On series of translates of positive functions III

Zoltán Buczolich[*], Department of Analysis, ELTE Eötvös Loránd University, Pázmány Péter Sétány 1/c, 1117 Budapest, Hungary
email: buczo@cs.elte.hu
www.cs.elte.hu/∼buczo
ORCID Id: 0000-0001-5481-8797

Balázs Maga[†], Department of Analysis, ELTE Eötvös Loránd University, Pázmány Péter Sétány 1/c, 1117 Budapest, Hungary
email: magab@cs.elte.hu
www.cs.elte.hu/∼magab

and

Gáspár Vértesy[‡], Department of Analysis, ELTE Eötvös Loránd University, Pázmány Péter Sétány 1/c, 1117 Budapest, Hungary
email: vertesy.gaspar@gmail.com

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Dedicated to the memory of Jean-Pierre Kahane

Abstract

Suppose Λ is a discrete infinite set of nonnegative real numbers. We say that Λ is of type 1 if the series $s(x) = \sum_{\lambda\in\Lambda} f(x+\lambda)$ satisfies a zero-one law. This means that for any non-negative measurable $f : \mathbb{R} \to [0, +\infty)$ either the convergence set $C(f, \Lambda) = \{x : s(x) < +\infty\} = \mathbb{R}$ modulo sets of Lebesgue zero, or its complement the divergence set $D(f, \Lambda) = \{x : s(x) = +\infty\} = \mathbb{R}$ modulo sets of measure zero. If Λ is not of type 1 we say that Λ is of type 2.

In this paper we show that there is a universal Λ with gaps monotone decreasingly converging to zero such that for any open subset $G\subset\mathbb{R}$ one can find a characteristic function f_G such that $G\subset D(f_G, \Lambda)$ and $C(f_G, \Lambda) = \mathbb{R}\setminus G$ modulo sets of measure zero.

We also consider the question whether $C(f, \Lambda)$ can contain non-degenerate intervals for continuous functions when $D(f, \Lambda)$ is of positive measure.

The above results answer some questions raised in a paper of Z. Buczolich, J-P. Kahane, and D. Mauldin.

1 Introduction

This paper was written for the Kahane memorial volume of Analysis Mathematica. We selected a topic related to Jean-Pierre Kahane's work and decided to answer some questions raised in paper [1] by Z. Buczolich, J-P. Kahane, and D. Mauldin.

This line of research was started in another joint paper with Dan Mauldin [3]. In that paper we considered a problem from 1970, originating from the Diplomarbeit of Heinrich von Weizsäker [8].

Suppose $f : (0, +\infty) \to \mathbb{R}$ is a measurable function. Is it true that $\sum_{n=1}^{\infty} f(nx)$ either converges (Lebesgue) almost everywhere or diverges almost everywhere, i.e. is there a zero-one law for $\sum f(nx)$?

This question also appeared in a paper of J. A. Haight [5].

In [5] it was proved that there exists a set $H\subset(0, \infty)$ of infinite measure, for which for all $x, y \in H,\ x \neq y$ the ratio x/y is not an integer, and furthermore

(†) *for all $x > 0$ $nx \notin H$ if n is sufficiently large.*

This implies that if $f(x) = \chi_H(x)$, the characteristic function of H then $\int_0^{\infty} f(x)dx = \infty$ and $\sum_{n=1}^{\infty} f(nx) < \infty$ everywhere.

Lekkerkerker in [7] started to study sets with property (†).

In [3] we answered the Haight–Weizsäker problem.

Theorem 1.1. *There exists a measurable function $f : (0, +\infty) \to \{0, 1\}$ and two nonempty intervals $I_F,\ I_\infty\subset[\frac{1}{2}, 1)$ such that for every $x \in I_\infty$ we have $\sum_{n=1}^{\infty} f(nx) =$*

$+\infty$ *and for almost every* $x \in I_F$ *we have* $\sum_{n=1}^{\infty} f(nx) < +\infty$. *The function* f *is the characteristic function of an open set* E.

Jean-Pierre Kahane was interested in this problem and soon after our paper had become available we started to receive faxes and emails from him. This cooperation lead to papers [1] and [2].

We considered a more general, additive version of the Haight–Weizsäker problem. Since $\sum_{n=1}^{\infty} f(nx) = \sum_{n=1}^{\infty} f(e^{\log x + \log n})$, that is using the function $h = f \circ \exp$ defined on $\mathbb{R}$ and $\Lambda = \{\log n : n = 1, 2, ...\}$ we were interested in almost everywhere convergence questions of the series $\sum_{\lambda\in\Lambda} h(x + \lambda)$.

Taking more general sets than $\Lambda = \{\log n : n = 1, 2, ...\}$ was also motivated by a paper, [6] of Haight. He proved, using the original multiplicative notation of our problem that if $\Lambda \subset [0, +\infty)$ is an arbitrary countable set such that its only accumulation point is $+\infty$ then there exists a measurable set $E \subset (0, +\infty)$ of infinite measure such that for all $x, y \in E$, $x \neq y$, $x/y \notin \Lambda$, and for a fixed x there exist only finitely many $\lambda \in \Lambda$ for which $\lambda x \in E$. This implies that choosing $f = \chi_E$ we have $\sum_{\lambda\in\Lambda} f(\lambda x) < \infty$, but $\int_{\mathbb{R}^+} f(x)dx = \infty$.

Next we recall from [1] the definition of type 1 and type 2 sets. Given Λ an unbounded, infinite discrete set of nonnegative numbers, and a measurable $f : \mathbb{R} \to [0, +\infty)$, we consider the sum

$$s(x) = \sum_{\lambda\in\Lambda} f(x + \lambda),$$

and the complementary subsets of $\mathbb{R}$:

$$C = C(f, \Lambda) = \{x : s(x) < \infty\}, \qquad D = D(f, \Lambda) = \{x : s(x) = \infty\}.$$

Definition 1.2. The set Λ is of type 1 if, for every f, either $C(f, \Lambda) = \mathbb{R}$ a.e. or $C(f, \Lambda) = \emptyset$ a.e. (or equivalently $D(f, \Lambda) = \emptyset$ a.e. or $D(f, \Lambda) = \mathbb{R}$ a.e.). Otherwise, Λ has type 2.

That is for type 1 sets we have a "zero-one" law for the almost everywhere convergence properties of the series $\sum_{\lambda\in\Lambda} f(x + \lambda)$, while for type 2 sets the situation is more complicated.

Definition 1.3. The unbounded, infinite discrete set $\Lambda = \{\lambda_1, \lambda_2, ...\}$, $\lambda_1 < \lambda_2 < ...$ is asymptotically dense if $d_n = \lambda_n - \lambda_{n-1} \to 0$, or equivalently:

$$\forall a > 0, \quad \lim_{x\to\infty} \#(\Lambda \cap [x, x + a]) = \infty.$$

If d_n tends to zero monotone decreasingly, we speak about decreasing gap asymptotically dense sets.

If Λ is not asymptotically dense we say that it is asymptotically lacunary.

We denote the non-negative continuous functions on $\mathbb{R}$ by $C^+(\mathbb{R})$, and if, in addition these functions tend to zero in $+\infty$ they belong to $C_0^+(\mathbb{R})$.

In [1] we gave some necessary and some sufficient conditions for a set Λ being of type 2. A complete characterization of type 2 sets is still unknown. We recall here from [1] the theorem concerning the Haight–Weizsäker problem. This contains the additive version of the result of Theorem 1.1 with some additional information.

Theorem 1.4. *The set $\Lambda = \{\log n : n = 1, 2, ...\}$ has type 2. Moreover, for some $f \in C_0^+(\mathbb{R})$, $C(f, \Lambda)$ has full measure on the half-line $(0, \infty)$ and $D(f, \Lambda)$ contains the half-line $(-\infty, 0)$. If for each c, $\int_c^{+\infty} e^y g(y)dy < +\infty$, then $C(g, \Lambda) = \mathbb{R}$ a.e. If $g \in C_0^+(\mathbb{R})$ and $C(g, \Lambda)$ is not of the first (Baire) category, then $C(g, \Lambda) = \mathbb{R}$ a.e. Finally, there is some $g \in C_0^+(\mathbb{R})$ such that $C(g, \Lambda) = \mathbb{R}$ a.e. and $\int_0^{+\infty} e^y g(y)dy = +\infty$.*

As Λ used in the above theorem is a decreasing gap asymptotically dense set and quite often it is much easier to construct examples with lacunary Λs, in our paper we try to give examples with a decreasing gap asymptotically dense Λ.

