

**BOUND FOR THE COCHARACTERS OF THE IDENTITIES OF
IRREDUCIBLE REPRESENTATIONS OF $\mathfrak{sl}_2(\mathbb{C})$**

MÁTYÁS DOMOKOS

Dedicated to Vesselin Drensky on his 70th birthday

ABSTRACT. For each irreducible finite dimensional representation of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ of 2×2 traceless matrices, an explicit uniform upper bound is given for the multiplicities in the cocharacter sequence of the polynomial identities satisfied by the given representation.

1. INTRODUCTION

Let $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ be a finite dimensional representation of the Lie algebra \mathfrak{g} over a field K of characteristic zero; that is, $\mathfrak{gl}(V) = \text{End}_K(V)$, the space of all K -linear transformations of the finite dimensional K -vector space V , viewed as a Lie algebra with Lie product $[A, B] := A \circ B - B \circ A$ for $A, B \in \text{End}_K(V)$, and ρ is a homomorphism of Lie algebras. Denote by $F_m := K\langle x_1, \dots, x_m \rangle$ the free associative K -algebra with m generators. Consider F_m a subalgebra of F_{m+1} in the obvious way, and write $F := \bigcup_{m=1}^{\infty} F_m$ for the free associative algebra of countable rank. We say that $f = 0$ is an identity of the representation ρ of \mathfrak{g} (or briefly, of the pair (\mathfrak{g}, ρ)) for some $f \in F_m$ if for any elements $A_1, \dots, A_m \in \mathfrak{g}$ we have the following equality in the associative K -algebra $\text{End}_K(V)$:

$$f(\rho(A_1), \dots, \rho(A_m)) = 0 \in \text{End}_K(V).$$

Note that an identity of the representation ρ of the Lie algebra \mathfrak{g} is also called in the literature a *weak polynomial identity* for the pair $(\text{End}_K(V), \rho(\mathfrak{g}))$. This notion was introduced and powerfully applied first by Razmyslov [13, 14, 15, 16] (see Drensky [8] for a recent survey on weak polynomial identities). Set

$$I(\mathfrak{g}, \rho) := \{f \in F \mid f = 0 \text{ is an identity of } (\mathfrak{g}, \rho)\}.$$

Clearly $I(\mathfrak{g}, \rho)$ is an ideal in F stable with respect to all K -algebra endomorphisms of F of the form $x_i \mapsto u_i$, where u_i for $i = 1, 2, \dots$ is an element of the Lie subalgebra of F generated by x_1, x_2, \dots . In particular, the general linear group $\text{GL}_m(K)$ acts on F_m via K -algebra automorphisms: for $g = (g_{ij})_{i,j=1}^m$ we have

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$g \cdot x_j = \sum_{i=1}^m g_{ij} x_i$, and $I(\mathfrak{g}, \rho) \cap F_m$ is a $\mathrm{GL}_m(K)$ -invariant subspace of F_m . The *multilinear component* of F_m is

$$P_m := \mathrm{Span}_K \{x_{\pi(1)} \cdots x_{\pi(m)} \mid \pi \in S_m\},$$

where S_m is the symmetric group of degree m . It is well known that when $\mathrm{char}(K) = 0$, the ideal $I(\mathfrak{g}, \rho)$ is determined by the multilinear components $I(\mathfrak{g}, \rho) \cap P_m$, $m = 1, 2, \dots$. Identifying S_m with the subgroup of permutation matrices in $\mathrm{GL}_m(K)$ we get its action on F_m via K -algebra automorphisms (more explicitly, $\pi \in S_m$ is the automorphism of F_m given by $x_i \mapsto x_{\pi(i)}$), and the subspaces P_m and $I(\mathfrak{g}, \rho) \cap P_m$ are S_m -invariant. Define the *m th cocharacter* of (\mathfrak{g}, ρ) as

$$\chi_m(\mathfrak{g}, \rho) := \text{the character of the } S_m\text{-module } P_m / (I(\mathfrak{g}, \rho) \cap P_m).$$

We call

$$\chi(\mathfrak{g}, \rho) := (\chi_m(\mathfrak{g}, \rho) \mid m = 1, 2, \dots)$$

the *cocharacter sequence* of (\mathfrak{g}, ρ) . The irreducible S_m -modules are labeled by partitions of m ; let χ^λ denote the character of the irreducible S_m -module associated to the partition $\lambda = (\lambda_1, \dots, \lambda_m) \vdash m$. We have

$$\chi_m(\mathfrak{g}, \rho) = \sum_{\lambda \vdash m} \mathrm{mult}_\lambda(\mathfrak{g}, \rho) \chi^\lambda,$$

and we are interested in the multiplicities $\mathrm{mult}_\lambda(\mathfrak{g}, \rho)$ of the irreducible S_m -characters in the cocharacter sequence. Note that the value of $\chi_m(\mathfrak{g}, \rho)$ on the identity element of S_m is

$$c_m(\mathfrak{g}, \rho) := \dim_K(P_m / (I(\mathfrak{g}, \rho) \cap P_m)),$$

and

$$(c_m(\mathfrak{g}, \rho) \mid m = 1, 2, \dots)$$

is called the *codimension sequence* of (\mathfrak{g}, ρ) . It was proved by Gordienko [10] that $\lim_{m \rightarrow \infty} \sqrt[m]{c_m(\mathfrak{g}, \rho)}$ exists and is an integer. As is observed in [10, Example 3], an obvious upper bound for $c_m(\mathfrak{g}, \rho)$ can be obtained from the fact that there is a natural K -linear embedding

$$(1) \quad P_m / (I(\mathfrak{g}, \rho) \cap P_m) \hookrightarrow \mathrm{Hom}_K(\rho(\mathfrak{g})^{\otimes m}, \mathrm{End}_K(V)).$$

