Classes and equivalence of linear sets in $\mathrm{PG}(1,q^n)$

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Abstract

The equivalence problem of $\mathbb{F}_q$-linear sets of rank n of $\mathrm{PG}(1,q^n)$ is investigated, also in terms of the associated variety, projecting configurations, $\mathbb{F}_q$-linear blocking sets of Rédei type and MRD-codes.

1 Introduction

Linear sets are natural generalizations of subgeometries. Let $\Lambda = \mathrm{PG}(W,\mathbb{F}_{q^n}) = \mathrm{PG}(r-1,q^n)$, where W is a vector space of dimension r over $\mathbb{F}_{q^n}$. A point set L of Λ is said to be an $\mathbb{F}_q$-*linear set* of Λ of rank k if it is defined by the non-zero vectors of a k-dimensional $\mathbb{F}_q$-vector subspace U of W, i.e.

$$L = L_U = \{\langle \mathbf{u}\rangle_{\mathbb{F}_{q^n}} : \mathbf{u} \in U \setminus \{\mathbf{0}\}\}.$$

The maximum field of linearity of an $\mathbb{F}_q$-linear set L_U is $\mathbb{F}_{q^t}$ if t is the largest integer such that L_U is an $\mathbb{F}_{q^t}$-linear set. In the recent years, starting from the paper [18] by Lunardon, linear sets have been used to construct or characterize various objects in finite geometry, such as blocking sets and multiple blocking sets in finite projective spaces, two-intersection sets in finite projective spaces, translation spreads of the Cayley Generalized Hexagon, translation ovoids of polar spaces, semifield flocks and finite semifields. For a survey on linear sets we refer the reader to [24], see also [14].

One of the most natural questions about linear sets is their equivalence. Two linear sets L_U and L_V of $\mathrm{PG}(r-1,q^n)$ are said to be $\mathrm{P\Gamma L}$-*equivalent* (or simply *equivalent*) if there is an element φ in $\mathrm{P\Gamma L}(r,q^n)$ such that $L_U^{\varphi} = L_V$. In the applications it is crucial to have methods to decide whether two linear

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sets are equivalent or not. For $f \in \Gamma\mathrm{L}(r, q^n)$ we have $L_{U^f} = L_U^{\varphi_f}$, where φ_f denotes the collineation of $\mathrm{PG}(W, \mathbb{F}_{q^n})$ induced by f. It follows that if U and V are $\mathbb{F}_q$-subspaces of W belonging to the same orbit of $\Gamma\mathrm{L}(r, q^n)$, then L_U and L_V are equivalent. The above condition is only sufficient but not necessary to obtain equivalent linear sets. This follows also from the fact that $\mathbb{F}_q$-subspaces of W with different ranks can define the same linear set, for example $\mathbb{F}_q$-linear sets of $\mathrm{PG}(r-1, q^n)$ of rank $k \geq rn - n + 1$ are all the same: they coincide with $\mathrm{PG}(r-1, q^n)$. As it was showed recently in [6], if $r = 2$, then there exist $\mathbb{F}_q$-subspaces of W of the same rank n but on different orbits of $\Gamma\mathrm{L}(2, q^n)$ defining the same linear set of $\mathrm{PG}(1, q^n)$.

Suppose that $L_U^{\varphi_f} = L_V$ for some collineation, but there is no $\mathbb{F}_{q^n}$-semilinear map between U and V. Then the $\mathbb{F}_q$-subspaces U^f and V define the same linear set, but there is no invertible $\mathbb{F}_{q^n}$-semilinear map between them. This observation motivates the following definition. An $\mathbb{F}_q$-linear set L_U with maximum field of linearity $\mathbb{F}_q$ is called *simple* if for each $\mathbb{F}_q$-subspace V of W with $\dim_q(U) = \dim_q(V)$, $L_U = L_V$ only if U and V are in the same orbit of $\Gamma\mathrm{L}(W, \mathbb{F}_{q^n})$. Natural examples of simple linear sets are the subgeometries (cf. [17, Theorem 2.6] and [13, Section 25.5]). In [5] it was proved that $\mathbb{F}_q$-linear sets of rank $n+1$ of $\mathrm{PG}(2, q^n)$ admitting $(q+1)$-secants are simple. This allowed the authors to translate the question of equivalence to the study of the orbits of the stabilizer of a subgeometry on subspaces and hence to obtain the complete classification of $\mathbb{F}_q$-linear blocking sets in $\mathrm{PG}(2, q^4)$. Until now, the only known examples of non-simple linear sets are those of pseudoregulus type of $\mathrm{PG}(1, q^n)$ for $n \geq 5$ and $n \neq 6$, see [6].

In this paper we focus on linear sets of rank n of $\mathrm{PG}(1, q^n)$. Such linear sets are related to $\mathbb{F}_q$-linear blocking sets of Rédei type, MRD-codes of size q^{2n} with minimum rank distance $n-1$ and projections of subgeometries. We first introduce a method which can be used to find non-simple linear sets of rank n of $\mathrm{PG}(1, q^n)$. Let L_U be a linear set of rank n of $\mathrm{PG}(W, \mathbb{F}_{q^n}) = \mathrm{PG}(1, q^n)$ and let β be a non-degenerate alternating form of W. Denote by $\perp$ the orthogonal complement map induced by $\mathrm{Tr}_{q^n/q} \circ \beta$ on W (considered as an $\mathbb{F}_q$-vector space). Then U and $U^\perp$ defines the same linear set (cf. Result 2.1) and if U and $U^\perp$ lie on different orbits of $\Gamma\mathrm{L}(W, \mathbb{F}_{q^n})$, then L_U is non-simple. Using this approach we show that there are non-simple linear sets of rank n of $\mathrm{PG}(1, q^n)$ for $n \geq 5$, not of pseudoregulus type (cf. Proposition 3.9). Contrary to what we expected initially, simple linear sets are harder to find. We prove that the linear set of $\mathrm{PG}(1, q^n)$ defined by the trace function is simple (cf. Theorem 3.7). We also show that linear sets of rank n of $\mathrm{PG}(1, q^n)$ are simple for $n \leq 4$ (cf. Theorem 4.5).

Moreover, in $\mathrm{PG}(1,q^n)$ we extend the definition of simple linear sets and introduce the $\mathcal{Z}(\Gamma\mathrm{L})$-class and the $\Gamma\mathrm{L}$-class for linear sets of rank n. In Section 5 we point out the meaning of these classes in terms of equivalence of the associated blocking sets, MRD-codes and projecting configurations.

2 Definitions and preliminary results

2.1 Dual linear sets with respect to a symplectic polarity of a line

For $\alpha \in \mathbb{F}_{q^n}$ and a divisor h of n we will denote by $\mathrm{Tr}_{q^n/q^h}(\alpha)$ the trace of α over the subfield $\mathbb{F}_{q^h}$, that is, $\mathrm{Tr}_{q^n/q^h}(\alpha) = \alpha + \alpha^{q^h} + \ldots + \alpha^{q^{n-h}}$. By $\mathrm{N}_{q^n/q^h}(\alpha)$ we will denote the norm of α over the subfield $\mathbb{F}_{q^h}$, that is, $\mathrm{N}_{q^n/q^h}(\alpha) = \alpha^{1+q^h+\ldots+q^{n-h}}$. Since in the paper we will use only norms over $\mathbb{F}_q$, the function $\mathrm{N}_{q^n/q}$ will be denoted simply by N.

Starting from a linear set L_U and using a polarity τ of the space it is always possible to construct another linear set, which is called *dual linear set of L_U with respect to the polarity τ* (see [24]). In particular, let L_U be an $\mathbb{F}_q$–linear set of rank n of a line $\mathrm{PG}(W,\mathbb{F}_{q^n})$ and let $\beta : W \times W \longrightarrow \mathbb{F}_{q^n}$ be a non-degenerate reflexive $\mathbb{F}_{q^n}$–sesquilinear form on the 2-dimensional vector space W over $\mathbb{F}_{q^n}$ determining a polarity τ. The map $\mathrm{Tr}_{q^n/q} \circ \beta$ is a non-degenerate reflexive $\mathbb{F}_q$–sesquilinear form on W, when W is regarded as a $2n$-dimensional vector space over $\mathbb{F}_q$. Let $\perp_\beta$ and $\perp'_\beta$ be the orthogonal complement maps defined by β and $\mathrm{Tr}_{q^n/q} \circ \beta$ on the lattices of the $\mathbb{F}_{q^n}$-subspaces and $\mathbb{F}_q$-subspaces of W, respectively. The dual linear set of L_U with respect to the polarity τ is the $\mathbb{F}_q$–linear set of rank n of $\mathrm{PG}(W,\mathbb{F}_{q^n})$ defined by the orthogonal complement $U^{\perp'_\beta}$ and it will be denoted by L_U^τ. Also, up to projectively equivalence, such a linear set does not depend on τ.

For a point $P = \langle \mathbf{z} \rangle_{\mathbb{F}_{q^n}} \in \mathrm{PG}(W,\mathbb{F}_{q^n})$ the *weight* of P with respect to the linear set L_U is $w_{L_U}(P) := \dim_q(\langle \mathbf{z} \rangle_{\mathbb{F}_{q^n}} \cap U)$. Note that when $P \in L_U$, then the weight depends on the subspace U and not only on the set of points of L_U. It can happen that for two $\mathbb{F}_q$-subspaces U and V of W we have $L_U = L_V$ with $w_{L_U}(P) \neq w_{L_V}(P)$. When we write "the weight of $P \in L_U$", then we always mean $w_{L_U}(P)$ and hence when we speak about the weight of a point, we will never omit the subscript.

Result 2.1. From [24, Property 2.6] (with $r = 2$, $s = 1$ and $t = n$) it can be easily seen that if L_U is an $\mathbb{F}_q$–linear set of rank n of a line $\mathrm{PG}(W,\mathbb{F}_{q^n})$ and L_U^τ is its dual linear set with respect to a polarity τ, then $w_{L_U^\tau}(P^\tau) = w_{L_U}(P)$

for each point $P \in \mathrm{PG}(W,\mathbb{F}_{q^n})$. If τ is a symplectic polarity of a line $PG(W,\mathbb{F}_{q^n})$, then $P^\tau = P$ and hence $L_U = L_U^\tau = L_{U^{\perp'_\beta}}$.

2.2 $\mathbb{F}_q$-linear sets of $\mathrm{PG}(1,q^n)$ of class r

In this paper we investigate the equivalence of $\mathbb{F}_q$-linear sets of rank n of the projective line $\mathrm{PG}(W,\mathbb{F}_{q^n}) = \mathrm{PG}(1,q^n)$. As we have seen in the introduction, two $\mathbb{F}_q$-linear sets L_U and L_V of rank n of $\mathrm{PG}(1,q^n)$ are equivalent if there is an element φ_f in $\mathrm{P\Gamma L}(2,q^n)$ such that $L_U^{\varphi_f} = L_{U^f} = L_V$, where $f \in \Gamma\mathrm{L}(W,\mathbb{F}_{q^n})$ is the semilinear map inducing φ_f. Hence the first step is to determine the $\mathbb{F}_q$-vector subspaces of W defining the same linear set. This motivates the definition of the $\mathcal{Z}(\Gamma\mathrm{L})$-class and $\Gamma\mathrm{L}$-class of a linear set L_U of $\mathrm{PG}(1,q^n)$ (cf. Definitions 2.3 and 2.4). The next proposition relies on the characterization of functions over $\mathbb{F}_q$ determining few directions. It states that the $\mathbb{F}_q$-rank of L_U of $\mathrm{PG}(1,q^n)$ is uniquely defined when the maximum field of linearity of L_U is $\mathbb{F}_q$. This will allow us to state our definitions and results without further conditions on the rank of the corresponding $\mathbb{F}_q$-subspaces.

