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# Improvements on the density of maximal 1-planar graphs

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**Abstract**

A graph is 1-planar if it can be drawn in the plane such that each edge is crossed at most once. A graph, together with a 1-planar drawing is called 1-plane. A graph is maximal 1-planar (1-plane), if we cannot add any missing edge so that the resulting graph is still 1-planar (1-plane). Brandenburg et al. showed that there are maximal 1-planar graphs with only $\frac{45}{17}n + O(1) \approx 2.647n$ edges and maximal 1-plane graphs with only $\frac{7}{3}n + O(1) \approx 2.33n$ edges. On the other hand, they showed that a maximal 1-planar graph has at least $\frac{28}{13}n - O(1) \approx 2.15n - O(1)$ edges, and a maximal 1-plane graph has at least $2.1n - O(1)$ edges. We improve both lower bounds to $\frac{20n}{9} \approx 2.22n$.

**KEYWORDS**

1-planar, edge density

## 1 | INTRODUCTION

In a *drawing* of a simple undirected graph $G$, vertices are represented by distinct points in the plane and edges are represented by simple continuous curves connecting the corresponding points. For simplicity, the points and curves are also called vertices and edges, and if it does not lead to confusion, we denote them the same way as the original vertices and edges of $G$, respectively. We assume that edges do not contain vertices in their interior and that edges with a common endpoint do not cross. In particular, an edge cannot cross itself. It follows from Euler's formula that a planar graph of $n$ vertices has at most $3n - 6$ edges. Also if a planar graph $G$ has fewer edges, then we can add edges to it so that the resulting graph has exactly $3n - 6$ edges and it is still planar. This holds even if we start with a fixed planar drawing of $G$.

A drawing of a graph is *1-planar* if each edge is crossed at most once. A graph is *1-planar* if it has a 1-planar drawing. It is *maximal* 1-planar, if we cannot add any edge to it so that the resulting graph is still 1-planar. A graph together with a 1-planar drawing is a *1-plane graph*. It is *maximal* 1-plane,

if we cannot add any edge to it so that the resulting drawing is still 1-plane. The maximum number of edges of a 1-planar or 1-plane graph is $4n - 8$ [4]. Recently, Brandenburg et al. [1,2] observed a very interesting phenomenon: there are maximal 1-planar and 1-plane graphs with much fewer edges.

**Theorem 1 (Brandenburg et al. [2]).**

 (i) *Let $e(n)$ be the minimum number of edges of a maximal 1-planar graph with $n$ vertices. The following holds*

$$\frac{28}{13}n - O(1) \approx 2.15n - O(1) \le e(n) \le \frac{45}{17}n + O(1) \approx 2.647n,$$

**(ii)** *Let $e'(n)$ be the minimum number of edges of a maximal 1-plane graph with $n$ vertices. The following holds*

$$2.1n - O(1) \le e(n) \le \frac{7}{3}n + O(1) \approx 2.33n.$$

The upper bound in part (ii) has been proved independently by Eades et al. [3]. In this note, we improve both lower bounds.

**Theorem 2.** *A maximal 1-planar or 1-plane graph has at least $\frac{20}{9}n - O(1) \approx 2.22n$ edges.*

That is, $e(n), e'(n) \ge \frac{20}{9}n - O(1)$.

## 2 | PRELIMINARIES

Our method is based on the ideas of Brandenburg et al. [2]. We also point out an error in [2], but with our approach their proof goes through as well. The following observations are essentially from their article. We include the proofs for completeness. Throughout this section, $G$ is a maximal 1-plane graph of at least four vertices. The edges of $G$ divide the plane into *faces*. A face is bounded by edges and edge segments. These edges and edge segments end in vertices or crossings.