Our starting observation is that the adjoint representation of \mathfrak{g} on itself induces a natural representation of \mathfrak{g} on $\rho(\mathfrak{g})^{\otimes m}$ (the m th tensor power of $\rho(\mathfrak{g})$) and on $\mathrm{End}_K(V)$, such that the image of the embedding (1) is contained in the subspace of \mathfrak{g} -module homomorphisms from $\rho(\mathfrak{g})^{\otimes m}$ to $\mathrm{End}_K(V)$. So (1) can be refined as

$$(2) \quad P_m / (I(\mathfrak{g}, \rho) \cap P_m) \hookrightarrow \mathrm{Hom}_{\mathfrak{g}}(\rho(\mathfrak{g})^{\otimes m}, \mathrm{End}_K(V)).$$

This will be used to give an upper bound for the multiplicities in the cocharacter sequence $\chi(\mathfrak{sl}_2(\mathbb{C}), \rho^{(d)})$ of the d -dimensional irreducible representation

$$\rho^{(d)} : \mathfrak{sl}_2(\mathbb{C}) \rightarrow \mathfrak{gl}(\mathbb{C}^d) = \mathbb{C}^{d \times d}$$

of $\mathfrak{sl}_2(\mathbb{C})$ for $d = 1, 2, \dots$. Note that throughout the paper we shall identify $\mathfrak{gl}(\mathbb{C}^d)$ with the associative algebra $\mathbb{C}^{d \times d}$ of $d \times d$ complex matrices, viewed as a Lie algebra with Lie bracket $[A, B] = AB - BA$.

Theorem 1.1. *The multiplicity $\mathrm{mult}_\lambda(\mathfrak{sl}_2(\mathbb{C}), \rho^{(d)})$ in $\chi(\mathfrak{sl}_2(\mathbb{C}), \rho^{(d)})$ is non-zero only if $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ (i.e. λ has at most 3 non-zero parts), and in this case we have the inequality*

$$\mathrm{mult}_\lambda(\mathfrak{sl}_2(\mathbb{C}), \rho^{(d)}) \leq 3^{d-2}.$$

Remark 1.2. (i) The exact values of $\text{mult}_\lambda(\mathfrak{sl}_2(\mathbb{C}), \rho^{(d)})$ are known for $d \leq 3$. For $d = 1$ all the multiplicities are obviously zero. It was proved in [12] (see also [7, Exercise 12.6.12]) that

$$\text{mult}_\lambda(\mathfrak{sl}_2(\mathbb{C}), \rho^{(2)}) = 1 \text{ for all } \lambda = (\lambda_1, \lambda_2, \lambda_3).$$

The multiplicities $\text{mult}_\lambda(\mathfrak{sl}_2(\mathbb{C}), \rho^{(3)})$ are computed in [5, Theorem 3.7, Proposition 3.8]. It turns out that $\text{mult}_\lambda(\mathfrak{sl}_2(\mathbb{C}), \rho^{(3)}) \in \{1, 2, 3\}$ for each $\lambda = (\lambda_1, \lambda_2, \lambda_3)$.

(ii) Theorem 1.1 shows in particular that for each dimension d , there is a uniform bound (depending on d only) for the multiplicities $\text{mult}_\lambda(\mathfrak{sl}_2(\mathbb{C}), \rho^{(d)})$. For comparison we mention that the multiplicities in the cocharacter sequence of the ordinary polynomial identities of 2×2 matrices are unbounded: see [6] and [9]. For example, for any partition $\lambda = (\lambda_1, \lambda_2)$ with $\lambda_2 > 0$, the multiplicity is $(\lambda_1 - \lambda_2 + 1)\lambda_2$. On the other hand, the cocharacter multiplicities of any PI algebra are polynomially bounded by [2].

(iii) There is no uniform upper bound independent of d for the multiplicities $\text{mult}_\lambda(\mathfrak{sl}_2(\mathbb{C}), \rho^{(d)})$, because by Proposition 4.1, $\max\{\text{mult}_\lambda(\mathfrak{sl}_2(\mathbb{C}), \rho^{(d)}) \mid m = 1, 2, \dots, \lambda \vdash m\} \geq d - 1$ for $d \geq 2$.

(iv) The irreducible representations of $\mathfrak{sl}_2(\mathbb{C})$ are defined over \mathbb{Q} . For any field K of characteristic zero and any positive integer d , the Lie algebra $\mathfrak{sl}_2(K)$ has a unique (up to isomorphism) d -dimensional irreducible representation $\rho_K^{(d)}$ over K . By well-known general arguments, the multiplicities $\text{mult}_\lambda(\mathfrak{sl}_2(K), \rho_K^{(d)})$ do not depend on K . Therefore Theorem 1.1 implies that $\text{mult}_\lambda(\mathfrak{sl}_2(K), \rho_K^{(d)}) \leq 3^{d-2}$ for any field K of characteristic zero.

(v) A different interpretation and approach to the study of $\text{Hom}_{\mathfrak{g}}(\rho(\mathfrak{g})^{\otimes m}, \text{End}_K(V))$ for $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ and $\rho = \rho^{(d)}$ is given in our parallel preprint [4], using classical invariant theory.

