

# A new family of MRD-codes

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## Abstract

We introduce a family of linear sets of  $\text{PG}(1, q^{2n})$  arising from maximum scattered linear sets of pseudoregulus type of  $\text{PG}(3, q^n)$ . For  $n = 3, 4$  and for certain values of the parameters we show that these linear sets of  $\text{PG}(1, q^{2n})$  are maximum scattered and they yield new MRD-codes with parameters  $(6, 6, q; 5)$  for  $q > 2$  and with parameters  $(8, 8, q; 7)$  for  $q$  odd.

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## 1 Introduction

Linear sets are natural generalizations of subgeometries. Let  $\Lambda = \text{PG}(V, \mathbb{F}_{q^n}) = \text{PG}(r-1, q^n)$ , where  $V$  is a vector space of dimension  $r$  over  $\mathbb{F}_{q^n}$ . A point set  $L$  of  $\Lambda$  is said to be an  $\mathbb{F}_q$ -linear set of  $\Lambda$  of rank  $k$  if it is defined by the non-zero vectors of a  $k$ -dimensional  $\mathbb{F}_q$ -vector subspace  $U$  of  $V$ , i.e.

$$L = L_U = \{\langle \mathbf{u} \rangle_{\mathbb{F}_{q^n}} : \mathbf{u} \in U \setminus \{\mathbf{0}\}\}.$$

The maximum field of linearity of an  $\mathbb{F}_q$ -linear set  $L_U$  is  $\mathbb{F}_{q^t}$  if  $t \mid n$  is the largest integer such that  $L_U$  is an  $\mathbb{F}_{q^t}$ -linear set.

Two linear sets  $L_U$  and  $L_W$  of  $\text{PG}(r-1, q^n)$  are said to be PTL-equivalent (or simply equivalent) if there is an element  $\phi$  in  $\text{PTL}(r, q^n)$  such that  $L_U^\phi = L_W$ . It may happen that two  $\mathbb{F}_q$ -linear sets  $L_U$  and  $L_W$  of  $\text{PG}(r-1, q^n)$  are equivalent even if the two  $\mathbb{F}_q$ -vector subspaces  $U$  and  $W$  are not in the same orbit of  $\text{GL}(r, q^n)$  (see [8] and [5] for further details).

In [35, Section 4], the author showed that scattered linear sets of  $\text{PG}(1, q^m)$  of rank  $m$  yield  $\mathbb{F}_q$ -linear MRD-codes of dimension  $2m$  and minimum distance  $m-1$ . Precisely, such codes are all  $\mathbb{F}_q$ -linear MRD-codes of dimension

$2m$ , minimum distance  $m - 1$  and middle nucleus of order  $q^m$  (cf. Proposition 6.1). This result has been recently generalized in [6]. The number of non-equivalent MRD-codes obtained from a scattered linear set of  $\text{PG}(1, q^m)$  of rank  $m$  was studied in [5, Section 5.4]. In [24] the author investigated in detail the relationship between linear sets of  $\text{PG}(n - 1, q^n)$  of rank  $n$  and  $\mathbb{F}_q$ -linear MRD-codes.

So far, the known non-equivalent families of  $\mathbb{F}_q$ -linear MRD-codes of dimension  $2m$ , minimum distance  $m - 1$  and with *middle nucleus*  $\mathbb{F}_{q^m}$  (which is an invariant with respect to the equivalence on MRD-codes, see Section 6) arise from the following maximum scattered  $\mathbb{F}_q$ -vector subspaces of  $\mathbb{F}_{q^m} \times \mathbb{F}_{q^m}$ :

1.  $U_1 := \{(x, x^{q^s}) : x \in \mathbb{F}_{q^m}\}$ ,  $1 \leq s \leq m - 1$   $\gcd(s, m) = 1$  ([4]) gives Gabidulin codes when  $s = 1$ , and generalized Gabidulin codes when  $s > 1$ ;
2.  $U_2 := \{(x, \delta x^{q^s} + x^{q^{m-s}}) : x \in \mathbb{F}_{q^m}\}$ ,  $N_{q^m/q}(\delta) \neq 1$  (<sup>1</sup>),  $\gcd(s, m) = 1$  ([27] for  $s = 1$ ) gives MRD-codes found by Sheekey in [35] as part of a larger family. The equivalence issue for these codes was studied also by Lunardon, Trombetti and Zhou in [28].

In this paper we present a family of  $\mathbb{F}_q$ -linear sets of rank  $m$  of  $\text{PG}(1, q^m)$ ,  $m = 2n$  and  $n > 1$ , arising from  $\mathbb{F}_q$ -linear sets of  $\text{PG}(3, q^n)$  of pseudoregulus type. These linear sets are defined by the following  $\mathbb{F}_q$ -vector subspaces of  $\mathbb{F}_{q^m} \times \mathbb{F}_{q^m}$ :

$$U_{b,s} := \{(x, bx^{q^s} + x^{q^{s+n}}) : x \in \mathbb{F}_{q^{2n}}\} \quad (1)$$

with  $N_{q^{2n}/q^n}(b) \neq 1$ ,  $1 \leq s \leq 2n - 1$  and  $\gcd(s, n) = 1$ .

We will show that each point of  $L_{U_{b,s}}$  has weight at most 2 (cf. Proposition 4.1) and when  $L_{U_{b,s}}$  is scattered and  $m > 4$ , then, as we will see in Section 6, the corresponding MRD-code is not equivalent to any previously known MRD-code with the same parameters. Finally, in the last section, we exhibit for  $m = 6$  and  $m = 8$  infinite examples of scattered  $\mathbb{F}_q$ -subspaces of type  $U_{b,s}$  and hence new infinite families of MRD-codes.

## 2 Linear sets

Let  $L_U$  be an  $\mathbb{F}_q$ -linear set of  $\Lambda = \text{PG}(r - 1, q^n)$ ,  $q = p^h$ ,  $p$  prime, of rank  $k$ . We point out that different vector subspaces can define the same linear

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<sup>1</sup> $N_{q^m/q}(\cdot)$  denotes the norm function from  $\mathbb{F}_{q^m}$  over  $\mathbb{F}_q$ .

set. For this reason a linear set and the vector space defining it must be considered as coming in pair.

Let  $\Omega = \text{PG}(W, \mathbb{F}_{q^n})$  be a subspace of  $\Lambda$ , then  $\Omega \cap L_U$  is an  $\mathbb{F}_q$ -linear set of  $\Omega$  defined by the  $\mathbb{F}_q$ -vector subspace  $U \cap W$  and, if  $w_{L_U}(\Omega) := \dim_{\mathbb{F}_q}(W \cap U) = i$ , we say that  $\Omega$  has *weight*  $i$  w.r.t.  $L_U$ . Hence a point of  $\Lambda$  belongs to  $L_U$  if and only if it has weight at least 1 and, if  $L_U$  has rank  $k$ , then  $|L_U| \leq q^{k-1} + q^{k-2} + \dots + q + 1$ . For further details on linear sets see [34] and [23].

An  $\mathbb{F}_q$ -linear set  $L_U$  of  $\Lambda$  of rank  $k$  is *scattered* if all of its points have weight 1, or equivalently, if  $L_U$  has maximum size  $q^{k-1} + q^{k-2} + \dots + q + 1$ . The associated  $\mathbb{F}_q$ -vector subspace  $U$  is said to be *scattered*. A scattered  $\mathbb{F}_q$ -linear set of  $\Lambda$  of highest possible rank is a *maximum scattered  $\mathbb{F}_q$ -linear set* of  $\Lambda$ ; see [4]. Maximum scattered linear sets have a lot of applications in Galois Geometry. For a recent survey on the theory of scattered spaces in Galois Geometry and its applications see [19].

The rank of a scattered  $\mathbb{F}_q$ -linear set of  $\text{PG}(r-1, q^n)$ ,  $rn$  even, is at most  $rn/2$  ([4, Theorems 2.1, 4.2 and 4.3]). For  $n = 2$  scattered  $\mathbb{F}_q$ -linear sets of  $\text{PG}(r-1, q^2)$  of rank  $r$  are the Baer subgeometries. When  $r$  is even there always exist scattered  $\mathbb{F}_q$ -linear sets of rank  $\frac{rn}{2}$  in  $\text{PG}(r-1, q^n)$ , for any  $n \geq 2$  (see [18, Theorem 2.5.5] for an explicit example). Existence results were proved for  $r$  odd,  $n-1 \leq r$ ,  $n$  even, and  $q > 2$  in [4, Theorem 4.4], but no explicit constructions were known for  $r$  odd, except for the case  $r = 3$ ,  $n = 4$ , see [2, Section 3]. Very recently in [3, Theorem 1.2] and in [6, Section 2] maximum scattered  $\mathbb{F}_q$ -linear sets of  $\text{PG}(r-1, q^n)$  of rank  $rn/2$  have been constructed for any integers  $r, n \geq 2$ ,  $rn$  even, and for any prime power  $q \geq 2$ .

