s, \theta_b \circ \Phi_s\} = \{\theta_a^+, \theta_b^+\} + \{\theta_a^+, (\mathcal{G}_b)_s\} + \{(\mathcal{G}_a)_s, \theta_b^+\} + \{(\mathcal{G}_a)_s, (\mathcal{G}_b)_s\}.$$

Now, recalling the definitions (4.16) and (4.148), at point p the Poisson bracket formula (1.8) allows us to write that

$$(4.157) \quad \begin{aligned} \{\theta_a^+, (\mathcal{G}_b)_s\}(p) &= \sum_{k,l=1}^N \Omega_{k,l} \frac{\partial \theta_a^+}{\partial x_k}(p) \frac{\partial (\mathcal{G}_b)_s}{\partial x_l}(p) = \sum_{k,l=1}^N \Omega_{k,l} \frac{\partial \theta_a^+}{\partial x_k}(p) \frac{\partial \mathcal{G}_b}{\partial \tilde{x}_l}(s, p) \\ &= \sum_{k,l=1}^N \Omega_{k,l} \frac{\partial \theta_a^+}{\partial x_k}(p) \mathcal{U}_b^{(l)}(s, p). \end{aligned}$$

However, on account of Proposition 21 it is clear that

$$(4.158) \quad \{\theta_a^+, (\mathcal{G}_b)_s\}(p) \rightarrow 0 \quad (s \rightarrow \infty).$$

It is obvious that these arguments also apply to the last two terms appearing on the right hand side of the equation (4.156). Indeed, one can easily verify that

$$(4.159) \quad \{(\mathcal{G}_a)_s, \theta_b^+\}(p) \rightarrow 0 \quad \text{and} \quad \{(\mathcal{G}_a)_s, (\mathcal{G}_b)_s\}(p) \rightarrow 0 \quad (s \rightarrow \infty),$$

and so from (4.156) we infer that

$$(4.160) \quad \{\theta_a^+, \theta_b^+\}(p) = 0.$$

Next, remembering (4.155), let us notice that $\forall s \in \mathbb{R}$ we can write

$$(4.161) \quad \begin{aligned} \delta_{a,b} &= \{\lambda_a, \theta_b\} \circ \Phi_s = \{\lambda_a \circ \Phi_s, \theta_b \circ \Phi_s\} \\ &= s\{\sinh(\theta_a^+), \theta_b^+\} + \{\lambda_a^+, \theta_b^+\} + \{(\mathcal{E}_a)_s, \theta_b^+\} \\ &\quad + s\{\sinh(\theta_a^+), (\mathcal{G}_b)_s\} + \{\lambda_a^+, (\mathcal{G}_b)_s\} + \{(\mathcal{E}_a)_s, (\mathcal{G}_b)_s\}. \end{aligned}$$

As before, we inspect the right hand side of the above equation on a term-by-term basis. Due to the relationship (4.160), at point p we can write

$$(4.162) \quad \{\sinh(\theta_a^+), \theta_b^+\}(p) = \cosh(\theta_a^+(p))\{\theta_a^+, \theta_b^+\}(p) = 0.$$

Repeating the ideas surrounding the equations (4.157) and (4.158), from Proposition 21 and the Poisson bracket formula (1.8) one can also infer that in the $s \rightarrow \infty$ limit we have

$$(4.163) \quad \{(\mathcal{E}_a)_s, \theta_b^+\}(p) \rightarrow 0, \quad s\{\sinh(\theta_a^+), (\mathcal{G}_b)_s\}(p) \rightarrow 0,$$

$$(4.164) \quad \{\lambda_a^+, (\mathcal{G}_b)_s\}(p) \rightarrow 0, \quad \{(\mathcal{E}_a)_s, (\mathcal{G}_b)_s\}(p) \rightarrow 0.$$

Thus, (4.161) immediately leads to the relationship

$$(4.165) \quad \{\lambda_a^+, \theta_a^+\}(p) = \delta_{a,b}.$$

Finally, we turn our attention to the remaining Poisson brackets involving only the asymptotic positions. From (4.155) it is clear that for each $s \in \mathbb{R}$ we have

$$(4.166) \quad \begin{aligned} 0 &= \{\lambda_a, \lambda_b\} \circ \Phi_s = \{\lambda_a \circ \Phi_s, \lambda_b \circ \Phi_s\} \\ &= \{\lambda_a^+, \lambda_b^+\} + s^2\{\sinh(\theta_a^+), \sinh(\theta_b^+)\} + s\left(\{\sinh(\theta_a^+), \lambda_b^+\} + \{\lambda_a^+, \sinh(\theta_b^+)\}\right) \\ &\quad + \{\sinh(\theta_a^+), s(\mathcal{E}_b)_s\} + \{s(\mathcal{E}_a)_s, \sinh(\theta_b^+)\} + \{\lambda_a^+, (\mathcal{E}_b)_s\} + \{(\mathcal{E}_a)_s, \lambda_b^+\} + \{(\mathcal{E}_a)_s, (\mathcal{E}_b)_s\}. \end{aligned}$$

Due to the equation (4.160) we can write

$$(4.167) \quad \{\sinh(\theta_a^+), \sinh(\theta_b^+)\}(p) = \cosh(\theta_a^+(p))\{\theta_a^+, \theta_b^+\}(p) \cosh(\theta_b^+(p)) = 0,$$

whilst the Poisson bracket (4.165) entails

$$(4.168) \quad \{\sinh(\theta_a^+), \lambda_b^+\}(p) + \{\lambda_a^+, \sinh(\theta_b^+)\}(p) = -\cosh(\theta_a^+(p))\delta_{b,a} + \delta_{a,b} \cosh(\theta_b^+(p)) = 0.$$

As before, the application of (1.8) and Proposition 21 immediately yields that at point p the last five terms appearing on the right hand side of (4.166) vanish as $s \rightarrow \infty$, and so we end up with the desired equation

$$(4.169) \quad \{\lambda_a^+, \lambda_b^+\}(p) = 0.$$

Now, on account of the relationships (4.160), (4.165) and (4.169) the proof is complete. \square

To proceed further, we still have to establish the relationships between the asymptotic and the dual variables. Recalling (4.33), we already know that $\theta_c^+ = 2\hat{\theta}_c$ ($c \in \mathbb{N}_n$), but the relationship (4.31) is less explicit. To make it more transparent, we shall need the smooth function $\Delta_c: Q \rightarrow \mathbb{R}$ defined on the configuration space Q (1.6) by the formula

$$(4.170) \quad \begin{aligned} \Delta_c(\boldsymbol{\xi}) = & -\frac{1}{2} \sum_{d=1}^{c-1} \ln \left(1 + \frac{\sin(\mu)^2}{\sinh(\xi_c - \xi_d)^2} \right) + \frac{1}{2} \sum_{d=c+1}^n \ln \left(1 + \frac{\sin(\mu)^2}{\sinh(\xi_c - \xi_d)^2} \right) \\ & + \frac{1}{2} \sum_{\substack{d=1 \\ (d \neq c)}}^n \ln \left(1 + \frac{\sin(\mu)^2}{\sinh(\xi_c + \xi_d)^2} \right) + \frac{1}{2} \ln \left(1 + \frac{\sin(\nu)^2}{\sinh(2\xi_c)^2} \right), \end{aligned}$$

where $c \in \mathbb{N}_n$ and $\boldsymbol{\xi} = (\xi_1, \dots, \xi_n) \in Q$.