Proposition 2.2. *Let L_U be an $\mathbb{F}_q$-linear set of* $\mathrm{PG}(W,\mathbb{F}_{q^n}) = \mathrm{PG}(1,q^n)$ *of rank n. The maximum field of linearity of L_U is $\mathbb{F}_{q^d}$, where*

$$d = \min\{w_{L_U}(P) \colon P \in L_U\}.$$

If the maximum field of linearity of L_U is $\mathbb{F}_q$, then the rank of L_U as an $\mathbb{F}_q$-linear set is uniquely defined, i.e. for each $\mathbb{F}_q$-subspace V of W if $L_U = L_V$, then $\dim_q(V) = n$.

Proof. First assume that $\langle(0,1)\rangle_{\mathbb{F}_{q^n}} \notin L_U$, i.e. $U = \{(x, f(x)) \colon x \in \mathbb{F}_{q^n}\}$ for some q-polynomial f over $\mathbb{F}_{q^n}$.

Consider the following map, $U \to \mathrm{PG}(2,q^n) \colon (x, f(x)) \mapsto \langle(x, f(x), 1)\rangle_{\mathbb{F}_{q^n}}$. We will call this q-set of $\mathrm{PG}(2,q^n)$ the graph of f and we will denote it by G_f. Let X_0, X_1, X_2 denote the coordinate functions in $\mathrm{PG}(2,q^n)$ and consider the line $X_2 = 0$ as the line at infinity, denoted by ℓ_∞. The points of ℓ_∞ are called directions, denoted by $(m) := \langle(1,m,0)\rangle_{\mathbb{F}_{q^n}}$ and by $(\infty) := \langle(0,1,0)\rangle_{\mathbb{F}_{q^n}}$. The set of directions determined by f is

$$D_f := \left\{ \left(\frac{f(x)-f(y)}{x-y} \right) : x, y \in \mathbb{F}_{q^n},\, x \neq y \right\} = \left\{ \left(\frac{f(z)}{z} \right) : z \in \mathbb{F}_{q^n}^* \right\}.$$

It follows that $\langle(x, f(x))\rangle_{q^n} \mapsto \langle(x, f(x), 0)\rangle_{\mathbb{F}_{q^n}}$ is a bijection between the point set of L_U and the set of directions determined by f. The point $P_m := \langle(1,m)\rangle_{\mathbb{F}_{q^n}}$ is mapped to the direction (m).

For each line ℓ through (m) if ℓ meets the graph of f, then it meets it in q^t points, where $t = w_{L_U}(P_m)$. Indeed, suppose that ℓ meets the graph of f in $\langle (x_0, f(x_0), 1)\rangle_{\mathbb{F}_{q^n}}$. To obtain the number of the other points of $\ell \cap G_f$ we have to count

$$\left|\left\{x \in \mathbb{F}_{q^n} \setminus \{x_0\} \colon \frac{f(x)-f(x_0)}{x-x_0} = m\right\}\right| = \left|\left\{z \in \mathbb{F}_{q^n}^* \colon \frac{f(z)}{z} = m\right\}\right|,$$

which is $q^t - 1$.

Let $d = \min\{w_{L_U}(P) \colon P \in L_U\}$. If $q = p^e$, p prime, then p^{de} is the largest p-power such that every line meets the graph of f in a multiple of p^{de} points. Then a result on the number of direction determined by functions over $\mathbb{F}_q$ due to Ball, Blokhuis, Brouwer, Storme and Szőnyi [2], and Ball [1] yields that either $d = n$ and $f(x) = \lambda x$ for some $\lambda \in \mathbb{F}_{q^n}$, or $\mathbb{F}_{q^d}$ is a subfield of $\mathbb{F}_{q^n}$ and

$$q^{n-d} + 1 \leq |D_f| \leq \frac{q^n - 1}{q^d - 1}. \tag{1}$$

Moreover, if $q^d > 2$, then f is $\mathbb{F}_{q^d}$-linear. In our case we already know that f is $\mathbb{F}_q$-linear, so even in the case $q^d = 2$ it follows that U is an $\mathbb{F}_{q^d}$-subspace of W and hence L_U is an $\mathbb{F}_{q^d}$-linear set. We show that $\mathbb{F}_{q^d}$ is the maximum field of linearity of L_U. Suppose, contrary to our claim, that L_U is $\mathbb{F}_{q^r}$-linear of rank z for some $r > d$. Then L_U is also $\mathbb{F}_q$-linear of rank rz. It follows that $rz \leq n$ since otherwise $L_U = \mathrm{PG}(1, q^n)$. Then for the size of L_U we get $|L_U| \leq (q^{rz} - 1)/(q^r - 1) \leq (q^n - 1)/(q^r - 1)$. To get a contradiction, we show that this is less than $q^{n-d} + 1$, which is the lower bound obtain for $|L_U|$ in (1). After rearranging we get

$$\frac{q^n - 1}{q^r - 1} < q^{n-d} + 1 \Leftrightarrow q^{n-d}(q^d + 1) < (q^{n-d} + 1)q^r.$$

The latter inequality always holds because of $r \geq d + 1$. This contradiction shows $r = n$.

Now suppose that $\mathbb{F}_q$ is the maximum field of linearity of L_U and let V be an r-dimensional $\mathbb{F}_q$-subspace of W such that $L_U = L_V$. We cannot have $r > n$ since $L_U \neq \mathrm{PG}(1, q^n)$. Suppose, contrary to our claim, that $r \leq n - 1$. Then $|L_U| \leq (q^{n-1} - 1)/(q - 1)$ contradicting (1) which gives $q^{n-1} + 1 \leq |L_U|$.

Now suppose that $\langle (0,1)\rangle_{\mathbb{F}_{q^n}} \in L_U$. After a suitable projectivity φ_f we have $\langle (0,1)\rangle_{\mathbb{F}_{q^n}} \notin L_{U^f}$. Of course the maximum field of linearity of L_U and L_{U^f} coincide and for each point P of L_U we have $w_{L_U}(P) = w_{L_{U^f}}(P^{\varphi_f})$. Hence the first part of the theorem follows. The second part also follows easily since $L_U = L_V$ with $\dim_q(U) \neq \dim_q(V)$ would yield $L_{U^f} = L_{V^f}$ with $\dim_q(U^f) \neq \dim_q(V^f)$, a contradiction. $\square$

Now we can give the following definitions of classes of an $\mathbb{F}_q$-linear set of a line.

Definition 2.3. *Let L_U be an $\mathbb{F}_q$-linear set of $\mathrm{PG}(W, \mathbb{F}_{q^n}) = \mathrm{PG}(1, q^n)$ of rank n with maximum field of linearity $\mathbb{F}_q$. We say that L_U is of $\mathcal{Z}(\Gamma\mathrm{L})$-class r if r is the largest integer such that there exist $\mathbb{F}_q$-subspaces $U_1, U_2, \ldots, U_r$ of W with $L_{U_i} = L_U$ for $i \in \{1, 2, \ldots, r\}$ and $U_i \neq \lambda U_j$ for each $\lambda \in \mathbb{F}^*_{q^n}$ and for each $i \neq j$, $i, j \in \{1, 2, \ldots, r\}$.*

Definition 2.4. *Let L_U be an $\mathbb{F}_q$-linear set of $\mathrm{PG}(W, \mathbb{F}_{q^n}) = \mathrm{PG}(1, q^n)$ of rank n with maximum field of linearity $\mathbb{F}_q$. We say that L_U is of $\Gamma\mathrm{L}$-class s if s is the largest integer such that there exist $\mathbb{F}_q$-subspaces $U_1, U_2, \ldots, U_s$ of W with $L_{U_i} = L_U$ for $i \in \{1, 2, \ldots, s\}$ and there is no $f \in \Gamma\mathrm{L}(2, q^n)$ such that $U_i = U_j^f$ for each $i \neq j$, $i, j \in \{1, 2, \ldots, s\}$.*

Simple linear sets (cf. Section 1) of $\mathrm{PG}(1, q^n)$ are exactly those of $\Gamma\mathrm{L}$-class one. The next propositions are easy to show.

Proposition 2.5. *Let L_U be an $\mathbb{F}_q$-linear set of $\mathrm{PG}(W, \mathbb{F}_{q^n}) = \mathrm{PG}(1, q^n)$ of rank n with maximum field of linearity $\mathbb{F}_q$ and let P be a point of $\mathrm{PG}(1, q^n)$. Then for each $f \in \Gamma\mathrm{L}(2, q^n)$ we have $w_{L_U}(P) = w_{L_{U^f}}(P^{\varphi_f})$.* $\square$

Proposition 2.6. *Let L_U be an $\mathbb{F}_q$-linear set of $\mathrm{PG}(W, \mathbb{F}_{q^n}) = \mathrm{PG}(1, q^n)$ of rank n with maximum field of linearity $\mathbb{F}_q$ and let φ be a collineation of $\mathrm{PG}(W, \mathbb{F}_{q^n})$. Then L_U and L_U^{φ} have the same $\mathcal{Z}(\Gamma\mathrm{L})$-class and $\Gamma\mathrm{L}$-class.* $\square$

Remark 2.7. *Let L_U be an $\mathbb{F}_q$-linear set of rank n of $\mathrm{PG}(1, q^n)$ with $\Gamma\mathrm{L}$-class s and let $U_1, U_2, \ldots, U_s$ be $\mathbb{F}_q$-subspaces belonging to different orbits of $\Gamma\mathrm{L}(2, q^n)$ and defining L_U. The $\mathrm{P}\Gamma\mathrm{L}(2, q^n)$-orbit of L_U is the set*

$$\bigcup_{i=1}^{s} \{L_{U_i^f} \colon f \in \Gamma\mathrm{L}(2, q^n)\}.$$

3 Examples of simple and non-simple linear sets of $\mathrm{PG}(1, q^n)$

Let $\mathbb{V} = \mathbb{F}_{q^n} \times \mathbb{F}_{q^n}$ and let L_U be an $\mathbb{F}_q$–linear set of rank n of $\mathrm{PG}(1, q^n) = \mathrm{PG}(\mathbb{V}, \mathbb{F}_{q^n})$. We can always assume (up to a projectivity) that L_U does not contain the point $\langle(0, 1)\rangle_{\mathbb{F}_{q^n}}$. Then $U = U_f = \{(x, f(x)) \colon x \in \mathbb{F}_{q^n}\}$, for some

q-polynomial $f(x)=\sum_{i=0}^{n-1} a_i x^{q^i}$ over $\mathbb{F}_{q^n}$. For the sake of simplicity we will write L_f instead of L_{U_f} to denote the linear set defined by U_f.

According to Result 2.1 and using the same notations as in Section 2.1 if L_U is an $\mathbb{F}_q$-linear set of rank n of $\mathrm{PG}(1,q^n)$ and τ is a symplectic polarity, then $U^{\perp'_\beta}$ defines the same linear set as U. Since in general $U^{\perp'_\beta}$ and U are not equivalent under the action of the group $\Gamma\mathrm{L}(2,q^n)$, simple linear sets of a line are harder to find.

Consider the non-degenerate symmetric bilinear form of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$ defined by the following rule

$$< x,y >:= \mathrm{Tr}_{q^n/q}(xy). \tag{2}$$

Then the *adjoint map* $\hat{f}$ of an $\mathbb{F}_q$-linear map $f(x)=\sum_{i=0}^{n-1} a_i x^{q^i}$ of $\mathbb{F}_{q^n}$ (with respect to the bilinear form $\langle,\rangle$) is

$$\hat{f}(x) := \sum_{i=0}^{n-1} a_i^{q^{n-i}} x^{q^{n-i}}. \tag{3}$$

Let $\eta : \mathbb{V}\times\mathbb{V} \longrightarrow \mathbb{F}_{q^n}$ be the non-degenerate alternating bilinear form of $\mathbb{V}$ defined by $\eta((x,y),(u,v)) = xv - yu$. Then η induces a symplectic polarity on the line $\mathrm{PG}(\mathbb{V},\mathbb{F}_{q^n})$ and

$$\eta'((x,y),(u,v)) = \mathrm{Tr}_{q^n/q}(\eta((x,y),(u,v))) \tag{4}$$

is a non-degenerate alternating bilinear form on $\mathbb{V}$, when $\mathbb{V}$ is regarded as a $2n$-dimensional vector space over $\mathbb{F}_q$. We will always denote in the paper by $\perp$ and $\perp'$ the orthogonal complement maps defined by η and η' on the lattices of the $\mathbb{F}_{q^n}$-subspaces and the $\mathbb{F}_q$-subspaces of $\mathbb{V}$, respectively. Direct calculation shows that

$$U_f^{\perp'} = U_{\hat{f}}. \tag{5}$$

Result 2.1 and (5) allow us to slightly reformulate [3, Lemma 2.6].