## 2.1 Scattered linear sets of pseudoregulus type in $\text{PG}(3, q^n)$

In [26], generalizing results contained in [32], [20] and [22], a family of maximum scattered linear sets of  $\text{PG}(2h-1, q^n)$  of rank  $hn$  ( $h, n \geq 2$ ), called of *pseudoregulus type*, is introduced. In particular, a maximum scattered  $\mathbb{F}_q$ -linear set  $L_U$  of  $\Lambda = \text{PG}(3, q^n)$  of rank  $2n$  is of *pseudoregulus type* if (i) there exist  $q^n + 1$  pairwise disjoint lines of  $L_U$  of weight  $n$  w.r.t.  $L_U$ , say  $s_1, s_2, \dots, s_{q^n+1}$ ;

(ii) there exist exactly two skew lines  $t_1$  and  $t_2$  of  $\Lambda$ , disjoint from  $L_U$ , such that  $t_j \cap s_i \neq \emptyset$  for each  $i = 1, \dots, q^n + 1$  and for each  $j = 1, 2$ .

The set of lines  $\mathcal{P}_{L_U} = \{s_i : i = 1, \dots, q^n + 1\}$  is called the  *$\mathbb{F}_q$ -pseudoregulus* (or simply *pseudoregulus*) of  $\Lambda$  associated with  $L_U$  and  $t_1$  and  $t_2$  are the *transversal lines* of  $\mathcal{P}_{L_U}$  (or *transversal lines* of  $L_U$ ). Note that by [26,

Corollary 3.3], if  $n > 2$  the pseudoregulus  $\mathcal{P}_{L_U}$  associated with  $L_U$  and its transversal lines are uniquely determined.

In [20, Sec. 2] and in [26, Theorems 3.5 and 3.9],  $\mathbb{F}_q$ -linear sets of pseudoregulus type of  $\text{PG}(2h-1, q^n)$  of rank  $hn$  ( $h, n \geq 2$ ) have been algebraically characterized. In particular, in  $\text{PG}(3, q^n)$  we have the following result.

**Theorem 2.1.** *Let  $t_1 = \text{PG}(U_1, \mathbb{F}_{q^n})$  and  $t_2 = \text{PG}(U_2, \mathbb{F}_{q^n})$  be two disjoint lines of  $\Lambda = \text{PG}(V, \mathbb{F}_{q^n}) = \text{PG}(3, q^n)$  and let  $\Phi_f$  be a strictly semilinear collineation between  $t_1$  and  $t_2$  defined by the  $\mathbb{F}_{q^n}$ -semilinear map  $f$  with companion automorphism an element  $\sigma \in \text{Aut}(\mathbb{F}_{q^n})$  such that  $\text{Fix}(\sigma) = \mathbb{F}_q$ . Then, for each  $\rho \in \mathbb{F}_{q^n}^*$ , the set*

$$L_{\rho, f} = \{\langle \mathbf{u} + \rho f(\mathbf{u}) \rangle_{\mathbb{F}_{q^n}} : \mathbf{u} \in U_1 \setminus \{\mathbf{0}\}\}$$

*is an  $\mathbb{F}_q$ -linear set of  $\Lambda$  of pseudoregulus type whose associated pseudoregulus is  $\mathcal{P}_{L_{\rho, f}} = \{\langle P, P^{\Phi_f} \rangle : P \in t_1\}$ , with transversal lines  $t_1$  and  $t_2$ .*

*Conversely, each  $\mathbb{F}_q$ -linear set of pseudoregulus type of  $\Lambda = \text{PG}(3, q^n)$  can be obtained as described above.*

In [26],  $\mathbb{F}_q$ -linear sets of pseudoregulus type of the projective line  $\Lambda = \text{PG}(V, \mathbb{F}_{q^n}) = \text{PG}(1, q^n)$  ( $n \geq 2$ ) are also introduced. Let  $P_1 = \langle \mathbf{w} \rangle$  and  $P_2 = \langle \mathbf{v} \rangle$  be two distinct points of the line  $\Lambda$  and let  $\tau$  be an  $\mathbb{F}_q$ -automorphism of  $\mathbb{F}_{q^n}$  such that  $\text{Fix}(\tau) = \mathbb{F}_q$ ; then for each  $\rho \in \mathbb{F}_{q^n}^*$  the set

$$W_{\rho, \tau} = \{\lambda \mathbf{w} + \rho \lambda^\tau \mathbf{v} : \lambda \in \mathbb{F}_{q^n}\}, \quad (2)$$

is an  $\mathbb{F}_q$ -vector subspace of  $V$  of dimension  $n$  and  $L_{\rho, \tau} := L_{W_{\rho, \tau}}$  is a maximum scattered  $\mathbb{F}_q$ -linear set of  $\Lambda$ . The linear sets  $L_{\rho, \tau}$  are called of *pseudoregulus type* and the points  $P_1$  and  $P_2$  are their *transversal points*. Also, if  $n > 2$ , then these transversal points are uniquely determined ([26, Prop. 4.3]). For more details on such linear sets see [9]. Also, by [26, Remark 4.5], if  $L_U$  is an  $\mathbb{F}_q$ -linear set of pseudoregulus type of  $\text{PG}(3, q^n)$ , and  $s$  is a line of weight  $n$  w.r.t.  $L_U$ , then  $L_U \cap s$  is an  $\mathbb{F}_q$ -linear set of pseudoregulus type of the line  $s$  whose transversal points are the intersection points of  $s$  with the transversal lines of  $\mathcal{P}_{L_U}$  (see also [21, Prop. 2.5] and [31, Theorem 2.8] for further details).

### 3 Linear sets and dual linear sets in $\text{PG}(1, q^n)$

Let  $\mathbb{V} = \mathbb{F}_{q^n} \times \mathbb{F}_{q^n}$  and let  $L_U$  be an  $\mathbb{F}_q$ -linear set of rank  $n$  of  $\text{PG}(1, q^n) = \text{PG}(\mathbb{V}, \mathbb{F}_{q^n})$ . We can always assume (up to a projectivity) that  $L_U$  does not

contain the point  $\langle(0, 1)\rangle_{\mathbb{F}_{q^n}}$ . Then  $U = U_f = \{(x, f(x)) : x \in \mathbb{F}_{q^n}\}$ , for some  $q$ -polynomial  $f(x) = \sum_{i=0}^{n-1} a_i x^{q^i}$  over  $\mathbb{F}_{q^n}$ . For the sake of simplicity we will write  $L_f$  instead of  $L_{U_f}$  to denote the linear set defined by  $U_f$ .

Consider the non-degenerate symmetric bilinear form of  $\mathbb{F}_{q^n}$  over  $\mathbb{F}_q$  defined by the following rule

$$\langle x, y \rangle := \text{Tr}_{q^n/q}(xy).^{(2)} \quad (3)$$

Then the *adjoint map*  $\hat{f}$  of an  $\mathbb{F}_q$ -linear map  $f(x) = \sum_{i=0}^{n-1} a_i x^{q^i}$  of  $\mathbb{F}_{q^n}$  (with respect to the bilinear form (3)) is

$$\hat{f}(x) := \sum_{i=0}^{n-1} a_i^{q^{n-i}} x^{q^{n-i}}. \quad (4)$$

Let  $\eta : \mathbb{V} \times \mathbb{V} \longrightarrow \mathbb{F}_{q^n}$  be the non-degenerate alternating bilinear form of  $\mathbb{V}$  defined by  $\eta((x, y), (u, v)) = xv - yu$ . Then  $\eta$  induces a symplectic polarity  $\tau$  on the line  $\text{PG}(\mathbb{V}, \mathbb{F}_{q^n})$  and

$$\eta'((x, y), (u, v)) := \text{Tr}_{q^n/q}(\eta((x, y), (u, v))) = \text{Tr}_{q^n/q}(xv - yu) \quad (5)$$

is a non-degenerate alternating bilinear form on  $\mathbb{V}$ , when  $\mathbb{V}$  is regarded as a  $2n$ -dimensional vector space over  $\mathbb{F}_q$ . We will always denote in the paper by  $\perp$  and  $\perp'$  the orthogonal complement maps defined by  $\eta$  and  $\eta'$  on the lattices of the  $\mathbb{F}_{q^n}$ -subspaces and the  $\mathbb{F}_q$ -subspaces of  $\mathbb{V}$ , respectively. Direct calculation shows that

$$U_f^{\perp'} = U_{\hat{f}}, \quad (6)$$

and the  $\mathbb{F}_q$ -linear set of rank  $n$  of  $\text{PG}(\mathbb{V}, \mathbb{F}_{q^n})$  defined by the orthogonal complement  $U^{\perp'}$  is called *the dual linear set of  $L_U$*  with respect to the polarity  $\tau$ .