We close the introduction by mentioning the recent paper of da Silva Macedo and Koshlukov [3, Theorem 3.7], where the codimension growth of polynomial identities of representations of Lie algebras is studied. In particular, in [3, Theorem 3.7] the identities of representations of $\mathfrak{sl}_2(\mathbb{C})$ play a decisive role.

2. MATRIX COMPUTATIONS

Denote by $\tilde{\rho}^{(d)} : \mathfrak{sl}_2(\mathbb{C}) \rightarrow \mathfrak{gl}(\mathbb{C}^{d \times d})$ the representation given by

$$(3) \quad \tilde{\rho}^{(d)}(A)(L) = \rho^{(d)}(A)L - L\rho^{(d)}(A) \text{ for } A \in \mathfrak{sl}_2(\mathbb{C}), L \in \mathbb{C}^{d \times d}.$$

We have $\tilde{\rho}^{(d)} \cong \rho^{(d)} \otimes \rho^{(d)*}$. The representations of $\mathfrak{sl}_2(\mathbb{C})$ are self-dual, and so by the Clebsch-Gordan rules we have

$$(4) \quad \tilde{\rho}^{(d)} \cong \rho^{(d)} \otimes \rho^{(d)} \cong \bigoplus_{n=1}^d \rho^{(2n-1)}.$$

We shall need an explicit decomposition of $\mathbb{C}^{d \times d}$ as a direct sum of minimal $\tilde{\rho}^{(d)}$ -invariant subspaces.

Set

$$e := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

so e, f, h is a \mathbb{C} -vector space basis of $\mathfrak{sl}_2(\mathbb{C})$, with $[h, e] = 2e$, $[h, f] = -2f$, and $[e, f] = h$.

Recall that given a representation $\psi : \mathfrak{sl}_2(\mathbb{C}) \rightarrow \mathfrak{gl}(V)$, by a *highest weight vector* we mean a non-zero element $w \in V$ such that $\psi(e)(w) = 0 \in V$ and $\psi(h)(w) = nw$ for some non-negative integer n (the non-negative integer λ is called the *weight* of w); in this case w generates a minimal $\mathfrak{sl}_2(\mathbb{C})$ -invariant subspace in V , on which the representation of $\mathfrak{sl}_2(\mathbb{C})$ is isomorphic to $\rho^{(n+1)}$. Moreover, any finite dimensional irreducible $\mathfrak{sl}_2(\mathbb{C})$ -module contains a unique (up to non-zero scalar multiples) highest weight vector.

Lemma 2.1. *Consider the $\mathfrak{sl}_2(\mathbb{C})$ -module $\mathbb{C}^{d \times d}$ via the representation $\tilde{\rho}^{(d)}$. To simplify notation set $\rho := \rho^{(d)}$ and $\tilde{\rho} := \tilde{\rho}^{(d)}$.*

- (i) $\rho(e)^n$ is a highest weight vector in $\mathbb{C}^{d \times d}$ of weight $2n$ for $n = 0, 1, \dots, d-1$.
- (ii) $\rho(e)^{n-1}$ generates a minimal $\tilde{\rho}$ -invariant subspace V_n on which $\mathfrak{sl}_2(\mathbb{C})$ acts via $\rho^{(2n-1)}$ for $n = 1, \dots, d$.
- (iii) $\mathbb{C}^{d \times d} = \bigoplus_{n=1}^d V_n$.
- (iv) For $L_1 \in V_{n_1}$ and $L_2 \in V_{n_2}$ with $1 \leq n_1 \neq n_2 \leq d$ we have $\text{Tr}(L_1 L_2) = 0$.

Proof. (i) We have $\tilde{\rho}(e)(\rho(e)^n) = \rho(e)\rho(e)^n - \rho(e)^n\rho(e) = 0$ and

$$\tilde{\rho}(h)(\rho(e)^n) = \rho([h, e])\rho(e)^{n-1} + \rho(e)\rho([h, e])\rho(e)^{n-2} + \dots + \rho(e)^{n-1}\rho([h, e]) = 2n\rho(e)^n.$$

This shows that $\rho(e)^n$ is a highest weight vector of weight $2n$ for the representation $\tilde{\rho}$.

- (ii) Statement (i) implies that $\rho(e)^{n-1}$ generates an irreducible $\mathfrak{sl}_2(\mathbb{C})$ -submodule of $\tilde{\rho}$ isomorphic to $\rho^{(2n-1)}$ for $n = 1, \dots, d$.
- (iii) follows from (ii) and (4).
- (iv) Consider the symmetric non-degenerate bilinear form

$$\beta : \mathbb{C}^{d \times d} \times \mathbb{C}^{d \times d} \rightarrow \mathbb{C}, \quad (L, M) \mapsto \text{Tr}(LM).$$

Note that β is $\tilde{\rho}$ -invariant:

$$\begin{aligned} \beta([\rho(A), L], M) + \beta(L, [\rho(A), M]) &= \text{Tr}([\rho(A), L]M) + \text{Tr}(L[\rho(A), M]) \\ &= \text{Tr}([\rho(A), LM]) = 0 \quad \text{for any } A \in \mathfrak{sl}_2(\mathbb{C}). \end{aligned}$$

