# On the local time of the Half-Plane Half-Comb walk

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Abstract The Half-Plane Half-Comb walk is a random walk on the plane, when we have a square lattice on the upper half-plane and a comb structure on the lower half-plane, i.e., horizontal lines below the x-axis are removed. We prove that the probability that this walk return to origin in 2N steps is asymptotically equal to  $2/(\pi N)$ . As a consequence we prove strong laws and a limit distribution for the local time.

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### 1 Introduction and main results

The properties of a simple symmetric random walk on the square lattice  $\mathbb{Z}^2$  have been extensively investigated in the literature since Dvoretzky and Erdős [11], and Erdős and Taylor [12]. For these and further results we refer to Révész [18].

Subsequent investigations concern random walks on other structures of the plane. For example, a simple random walk on the 2-dimensional comb lattice that is obtained from  $\mathbb{Z}^2$  by removing all horizontal lines off the *x*-axis was studied by Weiss and Havlin [21], Bertacchi and Zucca [4], Bertacchi [2], Csáki et al. [6], [7].

The latter are particular cases of the so-called anisotropic random walk on the plane. The general case is given by the transition probabilities

$$\mathbf{P}(\mathbf{C}(N+1) = (k+1,j)|\mathbf{C}(N) = (k,j)) = \mathbf{P}(\mathbf{C}(N+1) = (k-1,j)|\mathbf{C}(N) = (k,j)) = \frac{1}{2} - p_j,$$
$$\mathbf{P}(\mathbf{C}(N+1) = (k,j+1)|\mathbf{C}(N) = (k,j)) = \mathbf{P}(\mathbf{C}(N+1) = (k,j-1)|\mathbf{C}(N) = (k,j)) = p_j,$$

for  $(k, j) \in \mathbb{Z}^2$ ,  $N = 0, 1, 2, \ldots$  with  $0 < p_j \leq 1/2$  and  $\min_{j \in \mathbb{Z}} p_j < 1/2$ . See Seshadri et al. [19], Silver et al. [20], Heyde [13] and Heyde et al. [14]. The simple symmetric random walk corresponds

to the case  $p_j = 1/4$ ,  $j = 0, \pm 1, \pm 2, \ldots$ , while  $p_0 = 1/4$ ,  $p_j = 1/2$ ,  $j = \pm 1, \pm 2, \ldots$  defines the random walk on the comb.

In our paper [8] we combined the simple symmetric random walk with a random walk on a comb, when  $p_j = 1/4$ , j = 0, 1, 2, ... and  $p_j = 1/2$ , j = -1, -2, ..., i.e., we have a square lattice on the upper half-plane, and a comb structure on the lower half-plane. We call this model Half-Plane Half-Comb (HPHC) and denote the random walk on it by  $\mathbf{C}(N) = (C_1(N), C_2(N)), N = 0, 1, 2, ...$  Here, for convenient information, we first repeat the precise construction of this walk, as it was given in [7]:

On a suitable probability space consider two independent simple symmetric (one-dimensional) random walks  $S_1(\cdot)$ , and  $S_2(\cdot)$ . We may assume that on the same probability space we have a sequence of independent geometric random variables  $\{Y_i, i = 1, 2, ...\}$ , independent from  $S_1(\cdot), S_2(\cdot)$ , with distribution

$$\mathbf{P}(Y_i = k) = \frac{1}{2^{k+1}}, \ k = 0, 1, 2, \dots$$
(1.1)

Now horizontal steps will be taken consecutively according to  $S_1(\cdot)$ , and vertical steps consecutively according to  $S_2(\cdot)$  in the following way. Start from (0,0), take  $Y_1$  horizontal steps (possibly  $Y_1 = 0$ ) according to  $S_1(\cdot)$ , then take 1 vertical step. If this arrives to the upper half-plane ( $S_2(1) = 1$ ), then take  $Y_2$  horizontal steps. If, however, the first vertical step is in the negative direction ( $S_2(1) = -1$ ), then continue with another vertical step, and so on. In general, if the random walk is on the upper half-plane, ( $y \ge 0$ ) after a vertical step, then take a random number of horizontal steps according to the next (so far) unused  $Y_j$ , independent from the previous steps. On the other hand, if the random walk is on the lower half- plane (y < 0) then continue with vertical steps according to  $S_2(\cdot)$  until it reaches the x-axis, and so on.