One might believe that for type 2 Λs $C(f, \Lambda)$, or $D(f, \Lambda)$ are always half-lines if they differ from $\mathbb{R}$. Indeed in [1] we obtained results in this direction. A number $t > 0$ is called a translator of Λ if $(\Lambda + t)\backslash\Lambda$ is finite. Condition $(*)$ is said to be satisfied if $T(\Lambda)$, the countable additive semigroup of translators of Λ, is dense in $\mathbb{R}^+$. We showed that condition $(*)$ implies that $C(f, \Lambda)$ is either $\emptyset$, $\mathbb{R}$, or a right half-line modulo sets of measure zero.

In [4] we showed that this is not always the case. For a given $\alpha \in (0, 1)$ and a sequence of natural numbers $n_1 < n_2 < ...$ we put $\Lambda^{\alpha^k} := \cup_{k=1}^{\infty} \Lambda_k^{\alpha^k}$, $\Lambda_k^{\alpha^k} = \alpha^k \mathbb{Z} \cap [n_k, n_{k+1})$.

If $\alpha = \frac{1}{q}$ for some $q \in \{2, 3, ...\}$, then a slight modification of the proof of Theorem 1 of [1] shows that $\Lambda^{(\frac{1}{q})^k}$ is of type 1 and condition $(*)$ is satisfied.

If $\alpha \notin \mathbb{Q}$, then one can apply Theorem 5 of [1] to show that Λ^{α^k} is of type 2.

The difficult case is when $\alpha = \frac{p}{q}$ with $(p, q) = 1$, $p, q > 1$, $p < q$. In this case we showed that $\Lambda^{(\frac{p}{q})^k}$ is of type 2. In the cases $\Lambda^{(\frac{p}{q})^k}$, $(p > 1)$ condition $(*)$ is not satisfied and we also showed in [4] that there exists a characteristic function f such that $C(f, \Lambda)$ does not equal $\emptyset$, $\mathbb{R}$, or a right half-line modulo sets of measure zero. This structure of $C(f, \Lambda)$ had not been seen before our paper [4].

From the point of view of our current paper the following question (QUESTION 2 in [1]) is the most relevant:

Question 1.5. Given open sets G_1 and G_2 when is it possible to find Λ and f such that $C(f, \Lambda)$ contains G_1 and $D(f, \Lambda)$ contains G_2?

It was remarked in [1] that if the counting function of Λ, $n(x) = \#\{\Lambda \cap [0, x]\}$

satisfies a condition of the type

$$\forall \ell < 0 \ \forall a \in \mathbb{R} \quad \limsup_{x\to\infty} \frac{n(x+\ell+a) - n(x+a)}{n(x+\ell) - n(x)} < +\infty$$

(as is the case for $\Lambda = \{\log n\}$) then either $C(f, \Lambda)$ has full measure on $\mathbb{R}$ or $C(f, \Lambda)$ does not contain any interval.

It was also mentioned in [1] that if Λ is asymptotically lacunary then it is possible to construct $f \in C_0^+(\mathbb{R})$ such that both $C(f, \Lambda)$ and $D(f, \Lambda)$ have interior points.

In this paper we give an almost complete answer to Question 1.5. In Section 2 we prove Theorem 2.1. This theorem states that there is a universal decreasing gap asymptotically dense Λ such that for any open subset $G \subset \mathbb{R}$ one can find a characteristic function f_G such that $G \subset D(f_G, \Lambda)$ and $C(f_G, \Lambda) = \mathbb{R} \setminus G$ modulo sets of measure zero. We also show that one can also select a $g_G \in C_0^+(\mathbb{R})$ with similar properties.

In Section 3 we consider the question of subintervals in $C(f, \Lambda)$ when $f \in C_0^+(\mathbb{R})$. In Theorem 3.1 we prove that there exists a universal asymptotically dense infinite discrete set Λ such that for any open set $G \subset \mathbb{R}$ one can select an $f_G \in C_0^+(\mathbb{R})$ such that $D(f_G, \Lambda) = G$. In this case there is no exceptional set of measure zero, $D(f_G, \Lambda)$ equals G exactly. On the other hand, Λ is not of decreasing gap. As Theorem 3.4 shows it is impossible to find such a universal Λ with decreasing gaps. In Theorem 3.4 we prove that if Λ is a decreasing gap asymptotically dense set, $f \in C^+(\mathbb{R})$ and x is an interior point of $C(f, \Lambda)$ then $[x, +\infty) \cap D(f, \Lambda)$ is of zero Lebesgue measure.

The example provided in Theorem 3.3 demonstrates that there is a decreasing gap asymptotically dense Λ and an $f \in C_0^+(\mathbb{R})$ such that $D(f, \Lambda)$ and $C(f, \Lambda)$ both contain interior points. Of course, as Theorem 3.4 shows the interior points of $D(f, \Lambda)$ are to the left of those of $C(f, \Lambda)$.

2 A universal decreasing gap asymptotically dense Λ set

Let μ denote the one-dimensional Lebesgue measure.

We denote by $\mathbb{N} := \{n \in \mathbb{Z} : n \geq 1\}$ the set of natural numbers. For every $A, B \subset \mathbb{R}$ we put $A + B := \{a + b : a \in A \text{ and } b \in B\}$ and $A - B := \{a - b : a \in A \text{ and } b \in B\}$.

The integer, and fractional parts of $x \in \mathbb{R}$ are denoted by $\lfloor x \rfloor$ and $\{x\}$, respectively.

Theorem 2.1. *There is a strictly monotone increasing unbounded sequence* $(\lambda_0, \lambda_1, \dots) = \Lambda$ *in* $\mathbb{R}$ *such that* $\lambda_n - \lambda_{n-1}$ *tends to* 0 *monotone decreasingly, that is* Λ *is a decreasing gap asymptotically dense set, such that for every open set* $G \subset \mathbb{R}$ *there is a function* $f_G : \mathbb{R} \to [0, +\infty)$ *for which*

$$\mu\left(\left\{x \notin G : \sum_{n=0}^{\infty} f_G(x+\lambda_n) = \infty\right\}\right) = 0, \text{ and} \tag{1}$$

$$\sum_{n=0}^{\infty} f_G(x+\lambda_n) = \infty \text{ for every } x \in G, \tag{2}$$

moreover $f_G = \chi_{U_G}$ *for a closed set* $U_G \subset \mathbb{R}$. *By* (1) *and* (2) *we have* $D(f_G, \Lambda) \supset G$, *and* $C(f_G, \Lambda) = \mathbb{R} \backslash G$ *modulo sets of measure zero.*

One can also select a $g_G \in C_0^+(\mathbb{R})$ *satisfying* (1) *and* (2) *instead of* f_G.

Remark 2.2. Observe that in the above theorem we construct a universal Λ and for this set, depending on our choice of G we can select a suitable f_G such that $D(f_G, \Lambda) = G$ modulo sets of measure zero.

Proof. Let

$$\mathcal{I} := \{(j,k) : j \in \mathbb{N} \text{ and } k \in \mathbb{Z} \cap [0, 2j \cdot 2^j)\}$$

with the following lexicographical ordering: if $(j,k), (\widetilde{j}, \widetilde{k}) \in \mathcal{I}$ then

$$(j,k) <_{\mathcal{I}} (\widetilde{j}, \widetilde{k}) \Leftrightarrow \big(j < \widetilde{j} \text{ or } (j = \widetilde{j} \text{ and } k < \widetilde{k})\big).$$

Given $(j,k) \in \mathcal{I}$ we define its immediate successor $(\hat{j}, \hat{k})$ the following way: let $\hat{j} := j$ and $\hat{k} := k+1$ if $k < 2j \cdot 2^j - 1$, and let $\hat{j} := j+1$ and $\hat{k} := 0$ if $k = 2j \cdot 2^j - 1$. It is clear that starting with $(1,0)$ by repeated application of taking the immediate successor we can enumerate $\mathcal{I}$ and hence we will be able to do induction on $\mathcal{I}$. We will also introduce the operation of taking the predecessor of $(j,k) \neq (1,0)$ which will be denoted by $(\check{j}, \check{k})$ and which is defined by the property $(\hat{\check{j}}, \hat{\check{k}}) = (j,k)$.