The radical of the bilinear form $\beta_{V_n} : V_n \times V_n \mapsto \mathbb{C}$ (the restriction of β to $V_n \times V_n$) is a $\tilde{\rho}$ -invariant subspace in V_n , so it is either V_n or $\{0\}$. We claim that it is not V_n . Indeed, V_n contains a non-zero diagonal matrix D with real entries, since the zero weight subspace in $\mathbb{C}^{d \times d}$ (with respect to $\tilde{\rho}(h)$) is the subspace of diagonal matrices, and V_n intersects the zero-weight space in a 1-dimensional subspace (defined over the reals). Now being a sum of squares of non-zero real numbers, $0 \neq \text{Tr}(D^2) = \beta(D, D)$. Thus β_{V_n} is non-degenerate. The representation $\tilde{\rho}$ is multiplicity free by (4), and by (ii) and (iii), every $\tilde{\rho}$ -invariant subspace is of the form $\sum_{j \in J} V_j$ for some subset $J \subseteq \{1, 2, \dots, d\}$. As we showed above, the orthogonal complement of V_n (with respect to β) is disjoint from V_n , so it is the sum of the other minimal invariant subspaces V_j , $j \in \{1, \dots, d\} \setminus \{n\}$. \square

The representation $\rho^{(2)}$ is the defining representation of $\mathfrak{sl}_2(\mathbb{C})$ on \mathbb{C}^2 , and $\rho^{(d)}$ is the $(d-1)$ th symmetric tensor power of $\rho^{(2)}$. Denote by x, y the standard basis vectors in \mathbb{C}^2 , and take the basis $x^{d-1}, x^{d-1}y, \dots, y^{d-1}$ in the $(d-1)$ th symmetric tensor power of \mathbb{C}^2 . Then denoting by $E_{i,j}$ the matrix unit with entry 1 in the (i, j) position and zeros in all other positions, the representation $\rho^{(d)}$ as a matrix

representation $\mathfrak{sl}_2(\mathbb{C}) \rightarrow \mathbb{C}^{d \times d}$ is given as follows:

$$\rho^{(d)}(e) = \sum_{i=1}^{d-1} iE_{i,i+1}, \quad \rho^{(d)}(f) = \sum_{i=1}^{d-1} (d-i)E_{i+1,i}, \quad \rho^{(d)}(h) = \sum_{i=1}^d (d+1-2i)E_{i,i}$$

Lemma 2.2. *For $d \geq 3$ the \mathbb{C} -vector space $\mathbb{C}^{d \times d}$ is spanned by*

$$\{\rho^{(d)}(A_1) \cdots \rho^{(d)}(A_{d-1}) \mid A_1, \dots, A_{d-1} \in \mathfrak{sl}_2(\mathbb{C})\}.$$

Proof. To simplify the notation write $\rho := \rho^{(d)}$ and $\tilde{\rho} := \tilde{\rho}^{(d)}$. Let \mathcal{L} be the subspace of $\mathbb{C}^{d \times d}$ spanned by the products $\rho(A_1) \cdots \rho(A_{d-1})$, where $A_1, \dots, A_{d-1} \in \mathfrak{sl}_2(\mathbb{C})$. Clearly \mathcal{L} is a $\tilde{\rho}$ -invariant subspace of $\mathbb{C}^{d \times d}$. Since the representation $\tilde{\rho}$ is multiplicity free by (4), we have $\mathcal{L} = \sum_{j \in J} V_j$ for some subset $J \subseteq \{1, 2, \dots, d\}$ by Lemma 2.1 (ii) and (iii). Therefore to prove the equality $\mathcal{L} = \mathbb{C}^{d \times d}$ it is sufficient to show that $\mathcal{L} \cap V_n \neq \{0\}$ for each $n = 1, \dots, d$, or equivalently, that \mathcal{L} is not contained in $\sum_{j \in \{1, \dots, d\} \setminus \{n\}} V_j$. Since V_d is generated by $\rho(e)^{d-1} \in \mathcal{L}$, we have $V_d \subseteq \mathcal{L}$. Moreover, to prove $\mathcal{L} \not\subseteq \sum_{j \in \{1, \dots, d\} \setminus \{n+1\}} V_j$ for $n \in \{0, 1, \dots, d-2\}$, it is sufficient to present an element $L_n \in \mathcal{L}$ with $\text{Tr}(\rho(e)^n L_n) \neq 0$ by Lemma 2.1 (i), (ii) and (iv). We shall give below such elements $L_n \in \mathcal{L}$ for $n = 0, 1, \dots, d-2$.

For $n = 1, \dots, d-1$ we have

$$\rho(e)^n = \sum_{j=1}^{d-n} j \cdot (j+1) \cdots (j+n-1) E_{j,j+n}$$

$$\rho(f)^n = \sum_{j=1}^{d-n} (d-j) \cdot (d-j-1) \cdots (d-j-n+1) E_{j+n,j}$$

and $\rho(e)^0 = I_d = \rho(f)^0$, where I_d is the $d \times d$ identity matrix. It follows that for $n = 1, \dots, d-1$,

$$\rho(e)^n \rho(f)^n = \sum_{j=1}^{d-n} j(j+1) \cdots (j+n-1) \cdot (d-j)(d-j-1) \cdots (d-j-n+1) E_{j,j}$$