**Lemma 1.**

 (i) *There are at least two vertices on the boundary of each face.*
**(ii)** *If $u$ and $v$ are two vertices on the boundary of a face, then they are adjacent.*

*Proof.*

 (i) Each face is bounded by at least three edges or edge segments, and has at least three vertices or crossings on its boundary. Since there is at most one crossing on each edge, each edge segment contains a vertex as an endpoint. Therefore, there must be at least two vertices on the boundary of the face.
**(ii)** Suppose that there are two vertices, $u$ and $v$ on the boundary of a face. Now $u$ and $v$ could be connected by a curve in the face without creating any crossing. Therefore, by the maximality of $G$, $u$ and $v$ are already connected.                                                    ∎

**Lemma 2.** *There are neither isolated vertices nor vertices of degree 1 in $G$.*

*Proof.* Suppose that $v$ is an isolated vertex or a vertex of degree 1 in face $F$ of $G$. Now $G \setminus \{v\}$ is also maximal 1-plane, since if we can add an edge to $G \setminus \{v\}$, then we could have added it to $G$.
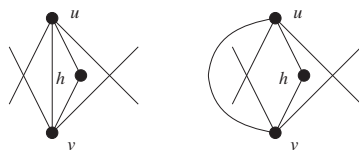
**FIGURE 1** Hermit $h$, surrounded by two pairs of crossing edges

Therefore, by Lemma 1, $F$ has at least two vertices on its boundary, different from $v$, and $v$ is adjacent to both of them, a contradiction. ∎

**Lemma 3.** *If $ab$ and $cd$ are crossing edges in $G$, then $a, b, c, d$ span a $K_4$ in $G$.*

*Proof.* Let $x$ be the crossing of $ab$ and $cd$. Since there are no other crossings on $ab$ and $cd$, there is a face bounded by $ax$ and $xc$. Now $a$ and $c$ are adjacent by Lemma 1. Similarly $a$ and $d$, $b$ and $c$, $b$ and $d$ are also adjacent. ∎

The smallest degree in $G$ is at least two by Lemma 2. Following [2], we call vertices of degree two *hermits*.

**Lemma 4.** *If a vertex $h$ has only two neighbors in $G$, say $u$ and $v$, then*

 **(i)** *$hu$ and $hv$ are not crossed by any edge,*

**(ii)** *$u$ and $v$ are adjacent in $G$.*

*Proof.*

 **(i)** Suppose to the contrary that $hu$ is crossed by an edge. By Lemma 3, vertex $h$ has degree at least three, a contradiction.

**(ii)** Since the only neighbors of $h$ are $u$ and $v$, and edges $hu$ and $hv$ are not crossed, there is a face that has $u$, $h$, and $v$ on its boundary. Therefore, $u$ and $v$ are adjacent by Lemma 1. ∎

**Lemma 5.** *Suppose that $h$ is a hermit, and its neighbors are $u$ and $v$. Delete $h$, $hu$, $hv$, $uv$ from $G$, and let $G'$ be the resulting graph with the original embedding. Let $F$ be the face of $G'$ that contains the point corresponding to vertex $h$. Then $F$ has only two vertices on its boundary, $u$ and $v$.*

*Proof.* If there was another vertex on the boundary of $F$, then we could connect it to $h$: either without any crossing or with exactly one crossing with edge $uv$ contradicting the maximality of $G$. ∎

Since $F$ has only two vertices on its boundary, $u$ and $v$, the boundary of $F$ consists of two pairs of crossing edge segments. The point corresponding to hermit $h$ is in $F$. We refer to this as *hermit $h$ is surrounded by two pairs of crossing edges*. See Figure 1. It follows from Lemma 5 that each hermit is surrounded by two pairs of crossing edges.

Remove all hermits from $G$. The resulting graph $\hat{G}$ with the inherited drawing is the *skeleton* of $G$. Notice that $\hat{G}$ is also maximal 1-plane and each vertex of $\hat{G}$ has degree at least three.

## 2.1 | A correction

In [2] the lower bound proofs rely on the following statement:

**Claim 1.** *[2] Every edge of $\hat{G}$ is covered by a $K_4$ in $\hat{G}$.*

However, this claim does not hold! See Figure 3 for a counterexample. Call an edge of $\hat{G}$ *exceptional* if it is not part of a $K_4$ in $\hat{G}$. We have to deal with exceptional edges as well.