For every $(j,k) \in \mathcal{I}$ let

$$I_{j,k} := \left[j - (k+1)2^{-j}, j - k2^{-j}\right] = [a_{I_{j,k}}, b_{I_{j,k}}].$$

In (6) a set $U_{j,k}$ will be defined such that with a properly selected Λ we have

$$I_{j,k} \subset U_{j,k} - \Lambda = \{x \in \mathbb{R} : \exists n \in \mathbb{N} \cup \{0\} \text{ such that } x + \lambda_n \in U_{j,k}\} \text{ and} \tag{3}$$

$$\mu(\{x \in [-j,j] : \exists \text{ infinitely many } (j^*, k^*) \in \mathcal{I} \text{ for which } x \in (U_{j^*,k^*} - \Lambda) \setminus I_{j^*,k^*}\}) = 0. \tag{4}$$

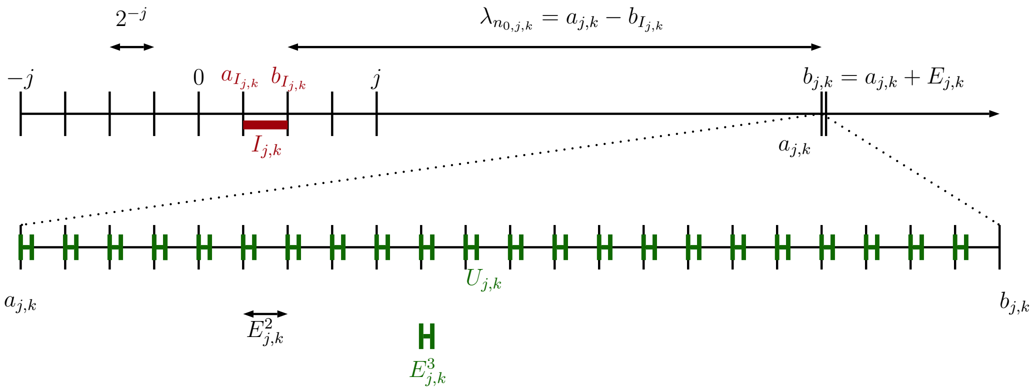

Figure 1: Definition of $I_{j,k}$ and $U_{j,k}$

Let G be an arbitrary open subset of $\mathbb{R}$ and let

$$U_G := \bigcup \{U_{j^*,k^*} : (j^*, k^*) \in \mathcal{I} \text{ and } I_{j^*,k^*} \subset G\}\,.$$

Put

$$f_G(x) := \begin{cases} 1 & \text{if } x \in U_G \\ 0 & \text{else}\,. \end{cases} \tag{5}$$

We will prove that Λ and f_G satisfy the conditions of the theorem.

Now we define the sets $U_{j,k}$. Before doing this we recall and introduce some notation. For every $(j,k) \in \mathcal{I}$ let

- $a_{I_{j,k}} := j - (k+1) \cdot 2^{-j}$ (that is $a_{I_{j,k}}$ is the left endpoint of $I_{j,k}$),
- $b_{I_{j,k}} := j - k \cdot 2^{-j}$ (that is $b_{I_{j,k}}$ is the right endpoint of $I_{j,k}$),
- $E_{j,k} := 2^{-2j \cdot 2^j - k}$,
- $a_{j,k} := 2^{2j \cdot 2^j + k}$,
- $b_{j,k} := a_{j,k} + E_{j,k}$.

See Figure 1. This and the other figure in this paper are to illustrate concepts and they are not drawn to illustrate a certain step, for example with a fixed j of our construction.

Let

$$U_{j,k} := \bigcup_{i=0}^{E_{j,k}^{-1}-1} [a_{j,k} + iE^2_{j,k}, a_{j,k} + iE^2_{j,k} + E^3_{j,k}] \subset [a_{j,k}, b_{j,k}]. \tag{6}$$

Next we prove a useful lemma:

Lemma 2.3. *For every* $(j,k) \in \mathcal{I}$ *we have*

$$a_{j,k} \le \frac{a_{\hat{\jmath},\hat{k}}}{2} \text{ and } E_{j,k} \ge 2E_{\hat{\jmath},\hat{k}}, \tag{7}$$

moreover,

$$E_{j,k}/2 \text{ is an integer multiple of } E_{\hat{\jmath},\hat{k}}. \tag{8}$$

Proof. It is enough to prove (7) for $a_{j,k}$ as $E_{j,k} = a_{j,k}^{-1}$.

First suppose that $k < 2j \cdot 2^j - 1$, then $\hat{\jmath} = j$, $\hat{k} = k+1$ and

$$a_{j,k} = 2^{2j\cdot 2^j + k} = \frac{2^{2j\cdot 2^j+(k+1)}}{2} = \frac{a_{\hat{\jmath},\hat{k}}}{2}. \tag{9}$$

If $k = 2j \cdot 2^j - 1$ then $\hat{\jmath} = j+1$, $\hat{k} = 0$ and

$$a_{j,k} = 2^{2j\cdot 2^j+k} = 2^{2j\cdot 2^j + 2j\cdot 2^j - 1} = 2^{4j2^j-1} = \frac{2^{2(j+1)\cdot 2^{(j+1)}}}{2^{2\cdot 2^{j+1}+1}} = \frac{a_{\hat{\jmath},\hat{k}}}{2^{2\cdot 2^{j+1}+1}}. \tag{10}$$

Using $E_{j,k} = a_{j,k}^{-1}$ from (9) and (10) it follows that (8) holds. □

Next we turn to the definition of Λ.

During the definition of Λ we will use the notation $d_n := \lambda_n - \lambda_{n-1}$, in fact, often we will define d_n and that will provide the value of λ_n given the already defined λ_{n-1}. Let $\lambda_0 := a_{1,0} - b_{I_{1,0}}$ and $n_{0,1,0} = 0$.

Suppose that for a $(j,k) \in \mathcal{I}$ we have already defined $n_{0,j,k}$ and λ_n for $n \le n_{0,j,k}$, $\lambda_{n_{0,j,k}} = a_{j,k} - b_{I_{j,k}}$ and $d_{n_{0,j,k}}/E_{j,k}^2$ is a positive integer (or $n_{0,j,k} = 0$). Now we need to do our next step to define these objects for $(\hat{\jmath},\hat{k})$.

Step $(\hat{\jmath},\hat{k})$**.** Let $n_{1,j,k} := n_{0,j,k} + 2^{-j}E_{j,k}^{-2} + 2E_{j,k}^{-1}$. For every integer $n \in [n_{0,j,k}+1, n_{1,j,k}]$ let $d_n := E_{j,k}^2 - E_{j,k}^3$. Thus we have

$$\begin{aligned} \lambda_{n_{1,j,k}} &= \lambda_{n_{0,j,k}} + (2^{-j}E_{j,k}^{-2} + 2E_{j,k}^{-1})(E_{j,k}^2 - E_{j,k}^3) \\ &= a_{j,k} - b_{I_{j,k}} + 2^{-j} - 2^{-j}E_{j,k} + 2E_{j,k} - 2E_{j,k}^2 \\ &= a_{j,k} - a_{I_{j,k}} + 2E_{j,k} - 2^{-j}E_{j,k} - 2E_{j,k}^2 \\ &= b_{j,k} - a_{I_{j,k}} + E_{j,k} - 2^{-j}E_{j,k} - 2E_{j,k}^2 \ge b_{j,k} - a_{I_{j,k}} \end{aligned} \tag{11}$$

and (from the second row of (11))

$$\lambda_{n_{1,j,k}} = a_{j,k} - b_{I_{j,k}} + 2^{-j} - 2^{-j}E_{j,k} + 2E_{j,k} - 2E_{j,k}^2 < a_{j,k} - b_{I_{j,k}} + 1. \tag{12}$$