Lemma 23. *For all $c \in \mathbb{N}_n$ we have $\lambda_c^+ = \frac{1}{2}\hat{\lambda}_c + \frac{1}{2}\Delta_c(\hat{\theta}_1, \dots, \hat{\theta}_n)$.*

Proof. From (3.31) and the definition (4.29) it is evident that $\forall a, b \in \mathbb{N}_n$ we can write that

$$(4.171) \quad \tilde{L}_{a,b} = \hat{L}_{a,b} = \hat{F}_a \frac{\sinh(i\hat{\mu})}{\sinh(i\hat{\mu} + \hat{\theta}_a - \hat{\theta}_b)} \bar{\tilde{F}}_b.$$

Thus, the $n \times n$ submatrix in the upper left hand corner of \tilde{L} is essentially a Cauchy type matrix multiplied by diagonal matrices from both sides. Therefore, by utilizing the determinant formula (3.37), it is straightforward to verify that for the c -th ($c \in \mathbb{N}_n$) leading principal minor (4.2) of \tilde{L} we have

$$(4.172) \quad \pi_c(\tilde{L}) = \prod_{d=1}^c |\hat{F}_d|^2 \prod_{1 \leq a < b \leq c} \frac{\sinh(\hat{\theta}_a - \hat{\theta}_b)^2}{|\sinh(i\hat{\mu} + \hat{\theta}_a - \hat{\theta}_b)|^2} = \prod_{d=1}^c |\hat{F}_d|^2 \prod_{1 \leq a < b \leq c} \left(1 + \frac{\sin(\mu)^2}{\sinh(\hat{\theta}_a - \hat{\theta}_b)^2} \right)^{-1}.$$

Recalling the definitions of m_j (4.4) and the components of \hat{F}_j (3.101), it is clear that

$$(4.173) \quad m_1(\tilde{L}) = |\hat{F}_1|^2 = e^{\hat{\lambda}_1} \hat{u}_1.$$

Moreover, for any $c \in \{2, \dots, n\}$ we can write that

$$(4.174) \quad m_c(\tilde{L}) = \frac{\pi_c(\tilde{L})}{\pi_{c-1}(\tilde{L})} = e^{\hat{\lambda}_c} \hat{u}_c \prod_{d=1}^{c-1} \left(1 + \frac{\sin(\mu)^2}{\sinh(\hat{\theta}_c - \hat{\theta}_d)^2} \right)^{-1}.$$

Thus, from the formula of \hat{u}_c (3.99) and the definition (4.31) the Lemma follows at once. \square

The above Lemma motivates the closer inspection of the smooth functions defined in (4.170). By taking the compositions $\Delta_c(\lambda_1, \dots, \lambda_n) \in C^\infty(P)$, elementary differentiations immediately reveal the symmetry property

$$(4.175) \quad \frac{\partial \Delta_a(\lambda_1, \dots, \lambda_n)}{\partial \lambda_b} = \frac{\partial \Delta_b(\lambda_1, \dots, \lambda_n)}{\partial \lambda_a} \quad (a, b \in \mathbb{N}_n).$$

Therefore, since the phase space P (1.2) is connected and simply connected, there is a globally defined function $\mathcal{S} \in C^\infty(Q)$, unique up to a constant, such that for the composition $\mathcal{S}(\lambda_1, \dots, \lambda_n)$ we have

$$(4.176) \quad d(\mathcal{S}(\lambda_1, \dots, \lambda_n)) = \sum_{c=1}^n \Delta_c(\lambda_1, \dots, \lambda_n) d\lambda_c.$$

At this point we are in a position to prove the most important result of the paper. Making use of the relationships displayed in equation (4.33) and in Lemma 23, the Theorem below can be seen as a corollary of our scattering theoretical analysis culminating in Lemma 22.

Theorem 24. *The dual variables $\hat{\lambda}_1, \dots, \hat{\lambda}_n, \hat{\theta}_1, \dots, \hat{\theta}_n$ defined in equations (3.100) and (3.1) form a global Darboux system on the phase space P (1.2). That is, we have*

$$(4.177) \quad \{\hat{\lambda}_a, \hat{\lambda}_b\} = 0, \quad \{\hat{\theta}_a, \hat{\theta}_b\} = 0, \quad \{\hat{\lambda}_a, \hat{\theta}_b\} = \delta_{a,b} \quad (a, b \in \mathbb{N}_n).$$

As a consequence, the duality map Ψ (3.102) is an anti-symplectomorphism, i.e., $\Psi^*\omega = -\omega$.

Proof. As a preliminary step, we introduce the temporary shorthand notation

$$(4.178) \quad \hat{\Delta}_c = \Delta_c(\hat{\theta}_1, \dots, \hat{\theta}_n) \in C^\infty(P) \quad (c \in \mathbb{N}_n).$$

To proceed further, take arbitrary indices $a, b \in \mathbb{N}_n$ and keep them fixed. Recalling (4.33) and Lemma 23, from the application of Lemma 22 it comes effortlessly that

$$(4.179) \quad \{\hat{\theta}_a, \hat{\theta}_b\} = \frac{1}{4}\{\theta_a^+, \theta_b^+\} = 0,$$

$$(4.180) \quad \{\hat{\lambda}_a, \hat{\theta}_b\} = \{\lambda_a^+, \theta_b^+\} - \{\hat{\Delta}_a, \hat{\theta}_b\} = \delta_{a,b} - \sum_{c=1}^n \frac{\partial \hat{\Delta}_a}{\partial \hat{\theta}_c} \{\hat{\theta}_c, \hat{\theta}_b\} = \delta_{a,b}.$$

Furthermore, the above two equations and the symmetry relation (4.175) entail

$$(4.181) \quad \begin{aligned} \{\hat{\lambda}_a, \hat{\lambda}_b\} &= 4\{\lambda_a^+, \lambda_b^+\} - 2\left(\{\lambda_a^+, \hat{\Delta}_b\} + \{\hat{\Delta}_a, \lambda_b^+\}\right) + \{\hat{\Delta}_a, \hat{\Delta}_b\} \\ &= -2\sum_{c=1}^n \left(\{\lambda_a^+, \hat{\theta}_c\} \frac{\partial \hat{\Delta}_b}{\partial \hat{\theta}_c} + \frac{\partial \hat{\Delta}_a}{\partial \hat{\theta}_c} \{\hat{\theta}_c, \lambda_b^+\} \right) + \sum_{c,d=1}^n \frac{\partial \hat{\Delta}_a}{\partial \hat{\theta}_c} \{\hat{\theta}_c, \hat{\theta}_d\} \frac{\partial \hat{\Delta}_b}{\partial \hat{\theta}_d} \\ &= -\sum_{c=1}^n \left(\{\lambda_a^+, \theta_c^+\} \frac{\partial \hat{\Delta}_b}{\partial \hat{\theta}_c} + \frac{\partial \hat{\Delta}_a}{\partial \hat{\theta}_c} \{\theta_c^+, \lambda_b^+\} \right) = \frac{\partial \hat{\Delta}_a}{\partial \hat{\theta}_b} - \frac{\partial \hat{\Delta}_b}{\partial \hat{\theta}_a} = 0, \end{aligned}$$

and so the proof of the fundamental Poisson brackets (4.177) is complete.