**Lemma 6.** *Suppose that edge $ab$ of $\hat{G}$ is exceptional. That is, $ab$ is not part of a $K_4$ in $\hat{G}$. Let $F_1$ and $F_2$ be the faces bounded by $ab$. Then*

(i)  $F_1 \neq F_2$,

(ii)  *for $i = 1, 2$, $F_i$ has exactly three vertices on its boundary, $a, b$ and let $f_i$ denote the third one. Moreover, $f_1 = f_2$.*

(iii)  *both $af_i$ and $bf_i$ are nonexceptional edges of $\hat{G}$.*

*Proof.*

(i)  Suppose that $ab$ is an exceptional edge of $\hat{G}$. If edge $ab$ is crossed by another edge, then it is part of a $K_4$ by Lemma 3. Therefore, $ab$ does not participate in a crossing. Let $F_1$ and $F_2$ be the faces bounded by $ab$. If $F_1 = F_2$, then $ab$ is a cut edge. In this case, by Lemma 1, both components have at least one other vertex on the boundary of $F_1 = F_2$, and they can be connected. This contradicts the maximality of $\hat{G}$. Consequently, $F_1 \neq F_2$.

(ii)  If there is an edge from $a$ and an edge from $b$ which cross, then $ab$ is part of a $K_4$ by Lemma 3. If there are at least four different vertices on the boundary of $F_i$, say, $a, b, x$, and $y$, then they form a $K_4$ by Lemma 1. We conclude that if $ab$ is exceptional, then $F_i$ has exactly three vertices on its boundary, $a, b$ and, say, $f_i$. If $f_1 \neq f_2$ and they are not connected, then we can connect them through $F_1$ and $F_2$. So by the maximality of $\hat{G}$, $f_1$, and $f_2$ are connected. By Lemma 1, $(a, b, f_1)$ and $(a, b, f_2)$ are all adjacent, and together with edge $f_1 f_2$, these four vertices form a $K_4$ covering $ab$, a contradiction. Therefore, $f_1 = f_2$ and we denote it by $f$ for the rest of the proof.

(iii)  Vertices $a, b$, and $f$ divide the boundary of $F_1$ into three parts. Between $a$ and $b$, we have edge $ab$ by assumption. Between $a$ and $f$, we either have edge $af$, or two segments of edges. In the latter case, $af$ is an edge of $\hat{G}$ and part of a $K_4$ by Lemma 3. We can argue similarly for face $F_2$. We conclude that $af$ is an edge of $\hat{G}$ and part of a $K_4$, unless $af$ is on the boundary of both $F_1$ and $F_2$. In that case, the degree of $a$ would be 2 in $\hat{G}$, which is impossible. We can argue the same way for edge $bf$.  ∎

**Remark.** *Each of the drawings on Figure 2 can be extended to a maximal 1-plane graph so that $ab$ is not part of a $K_4$. See Figure 3 for an example.*

## 3 | IMPROVEMENT OF THE LOWER BOUNDS – PROOF OF THEOREM 2

Let $\hat{G}$ be the skeleton of a maximal 1-plane graph. Recall that the skeleton $\hat{G}$ arises by removing each hermit from $G$ together with its two incident edges. The skeleton inherits its drawing from $G$, it is maximal 1-plane and each vertex has degree at least three. We distinguish three types of edges in $\hat{G}$: crossing, plain, and exceptional. Edges that participate in a crossing are *crossing edges*. A crossing-free edge that is part of a $K_4$, is a *plain* edge. Any other edge is *exceptional*. Those edges are crossing-free and do not belong to a $K_4$. Let $n(H)$, $c(H)$, $p(H)$, $e(H)$ denote the number of vertices, crossing edges, plain edges, and exceptional edges of a graph $H$. In particular, let $n = n(\hat{G})$, $c = c(\hat{G})$, $p = p(\hat{G})$, $e = e(\hat{G})$. We prove the following crucial inequality involving these quantities.
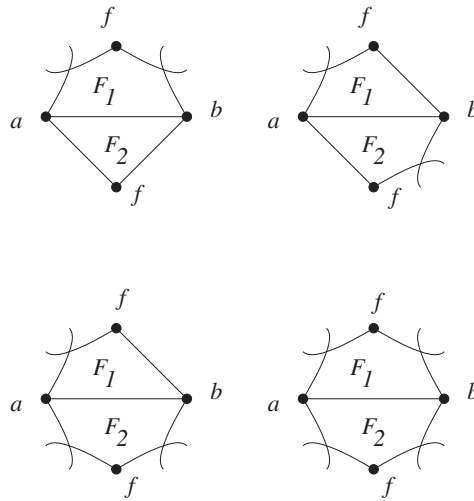
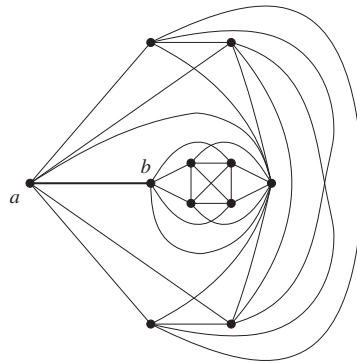**FIGURE 2** The four possible types of exceptional edges $ab$