Lemma 3.1 ([3]). *Let $L_f = \{\langle (x,f(x))\rangle_{\mathbb{F}_{q^n}} : x \in \mathbb{F}_{q^n}^*\}$ be an $\mathbb{F}_q$–linear set of $\mathrm{PG}(1,q^n)$ of rank n, with $f(x)$ a q-polynomial over $\mathbb{F}_{q^n}$, and let $\hat{f}$ be the adjoint of f with respect to the bilinear form (2). Then for each point $P\in\mathrm{PG}(1,q^n)$ we have $w_{L_f}(P) = w_{L_{\hat{f}}}(P)$. In particular, $L_f = L_{\hat{f}}$ and the maps defined by $f(x)/x$ and $\hat{f}(x)/x$ have the same image.*

Lemma 3.2. *Let φ be an $\mathbb{F}_q$-linear map of $\mathbb{F}_{q^n}$ and for $\lambda\in\mathbb{F}_{q^n}^*$ let φ_λ denote the $\mathbb{F}_q$-linear map: $x\mapsto \varphi(\lambda x)/\lambda$. Then for each point $P\in\mathrm{PG}(1,q^n)$ we have $w_{L_\varphi}(P) = w_{L_{\varphi_\lambda}}(P)$. In particular, $L_\varphi = L_{\varphi_\lambda}$.*

Proof. The statements follow from $\lambda U_{\varphi_\lambda} = U_\varphi$. $\square$

Remark 3.3. *The results of Lemmas 3.1 and 3.2 can also be obtained via Dickson matrices. For a q-polynomial f let D_f denote the Dickson matrix associated with f. When $f(x) = \lambda x$ for some $\lambda \in \mathbb{F}_{q^n}$ we will simply write D_λ. We will denote the point $\langle(1,\lambda)\rangle_{q^n}$ by P_λ.*

Transposition preserves the rank of matrices and $D_f^T = D_{\hat{f}}$, $D_\lambda^T = D_\lambda$. It follows that

$$\dim_q \ker(D_f - D_\lambda) = \dim_q \ker(D_f - D_\lambda)^T = \dim_q \ker(D_{\hat{f}} - D_\lambda),$$

and hence for each $\lambda \in \mathbb{F}_{q^n}$ we have $w_{L_f}(P_\lambda) = w_{L_{\hat{f}}}(P_\lambda)$.

Let $f_\mu(x) = f(x\mu)/\mu$. It is easy to see that $D_{1/\mu} D_f D_\mu = D_{f_\mu}$ and

$$\dim_q \ker(D_f - D_\lambda) = \dim_q \ker D_{1/\mu}(D_f - D_\lambda)D_\mu = \dim_q \ker(D_{f_\mu} - D_\lambda),$$

and hence $w_{L_f}(P_\lambda) = w_{L_{f_\mu}}(P_\lambda)$ for each $\lambda \in \mathbb{F}_{q^n}$.

From the previous arguments it follows that linear sets L_f with $f(x) = \hat{f}(x)$ are good candidates for being simple. In the next section we show that the trace function, which has the previous property, defines a simple linear set. We are going to use the following lemmas which will also be useful later.

Lemma 3.4. *Let f and g be two linearized polynomials. If $L_f = L_g$, then for each positive integer d the following holds*

$$\sum_{x\in\mathbb{F}_{q^n}^*} \left(\frac{f(x)}{x}\right)^d = \sum_{x\in\mathbb{F}_{q^n}^*} \left(\frac{g(x)}{x}\right)^d.$$

Proof. If $L_f = L_g =: L$, then $\{f(x)/x \colon x \in \mathbb{F}_{q^n}^*\} = \{g(x)/x \colon x \in \mathbb{F}_{q^n}^*\} =: H$. For each $h \in H$ we have $|\{x \colon f(x)/x = h\}| = q^i - 1$, where i is the weight of the point $\langle(1,h)\rangle_{q^n} \in L$ w.r.t. U_f, and similarly $|\{x \colon g(x)/x = h\}| = q^j - 1$, where j is the weight of the point $\langle(1,h)\rangle_{q^n} \in L$ w.r.t. U_g. Because of the characteristic of $\mathbb{F}_{q^n}$, we obtain:

$$\sum_{x\in\mathbb{F}_{q^n}^*} \left(\frac{f(x)}{x}\right)^d = -\sum_{h\in H} h^d = \sum_{x\in F_{q^n}^*} \left(\frac{g(x)}{x}\right)^d.$$

$\square$

Lemma 3.5 (Folklore)**.** *For any prime power q and integer d we have $\sum_{x\in\mathbb{F}_q^*} x^d = -1$ if $q-1 \mid d$ and $\sum_{x\in\mathbb{F}_q^*} x^d = 0$ otherwise.*

Lemma 3.6. *Let $f(x)=\sum_{i=0}^{n-1}a_ix^{q^i}$ and $g(x)=\sum_{i=0}^{n-1}b_ix^{q^i}$ be two q-polynomials over $\mathbb{F}_{q^n}$, such that $L_f=L_g$. Then*

$$a_0=b_0, \tag{6}$$

and for $k=1,2,\dots,n-1$ it holds that

$$a_ka_{n-k}^{q^k}=b_kb_{n-k}^{q^k}, \tag{7}$$

for $k=2,3,\dots,n-1$ it holds that

$$a_1a_{k-1}^{q}a_{n-k}^{q^k}+a_ka_{n-1}^{q}a_{n-k+1}^{q^k}=b_1b_{k-1}^{q}b_{n-k}^{q^k}+b_kb_{n-1}^{q}b_{n-k+1}^{q^k}. \tag{8}$$

Proof. We are going to use Lemma 3.5 together with Lemma 3.4 with different choices of d.

With $d=1$ we have

$$\sum_{x\in\mathbb{F}_{q^n}^*}\sum_{i=0}^{n-1}a_ix^{q^i-1}=\sum_{x\in\mathbb{F}_{q^n}^*}\sum_{i=0}^{n-1}b_ix^{q^i-1},$$

and hence

$$\sum_{i=0}^{n-1}a_i\sum_{x\in\mathbb{F}_{q^n}^*}x^{q^i-1}=\sum_{i=0}^{n-1}b_i\sum_{x\in\mathbb{F}_{q^n}^*}x^{q^i-1}.$$

Since q^n-1 cannot divide q^i-1 with $i=1,2,\dots,n-1$, $a_0=b_0=:c$ follows. Let φ denotes the $\mathbb{F}_q$-linear map which fixes $(0,1)$ and maps $(1,0)$ to $(1,-c)$. Then $U_f^{\varphi}=U_{f'}$ and $U_g^{\varphi}=U_{g'}$ with $f'=\sum_{i=1}^{n-1}a_ix^{q^i}$, $g'=\sum_{i=1}^{n-1}b_ix^{q^i}$ and of course with $L_{f'}=L_{g'}$. It follows that we may assume $c=0$.

First we show that (7) holds. With $d=q^k+1$, $1\le k\le n-1$ we obtain

$$\sum_{1\le i,j\le n-1}a_ia_j^{q^k}\sum_{x\in\mathbb{F}_{q^n}^*}x^{q^i-1+q^{j+k}-q^k}=\sum_{1\le i,j\le n-1}b_ib_j^{q^k}\sum_{x\in\mathbb{F}_{q^n}^*}x^{q^i-1+q^{j+k}-q^k}.$$

$\sum_{x\in\mathbb{F}_{q^n}^*}x^{q^i-1+q^{j+k}-q^k}=-1$ if and only if $q^i+q^{j+k}\equiv q^k+1 \pmod{q^n-1}$, and zero otherwise. Suppose that the former case holds.

First consider $j+k\le n-1$. Then $q^i+q^{j+k}\le q^{n-1}+q^{n-1}<q^k+1+2(q^n-1)$ hence one of the following holds.

- If $q^i+q^{j+k}=q^k+1$, then the right hand side is not divisible by q, a contradiction.
- If $q^i+q^{j+k}=q^k+1+(q^n-1)=q^n+q^k$, then $j+k=n$, a contradiction.

Now consider the case $j+k \geq n$. Then $q^i + q^{j+k} \equiv q^i + q^{j+k-n} \equiv q^k + 1 \pmod{q^n - 1}$. Since $j+k \leq 2(n-1)$, we have $q^i + q^{j+k-n} \leq q^{n-1} + q^{n-2} < q^k + 1 + 2(q^n - 1)$, hence one of the following holds.

- If $q^i + q^{j+k-n} = q^k + 1$, then $j+k = n$ and $i = k$.
- If $q^i + q^{j+k-n} = q^k + 1 + (q^n - 1) = q^n + q^k$, then there is no solution since $j+k-n \notin \{k, n\}$.

Hence (7) follows. Now we show that (8) also holds. Note that in this case $n \geq 3$, otherwise there is no k with $2 \leq k \leq n-1$. With $d = q^k + q + 1$, we obtain

$$\sum_{1 \leq i,j,m \leq n-1} a_i a_j^q a_m^{q^k} \sum_{x \in \mathbb{F}_{q^n}^*} x^{q^i - 1 + q^{j+1} - q + q^{m+k} - q^k} =$$

$$\sum_{1 \leq i,j,m \leq n-1} b_i b_j^q b_m^{q^k} \sum_{x \in \mathbb{F}_{q^n}^*} x^{q^i - 1 + q^{j+1} - q + q^{m+k} - q^k}.$$

$\sum_{x \in \mathbb{F}_{q^n}^*} x^{q^i - 1 + q^{j+1} - q + q^{m+k} - q^k} = -1$ if and only if $q^i + q^{j+1} + q^{m+k} \equiv q^k + q + 1 \pmod{q^n - 1}$, and zero otherwise. Suppose that the former case holds.

First consider $m+k \leq n-1$. Then $q^i + q^{j+1} + q^{m+k} \leq q^{n-1} + q^n + q^{n-1} < q^k + q + 1 + 2(q^n - 1)$ hence one of the following holds.

- If $q^i + q^{j+1} + q^{m+k} = q^k + q + 1$, then the right hand side is not divisible by q, a contradiction.
- If $q^i + q^{j+1} + q^{m+k} = q^k + q + 1 + (q^n - 1) = q^n + q^k + q$, then $m+k = n$, $j+1 = k$ and $i = 1$, a contradiction.

Now consider the case $m+k \geq n$. Then $q^i + q^{j+1} + q^{m+k} \equiv q^i + q^{j+1} + q^{m+k-n} \equiv q^k + q + 1 \pmod{q^n - 1}$. We have $q^i + q^{j+1} + q^{m+k-n} \leq q^{n-1} + q^n + q^{n-2} < q^k + q + 1 + 2(q^n - 1)$ hence one of the following holds.

- If $q^i + q^{j+1} + q^{m+k-n} = q^k + q + 1$, then $j+1 = k$, $i = 1$ and $m+k = n$.
- If $q^i + q^{j+1} + q^{m+k-n} = q^k + q + 1 + (q^n - 1) = q^n + q^k + q$, then $j+1 = n$, $i = k$ and $m+k = n+1$.