Since $a_{j,k} - a_{I_{j,k}} = 2^{2j\cdot 2^j+k} - (j - k\cdot 2^{-j})$ and $2^{-j}E_{j,k}$ are both integer multiples of $E_{j,k}^2 = (2^{-2j\cdot 2^j-k})^2$ from the third row of (11) we obtain that

$$\lambda_{n_{1,j,k}} \text{ is an integer multiple of } E_{j,k}^2. \tag{13}$$

By Lemma 2.3 and (12) we have

$$a_{\hat{\jmath},\hat{k}} - b_{I_{\hat{\jmath},\hat{k}}} \geq 2a_{j,k} - (j+1) \geq a_{j,k} + j + 1 > a_{j,k} - b_{I_{j,k}} + 1 > \lambda_{n_{1,j,k}}.$$

We set

$$n_{0,\hat{\jmath},\hat{k}} = n_{1,j,k} + \frac{a_{\hat{\jmath},\hat{k}} - b_{I_{\hat{\jmath},\hat{k}}} - \lambda_{n_{1,j,k}}}{2^{-1}E^2_{j,k}} \tag{14}$$

and

$$d_n = E^2_{j,k}/2 \text{ for every integer } n \in (n_{1,j,k}, n_{0,\hat{\jmath},\hat{k}}]. \tag{15}$$

We obtain by (14)

$$\lambda_{n_{0,\hat{\jmath},\hat{k}}} = \lambda_{n_{1,j,k}} + \frac{(n_{0,\hat{\jmath},\hat{k}} - n_{1,j,k})E^2_{j,k}}{2} = \lambda_{n_{1,j,k}} + a_{\hat{\jmath},\hat{k}} - b_{I_{\hat{\jmath},\hat{k}}} - \lambda_{n_{1,j,k}} = a_{\hat{\jmath},\hat{k}} - b_{I_{\hat{\jmath},\hat{k}}},$$

and by (8), $d_{n_{0,\hat{\jmath},\hat{k}}} = E^2_{j,k}/2$ is an integer multiple of $E^2_{\hat{\jmath},\hat{k}}$, hence (13) implies that

$$\lambda_n \text{ is an integer multiple of } E^2_{\hat{\jmath},\hat{k}} \text{ for } n \in (n_{1,j,k}, n_{0,\hat{\jmath},\hat{k}}]. \tag{16}$$

Thus we can proceed to the next step. By repeating this procedure we can carry out the above steps for all $(j,k) \in \mathcal{I}$ and hence we can define Λ.

Now we prove (3). We fix (j,k) and choose an arbitrary point x from $I_{j,k}$. Let n_x denote the smallest integer for which

$$x + \lambda_{n_x} > a_{j,k}. \tag{17}$$

Put $n'_x := n_x + \left\lfloor \frac{x+\lambda_{n_x} - a_{j,k}}{E^3_{j,k}} \right\rfloor$.

We have $x \in I_{j,k} \subset [-j, j]$. From $x + \lambda_{n_{0,j,k}} = x + a_{j,k} - b_{I_{j,k}}$ it follows that

$$x + \lambda_{n_{0,j,k}} - a_{j,k} = x - b_{I_{j,k}} \leq 0. \tag{18}$$

Therefore, $n_x > n_{0,j,k}$ and hence

$$d_n \leq d_{n_{0,j,k}+1} = E^2_{j,k} - E^3_{j,k} \text{ for every } n \in [n_x, \infty). \tag{19}$$

By minimality of n_x we have

$$x + \lambda_{n_x} - a_{j,k} \leq d_{n_x} \leq E^2_{j,k} - E^3_{j,k}. \tag{20}$$

Next we will show that $x + \lambda_{n'_x} \in U_{j,k}$. Using (19)

$$0 \leq \left\lfloor \frac{x + \lambda_{n_x} - a_{j,k}}{E^3_{j,k}} \right\rfloor \leq \frac{d_{n_x}}{E^3_{j,k}} \leq \frac{E^2_{j,k} - E^3_{j,k}}{E^3_{j,k}} = E^{-1}_{j,k} - 1. \tag{21}$$

We also infer

$$
\begin{aligned}
x+\lambda_{n'_x} &= x+\lambda_{n_x}+\sum_{n\in(n_x,n'_x]} d_n \le x+\lambda_{n_x}+\left\lfloor \frac{x+\lambda_{n_x}-a_{j,k}}{E^3_{j,k}}\right\rfloor (E^2_{j,k}-E^3_{j,k})\\
&= a_{j,k}+(x+\lambda_{n_x}-a_{j,k})+\left\lfloor \frac{x+\lambda_{n_x}-a_{j,k}}{E^3_{j,k}}\right\rfloor (E^2_{j,k}-E^3_{j,k})\\
&= a_{j,k}+\left\lfloor \frac{x+\lambda_{n_x}-a_{j,k}}{E^3_{j,k}}\right\rfloor E^2_{j,k}+E^3_{j,k}\left\{\frac{x+\lambda_{n_x}-a_{j,k}}{E^3_{j,k}}\right\}\\
&\quad\text{using (21)}\\
&\le a_{j,k}+(E^{-1}_{j,k}-1)E^2_{j,k}+E^3_{j,k}\le a_{j,k}+E_{j,k}=b_{j,k}.
\end{aligned}
\tag{22}
$$

From (11) and (22) we obtain

$$
\lambda_{n'_x}\le b_{j,k}-x\le b_{j,k}-a_{I_{j,k}}\le \lambda_{n_{1,j,k}},
$$

hence $n_x, n'_x \le n_{1,j,k}$, which means that $d_n = E^2_{j,k}-E^3_{j,k}$ for every $n\in(n_x,n'_x]$. This implies that the first inequality in (22) is, in fact an equality, that is

$$
x+\lambda_{n'_x}=a_{j,k}+\left\lfloor \frac{x+\lambda_{n_x}-a_{j,k}}{E^3_{j,k}}\right\rfloor E^2_{j,k}+E^3_{j,k}\left\{\frac{x+\lambda_{n_x}-a_{j,k}}{E^3_{j,k}}\right\}. \tag{23}
$$

Using (21) and (23) we can see that there exists an integer $i=\left\lfloor \frac{x+\lambda_{n_x}-a_{j,k}}{E^3_{j,k}}\right\rfloor \in [0,E^{-1}_{j,k}-1]$ such that

$$
a_{j,k}+iE^2_{j,k}\le x+\lambda_{n'_x}\le a_{j,k}+iE^2_{j,k}+E^3_{j,k}
$$

that is $x+\lambda_{n'_x}\in U_{j,k}$, which implies (3).

We continue with the proof of (4). Suppose $(\check{\jmath},\check{k}),(j,k),(\hat{\jmath},\hat{k})\in\mathcal{I}$. Then they are strictly monotone increasing in this order and are adjacent in the lexicographical ordering of $\mathcal{I}$. We have by Lemma 2.3 and the third row of (11)

$$
\begin{aligned}
j+\lambda_{n_{1,\check{\jmath},\check{k}}} &= j+a_{\check{\jmath},\check{k}}-a_{I_{\check{\jmath},\check{k}}}+2E_{\check{\jmath},\check{k}}-2^{-\check{\jmath}}E_{\check{\jmath},\check{k}}-2E^2_{\check{\jmath},\check{k}}\\
&< a_{\check{\jmath},\check{k}}+2j+1\le 2a_{\check{\jmath},\check{k}}\le a_{j,k},
\end{aligned}
\tag{24}
$$

that is $U_{j,k}-\lambda_{n_{1,\check{\jmath},\check{k}}}$ is to the right of j. By (16), $\lambda_n/E^2_{j,k}$ is an integer for every $n\in(n_{1,\check{\jmath},\check{k}},n_{0,j,k}]$. Therefore, (24) implies that

$$
\begin{aligned}
B_{j,k} :&= [b_{j,k}-\lambda_{n_{0,j,k}},j]\cap(U_{j,k}-\Lambda)\\
&= [b_{j,k}-\lambda_{n_{0,j,k}},j]\cap\big(U_{j,k}-\{\lambda_n : n\in(n_{1,\check{\jmath},\check{k}},n_{0,j,k}]\}\big)\\
&\subset[b_{j,k}-\lambda_{n_{0,j,k}},j]\cap\bigcup_{i\in\mathbb{Z}}[iE^2_{j,k},iE^2_{j,k}+E^3_{j,k}].
\end{aligned}
\tag{25}
$$