Recall the following lemma.

**Lemma 3.1** ([3, Lemma 2.6], [5, Lemma 3.1]). *Let  $L_f = \{\langle(x, f(x))\rangle_{\mathbb{F}_{q^n}} : x \in \mathbb{F}_{q^n}^*\}$  be an  $\mathbb{F}_q$ -linear set of  $\text{PG}(1, q^n)$  of rank  $n$ , with  $f(x)$  a  $q$ -polynomial over  $\mathbb{F}_{q^n}$ , and let  $\hat{f}$  be the adjoint of  $f$  with respect to the bilinear form (3). Then for each point  $P \in \text{PG}(1, q^n)$  we have  $w_{L_f}(P) = w_{L_{\hat{f}}}(P)$ . In particular,  $L_f = L_{\hat{f}}$  and the maps defined by  $f(x)/x$  and  $\hat{f}(x)/x$  have the same image.*

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<sup>2</sup> $\text{Tr}_{q^n/q}(\cdot)$  denotes the trace function from  $\mathbb{F}_{q^n}$  over  $\mathbb{F}_q$ .

## 4 From the geometry in $\text{PG}(3, q^n)$ to the geometry in $\text{PG}(1, q^{2n})$

From now on, we will consider  $\mathbb{V} = \mathbb{F}_{q^{2n}} \times \mathbb{F}_{q^{2n}}$  both as a 2-dimensional vector space over  $\mathbb{F}_{q^{2n}}$  and as a 4-dimensional vector space over  $\mathbb{F}_{q^n}$ . In the former case the linear set of  $\Sigma_1 := \text{PG}(\mathbb{V}, \mathbb{F}_{q^{2n}}) = \text{PG}(1, q^{2n})$  defined by an  $\mathbb{F}_q$ -subspace  $U \leq \mathbb{V}$  will be denoted as  $L_U$ , in the latter case the linear set of  $\Sigma_3 := \text{PG}(\mathbb{V}, \mathbb{F}_{q^n}) = \text{PG}(3, q^n)$  defined by  $U$  will be denoted by  $\bar{L}_U$ .

Consider the following two skew lines of  $\Sigma_3$ :  $\ell_0 := \{\langle (x, 0) \rangle_{\mathbb{F}_{q^n}} : x \in \mathbb{F}_{q^{2n}}^*\}$  and  $\ell_1 := \{\langle (0, y) \rangle_{\mathbb{F}_{q^n}} : y \in \mathbb{F}_{q^{2n}}^*\}$ . By Theorem 2.1,  $\mathbb{F}_q$ -linear sets of pseudoregulus type in  $\Sigma_3$  with transversal lines  $\ell_0$  and  $\ell_1$  are of the form  $\bar{L}_f := \bar{L}_{U_f}$ , where  $U_f = \{(x, f(x)) : x \in \mathbb{F}_{q^{2n}}\}$ , and  $f(x)$  is a strictly  $\mathbb{F}_{q^n}$ -semilinear invertible map of  $\mathbb{F}_{q^{2n}}$  with companion automorphism  $\sigma$ ,  $\text{Fix}(\sigma) = \mathbb{F}_q$ . It is easy to see that this happens if and only if  $f(x) = \alpha x^\sigma + \beta x^{\sigma q^n}$ , where  $\sigma : x \mapsto x^{q^s}$ ,  $1 \leq s \leq 2n - 1$ ,  $\gcd(s, n) = 1$ , and  $N_{q^{2n}/q^n}(\alpha) \neq N_{q^{2n}/q^n}(\beta)$ . That is,

$$U_f = \{(x, \alpha x^\sigma + \beta x^{\sigma q^n}) : x \in \mathbb{F}_{q^{2n}}\}, \quad (7)$$

with the same conditions as above. In  $\Sigma_1$  the  $\mathbb{F}_q$ -linear set  $L_f := L_{U_f}$  is not necessarily scattered, but as the next result shows, it cannot contain points with weight greater than two.

**Proposition 4.1.** *Each point of the  $\mathbb{F}_q$ -linear set  $L_f$  of  $\text{PG}(1, q^{2n})$ ,  $n \geq 2$ , where*

$$U_f = \{(x, f(x)) : x \in \mathbb{F}_{q^{2n}}\},$$

*with  $f(x) = \alpha x^\sigma + \beta x^{\sigma q^n}$ ,  $\sigma : x \mapsto x^{q^s}$ ,  $1 \leq s \leq 2n - 1$ ,  $\gcd(s, n) = 1$ , and  $N_{q^{2n}/q^n}(\alpha) \neq N_{q^{2n}/q^n}(\beta)$ , has weight at most two.*

*Proof.* We first recall that the pseudoregulus associated with  $\bar{L}_f$  in  $\Sigma_3 = \text{PG}(3, q^n)$  consists of  $q^n + 1$  lines, and these are the only lines with weight  $n$  w.r.t.  $\bar{L}_f$  ([26, Prop. 3.2]).

Let  $Q := \langle (x_0, f(x_0)) \rangle_{\mathbb{F}_{q^{2n}}}$  be a point of  $L_f$ . In  $\Sigma_3$  this point corresponds to a line  $\ell_Q$  disjoint from both  $\ell_0$  and  $\ell_1$  and meeting at least one line of the pseudoregulus associated with  $\bar{L}_f$ , say  $m$ . Note that  $w_{L_f}(Q) = w_{\bar{L}_f}(\ell_Q)$ . By [1, Theorem 5.1] a plane of  $\Sigma_3$  has weight either  $n$  or  $n + 1$  w.r.t.  $\bar{L}_f$ , hence if the weight of  $Q$  w.r.t.  $L_f$  is greater than one, then the plane  $\pi$  of  $\Sigma_3$  spanned by the lines  $\ell_Q$  and  $m$  has weight  $n + 1$ . Since  $\ell_Q \cap m$  is a point with weight one w.r.t.  $\bar{L}_f$ , the Grassmann formula gives that the weight of  $\ell_Q$  w.r.t.  $\bar{L}_f$  is two and hence the weight of  $Q$  w.r.t.  $L_f$  is two.  $\square$

## 5 A family of $\mathbb{F}_q$ -linear sets of $\text{PG}(1, q^{2n})$

In this section we investigate the family of  $\mathbb{F}_q$ -linear sets of  $\text{PG}(1, q^{2n})$  defined by  $\mathbb{F}_q$ -vector subspaces of form (7). Let  $U_f$  and  $U_g$  be two  $\mathbb{F}_q$ -vector subspaces of  $\mathbb{V} = \mathbb{F}_{q^{2n}} \times \mathbb{F}_{q^{2n}}$  of form (7), where  $f(x) = \alpha x^{q^s} + \beta x^{q^{s+n}}$  and  $g(x) = \alpha' x^{q^s} + \beta' x^{q^{s+n}}$ , with  $1 \leq s \leq 2n-1$  and  $\gcd(s, n) = 1$ . Since we are interested in the study of scattered linear sets of  $\text{PG}(1, q^{2n})$  not of pseudoregulus type, we can assume  $\alpha\beta \neq 0$  (cf. [26, Sec. 4]). If  $N_{q^{2n}/q^n}(\alpha\beta') = N_{q^{2n}/q^n}(\alpha'\beta)$  then there exists  $a \in \mathbb{F}_{q^{2n}}^*$  such that  $\beta\alpha' = \beta'\alpha a^{q^s(q^n-1)}$  and direct computations show that  $U_f^\varphi = U_g$ , where

$$\varphi: (x, y) \in \mathbb{V} \mapsto (xa, ya^{q^s} \alpha' / \alpha) \in \mathbb{V}.$$

From the previous arguments it follows that  $L_f$  is defined, up to the action of the group  $\text{GL}(2, q^n)$ , by an  $\mathbb{F}_q$ -vector subspace of  $\mathbb{V}$  of type

$$U_{b,s} := \{(x, bx^{q^s} + x^{q^{s+n}}) : x \in \mathbb{F}_{q^{2n}}\}, \quad (8)$$

with  $b \in \mathbb{F}_{q^{2n}}^*$  and  $1 \leq s \leq 2n-1$  such that  $N_{q^{2n}/q^n}(b) \neq 1$  and  $\gcd(s, n) = 1$ . We will denote by  $L_{b,s}$  the corresponding  $\mathbb{F}_q$ -linear set  $L_{U_{b,s}}$ .