This concludes the proof. □

3.1 Linear sets defined by the trace function

We show that there exist at least one simple $\mathbb{F}_q$-linear set in $\mathrm{PG}(1,q^n)$ for each q and n. Let $V=\{(x,\mathrm{Tr}_{q^n/q}(x))\colon x\in\mathbb{F}_{q^n}\}$. We show that $L_U=L_V$ occurs for an $\mathbb{F}_q$-subspace U of W if and only if $V=\lambda U$ for some $\lambda\in\mathbb{F}_{q^n}^*$, i.e. L_V is of $\mathcal{Z}(\Gamma\mathrm{L})$-class one. For the special case when L_U has a point of weight $n-1$ see also [7, Theorem 2.3].

Theorem 3.7. *The $\mathbb{F}_q$-subspace $U_f=\{(x,f(x))\colon x\in\mathbb{F}_{q^n}\}$ defines the same linear set of $\mathrm{PG}(1,q^n)$ as the $\mathbb{F}_q$-subspace $V=\{(x,\mathrm{Tr}_{q^n/q}(x))\colon x\in\mathbb{F}_{q^n}\}$ if and only if $\lambda U_f=V$ for some $\lambda\in\mathbb{F}_{q^n}^*$, i.e. L_V is simple.*

Proof. Let $f(x)=\sum_{i=0}^{n-1}a_ix^{q^i}$. We are going to use Lemma 3.6 with $g(x)=\mathrm{Tr}_{q^n/q}(x)$. The coefficients $b_0,b_1,\ldots,b_{n-1}$ of $g(x)$ are 1, hence $a_0=1$, and for $k=1,2,\ldots,n-1$

$$a_ka_{n-k}^{q^k}=1, \tag{9}$$

for $k=2,3,\ldots,n-1$

$$a_1a_{k-1}^{q}a_{n-k}^{q^k}+a_ka_{n-1}^{q}a_{n-k+1}^{q^k}=2. \tag{10}$$

Note that (9) implies $a_i\neq 0$ for $i=1,2,\ldots,n-1$. First we prove

$$a_i=a_1^{1+q+\ldots+q^{i-1}} \tag{11}$$

by induction on i for each $0<i<n$. The assertion holds for $i=1$. Suppose that it holds for some integer $i-1$ with $1<i<n$. We prove that it also holds for i. Then (10) with $k=i$ gives

$$a_1a_{i-1}^{q}a_{n-i}^{q^i}+a_ia_{n-1}^{q}a_{n-i+1}^{q^i}=2. \tag{12}$$

Also, (9) with $k=i$, $k=i-1$ and $k=1$, respectively, gives

$$a_{n-i}^{q^i}=1/a_i,$$

$$a_{n-i+1}^{q^i}=1/a_{i-1}^{q},$$

$$a_{n-1}^{q}=1/a_1.$$

Then (12) gives

$$a_1a_{i-1}^{q}/a_i+a_i/\left(a_1a_{i-1}^{q}\right)=2. \tag{13}$$

It follows that $a_1a_{i-1}^{q}/a_i=1$ and hence the induction hypothesis on a_{i-1} yields $a_i=a_1^{1+q+\ldots+q^{i-1}}$.

Finally we show $\mathrm{N}(a_1) = 1$. First consider n even. Then (9) with $k = n/2$ gives $a_{n/2}^{q^{n/2}+1} = 1$. Applying (11) yields $\mathrm{N}(a_1) = 1$. If n is odd, then (9) with $k = (n-1)/2$ gives $a_{(n-1)/2}a_{(n+1)/2}^{q^{(n-1)/2}} = 1$. Applying (11) yields $\mathrm{N}(a_1) = 1$. It follows that $a_1 = \lambda^{q-1}$ for some $\lambda \in \mathbb{F}_{q^n}^*$ and hence $f(x) = \sum_{i=0}^{n-1} \lambda^{q^i-1}x^{q^i}$. Then $\lambda U_f = \{(x, \mathrm{Tr}_{q^n/q}(x)) \colon x \in \mathbb{F}_{q^n}^*\}$. □

3.2 Non-simple linear sets

So far, the only known non-simple linear sets of $\mathrm{PG}(1, q^n)$ are those of pseudoregulus type when $n = 5$, or $n > 6$, see Remark 5.6. Now we want to show that $\mathbb{F}_q$-linear sets L_f of $\mathrm{PG}(1, q^n)$ introduced by Lunardon and Polverino, which are not of pseudoregulus type ([21, Theorems 2 and 3], are non-simple as well. Let start by proving the following preliminary result.

Proposition 3.8. *Let $f(x) = \sum_{i=0}^{n-1} a_i x^{q^i}$. There is an $\mathbb{F}_{q^n}$-semilinear map between U_f and $U_{\hat{f}}$ if and only if the following system of n equations has a solution $A, B, C, D \in \mathbb{F}_{q^n}$, $AD - BC \neq 0$, $\sigma = p^k$:*

$$C + Da_0^\sigma - a_0 A = \sum_{i=0}^{n-1} (Ba_i a_i^\sigma)^{q^{n-i}},$$

$$\dots$$

$$Da_m^\sigma - (a_{n-m}A)^{q^m} = \sum_{i=0}^{n-1} (Ba_i a_{i+m}^\sigma)^{q^{n-i}},$$

$$\dots$$

$$Da_{n-1}^\sigma - (a_1 A)^{q^{n-1}} = \sum_{i=0}^{n-1} (Ba_i a_{i+n-1}^\sigma)^{q^{n-i}},$$

where the indices are taken modulo n.

Proof. Because of cardinality reasons the condition $AD - BC \neq 0$ is necessary. Then

$$\{(x, \hat{f}(x)) \colon x \in \mathbb{F}_{q^n}\} = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x^\sigma \\ f(x)^\sigma \end{pmatrix} : x \in \mathbb{F}_{q^n} \right\}$$

holds if and only if

$$Cx^\sigma + D\sum_{j=0}^{n-1} a_j^\sigma x^{\sigma q^j} = \sum_{i=0}^{n-1} a_{n-i}^{q^i} \left(Ax^\sigma + B\sum_{j=0}^{n-1} a_j^\sigma x^{\sigma q^j} \right)^{q^i}$$

for each $x \in \mathbb{F}_{q^n}$. After reducing modulo $x^{q^n} - x$, this is a polynomial equation of degree q^{n-1} in the variable x^σ. It follows that it holds for each $x \in \mathbb{F}_{q^n}$ if and only if it is the zero polynomial. Comparing coefficients on both sides yields the assertion. □

We are able to prove the following.

Proposition 3.9. *Consider a polynomial of the form $f(x) = \delta x^q + x^{q^{n-1}}$, where $q > 4$ is a power of the prime p. If $n > 4$, then for each generator δ of the multiplicative group of $\mathbb{F}_{q^n}$ the linear set L_f is not simple.*

Proof. Lemma 3.1 yields $L_f = L_{\hat{f}}$ thus it is enough to show the existence of δ such that there is no $\mathbb{F}_{q^n}$-semilinear map between U_f and $U_{\hat{f}}$. In the equations of Proposition 3.8 we have $a_1 = \delta$, $a_{n-1} = 1$ and $a_0 = a_2 = \ldots = a_{n-2} = 0$, thus

$$C = (B\delta^{\sigma+1})^{q^{n-1}} + B^q,$$
$$D\delta^\sigma - A^q = 0,$$
$$0 = (B\delta)^{q^{n-1}},$$
$$D - (\delta A)^{q^{n-1}} = 0,$$

where $\sigma = p^k$ for some integer k. If there is a solution, then $B = C = 0$ and $(\delta A)^{q^{n-1}}\delta^\sigma = A^q$. Taking q-th powers on both sides yield

$$\delta^{\sigma q+1} = A^{q^2-1} \tag{14}$$

and hence

$$\delta^{\frac{(\sigma q+1)(q^n-1)}{q-1}} = 1. \tag{15}$$

For each σ let G_σ be the set of elements δ of $\mathbb{F}_{q^n}$ satisfying (15). For each σ, G_σ is a subgroup of the multiplicative group M of $\mathbb{F}_{q^n}$. We show that these are proper subgroups of M. We have $G_{p^k} = M$ if and only if $q^n - 1$ divides $\frac{(p^k q+1)(q^n-1)}{q-1}$, i.e. when $q - 1$ divides $p^k q + 1$. Since $\gcd(p^w + 1, p^v - 1)$ is always 1,2, or $p^{\gcd(w,v)} + 1$, it follows that for $q > 4$ we cannot have $q - 1$ as a divisor of $p^k q + 1$.

It follows that for any generator δ of M we have $\delta \notin \cup_j G_{p^j}$ and hence $\delta^{\sigma q+1} \neq A^{q^2-1}$ for each σ and for each A. □

Remark 3.10. *If $q = 4$, then (14) with $k = 2(n-1)+1$ asks for the solution of $\delta^3 = A^{15}$. When 5 does not divide $4^n - 1$, then $\{x^3 \colon x \in \mathbb{F}_{4^n}\} = \{x^{15} \colon x \in \mathbb{F}_{4^n}\}$ and hence for each δ there exists A such that $\delta^3 = A^{15}$.*

If $q=3$, then (14) *with $k=n-1$ asks for the solution of $\delta^2=A^8$. When 4 does not divide 3^n-1, then $\{x^2\colon x\in\mathbb{F}_{3^n}\}=\{x^8\colon x\in\mathbb{F}_{3^n}\}$ and hence for each δ there exists A such that $\delta^2=A^8$.*

If $q=2$, then (14) *with $k=0$ asks for the solution of $\delta^3=A^3$. This equation always has a solution.*

4 Linear sets of rank 4 of $\mathrm{PG}(1,q^4)$

$\mathbb{F}_q$-linear sets of rank two of $\mathrm{PG}(1,q^2)$ are the Baer sublines, which are equivalent. As we have mentioned in the introduction, subgeometries are simple linear sets, in fact they have $\mathcal{Z}(\Gamma\mathrm{L})$-class one (cf. [17, Theorem 2.6] and [13, Section 25.5]). There are two non-equivalent $\mathbb{F}_q$-linear sets of rank 3 of $\mathrm{PG}(1,q^3)$, the linear sets of size q^2+q+1 and those of size q^2+1. Linear sets in both families are equivalent, since the stabilizer of a q-order subgeometry Σ of $\Sigma^*=\mathrm{PG}(2,q^3)$ is transitive on the set of those points of $\Sigma^*\setminus\Sigma$ which are incident with a line of Σ and on the set of points of Σ^* not incident with any line of Σ (cf. Section 5.2 and [16]). In the first case we have the linear sets of pseudoregulus type with $\Gamma\mathrm{L}$-class 1 and $\mathcal{Z}(\Gamma\mathrm{L})$-class 2 (cf. Remark 5.6 and Example 5.1). In the second case we have the linear sets defined by $\mathrm{Tr}_{q^3/q}$ with $\Gamma\mathrm{L}$-class and $\mathcal{Z}(\Gamma\mathrm{L})$-class 1 (cf. Theorem 3.7, see also [11, Corollary 6]).

The main result of this section is that each $\mathbb{F}_q$-linear set of rank 4 of $\mathrm{PG}(1,q^4)$, with maximum field of linearity $\mathbb{F}_q$, is simple (cf. Theorem 4.5).