In paper [8] we investigated the almost sure limit properties of this walk by using strong approximation methods. Our first result was a strong approximation of both components of the random walk  $\mathbf{C}(\cdot)$  by certain time-changed Wiener processes (Brownian motions) with rates of convergence. Before stating it, we need some definitions. Assume that we have two independent standard Wiener processes  $W_1(t), W_2(t), t \ge 0$ , and consider

$$\alpha_2(t) := \int_0^t I\{W_2(s) \ge 0\} \, ds,$$

i.e., the time spent by  $W_2(\cdot)$  on the non-negative side during the interval [0, t]. The process  $\gamma_2(t) := \alpha_2(t) + t$  is strictly increasing, hence we can define its inverse:  $\beta_2(t) := \gamma_2^{-1}(t)$ . Observe that the processes  $\alpha_2(t)$ ,  $\beta_2(t)$  and  $\gamma_2(t)$  are defined in terms of  $W_2(t)$ , so they are independent from  $W_1(t)$ . It can be seen moreover that  $0 \le \alpha_2(t) \le t$ , and  $t/2 \le \beta_2(t) \le t$ .

**Theorem A** On an appropriate probability space for the HPHC random walk  $\{\mathbf{C}(N) = (C_1(N), C_2(N)); N = 0, 1, 2, ...\}$  one can construct two independent standard Wiener processes  $\{W_1(t); t \ge 0\}, \{W_2(t); t \ge 0\}$  such that, as  $N \to \infty$ , we have with any  $\varepsilon > 0$ 

$$|C_1(N) - W_1(N - \beta_2(N))| + |C_2(N) - W_2((\beta_2(N)))| = O(N^{3/8 + \varepsilon}) \quad a.s.$$

Our second result in paper [8] was the following LIL.

**Theorem B** We have

$$\limsup_{N \to \infty} \frac{C_1(N)}{\sqrt{N \log \log N}} = \limsup_{N \to \infty} \frac{C_2(N)}{\sqrt{N \log \log N}} = 1 \quad a.s.$$

Furthermore

$$\liminf_{N \to \infty} \frac{C_1(N)}{\sqrt{N \log \log N}} = -1 \quad a.s., \qquad \liminf_{N \to \infty} \frac{C_2(N)}{\sqrt{N \log \log N}} = -\sqrt{2} \quad a.s$$

Moreover we gave an explicit formula for the *N*-step return probability of the walk, which however was too complicated to conclude the asymptotic limit. The aim of the present paper is to study the local time of this walk. Based on the just mentioned formula and a beautiful result of Sparre Andersen, we first get the asymptotic limit of this return probability, and then use it for getting local time results.

# 2 Preliminaries

Let  $X_1, X_2...$  be i.i.d. random variables with  $\mathbf{P}(X_1 = \pm 1) = 1/2$ , and define  $S(0) = 0, S(i) = \sum_{i=1}^{i} X_i$ . Then  $\{S(n), n = 0, 1...\}$  is a simple symmetric random walk on the line with local time

$$\xi(x,n) = \#\{j: 0 \le j \le n, S(j) = x\}, \qquad x \in \mathbb{Z},$$

and put

$$A(n) = \sum_{j=0}^{\infty} \xi(j, n-1), \quad n = 1, 2, \dots$$
$$\mathbf{P}(2n, r) = \mathbf{P}(A(2n) = r, S(2n) = 0), \quad r = 1, 2, \dots, 2n.$$
(2.1)

Define

$$G_n = \#\{j : 0 \le j < n, S(j) \ge 0\}.$$

Then we can rephrase the definition of  $\mathbf{P}(2n, r)$  as follows:

$$\mathbf{P}(2n,r) = \mathbf{P}(G_{2n} = r, S(2n) = 0).$$
(2.2)