Also we can restrict our study to the choice of the integers  $s$  such that  $1 \leq s \leq n$  and  $\gcd(s, n) = 1$ . Indeed, by using the notation of Section 3, we have

$$U_{b,s}^{\perp'} = \{(x, b^{q^{2n-s}} x^{q^{2n-s}} + x^{q^{n-s}}) : x \in \mathbb{F}_{q^{2n}}\} = U_{b^{q^{2n-s}}, 2n-s}$$

and it can be easily seen that  $U_{b,s}$  and  $U_{b,s}^{\perp'}$  are equivalent via the linear invertible map  $\phi: (x, y) \in \mathbb{V} \mapsto (\alpha y, \beta x) \in \mathbb{V}$ , where  $\alpha$  is any element satisfying  $\alpha^{q^n-1} = -\frac{1}{b^{q^n-1}}$  and  $\beta = (b^{2q^n} \alpha^{q^n} + \alpha)^{q^{n-s}}$ .

Moreover we have the following result.

**Proposition 5.1.** *Two  $\mathbb{F}_q$ -subspaces  $U_{b,s}$  and  $U_{\bar{b},\bar{s}}$  of  $\mathbb{V} = \mathbb{F}_{q^{2n}} \times \mathbb{F}_{q^{2n}}$  of form (8) with  $b, \bar{b} \in \mathbb{F}_{q^{2n}}^*$ ,  $N_{q^{2n}/q^n}(b) \neq 1$ ,  $N_{q^{2n}/q^n}(\bar{b}) \neq 1$ ,  $1 \leq s, \bar{s} < n$  and  $\gcd(n, s) = \gcd(n, \bar{s}) = 1$ , are  $\text{GL}(2, q^{2n})$ -equivalent if and only if either*

$$s = \bar{s} \quad \text{and} \quad N_{q^{2n}/q^n}(\bar{b}) = N_{q^{2n}/q^n}(b)^\sigma$$

or

$$s + \bar{s} = n \quad \text{and} \quad N_{q^{2n}/q^n}(\bar{b}) N_{q^{2n}/q^n}(b)^\sigma = 1,$$

for some automorphism  $\sigma \in \text{Aut}(\mathbb{F}_{q^n})$ .

*Proof.*  $U_{b,s}$  and  $U_{\bar{b},\bar{s}}$  are  $\Gamma\text{L}(2, q^{2n})$ -equivalent if and only if there exist elements  $\alpha, \beta, \gamma, \delta \in \mathbb{F}_{q^{2n}}$ , with  $\alpha\delta \neq \beta\gamma$  and an automorphism  $\sigma \in \text{Aut}(\mathbb{F}_{q^{2n}})$  such that

$$\forall x \in \mathbb{F}_{q^{2n}}, \exists y \in \mathbb{F}_{q^{2n}} : \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} x^\sigma \\ (bx^{q^s} + x^{q^{s+n}})^\sigma \end{pmatrix} = \begin{pmatrix} y \\ \bar{b}y^{q^{\bar{s}}} + y^{q^{\bar{s}+n}} \end{pmatrix}.$$

Put  $z := x^\sigma$ , the last equation implies that for each  $z \in \mathbb{F}_{q^{2n}}$ , there exists  $y \in \mathbb{F}_{q^{2n}}$  such that

$$\begin{cases} \alpha z + \beta(b^\sigma z^{q^s} + z^{q^{n+s}}) = y, \\ \gamma z + \delta(b^\sigma z^{q^s} + z^{q^{n+s}}) = \bar{b}y^{q^{\bar{s}}} + y^{q^{n+\bar{s}}}. \end{cases} \quad (9)$$

Putting the first in the second equation of System (9), we get that

$$\gamma z + \delta(b^\sigma z^{q^s} + z^{q^{n+s}}) = \bar{b}(\alpha z + \beta(b^\sigma z^{q^s} + z^{q^{n+s}}))^{q^{\bar{s}}} + (\alpha z + \beta(b^\sigma z^{q^s} + z^{q^{n+s}}))^{q^{n+\bar{s}}} \quad (10)$$

for each  $z \in \mathbb{F}_{q^{2n}}$ .

If  $s = \bar{s}$ , since the monomials  $z, z^{q^s}, z^{q^{2s}}, z^{q^{n+s}}, z^{q^{n+2s}}$  are pairwise distinct modulo  $z^{q^{2n}} - z$ , from the previous polynomial identity we get

$$\begin{cases} \gamma = 0 \\ \delta b^\sigma = \bar{b}\alpha^{q^s} \\ \delta = \alpha^{q^{n+s}} \\ \bar{b}\beta^{q^s} b^\sigma q^s + \beta^{q^{n+s}} = 0 \\ \bar{b}\beta^{q^s} + \beta^{q^{n+s}} b^\sigma q^{n+s} = 0. \end{cases} \quad (11)$$

Since  $N_{q^{2n}/q^n}(b) \neq 1$ , System (11) is equivalent to

$$\begin{cases} \gamma = 0 \\ \beta = 0 \\ \delta b^\sigma = \bar{b}\alpha^{q^s} \\ \delta = \alpha^{q^{n+s}}, \end{cases}$$

which admits solutions if and only if  $N_{q^{2n}/q^n}(\bar{b}) = N_{q^{2n}/q^n}(b)^\sigma$ , with  $\sigma \in \text{Aut}(\mathbb{F}_{q^n})$ .

If  $s \neq \bar{s}$ , since  $1 \leq s, \bar{s} < n$  and  $\gcd(s, n) = \gcd(\bar{s}, n) = 1$ , we get

$$\{z^{q^s}, z^{q^{\bar{s}}}\} \cap \{z, z^{q^{n+s}}, z^{q^{n+\bar{s}}}, z^{q^{s+\bar{s}}}, z^{q^{n+s+\bar{s}}}\} = \emptyset$$

modulo  $z^{q^{2n}} - z$ . Hence polynomial identity (10) yields  $\alpha = \delta = 0$  and Equation (10) becomes

$$\gamma z = (\bar{b}\beta^{q^{\bar{s}}} b^\sigma q^{\bar{s}} + \beta^{q^{n+\bar{s}}}) z^{q^{s+\bar{s}}} + (\bar{b}\beta^{q^s} + \beta^{q^{n+s}} b^\sigma q^{n+s}) z^{q^{n+s+\bar{s}}}$$



for each  $z \in \mathbb{F}_{q^{2n}}$ . Also, since  $s + \bar{s} < 2n$ , the monomials  $z$  and  $z^{q^{s+\bar{s}}}$  are different modulo  $z^{q^{2n}} - z$ . Hence, if  $s + \bar{s} \neq n$  we immediately get  $\gamma = 0$ , a contradiction. It follows that  $s + \bar{s} = n$  and comparing the coefficients of the terms of degree 1 and  $q^{s+\bar{s}}$  we get

$$\begin{cases} \gamma = \bar{b}\beta^{q^{\bar{s}}} + \beta^{q^{n+\bar{s}}}b^{\sigma q^{n+\bar{s}}} \\ \bar{b}\beta^{q^{\bar{s}}}b^{\sigma q^{\bar{s}}} + \beta^{q^{n+\bar{s}}} = 0, \end{cases}$$

which admits solutions if and only if  $N_{q^{2n}/q^n}(\bar{b}b^{\sigma q^{\bar{s}}}) = 1$ , i.e. if and only if  $N_{q^{2n}/q^n}(\bar{b})N_{q^{2n}/q^n}(b^{q^{\bar{s}}})^\sigma = 1$ , for some automorphism  $\sigma \in \text{Aut}(\mathbb{F}_{q^n})$ .  $\square$

We finish this section by determining the linear automorphism group of  $U_{b,s}$  and with some results on the geometric structure of a linear set  $L_{b,s}$ .

**Corollary 5.2.** *The  $\mathbb{F}_{q^{2n}}$ -linear automorphism group  $\mathcal{G}_{b,s}$  of an  $\mathbb{F}_q$ -vector subspace  $U_{b,s}$  of  $\mathbb{V} = \mathbb{F}_{q^{2n}} \times \mathbb{F}_{q^{2n}}$  of form (8) consists of the following matrices*

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{q^s} \end{pmatrix},$$

with  $\alpha \in \mathbb{F}_{q^n}^*$ .

*Proof.* In the previous theorem choosing  $s = \bar{s}$  and  $b = \bar{b}$ , by System (11) we get  $\beta = \gamma = 0$  and  $\delta = \alpha^{q^s} = \alpha^{q^{n+s}}$ . The assertion follows.  $\square$

The previous corollary allows us to prove the following result.