is a diagonal matrix with non-negative integer entries, and the $(1, 1)$ -entry is positive. The same holds for $\rho(e)^0 \rho(f)^0 = I_d$. For n with $d-1-n$ even, $\rho(h)^{d-1-n}$ is the square of a diagonal matrix with integer entries, and its $(1, 1)$ -entry is positive. Hence $\text{Tr}(\rho(e)^n \rho(f)^n \rho(h)^{d-1-n}) \neq 0$, being a positive integer. So in this case we may take $L_n := \rho(f)^n \rho(h)^{d-1-n}$. For $n < d-2$ with $d-1-n$ odd, note that $\rho(e)\rho(f) - \rho(f)\rho(e) = \rho([e, f]) = \rho(h)$, and thus

$$\rho(f)^n \rho(h)^{d-2-n} = \rho(f)^n \rho(h)^{d-3-n} (\rho(e)\rho(f) - \rho(f)\rho(e))$$

also belongs to \mathcal{L} . Since $\rho(h)^{d-2-n}$ is a diagonal matrix with non-negative integer entries, and with a positive $(1, 1)$ -entry, we may take $L_n := \rho(f)^n \rho(h)^{d-2-n}$ in this case. It remains to deal with the case $n = d-2$. Then

$$\rho(e)^{d-2} \rho(f)^{d-2} = (d-1)((d-2)!)^2 \cdot (E_{1,1} + E_{2,2}),$$

hence taking $L_{d-2} := \rho(f)^{d-2} \rho(h)$ we get

$$\begin{aligned} \text{Tr}(\rho(e)^{d-2} L_{d-2}) &= \text{Tr}((d-1)((d-2)!)^2 \cdot ((d-1)E_{11} + (d-3)E_{22})) \\ &= (2d-4)(d-1)((d-2)!)^2, \end{aligned}$$

which is non-zero for $d \geq 3$. This finishes the proof of the equality $\mathcal{L} = \mathbb{C}^{d \times d}$. \square

3. ADJOINT INVARIANTS

Denote by $\text{ad} : \mathfrak{sl}_2(\mathbb{C}) \rightarrow \mathfrak{gl}(\mathfrak{sl}_2(\mathbb{C}))$ the *adjoint representation* of $\mathfrak{sl}_2(\mathbb{C})$ on itself, so $\text{ad}(A)(B) = [A, B]$ for $A, B \in \mathfrak{sl}_2(\mathbb{C})$. Take the n -fold direct sum $\text{ad}^{\oplus n} : \mathfrak{sl}_2(\mathbb{C}) \rightarrow \mathfrak{gl}(\mathfrak{sl}_2(\mathbb{C})^{\oplus n})$ of the adjoint representation, and write $\mathcal{O}[\mathfrak{sl}_2(\mathbb{C})^n]^{\mathfrak{sl}_2(\mathbb{C})}$ for the algebra of $\text{ad}^{\oplus n}$ -invariant polynomial functions on $\mathfrak{sl}_2(\mathbb{C})^{\oplus n}$. There is a right action of $\text{GL}_n(\mathbb{C})$ on $\mathfrak{sl}_2(\mathbb{C})^n$ that commutes with $\text{ad}^{\oplus n}$: for $g = (g_{ij})_{i,j=1}^n$ and $(A_1, \dots, A_n) \in \mathfrak{sl}_2(\mathbb{C})^n$ we have

$$(A_1, \dots, A_n) \cdot g := \left(\sum_{i=1}^n g_{i1} A_i, \dots, \sum_{i=1}^n g_{in} A_i \right).$$

This induces a left $\text{GL}_n(\mathbb{C})$ -action on the coordinate ring $\mathcal{O}[\mathfrak{sl}_2(\mathbb{C})^n]$: for $g \in \text{GL}_n(\mathbb{C})$, $f \in \mathcal{O}[\mathfrak{sl}_2(\mathbb{C})^n]$ and $(A_1, \dots, A_n) \in \mathfrak{sl}_2(\mathbb{C})^n$ we have $(g \cdot f)(A_1, \dots, A_n) = f((A_1, \dots, A_n) \cdot g)$.

Lemma 3.1. *Consider the linear map $\iota : F_m = \mathbb{C}\langle x_1, \dots, x_m \rangle \rightarrow \mathcal{O}[\mathfrak{sl}_2(\mathbb{C})^{m+d-1}]$ given by*

$$\iota(f)(A_1, \dots, A_{m+d-1}) = \text{Tr}(f(\rho^{(d)}(A_1), \dots, \rho^{(d)}(A_m)) \cdot \rho^{(d)}(A_{m+1}) \cdots \rho^{(d)}(A_{m+d-1}))$$

for $f \in F_m$ and $(A_1, \dots, A_{m+d-1}) \in \mathfrak{sl}_2(\mathbb{C})^{m+d-1}$. It has the following properties:

- (i) *The image of ι is contained in the subalgebra $\mathcal{O}[\mathfrak{sl}_2(\mathbb{C})^{m+d-1}]^{\mathfrak{sl}_2(\mathbb{C})}$ of $\mathfrak{sl}_2(\mathbb{C})$ -invariants.*
- (ii) *For $d \geq 3$ the kernel of ι is the ideal $I(\mathfrak{sl}_2(\mathbb{C}), \rho^{(d)}) \cap F_m$.*
- (iii) *The map ι is $\text{GL}_m(\mathbb{C})$ -equivariant, where we restrict the $\text{GL}_{m+d-1}(\mathbb{C})$ -action on $\mathcal{O}[\mathfrak{sl}_2(\mathbb{C})^{m+d-1}]$ to the subgroup $\text{GL}_m(\mathbb{C}) \cong \left\{ \begin{pmatrix} g & 0 \\ 0 & I_{d-1} \end{pmatrix} \mid g \in \text{GL}_m(\mathbb{C}) \right\}$ in $\text{GL}_{m+d-1}(\mathbb{C})$.*

Proof. For notational simplicity we shall write ρ instead of $\rho^{(d)}$.