**FIGURE 3** A maximal 1-plane graph. Edge $ab$ is not part of a $K_4$

**Lemma 7.** *If $\hat{G}$ is the skeleton of any drawing of a maximal 1-planar graph $G$ and $n(\hat{G}) \geq 4$, then*

$$9p(\hat{G}) + 10e(\hat{G}) + 7c(\hat{G}) \geq 20n(\hat{G}) - 30. \tag{1}$$

*Proof.* First we use induction on $e$. If there is an exceptional edge in $\hat{G}$, then we use the induction hypothesis for graphs with smaller $e$. If $e(\hat{G}) = 0$, then we prove the statement directly.

Suppose that there is an exceptional edge $ab$ in $\hat{G}$. Let $F_1$ and $F_2$ be the two faces bounded by $ab$. By Lemma 6, $F_1 \neq F_2$, and both $F_1$ and $F_2$ have exactly three vertices on their boundaries, $a$, $b$, and $f$, see Figure 2. The closure of $F_1 \cup F_2$ divides the plane into two parts, say $S_1, S_2$. Now $\overline{S_i}$, the closure of $S_i$, intersects $\hat{G}$ in $G_i$ for $i = 1, 2$. Remove the edge $ab$ and the interior of $F_1$, $F_2$ from $\hat{G}$. Now two almost disjoint subgraphs $G_1$ and $G_2$ arise such that they have exactly one vertex $f$ in common, $a \in G_1$ and $b \in G_2$. Both $G_1$ and $G_2$ are maximal 1-plane and both have at least four vertices. Therefore, we can use the induction hypothesis on $G_1$ and $G_2$. For $i = 1, 2$, let $n_i, c_i, p_i, e_i$ denote the number of vertices, crossing edges, plain edges, and exceptional edges of $G_i$. Now $9p_1 + 10e_1 + 7c_1 \geq 20n_1 - 30$ and $9p_2 + 10e_2 + 7c_2 \geq 20n_2 - 30$, where $e_1 + e_2 + 1 = e$, $n_1 + n_2 - 1 = n$ and $p_1 + p_2 = p$, $c_1 + c_2 = c$. Therefore, $9p + 10e - 10 + 7c \geq 20n + 20 - 60$, and the statement follows.
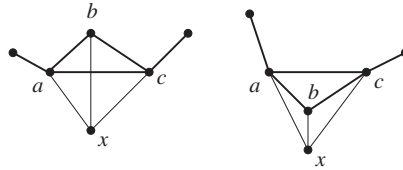
**FIGURE 4**   Operation 2

We may now assume that $e(\hat{G}) = 0$, and we should prove that $9p(\hat{G}) + 7c(\hat{G}) \geq 20n - 30$, where $n \geq 4$. In what follows, we define an increasing sequence of subgraphs $G_0 \subset G_1 \subset \cdots \subset \hat{G}$ recursively and keep track of the number of vertices and edges of $G_i$. In every step, we maintain the inequality $9p + 9e + 7c \geq 20n - 30$. Note that it is stronger than our target inequality $9p + 10e + 7c \geq 20n - 30$.

We can use the idea of Brandenburg et al. [2]. They defined the $K_4$-*network* of $G$, which is an auxiliary graph $\mathcal{K}$. Its vertex set corresponds to the $K_4$ subgraphs of $\hat{G}$. Two vertices in $\mathcal{K}$ are adjacent if the corresponding subgraphs in $\hat{G}$ share a vertex. Since $\hat{G}$ is connected and every edge is contained in a $K_4$, the graph $\mathcal{K}$ is connected. Brandenburg et al. proved a lower bound on the number of edges of $\hat{G}$ by building a certain spanning tree of $\mathcal{K}$ by an algorithm and investigating the number of edges of $\hat{G}$ involved in each step of the algorithm.