Similarly, by using (7)

$$-j+\lambda_{n_{0,\hat{j},\hat{k}}} = -j + a_{\hat{j},\hat{k}} - b_{I_{\hat{j},\hat{k}}} > a_{\hat{j},\hat{k}} - (2j+1) \\ \geq 2a_{j,k} - (2j+1) \geq a_{j,k} + E_{j,k} = b_{j,k}, \tag{26}$$

that is $U_{j,k} - \lambda_{n_{0,\hat{j},\hat{k}}}$ is to the left of $-j$. Since by (13) and (15) $\lambda_n / \left(E_{j,k}^2/2\right)$ is an integer for every $n \in [n_{1,j,k}, n_{0,\hat{j},\hat{k}}]$, (26) implies that

$$\begin{aligned} A_{j,k} :&= [-j, a_{j,k} - \lambda_{n_{1,j,k}}] \cap (U_{j,k} - \Lambda) \\ &= [-j, a_{j,k} - \lambda_{n_{1,j,k}}] \cap \Big(U_{j,k} - \{\lambda_n : n \in [n_{1,j,k}, n_{0,\hat{j},\hat{k}}]\}\Big) \\ &\subset [-j, a_{j,k} - \lambda_{n_{1,j,k}}] \cap \bigcup\nolimits_{i\in\mathbb{Z}} [iE_{j,k}^2/2, iE_{j,k}^2/2 + E_{j,k}^3]. \end{aligned} \tag{27}$$

We want to estimate the following expression from above:

$$\mu\left([-j,j] \cap (U_{j,k} - \Lambda) \setminus I_{j,k}\right) \\ \leq \mu\left(A_{j,k} \cup [a_{j,k} - \lambda_{n_{1,j,k}}, a_{I_{j,k}}] \cup [b_{I_{j,k}}, b_{j,k} - \lambda_{n_{0,j,k}}] \cup B_{j,k}\right). \tag{28}$$

By (25) and (27) we have

$$\begin{aligned} &\mu\left(A_{j,k} \cup B_{j,k}\right) \\ &\leq \mu\left([-j,j] \cap \Big(\bigcup\nolimits_{i\in\mathbb{Z}} [iE_{j,k}^2/2, iE_{j,k}^2/2 + E_{j,k}^3]\Big)\right) \\ &= E_{j,k}^3 \frac{2j}{E_{j,k}^2/2} = 4j \cdot E_{j,k}, \end{aligned} \tag{29}$$

and using the third row of (11)

$$\begin{aligned} \mu\left([a_{j,k} - \lambda_{n_{1,j,k}}, a_{I_{j,k}}]\right) &= a_{I_{j,k}} - \left(a_{j,k} - (a_{j,k} - a_{I_{j,k}} + 2E_{j,k} - 2^{-j}E_{j,k} - 2E_{j,k}^2)\right) \\ &= 2E_{j,k} - 2^{-j}E_{j,k} - 2E_{j,k}^2 \leq 2E_{j,k}. \end{aligned} \tag{30}$$

Moreover,

$$\mu[b_{I_{j,k}}, b_{j,k} - \lambda_{n_{0,j,k}}] = b_{j,k} - (a_{j,k} - b_{I_{j,k}}) - b_{I_{j,k}} = b_{j,k} - a_{j,k} = E_{j,k}. \tag{31}$$

Writing (29), (30) and (31) into (28) yields

$$\mu\left([-j,j] \cap (U_{j,k} - \Lambda) \setminus I_{j,k}\right) \leq (4j+3) \cdot E_{j,k}. \tag{32}$$

Thus

$$\sum_{(j^*,k^*)\in\mathcal{I}} \mu\left([-j,j] \cap (U_{j^*,k^*} - \Lambda) \setminus I_{j^*,k^*}\right) \tag{33}$$

$$\leq \sum_{\substack{(j^*,k^*)\in\mathcal{I}\\ j^*<j}} \mu\left([-j,j]\cap(U_{j^*,k^*}-\Lambda)\setminus I_{j^*,k^*}\right)$$

$$+ \sum_{(j^*,k^*)\in\mathcal{I}} \mu\left([-j^*,j^*]\cap(U_{j^*,k^*}-\Lambda)\setminus I_{j^*,k^*}\right)$$

$$\leq \sum_{\substack{(j^*,k^*)\in\mathcal{I}\\ j^*<j}} 2j + \sum_{(j^*,k^*)\in\mathcal{I}} (4j^*+3)\cdot E_{j^*,k^*}$$

$$\leq 2j\cdot 2j(2^{j-1}+...+1) + \sum_{j^*=1}^{\infty}\sum_{k^*=0}^{2j^*\cdot 2^{j^*}-1}(4j^*+3)E_{j^*,k^*}$$

$$\leq 4j^2\cdot 2^j + \sum_{j^*=1}^{\infty} 2j^*\cdot 2^{j^*}(4j^*+3)2^{-2j^*\cdot 2^{j^*}}$$

$$\leq 4j^2\cdot 2^j + \sum_{j^*=1}^{\infty}\left(8(j^*)^2+6j^*\right)2^{-2j^*\cdot 2^{j^*}+j^*} < \infty,$$

which by the Borel–Cantelli lemma implies (4).

Let G be a fixed open subset of $\mathbb{R}$. If $x\in G$, then $\{(j,k)\in\mathcal{I} : x\in I_{j,k}\subset G\}$ is an infinite set, hence according to (3) and (5)

$$\sum_{n=0}^{\infty} f_G(x+\lambda_n) = \infty.$$

If $x\in\mathbb{R}\setminus G$ and $\sum_{n=0}^{\infty} f_G(x+\lambda_n)=\infty$, then $\{n\in\mathbb{N} : x+\lambda_n\in U_G\}$ is an infinite set, which implies that $\{(j^*,k^*)\in\mathcal{I} : I_{j^*,k^*}\subset G \text{ and } x\in(U_{j^*,k^*}-\Lambda)\}$ is also infinite, thus (4) implies (1).

Next we see how one can modify f_G to obtain a $g_G\in C_0^+(\mathbb{R})$ still satisfying (1) and (2). In [1] there is Proposition 1, which says that one can modify f_G to obtain a $g_G\in C_0^+(\mathbb{R})$ such that $C(f_G,\lambda)=C(g_G,\lambda)$ a.e. and $D(f_G,\lambda)=D(g_G,\lambda)$ a.e. Since we want to preserve (2) we cannot change $D(f_G,\lambda)$ by an arbitrary set of measure zero. Hence in the next construction a little extra care is needed.

$$\text{Put } \Lambda_N=\{\lambda\in\Lambda : \lambda\leq 10N\} \text{ and } L_N=\#\Lambda_N. \tag{34}$$

Observe that $U_G\cap(-\infty,0]=\emptyset$, U_G does not contain a half-line, and $U_G\cap[0,N]$ is the union of finitely many disjoint closed intervals for any $N\in\mathbb{N}$.

Choose an open $\widetilde{U}_G\supset U_G$ such that it does not contain a half-line, and

$$\mu((\widetilde{U}_G\setminus U_G)\cap[N-1,N]) < \frac{2^{-N}}{L_N} \text{ for any } N\in\mathbb{N}. \tag{35}$$

Select a continuous function $\widetilde{g}_G$ such that $\widetilde{g}_G(x) = f_G(x)$ for $x \in U_G$, $\widetilde{g}_G(x) = 0$ if $x \notin \widetilde{U}_G$ and $|\widetilde{g}_G| \le 1$. Hence $\widetilde{g}_G \ge f_G$ on $\mathbb{R}$, and $D(\widetilde{g}_G, \Lambda) \supset D(f_G, \Lambda) \supset G$.