(i) By linearity of ι it is sufficient to show that $\iota(x_{i_1} \cdots x_{i_k})$ is an $\mathfrak{sl}_2(\mathbb{C})$ -invariant for any $i_1, \dots, i_k \in \{1, \dots, m\}$. Setting $n = k + d - 1$, $B_1 = A_{i_1}, \dots, B_k = A_{i_k}$, $B_{k+1} = A_{m+1}, \dots, B_n = A_{m+d-1}$ we have

$$(5) \quad \iota(x_{i_1} \cdots x_{i_k})(A_1, \dots, A_{m+d}) = \text{Tr}(\rho(B_1) \cdots \rho(B_n)).$$

For any $X \in \mathfrak{sl}_2(\mathbb{C})$ we have

$$\begin{aligned} 0 &= \text{Tr}([\rho(X), \rho(B_1) \cdots \rho(B_n)]) \\ &= \text{Tr}\left(\sum_{j=1}^n \rho(B_1) \cdots \rho(B_{j-1}) [\rho(X), \rho(B_j)] \rho(B_{j+1}) \cdots \rho(B_n)\right) \\ &= \text{Tr}\left(\sum_{j=1}^n \rho(B_1) \cdots \rho(B_{j-1}) \rho([X, B_j]) \rho(B_{j+1}) \cdots \rho(B_n)\right) \\ &= \sum_{j=1}^n \text{Tr}(\rho(B_1) \cdots \rho(B_{j-1}) \rho([X, B_j]) \rho(B_{j+1}) \cdots \rho(B_n)). \end{aligned}$$

The equalities (5) and

$$\sum_{j=1}^n \text{Tr}(\rho(B_1) \cdots \rho(B_{j-1}) \rho([X, B_j]) \rho(B_{j+1}) \cdots \rho(B_n)) = 0$$

mean that $\iota(x_{i_1} \cdots x_{i_k})$ is $\mathfrak{sl}_2(\mathbb{C})$ -invariant, so (i) holds.

(ii) Suppose that $f \in \ker(\iota)$. Then $\text{Tr}(f(\rho(A_1), \dots, \rho(A_m))B) = 0$ for all $B \in \mathbb{C}^{d \times d}$ by Lemma 2.2. By non-degeneracy of the trace we get $f(\rho(A_1), \dots, \rho(A_m)) = 0$. That is, $f \in I(\mathfrak{sl}_2(\mathbb{C}), \rho)$. Thus $\ker(\iota) \subseteq I(\mathfrak{sl}_2(\mathbb{C}), \rho) \cap F_m$. The reverse inclusion $I(\mathfrak{sl}_2(\mathbb{C}), \rho) \cap F_m \subseteq \ker(\iota)$ is obvious.

(iii) Take $g = (g_{ij})_{i,j=1}^m \in \text{GL}_m(\mathbb{C})$. For $f \in F_m$ and $(A_1, \dots, A_m) \in \mathfrak{sl}_2(\mathbb{C})^m$ we have (by linearity of ρ)

$$\begin{aligned} & \iota(g \cdot f)(A_1, \dots, A_m) \\ &= \text{Tr}(f(\sum_{i=1}^m g_{i1} \rho(A_i), \dots, \sum_{i=1}^m g_{im} \rho(A_i)) \cdot \rho(A_{m+1}) \cdots \rho(A_{m+d})) \\ &= \text{Tr}(f(\rho(\sum_{i=1}^m g_{i1} (A_i)), \dots, \rho(\sum_{i=1}^m g_{im} (A_i)) \cdot \rho(A_{m+1}) \cdots \rho(A_{m+d})) \\ &= (g \cdot \iota(f))(A_1, \dots, A_m). \end{aligned}$$

This shows (iii). \square

Restricting the action of $\text{GL}_n(\mathbb{C})$ on $\mathcal{O}[\mathfrak{sl}_2(\mathbb{C})^n]$ to the subgroup of diagonal matrices we get an \mathbb{N}_0^n -grading on $\mathcal{O}[\mathfrak{sl}_2(\mathbb{C})^n]$, preserved by the action of $\mathfrak{sl}_2(\mathbb{C})$. Denote by $\mathcal{O}[\mathfrak{sl}_2(\mathbb{C})^n]_{(1^n)}$ the multihomogeneous component of multidegree $(1, \dots, 1)$; this is the space of n -linear functions on $\mathfrak{sl}_2(\mathbb{C})$. The spaces $\mathcal{O}[\mathfrak{sl}_2(\mathbb{C})^n]_{(1^n)}$ and $\mathcal{O}[\mathfrak{sl}_2(\mathbb{C})^n]_{(1^n)}^{\mathfrak{sl}_2(\mathbb{C})}$ are S_n -invariant (where we restrict the $\text{GL}_n(\mathbb{C})$ -action to its subgroup S_n of permutation matrices). Lemma 3.1 has the following immediate consequence:

Corollary 3.2. *For $d \geq 3$ the restriction of ι to the multilinear component P_m of $\mathbb{C}\langle x_1, \dots, x_m \rangle$ factors through an S_m -equivariant \mathbb{C} -linear embedding*

$$\bar{\iota} : P_m / (I(\mathfrak{sl}_2(\mathbb{C})) \cap P_m) \rightarrow \mathcal{O}[\mathfrak{sl}_2(\mathbb{C})^{m+d-1}]_{(1^{m+d-1})}^{\mathfrak{sl}_2(\mathbb{C})}$$

where on the right hand side we consider the restriction of the S_{m+d-1} -action to its subgroup S_m (the stabilizer in S_{m+d-1} of the elements $m+1, m+2, \dots, m+d-1$).