We go in their footsteps, but take a closer look. We use a slightly more complex algorithm that sweeps through $\hat{G}$ rather than $\mathcal{K}$.

Let $G_0$ be a $K_4$ subgraph of $\hat{G}$. Suppose that we have already defined $G_{i-1}$, a connected subgraph of $\hat{G}$, and now we construct $G_i$. Therefore, the vertices, edges, subgraphs of $G_{i-1}$ are *old* and the ones of $G_i$ are *new*. To construct $G_i$ from $G_{i-1}$, we use one of the following operations in this order of preference.

1.  Adding an edge between two old vertices.

2.  Adding a new vertex $x$ and all $K_4$'s spanned by $x$ and three old vertices.

3.  Adding two new vertices, $x$ and $y$ and all $K_4$'s spanned by $x$, $y$ and two old vertices.

4.  Adding two new $K_4$'s such that they share a new vertex and each of them has a vertex in common with the old subgraph.

If none of these operations can be executed, then let $G_{\text{final}} = G_{i-1}$ and the algorithm terminates. Since the $K_4$-network $\mathcal{K}$ of $\hat{G}$ is connected, $G_{\text{final}} = \hat{G}$.

Although $\hat{G}$ does not have exceptional edges, for some $i$, $G_i$ might have one (or more). However, we could create an exceptional edge only with operation 1. Observe that $K_4$ has two different 1-planar drawings. Either all edges are crossing-free, or there is exactly one crossing.

We show by induction, that

$$9p(G_i) + 9e(G_i) + 7c(G_i) \geq 20n(G_i) - 30 \qquad (2)$$

for every $i$. This is certainly true for $i = 0$. Suppose $9p(G_{i-1}) + 9e(G_i) + 7c(G_{i-1}) \geq 20n(G_{i-1}) - 30$ for some $i$, and now we construct $G_i$. If we apply operation 1, then the left side of inequality (2) increases by at least five (it is the case when we add an edge that crosses a previously plain or exceptional edge), while the right side does not change, so (2) holds for $G_i$ as well.

Suppose that we executed the second operation and we added exactly one $K_4$ with new vertex $x$ and old vertices $a$, $b$, $c$, see Figure 4. Now either $xa$, $xb$, and $xc$ are all plain edges in $G_i$, or one of them, say $xc$, crosses $ab$. The other two edges, $xa$ and $xb$ do not cross an old edge since in this case there

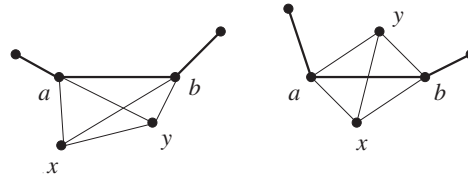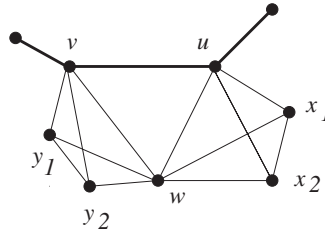**FIGURE 5**   Operation 3



**FIGURE 6**   Operation 4

would be more $K_4$'s involving $x$. The left side of inequality (2) increased by 27 or 23 while the right side increased by 20. If we added more than one $K_4$, then we had to add at least four edges adjacent to $x$. The addition of an edge increases the left side of the inequality by at least five, so it increased by at least 20.

Suppose that we executed the third operation, see Figure 5. Let $x$ and $y$ be the new vertices, $a$ and $b$ the old vertices of a new $K_4$. Edges $xa$, $xb$, $ya$, $yb$ cannot cross an old edge, since in that case we find a $K_4$ with exactly three old vertices contradicting the preference order of the operations. If $xy$ is not crossed by an old edge, then the left side of inequality (2) increases by 41 or 45, while the right side increases by 40. Suppose that the edge $xy$ crosses an old edge $cd$. Now $x, y, c, d$ form another $K_4$, and again none of the other new edges crosses an old edge. We added at least eight new edges, so the left-hand side increased by at least 40 again.