In terms of the symplectic form (1.5), the canonicity of the dual coordinates can be rephrased as

$$(4.182) \quad \omega = \sum_{c=1}^n d\hat{\lambda}_c \wedge d\hat{\theta}_c.$$

Recalling Ψ (3.102) and (1.3), it is clear that $\forall c \in \mathbb{N}_n$ we have $\Psi^*\lambda_c = \hat{\theta}_c$ and $\Psi^*\theta_c = \hat{\lambda}_c$, whence

$$(4.183) \quad \Psi^*\omega = \Psi^* \sum_{c=1}^n d\lambda_c \wedge d\theta_c = \sum_{c=1}^n d(\Psi^*\lambda_c) \wedge d(\Psi^*\theta_c) = \sum_{c=1}^n d\hat{\theta}_c \wedge d\hat{\lambda}_c = -\omega$$

also follows immediately. \square

4.7. The wave and the scattering maps. So far we have analyzed the asymptotics of the trajectories only for large positive values of time. Recalling (4.155), for any $a \in \mathbb{N}_n$ our results can be succinctly summarized as

$$(4.184) \quad \lambda_a \circ \Phi_s \sim s \sinh(\theta_a^+) + \lambda_a^+ \quad \text{and} \quad \theta_a \circ \Phi_s \sim \theta_a^+ \quad (s \rightarrow \infty).$$

However, a thorough analysis of the scattering properties does require the study of the asymptotics for $s \rightarrow -\infty$, too. For this reason, let us conjugate the matrix flow (4.34) with \mathcal{R}_N (4.26). Thereby from (4.30) we obtain

$$(4.185) \quad \{e^{\pm 2\lambda_a \circ \Phi(s,p)} \mid a \in \mathbb{N}_n\} = \text{Spec} \left(\mathcal{R}_N \tilde{L}(p) \mathcal{R}_N^{-1} e^{2s \sinh(\mathcal{R}_N \Theta^+(p) \mathcal{R}_N^{-1})} \right) \quad (s \in \mathbb{R}, p \in P),$$

where

$$(4.186) \quad \mathcal{R}_N \Theta^+ \mathcal{R}_N^{-1} = -\Theta^+.$$

The point is that, due to the appearance of the negative sign in the above equation, by applying Theorem 7 to the matrix flow

$$(4.187) \quad \mathbb{R} \ni s \mapsto \mathcal{R}_N \tilde{L}(p) \mathcal{R}_N^{-1} e^{-2s \sinh(\Theta^+(p))} \in GL(N, \mathbb{C})$$

we can infer the desired asymptotic results for $s \rightarrow -\infty$ as well. More precisely, by the methods of Subsections 4.2 and 4.3, for any $a \in \mathbb{N}_n$ we can easily establish the asymptotics

$$(4.188) \quad \lambda_a \circ \Phi_s \sim s \sinh(\theta_a^-) + \lambda_a^- \quad \text{and} \quad \theta_a \circ \Phi_s \sim \theta_a^- \quad (s \rightarrow -\infty),$$

where the asymptotic rapidities obey

$$(4.189) \quad \theta_a^- = -\theta_a^+ = -2\hat{\theta}_a,$$

whereas, in complete analogy with (4.31), for the asymptotic positions we can write

$$(4.190) \quad \lambda_a^- = \frac{1}{2} \ln(m_a(\mathcal{R}_N \tilde{L} \mathcal{R}_N^{-1})).$$

As a matter of fact, mimicking the proof of Lemma 23 one finds immediately that

$$(4.191) \quad \lambda_a^- = -\frac{1}{2} \hat{\lambda}_a + \frac{1}{2} \Delta_a(\hat{\theta}_1, \dots, \hat{\theta}_n).$$

Now, let us introduce the smooth manifolds

$$(4.192) \quad P^\pm = \{\zeta = (\boldsymbol{\xi}, \boldsymbol{\eta}) = (\xi_1, \dots, \xi_n, \eta_1, \dots, \eta_n) \in \mathbb{R}^N \mid \eta_1 \gtrsim \dots \gtrsim \eta_n \gtrsim 0\}$$

endowed with the symplectic forms

$$(4.193) \quad \omega^\pm = \sum_{a=1}^n dx_a^\pm \wedge dy_a^\pm,$$

where $x_a^\pm, y_a^\pm \in C^\infty(P^\pm)$ are convenient global coordinates on P^\pm defined by the formulae

$$(4.194) \quad x_a^\pm(\zeta) = \xi_a \quad \text{and} \quad y_a^\pm(\zeta) = \eta_a \quad (a \in \mathbb{N}_n, \zeta \in P^\pm).$$

Utilizing the asymptotic positions and rapidities, at this point we define the wave maps

$$(4.195) \quad W_\pm: P \rightarrow P^\pm, \quad p \mapsto (\lambda_1^\pm(p), \dots, \lambda_n^\pm(p), \theta_1^\pm(p), \dots, \theta_1^\pm(p)),$$

that are of central interest in scattering theory.

Theorem 25. *Both wave maps W_\pm (4.195) are symplectomorphisms. Moreover, the corresponding scattering map*

$$(4.196) \quad S = W_+ \circ W_-^{-1}: P^- \rightarrow P^+$$

is also a symplectomorphism of the form

$$(4.197) \quad S(\boldsymbol{\xi}, \boldsymbol{\eta}) = (-\xi_1 + \Delta_1(-\boldsymbol{\eta}/2), \dots, -\xi_n + \Delta_n(-\boldsymbol{\eta}/2), -\eta_1, \dots, -\eta_n),$$

where $(\boldsymbol{\xi}, \boldsymbol{\eta}) = (\xi_1, \dots, \xi_n, \eta_1, \dots, \eta_n) \in P^-$.

Proof. Let us define the auxiliary maps $\Upsilon_\pm: P \rightarrow P^\pm$ by the formulae

$$(4.198) \quad \Upsilon_\pm(p) = \left(\pm \frac{1}{2} \eta_1 + \frac{1}{2} \Delta_1(\boldsymbol{\xi}), \dots, \pm \frac{1}{2} \eta_n + \frac{1}{2} \Delta_n(\boldsymbol{\xi}), \pm 2\xi_1, \dots, \pm 2\xi_n \right),$$

where $p = (\boldsymbol{\xi}, \boldsymbol{\eta}) = (\xi_1, \dots, \xi_n, \eta_1, \dots, \eta_n) \in P$. It is easy to see that both Υ_+ and Υ_- are diffeomorphisms with inverses $\Upsilon_\pm^{-1}: P^\pm \rightarrow P$ given by

$$(4.199) \quad \Upsilon_\pm^{-1}(\zeta) = \left(\pm \frac{1}{2} \eta_1, \dots, \pm \frac{1}{2} \eta_n, \pm 2\xi_1 \mp \Delta_1(\pm \boldsymbol{\eta}/2), \dots, \pm 2\xi_n \mp \Delta_n(\pm \boldsymbol{\eta}/2) \right),$$

where $\zeta = (\boldsymbol{\xi}, \boldsymbol{\eta}) = (\xi_1, \dots, \xi_n, \eta_1, \dots, \eta_n) \in P^\pm$.