For a partition $\lambda \vdash m$ denote by $r(\lambda)$ the multiplicity of χ^λ in the restriction to S_m of the S_{m+d-1} -module $\mathcal{O}[\mathfrak{sl}_2(\mathbb{C})^{m+d-1}]_{(1^{m+d-1})}^{\mathfrak{sl}_2(\mathbb{C})}$. Corollary 3.2 immediately implies the following:

Corollary 3.3. *For $d \geq 3$ and any partition $\lambda \vdash m$ we have the inequality*

$$\text{mult}_\lambda(\mathfrak{sl}_2(\mathbb{C}), \rho^{(d)}) \leq r(\lambda).$$

The S_n -character of $\mathcal{O}[\mathfrak{sl}_2(\mathbb{C})^n]_{(1^n)}^{\mathfrak{sl}_2(\mathbb{C})}$ is known:

Proposition 3.4. *For a partition $\lambda \vdash n$ denote by $\nu(\lambda)$ the multiplicity of χ^λ in the S_n -character of $\mathcal{O}[\mathfrak{sl}_2(\mathbb{C})^n]_{(1^n)}^{\mathfrak{sl}_2(\mathbb{C})}$. Then we have*

$$\nu(\lambda) = \begin{cases} 1 & \text{for } \lambda = (\lambda_1, \lambda_2, \lambda_3) \text{ with } \lambda_1 \equiv \lambda_2 \equiv \lambda_3 \text{ modulo } 2 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The $\text{GL}_n(\mathbb{C})$ -module structure of $\mathcal{O}[\mathfrak{sl}_2(\mathbb{C})^n]_{(1^n)}^{\mathfrak{sl}_2(\mathbb{C})}$ is given for example in [12, Theorem 2.2]. The isomorphism types of the irreducible $\text{GL}_n(\mathbb{C})$ -module

direct summands of the degree n homogeneous component of $\mathcal{O}[\mathfrak{sl}_2(\mathbb{C})^n]$ are labeled by partitions of n with at most 3 non-zero parts. The multiplicity $\mu(\lambda)$ of the irreducible $\mathrm{GL}_n(\mathbb{C})$ -module W_λ in the degree n homogeneous component of $\mathcal{O}[\mathfrak{sl}_2(\mathbb{C})^n]^{\mathfrak{sl}_2(\mathbb{C})}$ is 1 if $\lambda_1, \lambda_2, \lambda_3$ have the same parity and is zero otherwise. Note finally that the multilinear component of W_λ is S_n -stable, and its S_n -character is χ^λ (see for example [1, Corollary 6.3.11]). \square

Following [11, Section I.1] for partitions $\lambda \vdash n$ and $\mu \vdash k$ we write $\lambda \subset \mu$ if $\lambda_i \leq \mu_i$ for all i . Moreover, given $\lambda \vdash m$ and $\mu \vdash m + d - 1$ with $\lambda \subset \mu$, by a *standard tableau of shape μ/λ* we mean a sequence $\lambda^{(0)} \subset \lambda^{(1)} \subset \dots \subset \lambda^{(d-1)}$ of partitions $\lambda^{(i)} \vdash m + i$, where $\lambda^{(0)} = \lambda$, $\lambda^{(d-1)} = \mu$. By the well-known branching rules for the symmetric group, for $\lambda \vdash m$ the multiplicity of χ^λ in the restriction to S_m of the irreducible S_{m+d-1} -character χ^μ equals the number of standard tableaux of shape μ/λ (see for example [1, Theorem 6.4.11]). Therefore Proposition 3.4 has the following consequence.

Corollary 3.5. *We have the equality*

$$r(\lambda) = |\{T \mid T \text{ is a standard skew tableau of shape } \mu/\lambda, \\ \mu \vdash m + d - 1, \mu = (\mu_1, \mu_2, \mu_3), \mu_1 \equiv \mu_2 \equiv \mu_3 \text{ modulo } 2\}|.$$

Corollary 3.6. *For $d \geq 3$ we have the inequality $r(\lambda) \leq 3^{d-2}$.*