4.1 Subspaces defining the same linear set

Lemma 4.1. *Let $f(x)=\sum_{i=0}^3 a_i x^{q^i}$ and $g(x)=\sum_{i=0}^3 b_i x^{q^i}$ be two q-polynomials over $\mathbb{F}_{q^4}$, such that $L_f=L_g$. Then*

$$\mathrm{N}(a_1)+\mathrm{N}(a_2)+\mathrm{N}(a_3)+a_1^{1+q^2}a_3^{q+q^3}+a_1^{q+q^3}a_3^{1+q^2}+\mathrm{Tr}\,q^4/q\left(a_1a_2^{q+q^2}a_3^{q^3}\right)=$$

$$\mathrm{N}(b_1)+\mathrm{N}(b_2)+\mathrm{N}(b_3)+b_1^{1+q^2}b_3^{q+q^3}+b_1^{q+q^3}b_3^{1+q^2}+\mathrm{Tr}\,q^4/q\left(b_1b_2^{q+q^2}b_3^{q^3}\right).$$

Proof. We are going to follow the proof of Lemma 3.6. As in that proof, we may assume $a_0=b_0=0$. In Lemma 3.4 take $d=1+q+q^2+q^3$. We obtain

$$\sum_{1\le i,j,k,m\le 3} a_i a_j^q a_k^{q^2} a_m^{q^3} \sum_{x\in\mathbb{F}_{q^4}^*} x^{q^i-1+q^{j+1}-q+q^{k+2}-q^2+q^{m+3}-q^3}=$$

$$\sum_{1\le i,j,k,m\le 3} b_i b_j^q b_k^{q^2} b_m^{q^3} \sum_{x\in\mathbb{F}_{q^4}^*} x^{q^i-1+q^{j+1}-q+q^{k+2}-q^2+q^{m+3}-q^3}.$$

$\sum_{x\in\mathbb{F}^*_{q^4}} x^{q^i-1+q^{j+1}-q+q^{k+2}-q^2+q^{m+3}-q^3} = -1$ if and only if

$$q^i+q^{j+1}+q^{k+2}+q^{m+3} \equiv q^i+q^{j+1}+q^{k+2}+q^{m-1} \equiv 1+q+q^2+q^3 \pmod{q^4-1},$$

and zero otherwise. Suppose that the former case holds.

First consider $k=1$. Then $q^i+q^{j+1}+q^{k+2}+q^{m-1} \leq q^3+q^4+q^3+q^2 < 1+q+q^2+q^3+2(q^4-1)$ hence one of the following holds.

- If $q^i+q^{j+1}+q^{k+2}+q^{m-1} = 1+q+q^2+q^3$, then $m=i=j=k=1$.
- If $q^i+q^{j+1}+q^{k+2}+q^{m-1} = 1+q+q^2+q^3+q^4-1 = q+q^2+q^3+q^4$, then $\{i, j+1, k+2, m-1\} = \{1,2,3,4\}$, hence one of the following holds

$$i=1,\ j=3,\ k=1,\ m=3,$$

$$i=2,\ j=3,\ k=1,\ m=2.$$

Now consider the case $k\geq 2$. Then $q^i+q^{j+1}+q^{k+2}+q^{m-1} \equiv q^i+q^{j+1}+q^{k-2}+q^{m-1} \leq q^3+q^4+q+q^2 < 1+q+q^2+q^3+2(q^4-1)$ hence one of the following holds.

- If $q^i+q^{j+1}+q^{k-2}+q^{m-1} = 1+q+q^2+q^3$, then $\{i, j+1, k-2, m-1\} = \{0,1,2,3\}$, hence one of the following holds

$$i=1,\ j=2,\ k=2,\ m=3,$$

$$i=2,\ j=2,\ k=2,\ m=2,$$

$$i=2,\ j=2,\ k=3,\ m=1,$$

$$i=3,\ j=1,\ k=2,\ m=2,$$

$$i=3,\ j=1,\ k=3,\ m=1.$$

- If $q^i+q^{j+1}+q^{k-2}+q^{m-1} = 1+q+q^2+q^3+q^4-1 = q+q^2+q^3+q^4$, then $i=j=k=m=3$.

□

Proposition 4.2. *Let $f(x)$ and $g(x)$ be two q-polynomials over $\mathbb{F}_{q^4}$ such that $L_f = L_g$. If the maximum field of linearity of f is $\mathbb{F}_q$, then*

$$g(x) = f(\lambda x)/\lambda,$$

or

$$g(x) = \hat{f}(\lambda x)/\lambda.$$

Proof. By Proposition 2.2, the maximum field of linearity of g is also $\mathbb{F}_q$. First note that $L_g = L_f$ when g is as in the assertion (cf. Lemmas 3.1 and 3.2). Let $f(x) = \sum_{i=0}^{3} a_i x^{q^i}$ and $g(x) = \sum_{i=0}^{3} b_i x^{q^i}$.

First we are going to use Lemma 3.6. From (6) we have $a_0 = b_0$. From (7) with $n = 4$ and $k = 1, 2$ we have $a_1 a_3^q = b_1 b_3^q$ and $a_2^{1+q^2} = b_2^{1+q^2}$, respectively. From (8) with $n = 4$ and $k = 2$ we obtain

$$a_1^{q+1} a_2^{q^2} + a_2 a_3^{q+q^2} = b_1^{q+1} b_2^{q^2} + b_2 b_3^{q+q^2}. \tag{16}$$

Note that $a_1 a_3^q = b_1 b_3^q$ implies

$$\mathrm{N}(b_1)\,\mathrm{N}(b_3) = \mathrm{N}(a_1)\,\mathrm{N}(a_3). \tag{17}$$

Multiplying (16) by b_2 and applying $a_2^{1+q^2} = b_2^{1+q^2}$ yields:

$$b_2^2 b_3^{q^2+q} - b_2(a_1^{q+1} a_2^{q^2} + a_2 a_3^{q^2+q}) + b_1^{q+1} a_2^{q^2+1} = 0. \tag{18}$$

First suppose $b_1 b_2 b_3 \neq 0$. Then (18) is a second degree polynomial in b_2. Applying $a_1 a_3^q = b_1 b_3^q$ it is easy to see that the roots of (18) are

$$b_{2,1} = \frac{a_1^{q+1} a_2^{q^2}}{b_3^{q^2+q}},$$

$$b_{2,2} = \frac{a_2 a_3^{q^2+q}}{b_3^{q^2+q}}.$$

First we consider $b_2 = b_{2,1}$. Then $a_2^{1+q^2} = b_2^{1+q^2}$ yields $\mathrm{N}(a_1) = \mathrm{N}(b_3)$ and hence $\mathrm{N}(b_1) = \mathrm{N}(a_3)$. In particular, $\mathrm{N}(b_1/a_3^q) = 1$ and hence $b_1 = a_3^q \lambda^{q-1}$ for some $\lambda \in \mathbb{F}_{q^4}^*$. From $a_1 a_3^q = b_1 b_3^q$ we obtain $b_3 = a_1^{q^3} a_3 / b_1^{q^3} = a_1^{q^3} \lambda^{q^3-1}$. Applying this we get $b_2 = a_1^{q+1} a_2^{q^2} / b_3^{q^2+q} = a_2^{q^2} \lambda^{q^2-1}$ and hence

$$g(x) = a_0 x + a_3^q \lambda^{q-1} x^q + a_2^{q^2} \lambda^{q^2-1} x^{q^2} + a_1^{q^3} \lambda^{q^3-1} x^{q^3}.$$

as we claimed.

Now consider $b_2 = b_{2,2}$. Then $a_2^{1+q^2} = b_2^{1+q^2}$ yields $\mathrm{N}(a_3) = \mathrm{N}(b_3)$ and hence $\mathrm{N}(a_1) = \mathrm{N}(b_1)$. Hence $b_1 = a_1 \lambda^{q-1}$ for some $\lambda \in \mathbb{F}_{q^4}^*$. From $a_1 a_3^q = b_1 b_3^q$ we obtain $b_3 = a_1^{q^3} a_3 / b_1^{q^3} = a_3 \lambda^{q^3-1}$. Applying this we obtain $b_2 = a_2 a_3^{q^2+q} / b_3^{q^2+q} = a_2 \lambda^{q^2-1}$ and hence

$$g(x) = a_0 x + a_1 \lambda^{q-1} x^q + a_2^{q^2} \lambda^{q^2-1} x^{q^2} + a_3^{q^3} \lambda^{q^3-1} x^{q^3}.$$

If $b_1 = b_3 = 0$, then either $b_2 = 0$ and the maximum field of linearity of $g(x)$ is $\mathbb{F}_{q^4}$, or $b_2 \neq 0$ and the maximum field of linearity of $g(x)$ is $\mathbb{F}_{q^2}$. Thus we may assume $b_1 \neq 0$ or $b_3 \neq 0$.

First assume $b_2 \neq 0$ and $b_1 = 0$. Then $b_3 \neq 0$ and (18) gives

$$b_2 b_3^{q^2+q} = a_1^{q+1} a_2^{q^2} + a_2 a_3^{q^2+q}.$$

Then $a_1 a_3^q = b_1 b_3^q$ yields either $a_1 = 0$ and $b_2 b_3^{q^2+q} = a_2 a_3^{q^2+q}$, or $a_3 = 0$ and $b_2 b_3^{q^2+q} = a_1^{q+1} a_2^{q^2}$. Taking (q^2+1)-powers on both sides gives $b_2^{q^2+1}\mathrm{N}(b_3) = a_2^{q^2+1}\mathrm{N}(a_3)$, or $b_2^{q^2+1}\mathrm{N}(b_3) = \mathrm{N}(a_1) a_2^{q^2+1}$, respectively. Applying $b_2^{q^2+1} = a_2^{q^2+1}$ we get $\mathrm{N}(b_3) = \mathrm{N}(a_3)$, or $\mathrm{N}(b_3) = \mathrm{N}(a_1)$, respectively. Note that the set of elements with norm 1 in $\mathbb{F}_{q^4}$ is $\{x^{q^3-1} \colon x \in \mathbb{F}_{q^4}^*\}$, thus in the first case there exists $\lambda \in \mathbb{F}_{q^4}^*$ such that $b_3 = a_3 \lambda^{q^3-1}$. Then $b_2 b_3^{q^2+q} = a_2 a_3^{q^2+q}$ yields $b_2 = a_2 \lambda^{q^2-1}$ and hence $g(x) = a_0 x + a_2 \lambda^{q^2-1} x^{q^2} + a_3 \lambda^{q^3-1} x^{q^3}$. In the second case the same reasoning yields $g(x) = a_0 x + a_2^{q^2} \lambda^{q^2-1} x^{q^2} + a_1^{q^3} \lambda^{q^3-1} x^{q^3}$.

If $b_2 \neq 0$ and $b_3 = 0$, then the coefficient of x^q in $\hat{g}(x)$ is zero and the assertion follows from the above arguments applied to $\hat{g}$ instead of g.

Now assume $b_2 = 0$ and $b_1 b_3 = 0$. Then $L_g = L_f$ is a linear set of pseudoregulus type and hence the assertion also follows from [15]. For the sake of completeness we present a proof also in this case. Equation $b_2^{q^2+1} = a_2^{q^2+1}$ yields $a_2 = 0$ and equation $a_1 a_3^q = b_1 b_3^q$ yields $a_1 a_3 = 0$. Then from Lemma 4.1 we have

$$\mathrm{N}(a_1) + \mathrm{N}(a_3) = \mathrm{N}(b_1) + \mathrm{N}(b_3). \tag{19}$$

If $b_1 = 0$, then $b_3 \neq 0$ and either $a_1 = 0$ and $\mathrm{N}(a_3) = \mathrm{N}(b_3)$, or $a_3 = 0$ and $\mathrm{N}(a_1) = \mathrm{N}(b_3)$. In the first case $g(x) = a_0 x + a_3 \lambda^{q^3-1} x^{q^3}$, in the second case $g(x) = a_0 x + a_1^q \lambda^{q^3-1} x^{q^3}$. If $b_3 = 0$, then $b_1 \neq 0$ and either $a_1 = 0$ and $\mathrm{N}(a_3) = \mathrm{N}(b_1)$, or $a_3 = 0$ and $\mathrm{N}(a_1) = \mathrm{N}(b_1)$. In the first case $g(x) = a_0 x + a_3^q \lambda^{q-1} x^q$, in the second case $g(x) = a_0 x + a_1 \lambda^{q-1} x^q$.