Next, recalling the coordinate functions displayed in (1.3) and (4.194), from the definition (4.198) it is clear that

$$(4.200) \quad \Upsilon_{\pm}^* x_a^{\pm} = x_a^{\pm} \circ \Upsilon_{\pm} = \pm \frac{1}{2} \theta_a + \frac{1}{2} \Delta_a(\lambda_1, \dots, \lambda_n) \quad \text{and} \quad \Upsilon_{\pm}^* y_a^{\pm} = y_a^{\pm} \circ \Upsilon_{\pm} = \pm 2\lambda_a.$$

Thus, focusing first only on Υ_+ , the observation we made in (4.176) allows us to write that

$$(4.201) \quad \Upsilon_+^* \sum_{a=1}^n x_a^+ dy_a^+ = \sum_{a=1}^n (\theta_a + \Delta_a(\lambda_1, \dots, \lambda_n)) d\lambda_a = \sum_{a=1}^n \theta_a d\lambda_a + d(\mathcal{S}(\lambda_1, \dots, \lambda_n)).$$

Now, by taking the exterior derivative of the above equation, from the definitions (1.5) and (4.193) it is clear that

$$(4.202) \quad \Upsilon_+^* \omega^+ = \sum_{a=1}^n d\theta_a \wedge d\lambda_a + d^2(\mathcal{S}(\lambda_1, \dots, \lambda_n)) = -\omega.$$

To put it simple, the diffeomorphism Υ_+ is an anti-symplectomorphism. The same technique allows us to infer the relationship $\Upsilon_-^* \omega^- = -\omega$, too. We mention in passing that, on account of (4.201), the composition $\mathcal{S}(\lambda_1, \dots, \lambda_n) \in C^\infty(P)$ can be seen as a generating function of Υ_+ .

Now, looking back to the definitions (3.102), (4.195) and (4.198), our key observation is that the wave maps can be realized as the compositions

$$(4.203) \quad W_{\pm} = \Upsilon_{\pm} \circ \Psi.$$

Therefore, by invoking Theorem 24, we conclude at once that both W_+ and W_- are symplectomorphisms. As concerns the scattering map (4.196), from (4.203) it is evident that

$$(4.204) \quad S = \Upsilon_+ \circ \Upsilon_-^{-1}.$$

Thus, by exploiting the explicit expressions appearing in (4.198) and (4.199), the formula (4.197) is also immediate. \square

5. DISCUSSION

Now, we are in a position to harvest some interesting consequences of our results. Take an arbitrary smooth function

$$(5.1) \quad \chi: \mathcal{P} \rightarrow \mathbb{R}, \quad Y \mapsto \chi(Y)$$

defined on the symmetric space \mathcal{P} (2.9), and suppose that it is *invariant* under the action of conjugations by the elements of the compact subgroup K (2.4); that is,

$$(5.2) \quad \chi(kYk^{-1}) = \chi(Y) \quad (Y \in \mathcal{P}, k \in K).$$

Recalling the Lax matrix L (2.33) and Lemma 1, it is immediate that for the function

$$(5.3) \quad H_{\chi}^g = \chi(L^g) \in C^\infty(P)$$

we can write that

$$(5.4) \quad H_{\chi}^g = \chi(\hat{y}^g e^{2\hat{\Theta}^g} (\hat{y}^g)^{-1}) = \chi(e^{2\hat{\Theta}^g}).$$

In other words, H_{χ}^g (5.3) can be seen as a function of the coordinates $\hat{\theta}_1^g, \dots, \hat{\theta}_n^g$. As a consequence, the members of the global Darboux system featuring in Theorem 24 provide *action-angle variables* for the Hamiltonian system (P, ω, H_{χ}^g) . Notice that the van Diejen type model (1.12) belongs to this distinguished family of Hamiltonian systems. Indeed, due to the invariance of the trace functional, the relationship (2.35) entails that

$$(5.5) \quad H^g = \frac{1}{2} \text{tr}(L^g) = \frac{1}{2} \text{tr}(e^{2\hat{\Theta}^g}) = \frac{1}{2} \sum_{j=1}^n e^{2\hat{\Theta}_j^g} = \sum_{a=1}^n \cosh(2\hat{\theta}_a^g).$$

From a practical point of view, the Hamiltonian H^g (1.12) is singled out only by the fact that it has a relatively simple form.

In the theory of integrable Hamiltonian systems one of the principal goals is the construction of action-angle variables. Due to the rather explicit descriptions of the variables $\hat{\theta}_a^g$ (3.2) and $\hat{\lambda}_a^g$ (3.100), in this paper this task is completely accomplished for the family of systems (P, ω, H_χ^g) , including the van Diejen model H^g (1.12). However, as discovered by Ruijsenaars [12, 13, 14], it is a remarkable feature of the CMS and the RSvD systems that they can be arranged into pairs based on their action-angle maps. Now we are in a position to reveal the duality property of the 2-parameter family of van Diejen type systems (1.12), too. For this reason, with the aid of the dual Lax matrix \hat{L} (3.24), for each K -invariant smooth function χ (5.1) we introduce the ‘dual Hamiltonian’

$$(5.6) \quad \hat{H}_\chi^g = \chi(\hat{L}^g) \in C^\infty(P).$$

Notice that

$$(5.7) \quad \hat{H}_\chi^g = \chi((y^g)^{-1} e^{2\Lambda} y^g) = \chi(e^{2\Lambda}),$$

whence the original canonical coordinates (1.3) provide action-angle variables for $(P, \omega, \hat{H}_\chi^g)$. Moreover, due to the relationship (3.104), we can write that

$$(5.8) \quad \hat{H}_\chi^g = H_\chi^{\hat{g}} \circ \Psi^g.$$

The upshot of this observation is that, up to the anti-symplectomorphism Ψ^g (3.102), the Hamiltonian systems $(P, \omega, \hat{H}_\chi^g)$ and $(P, \omega, H_\chi^{\hat{g}})$ can be identified. We also see that, at the level of the parameter space, the systems in duality are related by the involution (3.29). Since under this involution the Hamiltonian (1.12) transforms into itself, i.e., $H^{\hat{g}} = H^g$, we may say that the van Diejen systems of our interest are *self-dual*.

In order to uncover this new case of duality, in this paper we adapted Ruijsenaars’ ideas [12] to the geometric picture introduced in [28]. Indeed, the construction of the functions $\hat{\theta}_a$ (3.1) and $\hat{\lambda}_a$ (3.100) is built upon the careful analysis of the Ruijsenaars type commutation relation (2.37). However, to prove their smoothness and canonicity, we departed from the complex analytic approach advocated by [12]. We believe that our method presented in Section 4 for proving the canonicity is a bit more general in the sense that, once the temporal asymptotics with the uniformity assertion is established as in Lemma 8, this technique may be applied to a much wider class of Hamiltonian systems describing repulsive particles, even under weaker smoothness conditions. Of course, for real-analytic Hamiltonians, the question of the real-analyticity of the pertinent objects would require further analysis.