**Proposition 5.3.** *Let  $L_{b,s}$  be the  $\mathbb{F}_q$ -linear set of  $\text{PG}(1, q^{2n})$  of rank  $2n$  defined by an  $\mathbb{F}_q$ -vector subspace  $U_{b,s}$  of type (8) and let  $P\mathcal{G}_{b,s}$  be the projectivity group induced on the line  $\text{PG}(1, q^{2n})$  by  $\mathcal{G}_{b,s}$ . Then the following properties hold:*

- i) *the linear collineation group  $P\mathcal{G}_{b,s}$  preserves  $L_{b,s}$ , it has order  $\frac{q^n-1}{q-1}$ , fixes the two points  $\langle(1,0)\rangle_{\mathbb{F}_{q^{2n}}}$  and  $\langle(0,1)\rangle_{\mathbb{F}_{q^{2n}}}$  and any other point-orbit has size  $\frac{q^n-1}{q-1}$ ;*
- ii)  *$L_{b,s}$  is a union of orbits of points under the  $P\mathcal{G}_{b,s}$ -action;*
- iii) *all points of  $L_{b,s}$  belonging to the same  $P\mathcal{G}_{b,s}$ -orbit have the same weight w.r.t.  $L_{b,s}$ .*

*Proof.* Let  $\phi_\lambda$  be the linear collineation of  $P\mathcal{G}_{b,s}$  induced by the element  $\varphi_\lambda := \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{q^s} \end{pmatrix} \in \mathcal{G}_{b,s}$ , with  $\lambda \in \mathbb{F}_{q^n}^*$ . Since  $\text{Fix}(\sigma) \cap \mathbb{F}_{q^n}^* = \mathbb{F}_q$ , the group  $P\mathcal{G}_{b,s}$  has order  $\frac{q^n-1}{q-1}$ . Also, it can be easily seen that if  $P$  is a point of  $\text{PG}(1, q^{2n})$  different from  $\langle(1, 0)\rangle_{\mathbb{F}_{q^{2n}}}$  and  $\langle(0, 1)\rangle_{\mathbb{F}_{q^{2n}}}$ , then  $P^{\phi_\lambda} = P$  if and only if  $\phi_\lambda$  is the identity map. Hence Statements *i)* and *ii)* follow.

Let now  $P = \langle(x_0, f(x_0))\rangle_{\mathbb{F}_{q^{2n}}}$  be a point of  $L_{b,s}$ , i.e.  $f(x_0) = bx_0^{q^s} + x_0^{q^{n+s}}$ . Then  $P^{\phi_\lambda} = \langle(\lambda x_0, f(\lambda x_0))\rangle_{\mathbb{F}_{q^{2n}}}$  and

$$\begin{aligned} w_{L_{b,s}}(P) &= \dim_q(\langle(x_0, f(x_0))\rangle_{\mathbb{F}_{q^{2n}}} \cap U_{b,s}) = \dim_q \varphi_\lambda(\langle(x_0, f(x_0))\rangle_{\mathbb{F}_{q^{2n}}} \cap U_{b,s}) \\ &= \dim_q(\langle(\lambda x_0, f(\lambda x_0))\rangle_{\mathbb{F}_{q^{2n}}} \cap \varphi_\lambda(U_{b,s})) \\ &= \dim_q(\langle(\lambda x_0, f(\lambda x_0))\rangle_{\mathbb{F}_{q^{2n}}} \cap U_{b,s}) = w_{L_{b,s}}(P^{\phi_\lambda}), \end{aligned}$$

and Property *iii)* is proved.  $\square$

From the previous proposition we get the following result.

**Corollary 5.4.** *Let  $L_{b,s}$  be the  $\mathbb{F}_q$ -linear set of  $\text{PG}(1, q^{2n})$  of rank  $2n$  defined by an  $\mathbb{F}_q$ -vector subspace  $U_{b,s}$  of type (8). The size of  $L_{b,s}$  is a multiple of  $\frac{q^n-1}{q-1}$ . Furthermore, the set of points of weight 2 w.r.t.  $L_{b,s}$  is a union of orbits under the action of the linear collineation group  $P\mathcal{G}_{b,s}$ .  $\square$*

## 6 Scattered $\mathbb{F}_q$ -subspaces of type $U_{b,s}$ and the corresponding MRD-codes

The set of  $m \times n$  matrices  $\mathbb{F}_q^{m \times n}$  over  $\mathbb{F}_q$  is a rank metric  $\mathbb{F}_q$ -space with rank metric distance defined by  $d(A, B) = rk(A - B)$  for  $A, B \in \mathbb{F}_q^{m \times n}$ . A subset  $\mathcal{C} \subseteq \mathbb{F}_q^{m \times n}$  is called a rank distance code (RD-code for short). The minimum distance of  $\mathcal{C}$  is

$$d(\mathcal{C}) = \min_{A, B \in \mathcal{C}, A \neq B} \{d(A, B)\}.$$

In [11] the Singleton bound for an  $m \times n$  rank metric code  $\mathcal{C}$  with minimum rank distance  $d$  was proved:

$$\#\mathcal{C} \leq q^{\max\{m, n\}(\min\{m, n\} - d + 1)}. \quad (12)$$

If this bound is achieved, then  $\mathcal{C}$  is an MRD-code. MRD-codes have various applications in communications and cryptography; see for instance [12, 17]. More properties of MRD-codes can be found in [11, 12, 13, 33].

When  $\mathcal{C}$  is an  $\mathbb{F}_q$ -linear subspace of  $\mathbb{F}_q^{m \times n}$ , we say that  $\mathcal{C}$  is an  $\mathbb{F}_q$ -linear code and the dimension  $\dim_q(\mathcal{C})$  is defined to be the dimension of  $\mathcal{C}$  as a subspace over  $\mathbb{F}_q$ . If  $d$  is the minimum distance of  $\mathcal{C}$  we say that  $\mathcal{C}$  has parameters  $(m, n, q; d)$ .

The *middle nucleus* of a code  $\mathcal{C} \subseteq \mathbb{F}_q^{m \times n}$  (cf. [29], or [30] where the term *left idealiser* was used), is defined as

$$\mathcal{N}(\mathcal{C}) := \{Z \in \mathbb{F}_q^{m \times m} : ZC \in \mathcal{C} \text{ for all } C \in \mathcal{C}\},$$

and by [29, Theorem 5.4] it turns out to be a field of order at least  $q$ .

We will use the following equivalence definition for codes of  $\mathbb{F}_q^{m \times m}$ . If  $\mathcal{C}$  and  $\mathcal{C}'$  are two codes then they are equivalent if and only if there exist two invertible matrices  $A, B \in \mathbb{F}_q^{m \times m}$  and a field automorphism  $\sigma$  such that  $\{AC^\sigma B : C \in \mathcal{C}\} = \mathcal{C}'$ , or  $\{AC^{T\sigma} B : C \in \mathcal{C}\} = \mathcal{C}'$ , where  $T$  denotes transposition. The code  $\mathcal{C}^T$  is also called the *adjoint* of  $\mathcal{C}$ .

In [35, Section 5] Sheekey showed that scattered  $\mathbb{F}_q$ -linear sets of  $\text{PG}(1, q^m)$  of rank  $m$  yield  $\mathbb{F}_q$ -linear MRD-codes with parameters  $(m, m, q; m-1)$ . We briefly recall here the construction from [35]. Let  $U_f$  be a maximum scattered  $\mathbb{F}_q$ -vector subspace, defined as above. Then, after fixing an  $\mathbb{F}_q$ -bases for  $\mathbb{F}_{q^m}$ , the set of  $\mathbb{F}_q$ -linear maps of  $\mathbb{F}_{q^m}$

$$\mathcal{C}_f := \{x \mapsto af(x) + bx : a, b \in \mathbb{F}_{q^m}\} \quad (13)$$

corresponds to  $m \times m$  matrices over  $\mathbb{F}_q$  forming an  $\mathbb{F}_q$ -linear MRD-code with parameters  $(m, m, q; m-1)$ . Also, since  $\mathcal{C}_f$  is an  $\mathbb{F}_{q^m}$ -subspace of  $\text{End}(\mathbb{F}_{q^m}, \mathbb{F}_q)$ , its middle nucleus  $\mathcal{N}(\mathcal{C}_f)$  contains the set of scalar maps  $\mathcal{F}_m := \{x \in \mathbb{F}_{q^m} \mapsto \alpha x \in \mathbb{F}_{q^m} : \alpha \in \mathbb{F}_{q^m}\}$ , i.e.  $|\mathcal{N}(\mathcal{C}_f)| \geq q^m$ .

On the other hand  $\mathcal{N}(\mathcal{C}_f)$  is an  $\mathbb{F}_q$ -subspace of invertible maps together with the zero map (cf. [29, Corollary 5.6]), it is also an MRD-code with parameters  $(m, m, q; m)$ . Then (12) gives  $|\mathcal{N}(\mathcal{C}_f)| \leq q^m$ , thus  $\mathcal{N}(\mathcal{C}_f) = \mathcal{F}_m$ .

Regarding the converse we can state the following.