There is only one case left, when $b_2 = 0$ and $b_1 b_3 \neq 0$. Then from Lemma 4.1 and from $a_1 a_3^q = b_1 b_3^q$ it follows that

$$\mathrm{N}(a_1) + \mathrm{N}(a_3) = \mathrm{N}(b_1) + \mathrm{N}(b_3). \tag{20}$$

Together with (17) it follows that either $\mathrm{N}(a_1) = \mathrm{N}(b_1)$ and $\mathrm{N}(a_3) = \mathrm{N}(b_3)$, or $\mathrm{N}(a_1) = \mathrm{N}(b_3)$ and $\mathrm{N}(a_3) = \mathrm{N}(b_1)$. In the first case $g(x) = a_0 x + a_1 \lambda^{q-1} x^q + a_3 \lambda^{q^3-1} x^{q^3}$, in the second case $g(x) = a_0 x + a_3^q \lambda^{q-1} x^q + a_1^{q^3} \lambda^{q^3-1} x^{q^3}$, for some $\lambda \in \mathbb{F}_{q^4}^*$. $\square$

Now we are able to prove the following.

Theorem 4.3. *Let L_U be an $\mathbb{F}_q$–linear set of a line $\mathrm{PG}(W, \mathbb{F}_{q^4})$ of rank 4, with maximum field of linearity $\mathbb{F}_q$, and let β be a non–degenerate alternating form of W. If V is an $\mathbb{F}_q$–vector subspace of W such that $L_U = L_V$, then either*

$$V = \mu U,$$

or

$$V = \mu U^{\perp'_\beta},$$

for some $\mu \in \mathbb{F}^*_{q^4}$, where $\perp'_\beta$ is the orthogonal complement map induced by $\mathrm{Tr}_{q^4/q} \circ \beta$ on the lattice of the $\mathbb{F}_q$–subspaces of W.

Proof. First of all, observe that if β_1 is another non–degenerate alternating form of W and $\perp'_{\beta_1}$ is the corresponding orthogonal complement map induced on the lattice the $\mathbb{F}_q$-subspaces of W, direct computations show that there exists $a \in \mathbb{F}^*_{q^n}$ such that $\beta_1 = a\beta$ and for each $\mathbb{F}_q$–vector subspace S of W we get $S^{\perp'_\beta} = aS^{\perp'_{\beta_1}}$.

Let ϕ be the collineation of $\mathrm{PG}(W, \mathbb{F}_{q^4})$ such that L_U^ϕ does not contain the point $\langle(0, 1)\rangle_{\mathbb{F}_{q^4}}$. Then $L_{U^\varphi} = L_{V^\varphi}$, where φ is the invertible $\mathbb{F}_{q^4}$-semilinear map of W inducing ϕ, and σ is the associated field automorphism. Also, $U^\varphi = U_f$ and $V^\varphi = V_g$ for two q–polynomials f and g over $\mathbb{F}_{q^4}$. Since $L_f = L_g$, by Proposition 4.2 and by Lemma 3.2, taking also (5) into account, it follows that there exists $\lambda \in \mathbb{F}^*_{q^4}$ such that either $\lambda V_g = U_f$ or $\lambda V_g = U_{\hat{f}} = U_f^{\perp'}$, where $\perp'$ is the orthogonal complement map induced by the non-degenerate alternating form defined in (4). In the first case we have that $V = \mu U$, where $\mu = \frac{1}{\lambda^{\sigma^{-1}}}$. In the second case we have $V = \frac{1}{\lambda^{\sigma^{-1}}} U^{\varphi \perp' \varphi^{-1}}$. The map $\varphi \perp' \varphi^{-1}$ defines the orthogonal complement map on the lattice the $\mathbb{F}_q$-subspaces of W induced by another non–degenerate alternating form of W. As observed above, there exists $a \in \mathbb{F}^*_{q^4}$ such that $U^{\varphi \perp' \varphi^{-1}} = aU^{\perp'_\beta}$. The assertion follows with $\mu = \frac{a}{\lambda^{\sigma^{-1}}}$. □

4.2 Semilinear maps between U_f and $U_{\hat{f}}$

The next result is just Proposition 3.8 with $n = 4$.

Corollary 4.4. *Let $f(x) = a_0x + a_1x^q + a_2x^{q^2} + a_3x^{q^3}$. There is an $\mathbb{F}_{q^4}$-semilinear map between U_f and $U_{\hat{f}}$ if and only if the following system of four equations has a solution $A, B, C, D \in \mathbb{F}_{q^4}$, $AD - BC \neq 0$, $\sigma = p^k$.*

$$C + Da_0^\sigma - a_0A = Ba_0a_0^\sigma + (Ba_1a_1^\sigma)^{q^3} + (Ba_2a_2^\sigma)^{q^2} + (Ba_3a_3^\sigma)^q,$$

$$Da_1^\sigma - (a_3A)^q = Ba_0a_1^\sigma + (Ba_1a_2^\sigma)^{q^3} + (Ba_2a_3^\sigma)^{q^2} + (Ba_3a_0^\sigma)^q,$$
$$Da_2^\sigma - (a_2A)^{q^2} = Ba_0a_2^\sigma + (Ba_1a_3^\sigma)^{q^3} + (Ba_2a_0^\sigma)^{q^2} + (Ba_3a_1^\sigma)^q,$$
$$Da_3^\sigma - (a_1A)^{q^3} = Ba_0a_3^\sigma + (Ba_1a_0^\sigma)^{q^3} + (Ba_2a_1^\sigma)^{q^2} + (Ba_3a_2^\sigma)^q.$$

Theorem 4.5. *Linear sets of rank 4 of* $\mathrm{PG}(1, q^4)$*, with maximum field of linearity* $\mathbb{F}_q$*, are simple.*

Proof. Let $f = \sum_{i=0}^{3} a_i x^{q^i}$. After a suitable projectivity we may assume $a_0 = 0$. We will use Corollary 4.4 with $\sigma \in \{1, q^2\}$. We may assume that $a_1 = 0$ and $a_3 = 0$ do not hold at the same time since otherwise f is $\mathbb{F}_{q^2}$-linear.

First consider the case when $\mathrm{N}(a_1) = \mathrm{N}(a_3)$. Let $B = C = 0$, $D = A^{q^2}$ and take A such that $A^{q-1} = a_3/a_1^q$. This can be done since $\mathrm{N}(a_3/a_1^q) = 1$. Then Corollary 4.4 with $\sigma = q^2$ provides the existence of an $\mathbb{F}_{q^4}$-semilinear map between U_f and $U_{\hat{f}}$.

From now on we assume $\mathrm{N}(a_1) \neq \mathrm{N}(a_3)$.

If $a_2 = a_1 = 0$, then let $\sigma = 1$, $A = D = 0$, $B = 1$ and $C = a_3^{2q}$. If $a_2 = a_3 = 0$, then let $\sigma = 1$, $A = D = 0$, $B = 1$ and $C = a_1^{2q^3}$.

Now consider the case $a_2 = 0$ and $a_1a_3 \neq 0$. Let $A = D = 0$. Then the equations of Corollary 4.4 with $\sigma = 1$ yield

$$C = B^{q^3}a_1^{2q^3} + B^q a_3^{2q}, \tag{21}$$

$$0 = B^q a_1^q a_3^q + B^{q^3} a_1^{q^3} a_3^{q^3}. \tag{22}$$

(22) is equivalent to $0 = (Ba_1a_3)^{q^2} + Ba_1a_3$. Since $X^{q^2} + X = 0$ has q^2 solutions in $\mathbb{F}_{q^4}$, for any a_1 and a_3 we can find $B \in \mathbb{F}_{q^4}^*$ such that (22) is satisfied. If $B^{q^3}a_1^{2q^3} + B^q a_3^{2q} \neq 0$, then let C be this field element. We show that this is always the case. Suppose, contrary to our claim, that $B^{q^3-q} = -a_3^{2q}/a_1^{2q^3}$. Because of the choice of B (22) yields $B^{q^3-q} = -a_1^{q-q^3}a_3^{q-q^3}$. Since $B \neq 0$ this implies

$$-a_3^{2q}/a_1^{2q^3} = -a_1^{q-q^3}a_3^{q-q^3},$$

and hence $a_1^{q^2+1} = a_3^{q^2+1}$. A contradiction since $\mathrm{N}(a_1) \neq \mathrm{N}(a_3)$. From now on we assume $a_2 \neq 0$, we may also assume $a_2 = 1$ after a suitable projectivity.

Corollary 4.4 with $\sigma = 1$ yields

$$C = (Ba_1^2)^{q^3} + B^{q^2} + (Ba_3^2)^q, \tag{23}$$

$$Da_1 - (a_3A)^q = (Ba_1)^{q^3} + (Ba_3)^{q^2}, \tag{24}$$

$$D - A^{q^2} = (Ba_1a_3)^{q^3} + (Ba_3a_1)^q, \tag{25}$$

$$Da_3 - (a_1A)^{q^3} = (Ba_1)^{q^2} + (Ba_3)^q. \tag{26}$$

The right hand side of (24) is the q-th power of the right hand side of (26) and hence $D^q a_3^q - a_1 A = Da_1 - a_3^q A^q$, i.e.

$$a_3^q (D+A)^q = a_1(D+A).$$

Since a_1 or a_3 is non-zero, we have either $D = -A$, or $(D+A)^{q-1} = a_1/a_3^q$. The latter case can be excluded since in that case $\mathrm{N}(a_1) = \mathrm{N}(a_3)$. Let $D = -A$. Then the left hand side of (24) is $w(A) := -Aa_1 - a_3^q A^q$. The kernel of w is trivial and hence B uniquely determines A. The inverse of w is

$$w^{-1}(x) = \frac{-xa_1^{q+q^2+q^3} + x^q a_1^{q^2+q^3} a_3^q - x^{q^2} a_1^{q^3} a_3^{q+q^2} + x^{q^3} a_3^{q+q^2+q^3}}{\mathrm{N}(a_1) - \mathrm{N}(a_3)}.$$

Denote the right hand side of (24) by $r(B)$, the right hand side of (25) by $t(B)$. Then B has to be in the kernel of

$$K(x) := w^{-1}(r(x)) + (w^{-1}(r(x)))^{q^2} + t(x).$$

If $B = 0$, then $A = B = D = 0$ and hence this is not a suitable solution. It is easy to see that $Im\, t \subseteq \mathbb{F}_{q^2}$ and hence also $Im\, K \subseteq \mathbb{F}_{q^2}$, so the kernel of K has at least dimension 2.

Let $B \in \ker K$, $B \neq 0$, $A := w^{-1}(r(B))$ and $C := (Ba_1^2)^{q^3} + B^{q^2} + (Ba_3^2)^q$ (we recall $D = -A$). This gives a solution. We have to check that B can be chosen such that $AD - BC \neq 0$, i.e.

$$Q(B) := \left(w^{-1}(r(B))\right)^2 + B\left((Ba_1^2)^{q^3} + B^{q^2} + (Ba_3^2)^q\right),$$

is non-zero. We have $w^{-1}(r(x))(\mathrm{N}(a_1) - \mathrm{N}(a_3)) = \sum_{i=0}^{3} c_i x^{q^i}$, where

$$c_0 = a_1^{1+q^2+q^3} a_3^q - a_1^{q^3} a_3^{1+q+q^2},$$

$$c_1 = a_3^{2q+q^2+q^3} - a_1^{q+q^3} a_3^{q+q^2},$$

$$c_2 = a_3^{q+q^2+q^3} a_1^{q^2} - a_1^{q+q^2+q^3} a_3^{q^2},$$

$$c_3 = a_1^{q^2+q^3} a_3^{q+q^3} - a_1^{q+q^2+2q^3}.$$

If X_0, X_1, X_2, X_3 denote the coordinate functions in $\mathrm{PG}(3,q^4)$ and $Q(B)=0$ for some $B \in \mathbb{F}_{q^4}$, then the point $\langle (B, B^q, B^{q^2}, B^{q^3})\rangle_{q^4}$ is contained in the the quadric $\mathcal{Q}$ of $\mathrm{PG}(3,q^4)$ defined by the equation

$$\left(\sum_{i=0}^{3} c_i X_i\right)^2 + X_0(X_1 a_3^{2q} + X_2 + X_3 a_1^{2q^3})(\mathrm{N}(a_1) - \mathrm{N}(a_3))^2 = 0.$$

We can see that the equation of $\mathcal{Q}$ is the linear combination of the equations of two degenerate quadrics, a quadric of rank 1 and a quadric of rank 2. It follows that $\mathcal{Q}$ is always singular and it has rank 2 or 3. In particular, the rank of $\mathcal{Q}$ is 2 when the intersection of the planes $\mathcal{A} : X_0 = 0$ and $\mathcal{B} : X_1 a_3^{2q} + X_2 + X_3 a_1^{2q^3} = 0$ is contained in the plane $\mathcal{C} : \sum_{i=0}^{3} c_i X_0 = 0$. Straightforward calculations show that under our hypothesis ($a_1 \neq 0$ or $a_3 \neq 0$, $\mathrm{N}(a_1) \neq \mathrm{N}(a_3)$) this happens if only if $1 = a_1^q a_3$.
We recall that the kernel of K has dimension at least two. Let

$$H = \{\langle (x, x^q, x^{q^2}, x^{q^3})\rangle_{q^4} \colon K(x) = 0\}.$$

Our aim is to prove that H has points not belonging to the quadric $\mathcal{Q}$, i.e. $H \nsubseteq \mathcal{Q}$.