We proved in [8], that

$$\mathbf{P}(\mathbf{C}(2N) = (0,0)) = \binom{2N}{N} \frac{1}{4^{2N}} + \sum_{n=1}^{N} \sum_{r=1}^{2n} \mathbf{P}(2n,r) \binom{2N-2n}{N-n} \frac{1}{2^{2N-2n}} \binom{2N-2n+r}{r} \frac{1}{2^{2N-2n+r}}, (2.3)$$

where it was shown that

$$\mathbf{P}(2n, 2r-1) = \mathbf{P}(2n, 2r), \tag{2.4}$$

and we concluded the following complicated formula for  $\mathbf{P}(2n, 2r)$  (see Lemma 5.2 in [8])

$$\mathbf{P}(2n,2r) = \frac{1}{2^{2n}} \sum_{j=1}^{r} \frac{1}{2j-1} \binom{2j-1}{j} \frac{1}{2n+1-2j} \binom{2n+1-2j}{n+1-j}.$$

However, in order to proceed, we need a closed form for  $\mathbf{P}(2n, 2r)$ .

Sparre Andersen [1] proved some elegant results about the fluctuation of the sums of random variables. We only quote the case of simple symmetric random walk of his much more general results. In his formula (5.12) he defines

$$K_n = \#\{j : 0 < j \le n, S(j) > 0\},\$$

and gives the probability of

$$\mathbf{P}(K_{2n-1} = 2r, S(2n) = 0) = \mathbf{P}(K_{2n-1} = 2r+1, S(2n) = 0)$$
$$= \frac{1}{2}c_{2n}\frac{1}{n+1}\left(1 + \frac{n-2r}{n}\binom{-\frac{1}{2}}{r}\binom{-\frac{1}{2}}{n-r}\binom{-\frac{1}{2}}{n}^{-1}\right), \qquad r = 0, ..., n-1,$$

where

$$c_{2n} := \mathbf{P}(S(2n) = 0) = (-1)^n \binom{-\frac{1}{2}}{n} = \frac{1}{2^{2n}} \binom{2n}{n}.$$
$$\mathbf{P}(K_{2n-1} = 2r, S(2n) = 0) = \frac{1}{2^{2n+1}} \binom{2n}{n} \frac{1}{n+1} \left( 1 + \frac{n-2r}{n} \frac{\binom{2r}{r}\binom{2n-2r}{n-r}}{\binom{2n}{n}} \right).$$
(2.5)

In the above formulas for any real number  $\alpha$  we used the notation  $\binom{\alpha}{r} = \frac{\alpha(\alpha-1)(\alpha-2)\dots(\alpha-r+1)}{r!}$ .

However  $\mathbf{P}(2n, 2r) = \mathbf{P}(G_{2n} = 2r, S(2n) = 0)$ , given in (2.2) is slightly different from the above one. We will show the following

#### Lemma 2.1

$$\mathbf{P}(2n,2r) = \mathbf{P}(G_{2n} = 2r, S(2n) = 0) = \frac{1}{2^{2n+1}} \binom{2n}{n} \frac{1}{n+1} \left( 1 + \frac{2r-n}{n} \frac{\binom{2r}{r}\binom{2n-2r}{n-r}}{\binom{2n}{n}} \right).$$
(2.6)

**Proof**: Recall the definition of  $K_n$  and  $G_n$  and let

$$M_n = \#\{j : 0 < j \le n, S(j) \le 0\}.$$

Then observe that

$$\mathbf{P}(M_{2n} = k, S(2n) = 0) = \mathbf{P}(G_{2n} = k, S(2n) = 0).$$

Moreover, the following two events are the same:

$$\{M_{2n} = r, S(2n) = 0\} = \{K_{2n-1} = 2n - r, S(2n) = 0\}.$$
(2.7)

So

$$\mathbf{P}(2n,2r) = \mathbf{P}(G_{2n} = 2r, S(2n) = 0) = \mathbf{P}(M_{2n} = 2r, S(2n) = 0) = \mathbf{P}(K_{2n-1} = 2n - 2r, S(2n) = 0),$$
(2.8)

which immediately implies our lemma.  $\Box$ 

Recall now the definition of the sequence of i.i.d. geometric random variables given in the introduction

$$\mathbf{P}(Y_i = k) = 2^{-(k+1)} \quad i = 1, 2..., \quad k = 0, 1, 2...,$$
(2.9)

and let

$$U = U_K = \sum_{i=1}^{K} Y_i.$$
 (2.10)