Turning to the scattering properties of the 2-parameter family of hyperbolic van Diejen systems (1.12), from Theorem 25 we see that, up to an overall sign, the asymptotic rapidities are preserved. Moreover, from (4.170) it is also clear that the classical phase shifts are completely determined by the 1-particle and the 2-particle scattering processes. In other words, the scattering map (4.197) has a *factorized form*. Note that this peculiar feature seems to be characteristic to the CMS and the RSvD type many-particle systems. Indeed, for the models associated with the A -type root systems it is known for a long time (see e.g. [36, 37, 12]), and recently it has been proved for the hyperbolic BC_n Sutherland and the rational BC_n RSvD models, too [30, 38]. Thus, following [8], the content of Theorem 25 can be rephrased by saying that the hyperbolic van Diejen system (1.12) is also a *BC-type finite dimensional pure soliton system*.

To conclude the paper, let us recall that in [28] we conjectured a Lax matrix for a bit more general 3-parameter family of hyperbolic van Diejen systems (see equations (6.3-7) in [28]). Therefore, by generalizing the arguments of the present paper, it would be the natural next step to construct action-angle variables for these systems, too. We expect that the members of this 3-parameter family are also self-dual with factorized scattering maps.

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APPENDIX A. SPECTRAL ASYMPTOTICS REVISITED (BY SIMON RUIJSENAARS)

The results encoded in Theorem A2 of Ref. [12] concerning spectral asymptotics play a crucial role in Gábor's work on the van Diejen systems. My proof in Ref. [12] contains elaborate computations involving resolvents and nested Neumann expansions. By contrast, in this appendix the essence of Theorem A2 is recovered by exploiting the holomorphic implicit function theorem. The present method gives rise to a novel proof that is not only shorter and simpler, but also yields more explicit information. At the end of the appendix we also reconsider Theorem A1 of Ref. [12] along the same lines. Hence the counterparts of these older theorems are the present Theorems A.1 and A.2, resp.

The appendix is concerned with two families of complex $N \times N$ matrices. The first one consists of diagonal matrices

$$(A.1) \quad \mathcal{D} \equiv \{D = \text{diag}(d_1, \dots, d_N) \mid \text{Re}(d_N) < \dots < \text{Re}(d_1)\}.$$

We shall use the notation

$$(A.2) \quad \mu_j \equiv \text{Re}(d_j - d_{j+1}), \quad j = 1, \dots, N-1,$$

$$(A.3) \quad R \equiv \min(\mu_1, \dots, \mu_{N-1}).$$

The second family consists of matrices M that satisfy a restriction on their principal minors for our first theorem. Letting $1 \leq i_1 < \dots < i_j \leq N$, $j = 1, \dots, N$, we denote the $j \times j$ minor involving these indices by $M(i_1, \dots, i_j)$. Also, for the special case $i_k = k$, $k = 1, \dots, j$, we denote the corresponding principal minor by π_j . Thus we have in particular

$$(A.4) \quad \pi_1 = M_{11}, \quad \pi_2 = M_{11}M_{22} - M_{12}M_{21}, \quad \pi_N = |M|,$$

with $|M|$ the determinant of M . The second family is now given by

$$(A.5) \quad \mathcal{M} \equiv \{M \in \mathbb{C}^{N \times N} \mid \pi_j \neq 0, \quad j = 1, \dots, N\}.$$

As a consequence, for $M \in \mathcal{M}$ we may introduce nonzero complex numbers

$$(A.6) \quad m_1 \equiv M_{11}, \quad m_j \equiv \pi_j / \pi_{j-1}, \quad j = 2, \dots, N.$$

Our goal is now to elucidate the spectral asymptotics for matrices of the form

$$(A.7) \quad E(t) \equiv M \exp(tD), \quad M \in \mathcal{M}, \quad D \in \mathcal{D}, \quad t \in \mathbb{R},$$

as $t \rightarrow \infty$. As will transpire, the spectrum is simple for t sufficiently large, and the dominant asymptotics of the eigenvalues $\lambda_1(t), \dots, \lambda_N(t)$ is given by

$$(A.8) \quad \lambda_j(t) \sim m_j \exp(td_j), \quad j = 1, \dots, N, \quad t \rightarrow \infty.$$

(Note this yields an ordering $|\lambda_N(t)| < \dots < |\lambda_1(t)|$ for t large, due to the d -restriction in (A.1) and m_j being nonzero. Note also that (A.8) is plain for a diagonal $M \in \mathcal{M}$.)

For applications, however, it is crucial to improve considerably on the dominant asymptotics (A.8). For the $N = 2$ case this is readily done, since the eigenvalues can be calculated explicitly. However, this direct calculation yields no clue how to proceed for arbitrary N .

For a better understanding of the method followed for general N , we begin by detailing it for the $N = 2$ case. This serves to exemplify all steps of the flow chart without the inevitable notational clutter associated with the general case.

First, we define quantities $c_j(t)$, $j = 1, 2$, by setting

$$(A.9) \quad \lambda_j(t) = c_j(t)m_j \exp(td_j),$$

and note that the eigenvalues solve the system

$$(A.10) \quad \lambda_1 + \lambda_2 = M_{11} \exp(td_1) + M_{22} \exp(td_2), \quad \lambda_1 \lambda_2 = |M| \exp(td_1 + td_2).$$

Introducing

$$(A.11) \quad \epsilon \equiv \exp(td_2 - td_1),$$

this can be rewritten as

$$(A.12) \quad F_j(\epsilon; c_1, c_2) = 0, \quad j = 1, 2,$$

with

$$(A.13) \quad F_1 \equiv c_1 + c_2 \frac{|M|}{M_{11}^2} \epsilon - 1 - \frac{M_{22}}{M_{11}} \epsilon, \quad F_2 \equiv c_1 c_2 - 1.$$

It is plain that this system has a solution

$$(A.14) \quad F_j(0; 1, 1) = 0, \quad j = 1, 2.$$

Moreover, the matrix

$$(A.15) \quad D_c F(0; 1, 1) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

is regular and F is entire in ϵ, c_1 and c_2 . Therefore, we may invoke the holomorphic implicit function theorem. This theorem implies that for $|\epsilon| < a$ with a sufficiently small, there exists a unique holomorphic solution $c = s(\epsilon)$ with the properties

$$(A.16) \quad s(0) = (1, 1), \quad |s_j(\epsilon) - 1| \leq 1/2, \quad j = 1, 2, \quad |\epsilon| < a.$$

Defining T_0 by

$$(A.17) \quad \exp(-T_0 \mu_1) = a/2,$$

we now put

$$(A.18) \quad c_j(t) \equiv s_j(\exp(td_2 - td_1)), \quad t \geq T_0, \quad j = 1, 2.$$

By (A.16), this yields

$$(A.19) \quad |c_j(t)| \in [1/2, 3/2], \quad j = 1, 2, \quad t \geq T_0.$$

Choosing next $T_1 \geq T_0$ such that

$$(A.20) \quad |M_{11}| \exp(T_1 \operatorname{Re} d_1) > 3||M|/M_{11}| \exp(T_1 \operatorname{Re} d_2),$$

we get from (A.9) eigenvalues satisfying

$$(A.21) \quad |\lambda_2(t)| < |\lambda_1(t)|, \quad \forall t \in [T_1, \infty).$$

In particular, it follows that $\sigma(E(t))$ is simple for all $t \geq T_1$.