Suppose now that we arrive to a stage, where we cannot use any of the first three operations. Therefore, there is no $K_4$ in $\hat{G}$ that has exactly two or three old vertices. Let $u$ be an old vertex that has at least one neighbor not in $G_{i-1}$. Since $\hat{G}$ is connected, there is such a vertex. The graph $G_{i-1}$ is also connected, so $u$ has a neighbor in $G_{i-1}$ as well. Order all neighbors of $u$ in the circular order the corresponding edges emanate from $u$.

Recall that edges with a common endpoint do not cross. Let $v$ and $w$ be consecutive neighbors of $u$ such that $v \in G_{i-1}$ and $w \notin G_{i-1}$. We distinguish four cases.

**Case 1:** Both $uv$ and $uw$ are plain edges in $\hat{G}$. We consider a $K_4$ that contains the edge $uw$. By the assumptions, this $K_4$ has exactly one vertex ($u$) in $G_{i-1}$ and three vertices, say $w$, $x_1$, and $x_2$ not in $G_{i-1}$. The vertices $v$ and $w$ can be connected along $uv$ and $uw$, so by the maximality of $\hat{G}$, they are adjacent in $\hat{G}$. We consider a $K_4$ that contains the edge $vw$. By the assumptions, this $K_4$ has exactly one vertex ($v$) in $G_{i-1}$ and three vertices not in $G_{i-1}$: $w$, $y_1$, and $y_2$. By the assumptions, none of the new edges crosses an old edge. Add these two $K_4$'s to $G_{i-1}$ and let $G_i$ be the resulting graph. If $x_1, x_2, y_1, y_2$ are all different, then we added five new vertices so the right-hand side of (1) increased by 100, see Figure 6. The left-hand side increased by at least 100 since adding two crossing $K_4$'s means 4 crossing and eight plain edges and $4 \cdot 7 + 8 \cdot 9 = 100$. If $x_i = y_j$ for some $i, j$, then the situation is even better, the calculation is very similar.

**Case 2:** The edge $uv$ is plain in $\hat{G}$ and $uw$ is crossing. In this case, $uw$ is crossed by an edge $ab$ of $\hat{G}$. Now $a, b, u, w$ form a $K_4$, so $a$ and $b$ are not in $G_{i-1}$. Vertex $v$ can be connected to $a$ or $b$ along $uv, uw$, and $ab$. Suppose that it is $a$, so by the maximality of $\hat{G}$, $v$ and $a$ are adjacent. Consider a $K_4$ that contains edge $va$ and the one with vertices $a, b, u, w$. Add these two $K_4$'s to $G_{i-1}$ and let $G_i$ be the resulting graph. The calculation is very similar to the previous case.

**Case 3:** The edge $uw$ is plain in $\hat{G}$ and $uv$ is crossing. Let $ab$ be the edge that crosses $uv$. Now $a, b, u, v$ form a $K_4$, and $u, v$ are old vertices, so $a$ and $b$ are also old vertices. Vertex $w$ can be connected to $a$ or $b$, say, $a$, along $uw, uv$, and $ab$. So again by the maximality of $\hat{G}$, $w$ and $a$ are adjacent. Consider a $K_4$ that contains $wa$ and a $K_4$ that contains $uw$. Again, add these two $K_4$'s to $G_{i-1}$ and let $G_i$ be the resulting graph. The calculation is similar to the previous cases.

**Case 4:** Both $uv$ and $uw$ are crossing edges in $\hat{G}$. This case is the combination of the previous two cases. Edge $uw$ is crossed by $ab$, $uv$ is crossed by $cd$. Now one of $a$ and $b$, say, $a$, and one of $c$ and $d$, say, $c$, can be connected along $ab, uv, uw, cd$, so they are connected. Take the $K_4$ formed by $u, w, c, d$, and a $K_4$ the contains $ac$. Just like in the previous cases, add these two $K_4$'s to $G_{i-1}$ and let $G_i$ be the resulting graph. The calculation is the same again.

In summary, we proved that we can always apply one of the four operations in our algorithm, so the algorithm terminates when we obtain $\hat{G}$. On the other hand, we also proved that $9p(G_i) + 9e(G_i) + 7c(G_i) \geq 20n(G_i) - 30$ for every $i$. Therefore, $9p(\hat{G}) + 9e(\hat{G}) + 7c(\hat{G}) = 9p(\hat{G}) + 7c(\hat{G}) \geq 20n(\hat{G}) - 30$. This concludes the proof of Lemma 7. ■

## 3.1 | Proof of theorem 2

Recall that $e(n)$ ($e'(n)$) is the minimum number of edges of a maximal 1-planar (1-plane) graph with $n$ vertices. Since every maximal 1-planar graph with any 1-planar drawing is a maximal 1-plane graph, $e(n) \geq e'(n)$. Therefore, Theorem 2 follows immediately from the next result.