**Proposition 6.1.** *If  $\mathcal{C}$  is an MRD-code with parameters  $(m, m, q; m-1)$  and with middle nucleus isomorphic to  $\mathbb{F}_{q^m}$ , then  $\mathcal{C}$  is equivalent to some code  $\mathcal{C}_f$  (cf. (13)).*

*Proof.* By using a ring isomorphism between  $\mathbb{F}_q^{m \times m}$  and  $\text{End}(\mathbb{F}_{q^m}, \mathbb{F}_q)$ , we may suppose that  $\mathcal{C} \subset \text{End}(\mathbb{F}_{q^m}, \mathbb{F}_q)$ . Since  $\mathcal{N}(\mathcal{C}) \setminus \{\mathbf{0}\}$  and  $\mathcal{F}_m \setminus \{\mathbf{0}\}$  are two Singer cyclic subgroups of  $\text{GL}(\mathbb{F}_{q^m}, \mathbb{F}_q)$ , there exists  $H \in \text{GL}(\mathbb{F}_{q^m}, \mathbb{F}_q)$  such that

$$H^{-1} \circ \mathcal{N}(\mathcal{C}) \circ H = \mathcal{F}_m,$$

see for example [15, pg. 187]. With  $\mathcal{C}' := H^{-1} \circ \mathcal{C}$  we can see that  $\mathcal{N}(\mathcal{C}') = \mathcal{F}_m$ . It means that  $\mathcal{C}'$  is a 2-dimensional vector space over  $\mathcal{F}_m$  and hence it can be written as

$$\mathcal{C}' = \{\alpha r(x) + \beta s(x) : \alpha, \beta \in \mathbb{F}_{q^m}\},$$

for some  $q$ -polynomials  $r(x), s(x)$  over  $\mathbb{F}_{q^m}$ . Since each MRD-code with parameters  $(m, m, q; m-1)$  contains invertible elements (cf. [29, Lemma 2.1]), we may take  $h(x) \in \mathcal{C}'$  invertible. Then  $h^{-1} \circ \mathcal{C}'$  has the desired form, i.e.  $h^{-1} \circ \mathcal{C}' = \mathcal{C}_f$  for some  $q$ -polynomial  $f(x)$  over  $\mathbb{F}_{q^m}$ .  $\square$

**Proposition 6.2.** *The known  $\mathbb{F}_q$ -linear MRD-codes with parameters  $(m, m, q; m-1)$  and with middle nucleus isomorphic to  $\mathbb{F}_{q^m}$ , up to equivalence, arise from one of the following maximum scattered subspaces of  $\mathbb{F}_{q^m} \times \mathbb{F}_{q^m}$ :*

1.  $U_1 = \{(x, x^{q^s}) : x \in \mathbb{F}_{q^m}\}, 1 \leq s \leq m-1, \gcd(s, m) = 1.$
2.  $U_2 = \{(x, \delta x^{q^s} + x^{q^{m-s}}) : x \in \mathbb{F}_{q^m}\}, N_{q^m/q}(\delta) \neq 1, \gcd(s, m) = 1.$

*Proof.* The known  $\mathbb{F}_q$ -linear MRD-codes with parameters  $(m, m, q; m-1)$ , written as  $\mathbb{F}_q$ -linear maps over  $\mathbb{F}_{q^m}$ , are of the form

$$\mathcal{H}_{2,s}(\mu, h) := \{x \mapsto a_0 x + a_1 x^{q^s} + \mu a_0^{q^h} x^{q^{2s}} : a_0, a_1 \in \mathbb{F}_{q^m}\},$$

with  $\gcd(s, m) = 1$  and  $N_{q^{sm}/q^s}(\mu) \neq 1$ .

By [29, Corollary 5.9] the middle nuclei of the codes  $\mathcal{H}_{2,s}(\mu, h)$  are isomorphic to  $\mathbb{F}_{q^m}$  if and only if  $\mu = 0$  or  $m \mid 2s - h$ . In the former case we obtain generalized Gabidulin codes arising from maximum scattered linear sets of pseudoregulus type, i.e. from maximum scattered subspaces of  $\mathbb{F}_{q^m} \times \mathbb{F}_{q^m}$  of type  $U_1$ . If  $m \mid 2s - h$ , by [28, Proposition 4.3] the adjoint code of  $\mathcal{H}_{2,s}(\mu, h)$  is equivalent to  $\mathcal{H}_{2,s}(1/\mu, 2s-h) = \mathcal{H}_{2,s}(1/\mu, 0)$  and direct computations show that such a code is equivalent to a code arising from a maximum scattered subspace of type  $U_2$ . The assertion follows from the fact that the families of MRD-codes arising from maximum scattered subspaces of type  $U_1$  and  $U_2$ , respectively, are both closed under the adjoint operation (following the terminology of [35, 16, 25], the adjoint code of  $\mathcal{C}_f$  is  $\mathcal{C}_{\hat{f}}$ ).  $\square$

Put  $m = 2n, n > 1$  in the previous proposition. Note that if  $n = 2$  then a scattered  $\mathbb{F}_q$ -vector subspace  $U_{b,s}$  (which means  $N_{q^4/q}(b) \neq 1$ , cf. [10]) is of type either  $U_2$  or  $U_2^{\perp'}$ . Now, we are able to prove that MRD-codes arising from scattered subspaces of form (8) with  $n > 2$  are new.

By using the same arguments as in Corollary 5.2, the linear automorphism group  $\mathcal{G}_i$  of  $U_i$ ,  $i \in \{1, 2\}$ , is

$$\mathcal{G}_1 = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{q^s} \end{pmatrix} : a \in \mathbb{F}_{q^{2n}}^* \right\}, \quad \mathcal{G}_2 = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{q^s} \end{pmatrix} : a \in \mathbb{F}_{q^2}^* \right\}.$$

This allows us to prove the following:

**Theorem 6.3.** *If  $n > 2$ , the  $\mathbb{F}_q$ -vector subspace of  $\mathbb{F}_{q^{2n}} \times \mathbb{F}_{q^{2n}}$*

$$U_{b,s} = \{(x, bx^{q^s} + x^{q^{s+n}}) : x \in \mathbb{F}_{q^{2n}}\},$$

*with  $b \in \mathbb{F}_{q^{2n}}^*$  and  $1 \leq s \leq n-1$  such that  $N_{q^{2n}/q^n}(b) \neq 1$  and  $\gcd(s, n) = 1$ , is not equivalent to any subspace  $U_i$ ,  $i \in \{1, 2\}$ , under the action of the group  $\Gamma\text{L}(2, q^{2n})$ .*

*Proof.* If there exists an element  $\varphi \in \Gamma\text{L}(2, q^{2n})$  such that  $U_{b,s}^\varphi = U_i$ , for some  $i \in \{1, 2\}$ , then the corresponding linear automorphism groups will be isomorphic via the map

$$\omega \in \mathcal{G}_{b,s} \mapsto \varphi \circ \omega \circ \varphi^{-1} \in \mathcal{G}_i,$$

but this is a contradiction by comparing the sizes of the related groups (cf. Corollary 5.2).  $\square$

Let  $\mathcal{C}_f$  and  $\mathcal{C}_g$  be two MRD-codes arising from maximum scattered subspaces  $U_f$  and  $U_g$  of  $\mathbb{F}_{q^m} \times \mathbb{F}_{q^m}$ . In [35, Theorem 8] the author showed that there exist invertible matrices  $A, B$  such that  $AC_fB = \mathcal{C}_g$  if and only if  $U_f$  and  $U_g$  are  $\Gamma\text{L}(2, q^m)$ -equivalent. Hence, by Theorem 6.3, we get the following result.

**Theorem 6.4.** *If  $n > 2$ , the linear MRD-code of dimension  $4n$  and minimum distance  $2n - 1$  arising from a scattered  $\mathbb{F}_q$ -vector subspace  $U_{b,s} = \{(x, bx^{q^s} + x^{q^{s+n}}) : x \in \mathbb{F}_{q^{2n}}\}$  of  $\mathbb{F}_{q^{2n}} \times \mathbb{F}_{q^{2n}}$  is not equivalent to any previously known MRD-code with the same parameters.*  $\square$

In the next section we will show that when  $n = 3$  and  $q > 2$  and when  $n = 4$  and  $q$  is odd there exist values of  $b$  and  $s$  for which the  $\mathbb{F}_q$ -subspace  $U_{b,s}$  of  $\mathbb{F}_{q^{2n}} \times \mathbb{F}_{q^{2n}}$  is scattered, and from the above arguments the corresponding MRD-codes are new.

## 7 New maximum scattered subspaces

### 7.1 The $n = 3$ case

We want to show that there exists  $b \in \mathbb{F}_{q^6}^*$  such that

$$U_{b,1} := \{(x, bx^q + x^{q^4}) : x \in \mathbb{F}_{q^6}\}$$

is a maximum scattered  $\mathbb{F}_q$ -subspace.