It is also clear that $0 \le \widetilde{g}_G - f_G \le \chi_{\widetilde{U}_G \setminus U_G} =: h_G$, and

$$\sum_{\lambda \in \Lambda} \Big(\widetilde{g}_G(x+\lambda) - f_G(x+\lambda) \Big) \le \sum_{\lambda \in \Lambda} h_G(x+\lambda). \tag{36}$$

Next we prove that

$$\sum_{\lambda \in \Lambda} h_G(x+\lambda) \text{ is finite almost everywhere,} \tag{37}$$

yielding that $C(\widetilde{g}_G, \Lambda)$ equals $C(f_G, \Lambda)$ modulo a set of measure zero.

Put $H_{G,K,\infty} = \{x \in [-K, K] : \sum_{\lambda \in \Lambda} h_G(x+\lambda) = \infty\}$. We will show that

$$\text{for any } K > 1 \text{ we have } \mu(H_{G,K,\infty}) = 0. \tag{38}$$

This clearly implies (37).

Observe that if $x \in H_{G,K,\infty}$, then there are infinitely many λs such that $x + \lambda \in \widetilde{U}_G \setminus U_G$, that is, $x \in ((\widetilde{U}_G \setminus U_G) - \lambda) \cap [-K, K]$. Thus, by the Borel–Cantelli lemma to prove (38) it is sufficient to show that

$$\sum_{\lambda \in \Lambda} \mu\Big(\Big((\widetilde{U}_G \setminus U_G) - \lambda \Big) \cap [-K, K] \Big) < \infty. \tag{39}$$

This is shown by the following estimate

$$\sum_{\lambda \in \Lambda} \mu\Big(\Big((\widetilde{U}_G \setminus U_G) - \lambda \Big) \cap [-K, K] \Big) = \sum_{\lambda \in \Lambda} \sum_{N=1}^{\infty} \mu\Big(\Big(((\widetilde{U}_G \setminus U_G) \cap [N-1, N]) - \lambda \Big) \cap [-K, K] \Big)$$

$$= \sum_{N=1}^{\infty} \sum_{\lambda \in \Lambda} \mu\Big(\Big((\widetilde{U}_G \setminus U_G) \cap [N-1, N] \Big) \cap [\lambda - K, \lambda + K] \Big)$$

$$= \sum_{N=1}^{K} \sum_{\lambda \in \Lambda} \mu\Big(\Big((\widetilde{U}_G \setminus U_G) \cap [N-1, N] \Big) \cap [\lambda - K, \lambda + K] \Big)$$

$$+ \sum_{N=K+1}^{\infty} \sum_{\lambda \in \Lambda} \mu\Big(\Big((\widetilde{U}_G \setminus U_G) \cap [N-1, N] \Big) \cap [\lambda - K, \lambda + K] \Big)$$

(with a finite S_1)

$$= S_1 + \sum_{N=K+1}^{\infty} \sum_{\lambda \in \Lambda,\, \lambda \le 10N} \mu\Big(\Big((\widetilde{U}_G \setminus U_G) \cap [N-1, N] \Big) \cap [\lambda - K, \lambda + K] \Big)$$

(now using (34) and (35))

$$\leq S_1 + \sum_{N=K+1}^{\infty} L_N \cdot \frac{2^{-N}}{L_N} < \infty.$$

So far we have shown that $\widetilde{g}_G$ satisfies (1) and (2). Since $\widetilde{g}_G \in C^+(\mathbb{R})$, but not in $C_0^+(\mathbb{R})$. We need to adjust it a little further.

Since G is open choose an increasing sequence of compact sets $G_K \subset G \cap [-K, K]$ such that $\bigcup_{K=1}^{\infty} G_K = G$.

Put $M_0 = 0$. Choose $M_1 \in \mathbb{R}$ such that for any $x \in G_1$ we have

$$\sum_{\lambda \in \Lambda,\ M_0+10<\lambda<M_1} \widetilde{g}_G(x+\lambda) > 1,$$

and $\widetilde{g}_G(M_1+5) = 0$. This latter property can be satisfied since by assumption $\widetilde{U}_G$ does not contain a half-line.

In general, if we already have selected M_{K-1} such that $\widetilde{g}_G(M_{K-1}+5(K-1)) = 0$ then choose $M_K \in \mathbb{R}$ such that for any $x \in G_K$ we have

$$\sum_{\lambda \in \Lambda,\ M_{K-1}+10K<\lambda<M_K} \widetilde{g}_G(x+\lambda) > K, \tag{40}$$

and $\widetilde{g}_G(M_K + 5K) = 0$.

For $x \leq M_1 + 5$ we put $g_G(x) = \widetilde{g}_G(x)$. For $K > 1$ and $x \in (M_{K-1} + 5(K - 1), M_K + 5K]$ we put $g_G(x) = \frac{1}{K}\widetilde{g}_G(x)$.

It is clear that $g_G \in C_0^+(\mathbb{R})$.

Since $g_G \leq \widetilde{g}_G$ we have $C(g_G, \Lambda) \supset C(\widetilde{g}_G, \Lambda)$. If we can show that $G \subset D(g_G, \Lambda)$ then we are done. Suppose $x \in G$. Then there is a K_x such that $x \in G_K$ for any $K \geq K_x$. Therefore, for these K we have $x \in [-K_x, K_x] \subset [-K, K]$ and by using (40)

$$\sum_{\lambda \in \Lambda,\ M_{K-1}+6K<\lambda<M_K+4K} g_G(x+\lambda) = \sum_{\lambda \in \Lambda,\ M_{K-1}+6K<\lambda<M_K+4K} \frac{1}{K}\widetilde{g}_G(x+\lambda) > 1,$$

for any $K \geq K_x$ and hence $x \in D(g_G, \Lambda)$. □

3 Subintervals in $C(f, \Lambda)$

Theorem 3.1. *There exists an asymptotically dense infinite discrete set Λ such that for any open set $G \subset \mathbb{R}$ one can select an $f_G \in C_0^+(\mathbb{R})$ such that $D(f, \Lambda) = G$.*

Remark 3.2. As Theorem 3.4 shows in the above theorem we cannot assume that Λ is a decreasing gap set. On the other hand, in our claim we have $D(f,\Lambda) = G$, that is, there is no exceptional set of measure zero where we do not know what happens. This also implies that if the interior of $\mathbb{R}\backslash G$ is non-empty then $C(f,\Lambda)$ contains intervals.

Figure 2: Definition of I_j, U_j and related sets

Proof. Denote by $\mathcal{I}_D = \{[(k-1)/2^l, k/2^l] : \ k,l \in \mathbb{Z}, \ l \geq 0\}$ the system of dyadic intervals. It is clear that one can enumerate the elements of $\mathcal{I}_D$ in a sequence $\{I_j\}_{j=1}^{\infty}$ which satisfies the following properties

$$I_j = [a_{I_j}, b_{I_j}] = \left[\frac{k_j - 1}{2^{l_j}}, \frac{k_j}{2^{l_j}}\right] \subset [-j, j] \text{ and } \mu(I_j) = 2^{-l_j} \geq \frac{1}{j}. \tag{41}$$

We denote by $\overline{I}_j$ the closed interval which is concentric with I_j but is of length three times the length of I_j.

We put

$$U_j = [a_j, b_j] = [2^j, 2^j + 2^{-2^j}] \text{ and } \overline{U}_j = [a_j - 2^{-2^j-j-1}, b_j + 2^{-2^j-j-1}] = [\overline{a}_j, \overline{b}_j].$$

See Figure 2.

We suppose that $f_j(x) = 0$ if $x \notin \overline{U}_j$, $f_j(x) = 2^{-j}$ if $x \in U_j$, the function f_j is continuous on $\mathbb{R}$ and is linear on the connected components of $\overline{U}_j \backslash U_j$. We define

$$\begin{aligned} \Lambda_{1,j} &= \{k \cdot 2^{-2^j-j} : k \in \mathbb{Z}\} \cap [2^j - k_j 2^{-l_j}, 2^j + 2^{-2^j} - (k_j - 1)2^{-l_j}] \\ &= \{k \cdot 2^{-2^j-j} : k \in \mathbb{Z}\} \cap [a_j - b_{I_j}, b_j - a_{I_j}] \end{aligned} \tag{42}$$

and put $\Lambda_1 = \bigcup_{j=1}^{\infty} \Lambda_{1,j}$.