Proof. Associate to a standard skew tableau $T = \lambda^{(0)} \subset \lambda^{(1)} \subset \dots \subset \lambda^{(d-1)}$ of shape μ/λ , where $\mu = (\mu_1, \mu_2, \mu_3) \vdash m + d - 1$ and $\mu_1 \equiv \mu_2 \equiv \mu_3$ modulo 2 the function $f_T : \{1, \dots, d-1\} \rightarrow \{1, 2, 3\}$, which maps $j \in \{1, \dots, d-1\}$ to the unique $i \in \{1, 2, 3\}$ such that the i th component of the partition $\lambda^{(j)}$ is 1 greater than the i th component of $\lambda^{(j-1)}$. The assignment $T \mapsto f_T$ is obviously an injective map from the set of standard skew tableaux of shape μ/λ into the set of functions $\{1, \dots, d-1\} \rightarrow \{1, 2, 3\}$. We claim that at most 3^{d-2} functions are contained in the image of this map. Indeed, if the three parts of $\lambda^{(d-3)}$ have the same parity, then $(f_T(d-2), f_T(d-1)) \in \{(1, 1), (2, 2), (3, 3)\}$, since the three parts of $\mu = \lambda^{(d-1)}$ must have the same parity. If the three parts of $\lambda^{(d-3)}$ do not have the same parity, say the first two components of $\lambda^{(d-3)}$ have the same parity, and the third part has the opposite parity, then $(f_T(d-2), f_T(d-1)) \in \{(1, 2), (2, 1)\}$. Hence $r(\lambda)$ is not greater than 3-times the number of functions from a $(d-3)$ -element set to a 3-element set. Thus $r(\lambda) \leq 3^{d-2}$. \square

3.1. Proof of Theorem 1.1. For $d \geq 3$ the statement follows from Corollary 3.3 and Corollary 3.6. For the cases $d \leq 3$ see Remark 1.2 (i).

4. A LOWER BOUND

Proposition 4.1. *For $d \geq 2$ we have the equality*

$$\mathrm{mult}_{(d-1,1)}(\mathfrak{sl}_2(\mathbb{C}), \rho^{(d)}) = d - 1.$$

Proof. For $k = 0, 1, \dots, d-2$ consider the element

$$w_k := x_1^k [x_1, x_2] x_1^{d-2-k} \in \mathbb{C}\langle x_1, x_2 \rangle = F_2.$$

These elements are $\mathrm{GL}_2(\mathbb{C})$ -highest weight vectors with weight $(d-1, 1)$, hence each generates an irreducible $\mathrm{GL}_2(\mathbb{C})$ -submodule isomorphic to $W_{(d-1,1)}$ (see the proof of Proposition 3.4 for the notation W_λ : it is the polynomial $\mathrm{GL}_2(\mathbb{C})$ -module with

highest weight $\lambda = (\lambda_1, \lambda_2)$. Moreover, they are linearly independent modulo the ideal $I(\mathfrak{sl}_2(\mathbb{C}), \rho^{(d)})$: indeed, make the substitution $x_1 \mapsto \rho(h)$, $x_2 \mapsto \rho(e)$. Then we get

$$\begin{aligned} w_k(\rho(h), \rho(e)) &= \left(\sum_{i=1}^d (d+1-2i)E_{i,i} \right)^k \cdot \left(2 \sum_{i=1}^{d-1} iE_{i,i+1} \right) \cdot \left(\sum_{i=1}^d (d+1-2i)E_{i,i} \right)^{d-2-k} \\ &= 2 \sum_{i=1}^{d-1} i(d+1-2i)^k (d-1-2i)^{d-2-k} E_{i,i+1}. \end{aligned}$$

Denote by $Z = (Z_{i,j})_{i,j=1}^{d-1}$ the $(d-1) \times (d-1)$ matrix whose $(i, k+1)$ entry is the $(i, i+1)$ -entry of $w_k(\rho(h), \rho(e))$ (i.e. the coefficient of $E_{i,i+1}$ on the right hand side of the above equality). If $i \neq \frac{d-1}{2}$, then

$$Z_{i,k+1} = 2(d-1-2i)^{d-2} \cdot \left(\frac{d+1-2i}{d-1-2i} \right)^k.$$

Thus when d is even, Z is obtained from a Vandermonde matrix via multiplying each row by a non-zero integer. Since the numbers $\frac{d+1-2i}{d-1-2i}$, $i = 1, \dots, d-1$ are distinct, we conclude that $\det(Z) \neq 0$. When $d = 2f - 1$ is odd, the $(f-1)$ th row of Z is

$$(0, \dots, 0, 2(f-1)2^{d-2}).$$

Expand the determinant of Z along this row; the $(d-2) \times (d-2)$ minor of Z obtained by removing the $(f-1)$ th row and the last column of Z is again obtained from a Vandermonde matrix by multiplying each row by a non-zero integer. So $\det(Z)$ is non-zero also when d is odd. This shows that the elements $w_k(\rho(h), \rho(e))$, $k = 0, 1, \dots, d-2$ are linearly independent in $\mathbb{C}^{d \times d}$. Consequently, no non-trivial linear combination of w_0, w_1, \dots, w_{d-2} belongs to $I(\mathfrak{sl}_2(\mathbb{C}), \rho^{(d)})$. It follows that $F_2/I(\mathfrak{sl}_2(\mathbb{C}), \rho^{(d)}) \cap F_2$ contains the irreducible $\mathrm{GL}_2(\mathbb{C})$ -module $W_{(d-1,1)}$ with multiplicity $\geq d-1$. This multiplicity is in fact equal to $d-1$, because $d-1$ is the multiplicity of $W_{(d-1,1)}$ as a summand in F_2 . Recall finally that for $\lambda = (\lambda_1, \lambda_2) \vdash m$, the multiplicity of χ^λ in the cocharacter sequence coincides with the multiplicity of W_λ in $F_2/I(\mathfrak{sl}_2(\mathbb{C}), \rho^{(d)}) \cap F_2$. \square

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ALFRÉD RÉNYI INSTITUTE OF MATHEMATICS, REÁLTANODA UTCA 13-15, 1053 BUDAPEST, HUNGARY

Email address: domokos.matyas@renyi.hu