$U_{b,1}$  is scattered if and only if for each  $m \in \mathbb{F}_{q^6}$

$$\frac{bx^q + x^{q^4}}{x} = -m$$

has at most  $q$  solutions. Those  $m$  which admit exactly  $q$  solutions correspond to points  $\langle(1, -m)\rangle_{\mathbb{F}_{q^6}}$  of  $LU_{b,1}$  with weight one. It follows that  $U_{b,1}$  is scattered if and only if for each  $m \in \mathbb{F}_{q^6}$  the kernel of

$$r_{m,b}(x) := mx + bx^q + x^{q^4}$$

has dimension less than two, or, equivalently, the Dickson matrix

$$D_{m,b} := \begin{pmatrix} m & b & 0 & 0 & 1 & 0 \\ 0 & m^q & b^q & 0 & 0 & 1 \\ 1 & 0 & m^{q^2} & b^{q^2} & 0 & 0 \\ 0 & 1 & 0 & m^{q^3} & b^{q^3} & 0 \\ 0 & 0 & 1 & 0 & m^{q^4} & b^{q^4} \\ b^{q^5} & 0 & 0 & 1 & 0 & m^{q^5} \end{pmatrix}$$

associated to  $r_{m,b}(x)$  has rank at least five (cf. [36, Proposition 4.4]). Equivalently,  $D_{m,b}$  has a non-zero  $5 \times 5$  minor. We will denote by  $M_{i,j}$  the determinant of the matrix obtained from  $D_{m,b}$  by removing the  $i$ -th row and the  $j$ -th column. We will use the following:

$$M_{6,1} = b^{q^2} - b^{1+q^2+q^3} - b^{q+q^2+q^4} + b^{1+q+q^2+q^3+q^4} - b^{q^4} m^{q+q^2+q^3} - b m^{q^2+q^3+q^4}, \quad (14)$$

$$M_{6,5} = -b^{q^2} m + b^{q+q^2+q^4} m - b m^{q^3} + b^{1+q+q^4} m^{q^3} + b^{q^4} m^{1+q+q^2+q^3}. \quad (15)$$

We will show that for certain choices of  $b$  and  $q$  there is no  $m \in \mathbb{F}_{q^6}$  such that both of the above expressions are zero.

**Theorem 7.1.** *For  $q > 4$  we can always find  $b \in \mathbb{F}_{q^2}^*$ , such that  $U_{b,1}$  is a maximum scattered  $\mathbb{F}_q$ -subspace of  $\mathbb{F}_{q^6} \times \mathbb{F}_{q^6}$ .*

*Proof.* We want to find  $b \in \mathbb{F}_{q^2}^*$  such that at least one of (14) and (15) is non-zero. Suppose the contrary, i.e. for each  $b \in \mathbb{F}_{q^2}$ :

$$0 = b(1 - 2b^{q+1} + b^{2q+2} - m^{q+q^2+q^3} - m^{q^2+q^3+q^4}), \quad (16)$$

$$0 = b(-m + b^{q+1}m - m^{q^3} + b^{q+1}m^{q^3} + m^{1+q+q^2+q^3}). \quad (17)$$

Put  $x = m^{1+q+q^2}$  and  $z = 1 - b^{q+1}$ . Obviously  $z \neq 1$  and dividing (16) by  $b$  gives

$$z^2 = x^q + x^{q^2}, \quad (18)$$

multiplying (17) by  $m^{q^4+q^5}/b$  gives

$$z(x^{q^3} + x^{q^4}) = x^{q^3+1}. \quad (19)$$

Since  $b \in \mathbb{F}_{q^2}$ , it follows that  $b^{q+1} \in \mathbb{F}_q$  and hence  $z \in \mathbb{F}_q$ . Then (18) yields  $x^q + x^{q^2} \in \mathbb{F}_q$  and hence  $x \in \mathbb{F}_{q^2}$ . Then (18) and (19) give:

$$z^2 = x + x^q, \quad (20)$$

$$z^3 = x^{q+1}. \quad (21)$$

Thus  $x$  and  $x^q$  are roots of the equation

$$X^2 - z^2X + z^3 = 0. \quad (22)$$

From now on we distinguish two cases according to the parity of  $q$ . First suppose  $q$  odd. If (22) can be solved in  $\mathbb{F}_q$ , then  $x = x^q \in \mathbb{F}_q$  and hence (20) and (21) give  $z = x = 0$ , or  $z = 4$ ,  $x = 8$ . If we can find  $z \in \mathbb{F}_q \setminus \{0, 1, 4\}$  such that (22) has roots in  $\mathbb{F}_q$ , then we obtain a contradiction meaning that the two minors in consideration cannot vanish at the same time. Then  $U_{b,1}$  is scattered for each  $b \in \mathbb{F}_{q^2}$  which satisfies  $1 - b^{q+1} = z$ . Equation (22) has roots in  $\mathbb{F}_q$  if and only if  $z^4 - 4z^3$  is a square, hence, when  $z^2 - 4z$  is a square. Note that  $z = 2$  gives  $z^2 - 4z = -4$ , which is always a square when  $q \equiv 1 \pmod{4}$ . So from now on, we may assume  $q \equiv 3 \pmod{4}$  and hence  $q \geq 7$ . Consider the conic  $\mathcal{C}$  of  $\text{PG}(2, q)$  with equation  $X_0^2 - 4X_0X_2 - X_1^2 = 0$ . It is easy to see that  $\mathcal{C}$  is always non-singular, and that the line with equation  $X_0 = 0$  is a tangent to  $\mathcal{C}$ . For  $q \geq 7$   $\mathcal{C}$  has more than 7 points and hence we can find a point of  $\mathcal{C}$  not on the lines  $X_0 = 0$ ,  $X_0 - 4X_2 = 0$ ,  $X_0 - X_2 = 0$  and  $X_2 = 0$ . It means that we can always find a point  $\langle (x_0, x_1, 1) \rangle_{\mathbb{F}_q} \in \text{PG}(2, q)$  such that  $x_0^2 - 4x_0 = x_1^2$  and  $x_0 \in \mathbb{F}_q \setminus \{0, 1, 4\}$ . It follows that we can always find  $z$ , and hence  $b$ , with the given conditions.

Now consider the case when  $q$  is even. For  $z \neq 0$  (22) has a solution in  $\mathbb{F}_q$  if and only if the  $S$ -invariant of the equation, that is  $\text{Tr}_{q/2}(1/z)$ , equals to zero. If there is a solution in  $\mathbb{F}_q$ , then (20) and (21) give  $z = 0$ , so it is enough to prove that there exists  $z \in \mathbb{F}_q \setminus \{0, 1\}$ , such that  $\text{Tr}_{q/2}(1/z) = 0$ . The existence of such  $z$  gives a contradiction meaning that the two minors in consideration cannot vanish at the same time. The equation  $\text{Tr}_{q/2}(x) = 0$  has  $q/2$  pairwise distinct roots in  $\mathbb{F}_q$ , thus  $\text{Tr}_{q/2}(1/z) = 0$  has  $q/2 - 1$  non-zero solutions. It follows that for  $q \geq 8$  we can find such  $z$ .  $\square$

## 7.2 The $n = 4$ case

We will show that there exists  $b \in \mathbb{F}_{q^8}^*$  such that

$$U_{b,1} := \{(x, bx^q + x^{q^5}) : x \in \mathbb{F}_{q^8}\}$$

is a maximum scattered  $\mathbb{F}_q$ -subspace for each odd  $q$ .

$U_{b,1}$  is scattered if and only if for each  $m \in \mathbb{F}_{q^8}$

$$\frac{bx^q + x^{q^5}}{x} = -m$$

has at most  $q$  solutions. Those  $m$  which admit exactly  $q$  solutions correspond to points  $\langle (1, -m) \rangle_{\mathbb{F}_{q^8}}$  of  $L_{U_{b,1}}$  with weight one. It follows that  $U_{b,1}$  is scattered if and only if for each  $m \in \mathbb{F}_{q^8}$  the kernel of

$$r_{m,b}(x) := mx + bx^q + x^{q^5}$$

has dimension less than two, or, equivalently, the Dickson matrix

$$D_{m,b} := \begin{pmatrix} m & b & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & m^q & b^q & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & m^{q^2} & b^{q^2} & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & m^{q^3} & b^{q^3} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & m^{q^4} & b^{q^4} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & m^{q^5} & b^{q^5} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & m^{q^6} & b^{q^6} \\ b^{q^7} & 0 & 0 & 0 & 1 & 0 & 0 & m^{q^7} \end{pmatrix}$$

of  $r_{m,b}(x)$  has a non-zero  $7 \times 7$  minor. If we remove the first two columns and last two rows of the above matrix, then the remaining  $6 \times 6$  submatrix  $M$  has determinant  $(b^{q+q^5} - 1)m^{q^3+q^4}$ . It follows that with  $N_{q^8/q^4}(b) \neq 1$  the only



point of  $L_{U_{b,s}}$  with weight larger than 2 is  $\langle(1,0)\rangle_{\mathbb{F}_{q^8}}$ . On the other hand, it is easy to see that  $\langle(1,0)\rangle_{\mathbb{F}_{q^8}}$  is a point of  $L_{U_{b,s}}$  if and only if  $N_{q^8/q^4}(b) = 1$ .