Observe that if $x \in I_j$ then

$$x + \min \Lambda_{1,j} \leq b_{I_j} + \min \Lambda_{1,j} = b_{I_j} + a_j - b_{I_j} = a_j$$

and

$$x + \max \Lambda_{1,j} \geq a_{I_j} + \max \Lambda_{1,j} = a_{I_j} + b_j - a_{I_j} = b_j,$$

hence

$$\sum_{\lambda\in\Lambda_{1,j}} f_j(x+\lambda) \geq \frac{\operatorname{diam} U_j}{2^{-2^j-j}} 2^{-j} = \frac{2^{-2^j}}{2^{-2^j-j}} 2^{-j} = 1. \tag{43}$$

On the other hand, by (41)

$$\begin{aligned} &\overline{U}_j - \Lambda_{1,j} = \left[\min \overline{U}_j - \max \Lambda_{1,j}, \max \overline{U}_j - \min \Lambda_{1,j}\right] \\ &= [\overline{a}_j - b_j + a_{I_j}, \overline{b}_j - a_j + b_{I_j}] = \left[a_{I_j} - 2^{-2j} - 2^{-2^j-j-1}, b_{I_j} + 2^{-2j} + 2^{-2^j-j-1}\right] \\ &\subset \left[a_{I_j} - \frac{1}{j}, b_{I_j} + \frac{1}{j}\right] \subset \left[a_{I_j} - 2^{-l_j}, b_{I_j} + 2^{-l_j}\right] = \overline{I}_j \end{aligned}$$

thus

$$\sum_{\lambda\in\Lambda_{1,j}} f_j(x+\lambda) = 0 \text{ if } x \in [-j, j],\ x \notin \overline{I}_j. \tag{44}$$

Suppose $G\subset\mathbb{R}$ is a given open set and put $\mathcal{J}_G = \{j : \overline{I}_j \subset G\}$. Let $f_G(x) = \sum_{j\in\mathcal{J}_G} f_j(x)$. Then f_G is continuous and non-negative on $\mathbb{R}$ and clearly $\lim_{x\to\infty} f(x) = 0$.

We claim that

$$\sum_{\lambda\in\Lambda_1} f_G(x+\lambda) = +\infty \tag{45}$$

exactly on G.

Indeed, if $x \in G$ then there are infinitely many js such that $x \in I_j \subset \overline{I}_j \subset G$. This means that (43) holds for infinitely many $j \in \mathcal{J}_G$ and hence (45) is true when $x \in G$.

Next we need to verify that (45) does not hold for $x \notin G$. Suppose that $j_0 \geq 10$, $j_0 \in \mathcal{J}_G$, $x \notin G$ and $x \in [-j_0, j_0]$. Then $x \notin \overline{I}_{j_0}$ and by (44) we have

$$\sum_{\lambda\in\Lambda_{1,j_0}} f_{j_0}(x+\lambda) = 0. \tag{46}$$

Next assume that $j < j_0$. Then by using (41) and (42)

$$\begin{aligned} \max\{x+\lambda :\ \lambda \in \Lambda_{1,j}\} &\leq j_0 + 2^j + 2^{-2^j} - (k_j - 1)2^{-l_j} \leq j_0 + 2^j + 2^{-2^j} + j \\ &< 2j_0 + 2^{j_0-1} + 1 < 2^{j_0} - 1 < 2^{j_0} - 2^{-2^{j_0}-j_0-1} = \overline{a}_{j_0}. \end{aligned}$$

Hence,

$$\sum_{\lambda\in\Lambda_{1,j}} f_{j_0}(x+\lambda) = 0. \tag{47}$$

If $j_0 < j$ then

$$\min\{x+\lambda :\ \lambda \in \Lambda_{1,j}\} \geq -j_0 + 2^j - j > 2^{j-1} - 2j - 1 + 2^{j-1} + 1 > 2^{j_0} + 1 > \overline{b}_{j_0},$$

and hence in this case we also have (47).

Therefore, from (46) and (47) it follows that

$$\sum_{\lambda\in\Lambda_1} f_{j_0}(x+\lambda)=0 \text{ for } j_0\in\mathcal{J}_G,\ j_0\geq 10,\ |x|\leq j_0. \tag{48}$$

This implies

$$\sum_{\lambda\in\Lambda_1} f_G(x+\lambda)\leq \sum_{\substack{\lambda\in\Lambda_{1,j}\\ j\leq\max\{10,|x|\}}} f_j(x+\lambda)<+\infty.$$

Since Λ_1 is not asymptotically dense we need to choose an asymptotically dense Λ_2 such that

$$\sum_{\lambda\in\Lambda_2}\sum_{j=1}^{\infty} f_j(x+\lambda)<+\infty \text{ holds for any } x\in\mathbb{R}. \tag{49}$$

Then for any open $G\subset\mathbb{R}$

$$\sum_{\lambda\in\Lambda_2} f_G(x+\lambda)\leq\sum_{\lambda\in\Lambda_2}\sum_{j=1}^{\infty} f_j(x+\lambda)<+\infty$$

holds and if we let $\Lambda=\Lambda_1\cup\Lambda_2$ then Λ is asymptotically dense and $D(f_G,\Lambda)=G$.

To complete the proof of this theorem we need to verify (49) for a suitable Λ_2. For $j\geq 10$ put

$$\Lambda_{2,j}=\{k\cdot 2^{-j}:k\in\mathbb{Z}\}\cap(2^{j-1}+2(j-1),2^j+2j],\ \text{and } \Lambda_2=\bigcup\nolimits_{j=10}^{\infty}\Lambda_{2,j}.$$

Suppose $x\in[-j_0,j_0]$ and $j_0\geq 10$. Then for $j\geq j_0$ from $x+\lambda\in\overline{U}_j$ it follows that $2^j-1<x+\lambda\leq j+\lambda$, and hence

$$\lambda>2^j-j-1>2^{j-1}+2(j-1).$$

Similarly, $x+\lambda\in\overline{U}_j$ implies $2^j+1>x+\lambda\geq -j+\lambda$, and hence

$$\lambda<2^j+j+1<2^j+2j.$$

Thus from $x+\lambda\in\overline{U}_j$ it follows that $\lambda\in\Lambda_{2,j}$. Since the length of $\overline{U}_j$ is less than $2\cdot 2^{-2^j}<2^{-j}$ there is at most one $\lambda\in\Lambda_{2,j}$ for which $f_j(x+\lambda)\neq 0$ and for this λ we have $f_j(x+\lambda)=2^{-j}$.

Put $M_x=\max\{10,|x|\}$. Then

$$\sum_{\lambda\in\Lambda_2}\sum_{j=1}^{\infty} f_j(x+\lambda)=\sum_{\lambda\in\Lambda_2}\sum_{j=1}^{M_x} f_j(x+\lambda)+\sum_{j=M_x+1}^{\infty}\sum_{\lambda\in\Lambda_2} f_j(x+\lambda)$$

$$\leq \sum_{\lambda\in\Lambda_2}\sum_{j=1}^{M_x} f_j(x+\lambda) + \sum_{j=M_x+1}^{\infty} 2^{-j} < +\infty.$$

□

In Theorem 2.1 we verified that for decreasing gap asymptotically dense sets $D(f,\Lambda)$ can contain an open set, while $C(f,\Lambda)$ equals the complement of this open set only almost everywhere.

The next example shows that one can define decreasing gap asymptotically dense Λs for which one can find nonnegative continuous fs such that both $C(f,\Lambda)$ and $D(f,\Lambda)$ have interior points.