Note that $x \in \mathbb{F}_{q^4} \mapsto (x, x^q, x^{q^2}, x^{q^3}) \in \mathbb{F}_{q^4}^4$ is a vector-space isomorphism between $\mathbb{F}_{q^4}$ and the 4-dimensional $\mathbb{F}_q$-space $\{(x, x^q, x^{q^2}, x^{q^3}) \colon x \in \mathbb{F}_{q^4}\} \subset \mathbb{F}_{q^4}^4$. Denote by $\bar{H}$ the $\mathbb{F}_{q^4}$-extension of H, i.e. the projective subspace of $\mathrm{PG}(3,q^4)$ generated by the points of H. Then the projective dimension of $\bar{H}$ is $\dim\ker K - 1$. Let σ denotes the collineation $(X_0, X_1, X_2, X_3) \mapsto (X_3^q, X_0^q, X_1^q, X_2^q)$ of $\mathrm{PG}(3,q^4)$. Then the points of H are fixed points of σ and hence σ fixes the subspace $\bar{H}$. Note that the vertex of $\mathcal{Q}$ is always disjoint from H since it is contained in $\mathcal{A}$, while H is disjoint from it.

First of all note that if $\dim\ker K = 4$, i.e. K is the zero polynomial, then H is a subgeometry of $\mathrm{PG}(3,q^4)$ isomorphic to $\mathrm{PG}(3,q)$, which clearly cannot be contained in $\mathcal{Q}$. It follows that $\dim\ker K$ is either 3 or 2, i.e. H is either a q-order subplane or a q-order subline.

First assume $1 \neq a_1^q a_3$, i.e. the case when $\mathcal{Q}$ has rank 3. If H is a q-order subplane, then H cannot be contained in $\mathcal{Q}$. To see this, suppose the contrary and take three non-concurrent q-order sublines of H. The $\mathbb{F}_{q^4}$-extensions of these sublines are also contained in $\mathcal{Q}$, but there is at least one of them which does not pass through the singular point of $\mathcal{Q}$, a contradiction. Now assume that H is a q-order subline. The singular point of $\mathcal{Q}$ is the intersection of the planes $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$. Straightforward calculations show that this point is $V = \langle (v_0, v_1, v_2, v_3) \rangle_{q^4}$, where

$$v_0 = 0,$$

$$v_1 = a_1^{q^2+q^3}(a_1^{q^3} a_3^{q^2} - 1),$$

$$v_2 = a_1^{q^3} a_3^{q}(a_1^{q^2} a_3^{q} - a_1^{q^3} a_3^{q^2}),$$

$$v_3 = a_3^{q+q^2}(1 - a_1^{q^2} a_3^{q}).$$

Suppose, contrary to our claim, that H is contained in $\mathcal{Q}$. Then $\bar{H}$ passes through the singular point V of $\mathcal{Q}$. Since $\bar{H}$ is fixed by σ, it follows that the points $V, V^\sigma, V^{\sigma^2}, V^{\sigma^3}$ have to be collinear ($v_0 = 0$ yields that these four points cannot coincide). Let M denote the 4×4 matrix, whose i-th row consists of the coordinates of $V^{\sigma^{i-1}}$ for $i = 1, 2, 3, 4$. The rank of M is two, thus each of its minors of order three is zero. Let $M_{i,j}$ denote the submatrix of M obtained by deleting the i-th row and j-th column of M. Then

$$\det M_{1,2} = a_1^{q+1}(a_1^q a_3 - 1)^{q^3+1}\alpha,$$

$$\det M_{1,4} = a_3^{q^3+1}(a_1^q a_3 - 1)^{q^3+1}\beta,$$

where

$$\alpha = \mathrm{N}(a_1)(a_1^{q^2} a_3^{q} - 1) + \mathrm{N}(a_3)(1 - a_1^q a_3 - a_1^{q^3} a_3^{q^2} + a_1 a_3^{q^3}),$$

$$\beta = \mathrm{N}(a_1)(a_1 a_3^{q^3} + a_1^{q^2} a_3^{q} - a_1^q a_3 - 1) + \mathrm{N}(a_3)(1 - a_1^{q^3} a_3^{q^2}).$$

Since a_1 and a_3 cannot be both zeros and $a_1^q a_3 - 1 \neq 0$, we have $\alpha = \beta = 0$. But $\alpha - \beta = (\mathrm{N}(a_1) - \mathrm{N}(a_3))(a_1^q a_3 - a_1 a_3^{q^3})$. It follows that $a_1^q a_3 \in \mathbb{F}_q$ and hence α can be written as $(\mathrm{N}(a_1) - \mathrm{N}(a_3))(a_1^q a_3 - 1)$, which is non-zero. This contradiction shows that V cannot be contained in a line fixed by σ and hence $\bar{H}$ cannot pass through V. It follows that $H \nsubseteq \mathcal{Q}$ and hence we can choose B such that $AD - BC \neq 0$.

Now consider the case $1 = a_1^q a_3$. Then $\mathcal{Q}$ is the union of two planes meeting each other in $\ell := \mathcal{A} \cap \mathcal{B}$. It is easy to see that $R := \langle (0, 1, -a_3^{2q}, 0) \rangle_{q^4}$ and R^σ are two distinct points of ℓ. Since $\mathrm{N}(a_1) \neq \mathrm{N}(a_3)$ and $\mathrm{N}(a_1)\,\mathrm{N}(a_3) = 1$, $\det\{R, R^\sigma, R^{\sigma^2}, R^{\sigma^3}\} = \mathrm{N}(a_3)^2 - 1$ cannot be zero and hence $R \notin H$, otherwise $\dim\langle R, R^\sigma, R^{\sigma^2}, R^{\sigma^3} \rangle \leq \dim \bar{H} \leq 2$. Suppose, contrary to our claim, that H is contained in one of the two planes of $\mathcal{Q}$. Since $R \notin H$, such a plane can be written as $\langle H, R \rangle$ and since H is fixed by σ and $\ell \subseteq \langle H, R \rangle$, we have $\langle H, R \rangle^\sigma = \langle H, R^\sigma \rangle = \langle H, R \rangle$. Thus $R, R^\sigma, R^{\sigma^2}, R^{\sigma^3}$ are coplanar, a contradiction. $\square$

5 Different aspects of the classes of a linear set

5.1 Class of a linear set and the associated variety

Let L_U be an $\mathbb{F}_q$-linear set of rank k of $\mathrm{PG}(W, \mathbb{F}_{q^n}) = \mathrm{PG}(r-1, q^n)$. Consider the projective space $\Omega = \mathrm{PG}(W, \mathbb{F}_q) = \mathrm{PG}(rn-1, q)$. For each point $P = \langle \mathbf{u} \rangle_{\mathbb{F}_{q^n}}$ of $\mathrm{PG}(W, \mathbb{F}_{q^n})$ there corresponds a projective $(n-1)$-subspace $X_P := \mathrm{PG}(\langle \mathbf{u} \rangle_{q^n}, \mathbb{F}_q)$ of Ω. The variety of Ω associated to L_U is

$$\mathcal{V}_{r,n,k}(L_U) = \bigcup_{P \in L_U} X_P. \tag{27}$$

A $(k-1)$-space $\mathcal{H} = \mathrm{PG}(V, \mathbb{F}_q)$ of Ω is said to be a *transversal* space of $\mathcal{V}(L_U)$ if $\mathcal{H} \cap X_P \neq \emptyset$ for each point $P \in L_U$, i.e. $L_U = L_V$.

The $\mathcal{Z}(\Gamma\mathrm{L})$-class of an $\mathbb{F}_q$-linear set L_U of rank n of $\mathrm{PG}(W, \mathbb{F}_{q^n}) = \mathrm{PG}(1, q^n)$, with maximum field of linearity $\mathbb{F}_q$, is the number of transversal spaces of $\mathcal{V}_{2,n,n}(L_U)$ up to the action of the subgroup G of $\mathrm{PGL}(2n-1, q)$ induced by the maps $\mathbf{x} \in W \mapsto \lambda \mathbf{x} \in W$, with $\lambda \in \mathbb{F}_{q^n}^*$. Note that G fixes X_P for each point $P \in \mathrm{PG}(1, q^n)$ and hence fixes the variety.

The maximum size of an $\mathbb{F}_q$-linear set L_U of rank n of $\mathrm{PG}(1, q^n)$ is $(q^n - 1)/(q - 1)$. If this bound is attained (hence each point of L_U has weight one), then L_U is a *maximum scattered* linear set of $\mathrm{PG}(1, q^n)$. For maximum scattered linear sets, the number of transversal spaces through $Q \in \mathcal{V}(L_U)$ does not depend on the choice of Q and this number is the $\mathcal{Z}(\Gamma\mathrm{L})$-class of L_U.

Example 5.1. *Let $U = \{(x, x^q) \colon x \in \mathbb{F}_{q^n}\}$ and consider the linear set L_U. In [15] the variety $\mathcal{V}_{2,n,n}(L_U)$ was studied, and the transversal spaces were determined. It follows that the $\mathcal{Z}(\Gamma\mathrm{L})$-class of L_U is $\varphi(n)$, where φ is the Euler's phi function.*

5.2 Classes of linear sets as projections of subgeometries

Let $\Sigma = \mathrm{PG}(k-1, q)$ be a canonical subgeometry of $\Sigma^* = \mathrm{PG}(k-1, q^n)$. Let $\Gamma \subset \Sigma^* \setminus \Sigma$ be a $(k-r-1)$-space and let $\Lambda \subset \Sigma^* \setminus \Gamma$ be an $(r-1)$-space of Σ^*. The projection of Σ from *center* Γ to *axis* Λ is the point set

$$L = p_{\Gamma, \Lambda}(\Sigma) := \{\langle \Gamma, P \rangle \cap \Lambda \colon P \in \Sigma\}. \tag{28}$$

In [22] Lunardon and Polverino characterized linear sets as projections of canonical subgeometries. They proved the following.