We will denote by  $M_{i,j}$  the determinant of the matrix obtained from  $D_{m,b}$  by cancelling the  $i$ -row and the  $j$ -th column. We will use the following:

$$M_{8,2} = (b^{1+q^4} - 1)^{q+q^2} (b^{q^3+q^4} m + m^{q^4}) + m^{1+q^3+q^4+q^5} (b^{q^6} m^{q^2} + b^q m^{q^6}). \quad (23)$$

**Theorem 7.2.** *For odd  $q$  and  $b^2 = -1$  the  $\mathbb{F}_q$ -subspace  $U_{b,1}$  is maximum scattered in  $\mathbb{F}_{q^8} \times \mathbb{F}_{q^8}$ .*

*Proof.* We will show that there is no  $m \in \mathbb{F}_{q^8}^*$  such that (23) vanishes. Applying  $b^2 = -1$ , the vanishing of (23) would give

$$0 = 4(b^{q+1} m + m^{q^4}) + m^{1+q^3+q^4+q^5} (bm^{q^2} + b^q m^{q^6}). \quad (24)$$

Now we distinguish two cases, according to  $b \in \mathbb{F}_q$  (i.e.,  $q \equiv 1 \pmod{4}$ ), or  $b \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$  (i.e.,  $q \equiv 3 \pmod{4}$ ). First suppose that the former case holds. Then

$$0 = 4(-m + m^{q^4}) + bm^{1+q^3+q^4+q^5} (m^{q^2} + m^{q^6}). \quad (25)$$

Considering the  $\mathbb{F}_{q^8} \rightarrow \mathbb{F}_{q^4}$  trace of both sides of (25) and using the  $\mathbb{F}_{q^4}$ -linearity of this function, it follows that  $\text{Tr}_{q^8/q^4}(m^{q^3+q^5}) = 0$ . It is easy to see that  $\text{Tr}_{q^8/q^4}(x) = \text{Tr}_{q^8/q^4}(y) = 0$  implies  $xy \in \mathbb{F}_{q^4}$  for any two  $x, y \in \mathbb{F}_{q^8}$ , thus  $m^{q^3+q^5} m^{q^2+q^4}$  and  $m^{q^3+q^5} m^{q^4+q^6}$  are in  $\mathbb{F}_{q^4}$ . It follows that  $bm^{1+q^3+q^4+q^5} (m^{q^2} + m^{q^6}) = m\lambda$  for some  $\lambda \in \mathbb{F}_{q^4}$  and hence (25) gives  $m^{q^4-1} \in \mathbb{F}_{q^4}$ . But also  $m^{q^4+1} \in \mathbb{F}_{q^4}$  and hence  $m^2 \in \mathbb{F}_{q^4}$  giving either  $m \in \mathbb{F}_{q^4}$ , or  $\text{Tr}_{q^8/q^4}(m) = 0$ , but (25) gives  $m = 0$  in both cases.

Now consider the  $b \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$  case. Then  $b^{q+1} = 1$  and  $b^q = -b$ , thus (24) gives

$$0 = 4(m + m^{q^4}) + bm^{1+q^3+q^4+q^5} (m^{q^2} - m^{q^6}). \quad (26)$$

Since  $4(m + m^{q^4}) \in \mathbb{F}_{q^4}$  and  $bm^{1+q^4} \in \mathbb{F}_{q^4}$ , it follows that  $m^{q^3+q^5} (m^{q^2} - m^{q^6}) \in \mathbb{F}_{q^4}$ . It is easy to see that  $\text{Tr}_{q^8/q^4}(x) = 0$  and  $xy \in \mathbb{F}_{q^4}$  implies  $\text{Tr}_{q^8/q^4}(y) = 0$  for any two  $x, y \in \mathbb{F}_{q^8}$ , thus  $\text{Tr}_{q^8/q^4}(m^{q^3+q^5}) = 0$ . Then, as in the previous case,  $m^2 \in \mathbb{F}_{q^4}$  follows, which gives a contradiction.  $\square$

**Remark 7.3.** *It follows from Theorem 6.3 that the maximum scattered subspaces of this section are new, i.e. they cannot be obtained from previously known maximum scattered subspaces under the action of  $\text{GL}(2, q^n)$ ,  $n = 6, 8$ .*

*The question of the equivalence of the corresponding linear sets under the action of the group  $\text{PGL}(2, q^n)$  is addressed in [7].*

**Remark 7.4.** Computations with GAP yield the following results.

With respect to the cases not covered by Theorem 7.1: there exist  $b \in \mathbb{F}_{q^6}^*$  such that the subspace  $\{(x, bx^q + x^{q^4}) : x \in \mathbb{F}_{q^6}\}$  is scattered in  $\mathbb{F}_{q^6} \times \mathbb{F}_{q^6}$  also for  $q \in \{3, 4\}$ , but not for  $q = 2$ .

With respect to Theorem 7.2: for  $q \leq 8$ ,  $q$  even, there is no  $b \in \mathbb{F}_{q^8}^*$  such that  $\{(x, bx^q + x^{q^5}) : x \in \mathbb{F}_{q^8}\}$  is scattered in  $\mathbb{F}_{q^8} \times \mathbb{F}_{q^8}$  and for  $q \leq 11$ ,  $q$  odd, the corresponding subspace is scattered if and only if  $b^{q^4+1} = -1$ . According to the first paragraph of Section 5, each of these subspaces is equivalent to the scattered subspace found in Theorem 7.2.

There is no  $b \in \mathbb{F}_{q^{2n}}^*$  such that  $\{(x, bx^{q^s} + x^{q^{n+s}}) : x \in \mathbb{F}_{q^{2n}}\}$ ,  $\gcd(s, n) = 1$ , is scattered in  $\mathbb{F}_{q^{2n}} \times \mathbb{F}_{q^{2n}}$  when  $q \leq 5$  and  $n \in \{5, 6, 7, 8\}$ , or  $q = 7$  and  $n \in \{5, 6, 7\}$ , or  $q = 7$  and  $n = 8$ , or  $q = 8$  and  $n = 5$ .

**Conjecture 7.5.** According to the first paragraph of Section 5,  $f_1(x) = b_1x^q + x^{q^4} \in \mathbb{F}_{q^6}[x]$  and  $f_2(x) = b_2x^q + x^{q^4} \in \mathbb{F}_{q^6}[x]$  define equivalent subspaces when  $N_{q^6/q^3}(b_1) = N_{q^6/q^3}(b_2)$ . We conjecture that the size of the set

$$\{N_{q^6/q^3}(b) : f(x) = bx^q + x^{q^4} \text{ defines a maximum scattered } \mathbb{F}_q\text{-space } U_{b,1}\}$$

is  $\lfloor (q^2 + q + 1)(q - 2)/2 \rfloor$ , and hence there might be further examples of maximum scattered subspaces in this family. By GAP we verified this conjecture for  $q \leq 32$ .

**Remark 7.6.** The maximum number of directions determined by an  $\mathbb{F}_q$ -linear function over  $\mathbb{F}_{q^n}$  is  $(q^n - 1)/(q - 1)$ . Also, the maximum size of an  $\mathbb{F}_q$ -linear blocking set of Rédei type of  $\text{PG}(2, q^n)$  is  $q^n + (q^n - 1)/(q - 1)$ . According to [5, Section 5.3] our new examples of maximum scattered spaces yield new examples of functions and of blocking sets which attain these bounds.

In [14, pg. 132] the maximal cardinality of the image set  $\text{Im}(L(x)/x)$  is considered (with  $x \mapsto 1/x$  defined to take 0 to 0), where  $L(x)$  is an  $\mathbb{F}_p$ -linear function over  $\mathbb{F}_q$ ,  $p$  is a prime and  $q$  is a power of  $p$ . If for some invertible  $p$ -polynomial  $f$ , the subspace  $U_f = \{(x, f(x)) : x \in \mathbb{F}_q\}$  is scattered, then the cardinality of  $\text{Im}(L(x)/x)$  reaches its maximum, which is  $1 + (q - 1)/(p - 1)$ . It follows that the maximum scattered subspaces constructed in this paper yield such functions.

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