Theorem 3.3. *There exists a decreasing gap asymptotically dense Λ and an $f \in C_0^+(\mathbb{R})$ such that $I_1 = [0,1]\subset D(f,\Lambda)$ and $I_2 = [4,5]\subset C(f,\Lambda)$.*

Proof. Put $f(x) = 2^{-2^{j+1}}$ if $x \in [10j, 10j+1]$ for a $j \in \mathbb{N}$. Set $f(x) = 0$ if $x \in \{10j - 1/4, 10j + 5/4\}$ for a $j \in \mathbb{N}$, and also put $f(x) = 0$ for $x \leq 0$. We suppose that f is linear on the intervals where we have not defined it so far. Put $\Lambda_{1,j} = \{k \cdot 2^{-2^j} : k \in \mathbb{Z}\} \cap [10j-10, 10j-2)$ and $\Lambda_{2,j} = \{k \cdot 2^{-2^{j+1}} : k \in \mathbb{Z}\} \cap [10j-2, 10j)$. Let $\Lambda = \bigcup_{j=1}^{\infty}(\Lambda_{1,j} \cup \Lambda_{2,j})$. Observe that Λ is a decreasing gap asymptotically dense set.

One can see that for $x \in I_1$ we have

$$\sum_{\lambda\in\Lambda} f(x+\lambda) \geq \sum_{j=1}^{\infty} 2^{2^{j+1}} \cdot 2^{-2^{j+1}} = +\infty$$

and for $x \in I_2$

$$\sum_{\lambda\in\Lambda} f(x+\lambda) \leq \sum_{j=1}^{\infty} 2 \cdot 2^{2^j} \cdot 2^{-2^{j+1}} < +\infty.$$

It is also clear from the construction that $\lim_{x\to\infty} f(x) = 0$. □

Observe that in the above construction $I_1\subset D(f,\Lambda)$ was to the left of $I_2\subset C(f,\Lambda)$. The next theorem shows that for decreasing gap asymptotically dense Λs and continuous functions this situation cannot be improved. If x is an interior point of $C(f,\Lambda)$ then the half-line $[x,\infty)$ intersects $D(f,\Lambda)$ in a set of measure zero. As Theorem 3.1 shows if we do not assume that Λ is of decreasing gap then it is possible that $D(f,\Lambda)$ has a part of positive measure, even to the right of the interior points of $C(f,\Lambda)$.

Theorem 3.4. *Let Λ be a decreasing gap and asymptotically dense set, and let $f : \mathbb{R} \to [0,+\infty)$ be continuous. Then if x is an interior point of $C(f,\Lambda)$ then*

$$\mu\Big([x,+\infty) \cap D(f,\Lambda)\Big) = 0. \tag{50}$$

Proof. Proceeding towards a contradiction assume the existence of a non-degenerate closed interval $I \subset C(f,\Lambda)$. Suppose that there is a bounded subset $D_1(f,\Lambda) \subset D(f,\Lambda)$ with positive measure to the right of I. Choose an interval $J = [a_J, b_J]$ to the right of I such that

$$\mu(J) = \mu(I)/10, \text{ and } \mu(J \cap D(f,\Lambda)) = \alpha > 0. \tag{51}$$

We put $D_1(f,\Lambda) = J \cap D(f,\Lambda)$. We suppose that $\Lambda = \{\lambda_1, \lambda_2, ...\}$ is indexed in an increasing order. Select N such that

$$\lambda_n - \lambda_{n-1} < \frac{\mu(I)}{100} \text{ for } n \geq N. \tag{52}$$

We clearly have that $\sum_{i=N}^{\infty} f(x+\lambda_i)$ diverges on $D_1(f,\Lambda)$. Moreover, if $n \in \mathbb{N}$, which is to be fixed later, for large enough M we have $\sum_{i=N}^{M} f(x+\lambda_i) > n$ in a set $D_2(f,\Lambda) \subset D_1(f,\Lambda)$ of measure larger than $\frac{\alpha}{2}$. Hence we have

$$\int_{D_2(f,\Lambda)} \sum_{i=N}^{M} f(x+\lambda_i)dx \geq \frac{n\alpha}{2}. \tag{53}$$

Assume that $i \in \{N, N+1, ..., M\}$. We choose $\gamma(i)$ such that

$$a_J + \lambda_i - \lambda_{\gamma(i)} \in I, \text{ but } a_J + \lambda_i - \lambda_{\gamma(i)+1} \notin I. \tag{54}$$

Since a_J is to the right of I it is clear that $\lambda_{\gamma(i)} > \lambda_i$, therefore $\gamma(i) > i \geq N$ and hence (52) implies that $\gamma(i)$ is well-defined, that is (54) can be satisfied.

It is also clear that there exists $\widetilde{M}$ such that $\gamma(i) \leq \widetilde{M}$ holds for $i \in \{N, N+1, ..., M\}$.

By (51), (52), and (54) we have

$$J + \lambda_i - \lambda_{\gamma(i)} \subset I \text{ and hence } D_2(f,\Lambda) + \lambda_i - \lambda_{\gamma(i)} \subset I. \tag{55}$$

Next we verify that

$$\text{if } i' \neq i \text{ then } \gamma(i') \neq \gamma(i). \tag{56}$$

Indeed, we can suppose that $i' < i$, and proceeding towards a contradiction we also suppose that $\gamma(i') = \gamma(i)$. We know that $a_J + \lambda_i - \lambda_{\gamma(i)} \in I$, moreover $a_J + \lambda_{i'} - \lambda_{\gamma(i')} \in I$ holds as well. Since $\gamma(i) = \gamma(i')$ we have

$$a_J + \lambda_{i'} - \lambda_{\gamma(i')} = a_J + \lambda_i - \lambda_{\gamma(i)} - \lambda_i + \lambda_{i'} \in I.$$

Using the first half of (54) and $\lambda_{i'} \leq \lambda_{i-1} < \lambda_i$ we also obtain

$$a_J + \lambda_i - \lambda_{\gamma(i)} - \lambda_i + \lambda_{i'} \leq a_J + \lambda_i - \lambda_{\gamma(i)} - \lambda_i + \lambda_{i-1} \in I.$$

Since Λ is of decreasing gap and $\gamma(i) > i$ we have $\lambda_{\gamma(i)+1} - \lambda_{\gamma(i)} < \lambda_i - \lambda_{i-1}$, and hence

$$a_J + \lambda_i - \lambda_{\gamma(i)} - \lambda_i + \lambda_{i-1} < a_J + \lambda_i - \lambda_{\gamma(i)} - \lambda_{\gamma(i)+1} + \lambda_{\gamma(i)} \in I,$$

which contradicts (54).

By using (55) and (56) we infer

$$\int_{D_2(f,\Lambda)} \sum_{i=N}^{M} f(x+\lambda_i)dx = \sum_{i=N}^{M} \int_{D_2(f,\Lambda)} f(x+\lambda_i - \lambda_{\gamma(i)} + \lambda_{\gamma(i)})dx \tag{57}$$

$$= \sum_{i=N}^{M} \int_{D_2(f,\Lambda)+\lambda_i-\lambda_{\gamma(i)}} f(t+\lambda_{\gamma(i)})dt \leq \int_I \sum_{j=N}^{\widetilde{M}} f(t+\lambda_j)dt.$$

Thus by (53) we obtain

$$\int_I \sum_{i=N}^{\widetilde{M}} f(x+\lambda_i)dx \geq \frac{n\alpha}{2},$$

as the left-handside by (57) gives an upper bound for the integral in (53). However, $\sum_{i=N}^{\widetilde{M}} f(x+\lambda_i)$ is continuous, which yields that this integrand is at least $\frac{n\alpha}{4\mu(I)}$ in a non-degenerate closed subinterval $I_1 \subset I$. Thus we have $s(x) = \sum_{\lambda\in\Lambda} f(x+\lambda) > \frac{n\alpha}{4\mu(I)}$ in I_1. Hence, if we choose n to be large enough, we find that $s(x) > 1$ in I_1.

Now by applying the very same argument to I_1 instead of I, we might obtain that $s(x) > \frac{n_1\alpha}{4\mu(I_1)}$ in a non-degenerate closed subinterval $I_2 \subset I_1$. Thus if we choose n_1 to be large enough, we find that $s(x) > 2$ in I_2. Proceeding recursively we obtain a nested sequence of closed intervals $I_1, I_2, ...$ such that $s(x) > k$ for $x \in I_k$. As this system of intervals has a nonempty intersection, we find that there is a point in I with $s(x) = \infty$, a contradiction. □

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