To proceed, we note that by holomorphy of the functions $s_j(z)$ for $|z| < a$, we can find $\delta \in (0, a/2]$ such that

$$(A.22) \quad \sup_{|z| \leq \delta} |s'_j(z)| \leq |s'_j(0)| + \eta, \quad j = 1, 2,$$

with η an arbitrary fixed positive number. Defining T_2 by

$$(A.23) \quad \exp(-T_2 \mu_1) = \delta,$$

we now put

$$(A.24) \quad T_E \equiv \max(T_1, T_2).$$

Then we conclude from (A.18) that for all $t \geq T_E$ we have

$$(A.25) \quad |c_j(t) - 1| \leq \exp(-t\mu_1)(|s'_j(0)| + \eta),$$

$$(A.26) \quad |\dot{c}_j(t)| \leq \exp(-t\mu_1)|d_2 - d_1|(|s'_j(0)| + \eta).$$

We can make these bounds more explicit by calculating $s'_j(0)$ from the system (A.12). Indeed, using

$$(A.27) \quad s_j(\epsilon) = 1 + \epsilon v_j + O(\epsilon^2), \quad \epsilon \rightarrow 0,$$

we deduce from (A.13) that v_2 equals $-v_1$ and that we have

$$(A.28) \quad v_1 = \frac{1}{M_{11}} \left(M_{22} - \frac{|M|}{M_{11}} \right).$$

The upshot of our reasoning is that for all $t \geq T_E$ the spectrum of $E(t)$ is simple, with eigenvalues of the form

$$(A.29) \quad \lambda_1(t) = M_{11} \exp(td_1)(1 + \rho_1(t)), \quad \lambda_2(t) = \frac{|M|}{M_{11}} \exp(td_2)(1 + \rho_2(t)).$$

Here, the remainder terms are majorized by

$$(A.30) \quad |\rho_j(t)| \leq \exp(-t\mu_1)(|v_1| + \eta), \quad j = 1, 2,$$

$$(A.31) \quad |\dot{\rho}_j(t)| \leq \exp(-t\mu_1)|d_2 - d_1|(|v_1| + \eta), \quad j = 1, 2,$$

with v_1 given by (A.28). (Note that v_1 vanishes when M is upper or lower triangular, as it should.)

We are now prepared for the general case.

Theorem A.1. *There exists $T_E \in \mathbb{R}$ such that the $N \times N$ matrix*

$$(A.32) \quad E(t) = M \exp(tD), \quad M \in \mathcal{M}, \quad D \in \mathcal{D}, \quad t \in \mathbb{R},$$

has nondegenerate eigenvalues $\lambda_1(t), \dots, \lambda_N(t)$ satisfying

$$(A.33) \quad |\lambda_N(t)| < \dots < |\lambda_1(t)|, \quad \forall t \geq T_E.$$

They are of the form

$$(A.34) \quad \lambda_j(t) = m_j \exp(td_j)[1 + \rho_j(t)], \quad t \geq T_E.$$

Here, the remainder functions are real-analytic on (T_E, ∞) and satisfy

$$(A.35) \quad |\rho_1(t)| \leq \exp(-t\mu_1)q_1,$$

$$(A.36) \quad |\rho_j(t)| \leq \exp(-t\mu_{j-1})q_{j-1} + \exp(-t\mu_j)q_j, \quad j = 2, \dots, N-1,$$

$$(A.37) \quad |\rho_N(t)| \leq \exp(-t\mu_{N-1})q_{N-1},$$

while their time derivatives satisfy

$$(A.38) \quad |\dot{\rho}_1(t)| \leq \exp(-t\mu_1)|d_2 - d_1|q_1,$$

$$(A.39) \quad |\dot{\rho}_j(t)| \leq \exp(-t\mu_{j-1})|d_j - d_{j-1}|q_{j-1} + \exp(-t\mu_j)|d_{j+1} - d_j|q_j, \quad j = 2, \dots, N-1,$$

$$(A.40) \quad |\dot{\rho}_N(t)| \leq \exp(-t\mu_{N-1})|d_N - d_{N-1}|q_{N-1}.$$

With η an arbitrary fixed positive number, the q_j 's are given by

$$(A.41) \quad q_j \equiv |p_j| + \eta, \quad j = 1, \dots, N-1,$$

where

$$(A.42) \quad p_1 \equiv \frac{1}{M_{11}} \left(M_{22} - \frac{M(1,2)}{M_{11}} \right),$$

and

$$(A.43) \quad p_j \equiv \frac{M(1, \dots, j-1, j+1)}{M(1, \dots, j)} - \frac{m_{j+1}}{m_j}, \quad j = 2, \dots, N-1.$$

Furthermore, T_E can be chosen uniformly for (M, D) varying over an arbitrary compact subset of $\mathcal{M} \times \mathcal{D}$.

Proof. We define c_1, \dots, c_N by

$$(A.44) \quad \lambda_j = c_j m_j \exp(td_j), \quad j = 1, \dots, N,$$

where $\lambda_1, \dots, \lambda_N$ are the solutions to $|E - \lambda \mathbf{1}_N| = 0$. Thus the latter solve the system

$$(A.45) \quad \sum_{i_1 < \dots < i_j} \lambda_{i_1} \cdots \lambda_{i_j} = \sum_{i_1 < \dots < i_j} M(i_1, \dots, i_j) \exp(td_{i_1} + \dots + td_{i_j}), \quad j = 1, \dots, N.$$

Introducing

$$(A.46) \quad \epsilon_j \equiv \exp(td_{j+1} - td_j), \quad j = 1, \dots, N-1,$$

this can be rewritten

$$(A.47) \quad F_j(\epsilon_1, \dots, \epsilon_{N-1}; c_1, \dots, c_N) = 0, \quad j = 1, \dots, N,$$

where

$$(A.48) \quad F_j \equiv \sum_{i_1 < \dots < i_j} \left(\frac{m_{i_1} \cdots m_{i_j}}{m_1 \cdots m_j} c_{i_1} \cdots c_{i_j} - \frac{M(i_1, \dots, i_j)}{\pi_j} \right) \prod_{k_1=1}^{i_1-1} \epsilon_{k_1} \cdots \prod_{k_j=j}^{i_j-1} \epsilon_{k_j}, \quad j = 1, \dots, N.$$

Clearly, this system has a solution

$$(A.49) \quad F_j(0, \dots, 0; 1, \dots, 1) = 0, \quad j = 1, \dots, N,$$

and it is not hard to verify

$$(A.50) \quad (D_c F)(0, \dots, 0; 1, \dots, 1)_{ij} = \begin{cases} 0, & i < j, \\ 1, & i \geq j. \end{cases}$$

Since this yields a regular matrix and F is entire in

$$(A.51) \quad \epsilon \equiv (\epsilon_1, \dots, \epsilon_{N-1}), \quad c \equiv (c_1, \dots, c_N),$$

the holomorphic implicit function theorem can be invoked. It entails that for ϵ in the polydisc

$$(A.52) \quad |\epsilon_1|, \dots, |\epsilon_{N-1}| < a,$$

with a small enough, we get a unique holomorphic solution $c = s(\epsilon)$ fulfilling

$$(A.53) \quad s_j(0) = 1, \quad |s_j(\epsilon) - 1| \leq 1/2, \quad j = 1, \dots, N.$$

We now choose T_0 satisfying

$$(A.54) \quad \exp(-T_0 R) = a/2,$$

with R given by (A.3). Defining

$$(A.55) \quad c_j(t) \equiv s_j(\exp(td_2 - td_1), \dots, \exp(td_N - td_{N-1})), \quad t \geq T_0, \quad j = 1, \dots, N,$$

we then deduce from (A.53) that we have

$$(A.56) \quad |c_j(t)| \in [1/2, 3/2], \quad j = 1, \dots, N, \quad t \geq T_0.$$

Next, we choose $T_1 \geq T_0$ such that

$$(A.57) \quad |m_j| \exp(T_1 \mu_j) > 3|m_{j+1}|, \quad j = 1, \dots, N-1.$$

Then we get from (A.44) the ordering (A.33), which implies that $\sigma(E(t))$ is simple for all $t \geq T_1$.