Theorem 5.2 ([22, Theorems 1 and 2]). *Let Σ^*, Σ, Λ, Γ and $L = p_{\Gamma,\Lambda}(\Sigma)$ be defined as above. Then L is an $\mathbb{F}_q$-linear set of rank k and $\langle L\rangle = \Lambda$. Conversely, if L is an $\mathbb{F}_q$-linear set of rank k of $\Lambda = \mathrm{PG}(r-1, q^n) \subset \Sigma^*$ and $\langle L\rangle = \Lambda$, then there is a $(k-r-1)$-space Γ disjoint from Λ and a canonical subgeometry $\Sigma = \mathrm{PG}(r-1, q)$ disjoint from Γ such that $L = p_{\Gamma,\Lambda}(\Sigma)$.*

Let L_U be an $\mathbb{F}_q$-linear set of rank k of $\mathbb{P} = \mathrm{PG}(W, \mathbb{F}_{q^n}) = \mathrm{PG}(r-1, q^n)$ such that for each k-dimensional $\mathbb{F}_q$-subspace V of W if $\mathrm{PG}(V, \mathbb{F}_q)$ is a transversal space of $\mathcal{V}_{r,n,k}(L_U)$, then there exists $\gamma \in \mathrm{P\Gamma L}(W, \mathbb{F}_q)$, such that γ fixes the Desarguesian spread $\{X_P : P \in \mathbb{P}\}$ and $\mathrm{PG}(U, \mathbb{F}_q)^\gamma = \mathrm{PG}(V, \mathbb{F}_q)$. This is condition (A) from [6], and it is equivalent to say that L_U is a simple linear set. Then the main results of [6] can be formalized as follows.

Theorem 5.3 ([6]). *Let $L_1 = p_{\Gamma_1,\Lambda_1}(\Sigma_1)$ and $L_2 = p_{\Gamma_2,\Lambda_2}(\Sigma_2)$ be two linear sets of rank k. If L_1 and L_2 are equivalent and one of them is simple, then there is a collineation mapping Γ_1 to Γ_2 and Σ_1 to Σ_2.*

Theorem 5.4 ([6]). *If L is a non-simple linear set of rank k in $\Lambda = \langle L\rangle$, then there are a subspace $\Gamma = \Gamma_1 = \Gamma_2$ disjoint from Λ, and two q-order canonical subgeometries Σ_1, Σ_2 such that $L = p_{\Gamma,\Lambda}(\Sigma_1) = p_{\Gamma,\Lambda}(\Sigma_2)$, and there is no collineation fixing Γ and mapping Σ_1 to Σ_2.*

Now we interpret the classes of linear sets, hence we are going to consider $\mathbb{F}_q$-linear sets of rank n of $\Lambda = \mathrm{PG}(1, q^n) = \mathrm{PG}(W, \mathbb{F}_{q^n})$, with maximum field of linearity $\mathbb{F}_q$. Arguing as in the proof of [6, Theorem 7], if L_U is non-simple, then for any pair U, V of n-dimensional $\mathbb{F}_q$-subspaces of W with $L_U = L_V$ such that $U^f \neq V$ for each $f \in \Gamma\mathrm{L}(2, q^n)$ we can find a q-order subgeometry Σ of $\Sigma^* = \mathrm{PG}(n-1, q^n)$ and two $(n-3)$-spaces Γ_1 and Γ_2 of Σ^*, disjoint from Σ and from Λ, lying on different orbits of $Stab(\Sigma)$. On the other hand, arguing as in [6, Theorem 6], if there exist two $(n-3)$-subspaces Γ_1 and Γ_2 of Σ^*, disjoint from Σ and from Λ, belonging to different orbits of $Stab(\Sigma)$ and such that $L = p_{\Lambda,\Gamma_1}(\Sigma) = p_{\Lambda,\Gamma_2}(\Sigma)$, then it is possible to construct two n-dimensional $\mathbb{F}_q$-subspaces U and V of W with $L_U = L_V$ such that $U^f \neq V$ for each $f \in \Gamma\mathrm{L}(2, q^n)$. Hence we can state the following.

The ΓL-class of L_U is the number of orbits of $Stab(\Sigma)$ on $(n-3)$-spaces of Σ^* containing a Γ disjoint from Σ and from Λ such that $p_{\Lambda,\Gamma}(\Sigma)$ is equivalent to L_U.

5.3 Class of linear sets and linear blocking sets of Rédei type

A *blocking set* $\mathcal{B}$ of $\mathrm{PG}(V, \mathbb{F}_{q^n}) = \mathrm{PG}(2, q^n)$ is a point set meeting every line of the plane. Blocking sets of size $q^n + N \leq 2q^n$ with an N-secant are called blocking sets of *Rédei type*, the N-secants of the blocking set are called *Rédei lines*. Let L_U be an $\mathbb{F}_q$-linear set of rank n of a line $\ell = \mathrm{PG}(W, \mathbb{F}_{q^n})$, $W \leq V$, and let $\mathbf{w} \in V \setminus W$. Then $\langle U, \mathbf{w}\rangle_{\mathbb{F}_q}$ defines an $\mathbb{F}_q$-linear blocking set of $\mathrm{PG}(2, q^n)$ with Rédei line ℓ. The following theorem tells us the number of inequivalent blocking sets obtained in this way.

Theorem 5.5. *The* $\Gamma\mathrm{L}$*-class of an* $\mathbb{F}_q$*-linear set* L_U *of rank* n *of* $\mathrm{PG}(W, \mathbb{F}_{q^n}) = \mathrm{PG}(1, q^n)$*, with maximum field of linearity* $\mathbb{F}_q$*, is the number of inequivalent* $\mathbb{F}_q$*-linear blocking sets of Rédei type of* $\mathrm{PG}(V, \mathbb{F}_{q^n}) = \mathrm{PG}(2, q^n)$ *containing* L_U*.*

Proof. $\mathbb{F}_q$-linear blocking sets of $\mathrm{PG}(2, q^n)$ with more than one Rédei line are equivalent to those defined by $\mathrm{Tr}_{q^n/q^m}(x)$ for some divisor m of n, see [20, Theorem 5]. Suppose first that L_U is equivalent to L_T, where $T = \{(x, \mathrm{Tr}_{q^n/q}(x)) \colon x \in \mathbb{F}_{q^n}\}$. According to Theorem 3.7 L_T, and hence also L_U, have $\mathcal{Z}(\Gamma\mathrm{L})$-class and $\Gamma\mathrm{L}$-class one. Proposition 2.5 yields the existence of a unique point $P \in L_U$ such that $w_{L_U}(P) = n - 1$. Then for each $\mathbf{v} \in V \setminus W$ the $\mathbb{F}_q$-linear blocking set defined by $\langle U, \mathbf{v}\rangle_{\mathbb{F}_q}$ has more than one Rédei line, each of them incident with P, and hence it is equivalent to the Rédei type blocking set obtained from $\mathrm{Tr}_{q^n/q}(x)$.

Now let $\mathcal{B}_1 = L_{V_1}$ and $\mathcal{B}_2 = L_{V_2}$ be two $\mathbb{F}_q$-linear blocking sets of Rédei type with $\mathrm{PG}(W, \mathbb{F}_{q^n})$ the unique Rédei line. Denote by U_1 and U_2 the $\mathbb{F}_q$-subspaces $W \cap V_1$ and $W \cap V_2$, respectively, and suppose $L_{U_1} = L_{U_2}$ with $\mathbb{F}_q$ the maximum field of linearity. Then $\mathcal{B}_1$ and $\mathcal{B}_2$ have $(q+1)$-secants and we have $V_1 = U_1 \oplus \langle \mathbf{u_1}\rangle_{\mathbb{F}_q}$ and $V_2 = U_2 \oplus \langle \mathbf{u_2}\rangle_{\mathbb{F}_q}$ for some $\mathbf{u_1}, \mathbf{u_2} \in V \setminus W$.

If $\mathcal{B}_1^{\varphi_f} = \mathcal{B}_2$, then [5, Proposition 2.3] implies $V_1^f = \lambda V_2$ for some $\lambda \in \mathbb{F}_{q^n}^*$. Such $f \in \Gamma\mathrm{L}(3, q^n)$ has to fix W and it is easy to see that $U_1^f = \lambda U_2$, i.e. U_1 and U_2 are $\Gamma\mathrm{L}(2, q^n)$-equivalent.

Conversely, if there exists $f \in \Gamma\mathrm{L}(W, \mathbb{F}_{q^n})$ such that $U_1^f = U_2$, then $\mathcal{B}_1^{\varphi_g} = \mathcal{B}_2$, where $g \in \Gamma\mathrm{L}(V, \mathbb{F}_{q^n})$ is the extension of f mapping $\mathbf{u_1}$ to $\mathbf{u_2}$. □

5.4 Class of linear sets and MRD-codes

In [25, Section 4] Sheekey showed that maximum scattered linear sets of $\mathrm{PG}(1, q^n)$ correspond to $\mathbb{F}_q$-linear maximum rank distance codes (MRD-codes) of dimension $2n$ and minimum distance $n-1$, that is, a set $\mathcal{M}$ of q^{2n} $n \times n$ matrices over $\mathbb{F}_q$ forming an $\mathbb{F}_q$-subspace of $\mathbb{F}_q^{n\times n}$ of dimension $2n$ such that the non-zero matrices of $\mathcal{M}$ have rank at least $n-1$. For definitions and properties on MRD-codes we refer the reader to [9] by Delsarte and [12] by Gabidulin. For $n \times n$ matrices there are two different definitions of equivalence for MRD-codes in the literature. The arguments of [25, Section 4] yield the following interpretation of the ΓL-class:

- $\mathcal{M}$ and $\mathcal{M}'$ are equivalent if there are invertible matrices A, $B \in \mathbb{F}_q^{n\times n}$ and a field automorphism σ of $\mathbb{F}_q$ such that $A\mathcal{M}^\sigma B = \mathcal{M}'$, see [25]. In this case the ΓL-class of L_U is the number of inequivalent MRD-codes obtained from the linear set L_U.

- $\mathcal{M}$ and $\mathcal{M}'$ are equivalent if there are invertible matrices A, $B \in \mathbb{F}_q^{n\times n}$ and a field automorphism σ of $\mathbb{F}_q$ such that $A\mathcal{M}^\sigma B = \mathcal{M}'$, or $A\mathcal{M}^{T\sigma} B = \mathcal{M}'$, see [8]. In this case the number of inequivalent MRD-codes obtained from the linear set L_U is between $\lceil s/2 \rceil$ and s, where s is the ΓL-class of L_U.

We summarize here the known non-equivalent families of MRD-codes arising from maximum scattered linear sets.

1. $L_{U_1} := \{\langle (x, x^q)\rangle_{\mathbb{F}_{q^n}} : x \in \mathbb{F}_{q^n}^*\}$ (found by Blokhuis and Lavrauw [4]) gives Gabidulin codes,

2. $L_{U_2} := \{\langle (x, x^{q^s})\rangle_{\mathbb{F}_{q^n}} : x \in \mathbb{F}_{q^n}^*\}$, $\gcd(s, n) = 1$ ([4]) gives generalized Gabidulin codes,

3. $L_{U_3} := \{\langle (x, \delta x^q + x^{q^{n-1}})\rangle_{\mathbb{F}_{q^n}} : x \in \mathbb{F}_{q^n}^*\}$ (found by Lunardon and Polverino [21]) gives MRD-codes found by Sheekey,

4. $L_{U_4} := \{\langle (x, \delta x^{q^s} + x^{q^{n-s}})\rangle_{\mathbb{F}_{q^n}} : x \in \mathbb{F}_{q^n}^*\}$, $\mathrm{N}(\delta) \neq 1$, $\gcd(s, n) = 1$ gives MRD-codes found by Lunardon, Trombetti and Zhou in [23].

Remark 5.6. *The linear sets L_{U_1} and L_{U_2} coincide, but when $s \notin \{1, n-1\}$, then there is no $f \in \Gamma\mathrm{L}(2, q^n)$ such that $U_1^f = U_2$. These linear sets are of* pseudoregulus type*, [19] (see also Example 5.1), and in [6] it was proved that the* ΓL*-class of these linear sets is $\varphi(n)/2$, hence they are examples of non-simple linear sets for $n = 5$ and $n > 6$.*

It can be proved that the family L_{U_4} contains linear sets non-equivalent to those from the other families. We will report on this elsewhere.

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