**Theorem 3.** *Every maximal $N$-vertex 1-plane graph has at least $\frac{20}{9}N - \frac{10}{3}$ edges, where $N \geq 4$.*

*Proof.* Let $G$ be a maximal 1-plane graph, $N$ and $E$ denote the number of vertices and edges, and $h$ denotes the number of hermits. Let $\hat{G}$ be the skeleton of $G$ and let $n = n(\hat{G})$, $c = c(\hat{G})$, $p = p(\hat{G})$, $e = e(\hat{G})$ denote the number of vertices, crossing edges, plain edges, and exceptional edges of $\hat{G}$.

Every hermit is surrounded by two pairs of crossing edges. A crossing pair of edges can participate in four such surroundings, on the four sides of the crossing. This gives us $c \geq h$.

More precisely, define a bipartite graph $H$ with vertex classes $A$ and $B$. Vertices in $A$ correspond to the crossings (or equivalently, crossing pairs of edges) in $G$, vertices in $B$ correspond to hermits. An edge in $H$ means that the corresponding crossing surrounds the corresponding hermit. Since every hermit is surrounded by two pairs of crossing edges, each vertex in $B$ has degree 2. On the other hand, a crossing pair can surround at most four hermits, so each vertex in $A$ has degree at most 4. Therefore, $4|A| \geq 2|B|$. But $|A| = c/2$ and $|B| = h$, so $c \geq h$.

By Lemma 6, for each exceptional edge, each of the two neighboring cells has a pair of crossing edges on its boundary. These two crossings cannot participate in a surrounding of a hermit in that direction. This shows $c \geq e$ and $c - e \geq h$. Now $N = n + h$, $E = p + e + c + 2h$.

Let us minimize

$$F(p, e, c, h, n) = E - \frac{20}{9}N$$

under the conditions

$$c - e \geq h, \tag{3}$$

$$9p + 10e + 7c \geq 20n - 30, \tag{4}$$

and

$$p, e, c, h, n \geq 0 \tag{5}$$

$F(p, e, c, h, n) = E - \frac{20}{9}N = p + e + c + 2h - \frac{20}{9}n - \frac{20}{9}h = p + e + c - \frac{20}{9}n - \frac{2}{9}h.$

First we apply the following transformation: $e' = e - \frac{9}{10}\varepsilon, p' = p + \varepsilon, h' = h + \frac{9}{10}\varepsilon, c' = c, n' = n.$ Notice that if conditions (3), (4), and (5) hold for $(p, e, c, h, n)$, then they also hold for $(p', e', c', h', n')$. On the other hand, $F(p', e', c', h', n') = F(p, e, c, h, n) - \frac{1}{10}\varepsilon$. Therefore, the five-tuple $(p, e, c, h, n)$ that minimizes $F(p, e, c, h, n)$ under conditions (3), (4), and (5) has $e = 0$.

For parameter $h$, the only condition is that $c \geq h$. If $c > h$ and we increase $h$, then $F(p, 0, c, h, n)$ decreases, and the conditions still hold. Therefore, we may assume $c = h$. Now we have to minimize $F(p, 0, c, c, n) = p + \frac{7}{9}c - \frac{20}{9}n$ under the condition $9p + 7c \geq 20n - 30$. We get immediately that the minimum of $F(p, 0, c, c, n)$ under the conditions is $-\frac{10}{3}$. Consequently $E - \frac{20}{9}N \geq -\frac{10}{3}$.

Therefore, $E \geq \frac{20}{9}N - \frac{10}{3}$ for any maximal 1-planar drawing with $N \geq 4$ vertices and $E$ edges. ∎

**Remark.** *We believe that our bound is far from optimal. If our bound was close to optimal, then for some maximal 1-plane graph we would have to use operation 4 in almost every step of the algorithm described in the proof of Lemma 7. However, this seems impossible.*

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