We now observe that since the functions $s_j(\epsilon)$ are holomorphic in the polydisc (A.52), for an arbitrary $\eta > 0$ we can find $\delta \in (0, a/2]$ with

$$(A.58) \quad \sup_{|z_1|, \dots, |z_{N-1}| \leq \delta} |\partial_k s_j(z)| \leq |\partial_k s_j(0)| + \eta, \quad k = 1, \dots, N-1, \quad j = 1, \dots, N.$$

Using

$$(A.59) \quad s_j(z_1, \dots, z_{N-1}) - s_j(0) = \int_0^{z_1} \partial_1 s_j(w_1, 0, \dots, 0) dw_1 \\ + \int_0^{z_2} \partial_2 s_j(z_1, w_2, 0, \dots, 0) dw_2 + \dots + \int_0^{z_{N-1}} \partial_{N-1} s_j(z_1, \dots, z_{N-2}, w_{N-1}) dw_{N-1},$$

we infer from this that we have bounds

$$(A.60) \quad |s_j(z) - 1| \leq \sum_{k=1}^{N-1} |z_k| (|\partial_k s_j(0)| + \eta), \quad |z_1|, \dots, |z_{N-1}| \leq \delta, \quad j = 1, \dots, N.$$

Defining now T_2 by

$$(A.61) \quad \exp(-T_2 R) = \delta,$$

we set

$$(A.62) \quad T_E \equiv \max(T_1, T_2).$$

Then we see from (A.55) that for all $t \geq T_E$ we have

$$(A.63) \quad |c_j(t) - 1| \leq \sum_{k=1}^{N-1} \exp(-t\mu_k) (|\partial_k s_j(0)| + \eta), \quad j = 1, \dots, N,$$

$$(A.64) \quad |\dot{c}_j(t)| \leq \sum_{k=1}^{N-1} |d_{k+1} - d_k| \exp(-t\mu_k) (|\partial_k s_j(0)| + \eta), \quad j = 1, \dots, N.$$

We proceed to calculate the partials $\partial_k s_j(0)$ from the system (A.47). To this end we write

$$(A.65) \quad s_j(\epsilon) = 1 + \sum_{k=1}^{N-1} v_{jk} \epsilon_k + \text{h. o.}, \quad v_{jk} \equiv \partial_k s_j(0),$$

where h. o. denotes terms of higher order in the power series expansion around $\epsilon = 0$. Substituting this for c_j in (A.47), we collect all terms linear in $\epsilon_1, \dots, \epsilon_{N-1}$. Defining p_j by (A.43), the resulting linear system can then be written as

$$(A.66) \quad \sum_{k=1}^{N-1} \sum_{i=1}^j v_{ik} \epsilon_k - p_j \epsilon_j = 0, \quad j = 1, \dots, N-1, \quad \sum_{k=1}^{N-1} \sum_{i=1}^N v_{ik} \epsilon_k = 0.$$

Now by holomorphy the coefficients of $\epsilon_1, \dots, \epsilon_{N-1}$ must vanish. From this we conclude first that for $j = 1$ the partials $\partial_k s_j(0)$ are given by

$$(A.67) \quad v_{1k} = \begin{cases} p_1, & k = 1, \\ 0, & k > 1, \end{cases}$$

and then recursively for $j = 2, \dots, N-1$, by

$$(A.68) \quad v_{jk} = \begin{cases} -p_{j-1}, & k = j-1, \\ p_j, & k = j, \\ 0, & \text{otherwise,} \end{cases}$$

while for $j = N$ we finally get

$$(A.69) \quad v_{Nk} = \begin{cases} -p_{N-1}, & k = N-1, \\ 0, & k < N-1. \end{cases}$$

Substituting this in (A.63)–(A.64), we arrive at (A.34)–(A.40). As a consequence, it remains to prove the uniformity claim.

To this end we begin by observing that the functions F_j defined by (A.48) are not only entire functions of ϵ and c , but also holomorphic functions of the matrix elements M_{ij} on the open subset \mathcal{M} of \mathbb{C}^{N^2} . Moreover, recalling (A.46), we see that they are entire in t and in d_1, \dots, d_N . We now fix $M_0 \in \mathcal{M}$ and $D_0 \in \mathcal{D}$ and view the set \mathcal{D} as an open subset of \mathbb{C}^N in the obvious way.

Next, we consider closed polydiscs in \mathcal{M} and \mathcal{D} around M_0 and D_0 , with nonzero radii r_1 and r_2 . We need only show that we can choose T_E uniformly for M and D varying over such polydiscs $P_0(r_1) \times Q_0(r_2) \subset \mathcal{M} \times \mathcal{D}$ when we suitably decrease r_1 and r_2 if need be. (Indeed, any compact subset of $\mathcal{M} \times \mathcal{D}$ is covered by a finite number of such polydiscs.)

To prove this, we retrace the above steps, as applied to $M_0 \exp(tD_0)$. First, we note that the holomorphic implicit function theorem implies that when we choose not only a in (A.52), but also r_1 small enough, then we get a unique solution $s_j(\epsilon, M)$ obeying (A.53) and holomorphic in the Cartesian product of the ϵ - and M -polydiscs.

Second, we define T_0 by

$$(A.70) \quad \exp(-T_0 m(r_2)) = a/2, \quad m(r_2) \equiv \min_{D \in Q_0(r_2)} (R(D)),$$

with $R(D)$ given by (A.3). Then (A.56) follows again.

Third, we choose $T_1 \geq T_0$ such that the inequalities (A.57) hold true on $P_0(r_1) \times Q_0(r_2)$. Then we obtain the ordering (A.33) and nondegeneracy of $\sigma(E(t))$ for all $(M, D) \in P_0(r_1) \times Q_0(r_2)$ and all $t \geq T_1$.

Fourth, choosing $\delta \in (0, a/2]$ and eventually decreasing r_1 such that (A.58) holds on the product of the closed ϵ -polydisc with radius δ and $P_0(r_1)$, the bounds (A.60) follow as before.

Finally, defining T_2 by

$$(A.71) \quad \exp(-T_2 m(r_2)) = \delta,$$

it follows that our uniformity assertion holds true for T_E (A.62). \square

In Appendix A of Ref. [12] we also studied the spectral asymptotics for $t \rightarrow \infty$ of matrices of the form $M + tD$, with M an arbitrary $N \times N$ matrix and $D \in \mathcal{D}$. We have meanwhile realised that (a slightly different version of) the pertinent result (namely, Theorem A1 in Ref. [12]) can be readily understood via Rayleigh–Schrödinger perturbation theory, cf. for example [39], p. 7.

Specifically, the spectral asymptotics of $M + tD$ for $t \rightarrow \infty$ can be readily deduced from the behavior of the spectrum $\sigma(D + \epsilon M)$ for $|\epsilon|$ small enough. Indeed, since D has simple spectrum, we can find $r > 0$ such that this is still true for $D + \epsilon M$ with $|\epsilon| \leq r$. Then Rayleigh–Schrödinger perturbation theory can be invoked to infer that $D + \epsilon M$ has eigenvalues of the form

$$(A.72) \quad d_j + \epsilon M_{jj} + \epsilon^2 \alpha_j + O(\epsilon^3), \quad \epsilon \rightarrow 0, \quad j = 1, \dots, N,$$

where

$$(A.73) \quad \alpha_j \equiv \sum_{k \neq j} \frac{M_{jk} M_{kj}}{d_j - d_k}, \quad j = 1, \dots, N.$$

As a consequence, $M + tD$ has eigenvalues that satisfy estimates

$$(A.74) \quad \lambda_j(t) = M_{jj} + t d_j + t^{-1} \alpha_j + O(t^{-2}), \quad t \rightarrow \infty, \quad j = 1, \dots, N.$$

This result can also be recovered and slightly improved by the method we followed to prove Theorem A.1. This yields the following theorem, which concludes this appendix.

Theorem A.2. *There exists $T_E \in \mathbb{R}$ such that the matrix*

$$(A.75) \quad E(t) = M + tD, \quad M \in \mathbb{C}^{N \times N}, \quad D \in \mathcal{D}, \quad t \in \mathbb{R},$$

has nondegenerate eigenvalues $\lambda_1(t), \dots, \lambda_N(t)$, satisfying

$$(A.76) \quad |\lambda_N(t)| < \dots < |\lambda_1(t)|, \quad \forall t \geq T_E.$$

They are of the form

$$(A.77) \quad \lambda_j(t) = M_{jj} + td_j + \rho_j(t), \quad j = 1, \dots, N,$$

where the $\rho_j(t)$'s are real-analytic functions on (T_E, ∞) such that

$$(A.78) \quad |\rho_j(t)| \leq \frac{1}{t}(|\alpha_j| + \eta), \quad |\dot{\rho}_j(t)| \leq \frac{1}{t^2}(|\alpha_j| + \eta), \quad j = 1, \dots, N.$$

Here, η is an arbitrary fixed positive number and the α_j are given by (A.73). Moreover, T_E can be chosen uniformly for (M, D) varying over an arbitrary compact subset of $\mathbb{C}^{N \times N} \times \mathcal{D}$.

Proof. The crux is that the holomorphic function theorem again applies to the case at hand, provided we choose a suitable starting point. Specifically, let $\lambda_j(t)$, $j = 1, \dots, N$, be the solutions to $|E(t) - \lambda \mathbf{1}_N| = 0$. Then we set

$$(A.79) \quad c_j(\epsilon) \equiv \epsilon \lambda_j(1/\epsilon), \quad j = 1, \dots, N.$$

In this case the associated spectral system is given by

$$(A.80) \quad F_j(\epsilon; c_1, \dots, c_N) = 0, \quad j = 1, \dots, N,$$

where the functions F_j are of the form

$$(A.81) \quad F_j = \sum_{i_1 < \dots < i_j} \left(c_{i_1} \cdots c_{i_j} - d_{i_1} \cdots d_{i_j} - \epsilon \sum_{k=1}^j M_{i_k i_k} d_{i_1} \cdots \hat{d}_{i_k} \cdots d_{i_j} \right. \\ \left. - \epsilon^2 \sum_{1 \leq k < l \leq j} M(i_k, i_l) d_{i_1} \cdots \hat{d}_{i_k} \cdots \hat{d}_{i_l} \cdots d_{i_j} \right) + O(\epsilon^3).$$

Here the hat signifies that the pertinent d_i should be omitted.

This system has an obvious solution

$$(A.82) \quad F_j(0; d_1, \dots, d_N) = 0, \quad j = 1, \dots, N,$$

with corresponding partial matrix

$$(A.83) \quad (D_c F)(0; d_1, \dots, d_N) = \begin{pmatrix} 1 & \cdots & 1 \\ d_2 + \cdots + d_N & \cdots & d_1 + \cdots + d_{N-1} \\ \vdots & \vdots & \vdots \\ d_2 \cdots d_N & \cdots & d_1 \cdots d_{N-1} \end{pmatrix},$$

and it is not difficult to verify

$$(A.84) \quad |(D_c F)(0; d_1, \dots, d_N)| = \prod_{1 \leq i < j \leq N} (d_i - d_j).$$

(The determinant vanishes for $d_i = d_j$ and it is a polynomial of degree $N(N-1)/2$ in d_1, \dots, d_N . By antisymmetry it then must be a nonzero multiple of the right-hand side. It is easy to see this multiple equals 1.)

Since the numbers d_1, \dots, d_N are distinct, the product on the right-hand side is nonzero. Hence the holomorphic implicit function theorem may be invoked. From this we deduce that for $|\epsilon| < a$ with a small enough, there exists a unique solution $c(\epsilon)$ satisfying

$$(A.85) \quad |c_j(\epsilon) - d_j| \leq r_j/3, \quad j = 1, \dots, N,$$

where

$$(A.86) \quad r_1 \equiv \mu_1, \quad r_N \equiv \mu_{N-1}, \quad r_j \equiv \min(\mu_{j-1}, \mu_j), \quad j = 2, \dots, N-1.$$

This entails that the corresponding $\lambda_j(t)$ (cf. (A.79)) satisfy (A.33). Hence $\sigma(E(t))$ is simple for all $t \geq T_0$, where

$$(A.87) \quad T_0 \equiv 1/a.$$

Writing next

$$(A.88) \quad c_j(\epsilon) = d_j + c'_j(0)\epsilon + c''_j(0)\epsilon^2/2 + O(\epsilon^3),$$

it is immediate from the above system that $c'_j(0)$ equals M_{jj} . To prove that we have

$$(A.89) \quad c''_j(0) = 2\alpha_j, \quad j = 1, \dots, N,$$

with α_j given by (A.73), is arduous, but straightforward. (One need only check that $F_j(\epsilon; c)$ vanishes to second order in ϵ when c_j is replaced by $d_j + \epsilon M_{jj} + \epsilon^2 \alpha_j$ in (A.81). Using permutations, it suffices to verify that the sum of all second-order terms involving $M_{11}M_{22}$ and $M_{12}M_{21}$ vanishes. Noting that the latter product can only arise from α_1 and α_2 , this can be readily achieved.)

Now by holomorphy there exists for a given $\eta > 0$ a number $\delta \in (0, a/2]$ such that

$$(A.90) \quad |c_j(\epsilon) - d_j - M_{jj}\epsilon| \leq (|\alpha_j| + \eta)|\epsilon|^2.$$

Defining

$$(A.91) \quad T_E \equiv 1/\delta,$$

we then obtain (A.77)–(A.78). Finally, the uniformity assertion follows as in the proof of the above theorem. \square

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