Then  $U_K$  is negative binomial with  $E(U_K) = K$ ,  $Var(U_K) = 2K$  and

$$\mathbf{P}(U_K = r) = \binom{K - 1 + r}{r} \frac{1}{2^{K + r}}, \ r = 0, 1, 2, \dots$$
(2.11)

We will need the following two well-known identities about the negative binomial distribution:

$$\sum_{r=0}^{a} \binom{a+r}{r} \frac{1}{2^{a+r}} = 1$$
 (2.12)

$$\sum_{r=0}^{\infty} \binom{a+r}{r} \frac{1}{2^{a+r}} = 2$$
(2.13)

See the first one, e.g., in Pitman [17] (page 220), while the second one is equivalent with  $\sum_{r=0}^{\infty} \mathbf{P}(U_K = r) = 1.$ 

**Lemma A** Berry-Esseen bound: [16] (page 150) Let  $X_1, ..., X_n$  be i.i.d. random variables. Let

$$E(X_1) = 0, \quad Var(X_1) = \sigma^2 > 0, \quad E(|X|^3) < \infty, \quad \rho = E(|X|^3)/\sigma^3$$

Then with some constant A > 0 we have

$$\sup_{x} \left| \mathbf{P} \left( \sigma^{-1} n^{-1/2} \sum_{j=1}^{n} X_{j} < x \right) - \Phi(x) \right| \leq A \rho \, n^{-1/2}, \tag{2.14}$$

where  $\Phi(\cdot)$  is the standard normal distribution function.

In what follows we will use a result of Chen [5], about Harris recurrent Markov chains, so here we recall his definition. Let  $\{X_n\}_{n\geq 0}$  be a recurrent Markov chain with state space  $(E, \mathcal{E})$ , transition probability P(x, A) and invariant measure  $\mu$ . Recall that  $\{X_n\}_{n\geq 0}$  is called Harris recurrent if it is irreducible and for any  $A \in \mathcal{E}^+$ , and initial distribution  $\nu$ ,

$$P_{\nu}(X_n \in A \quad \text{infinitely often}) = 1,$$

where  $\mathcal{E}^+ = \{A \in \mathcal{E}; \mu(a) > 0\}$ . By Harris recurrence the invariant measure  $\mu$  uniquely (up to a constant multiplier) exists. Obviously our HPHC walk is Harris recurrent.

## 3 Asymptotic return probability

We want to determine the asymptotic probability that the HPHC random walk returns to the starting point in 2N steps.

**Theorem 3.1** For the asymptotic return probability of the HPHC walk, starting at (0,0), we have

$$\mathbf{P}(\mathbf{C}(2N) = (0,0)) \sim \frac{2}{\pi N}, \quad as \quad N \to \infty.$$

**Proof:** Recall the definition of  $U_K$  in (2.10). In what follows let  $U = U_{2N-2n+1}$ . Introduce the notation

$$Q(r,n) := \frac{\binom{2r}{r}\binom{2n-2r}{n-r}}{\binom{2n}{n}}.$$

Combining formulas (2.3), (2.4) and (2.6) we have that

$$\mathbf{P}(\mathbf{C}(2N) = (0,0)) = {\binom{2N}{N}} \frac{1}{4^{2N}}$$

$$+ \sum_{n=1}^{N} {\binom{2N-2n}{N-n}} \frac{1}{2^{2N-2n}} \sum_{r=1}^{2n} \mathbf{P}(2n,r) {\binom{2N-2n+r}{r}} \frac{2}{2^{2N-2n+r+1}}$$

$$= {\binom{2N}{N}} \frac{1}{4^{2N}} + \sum_{n=1}^{N} {\binom{2N-2n}{N-n}} \frac{1}{2^{2N-2n}} \sum_{r=1}^{n} 2\mathbf{P}(2n,2r)\mathbf{P}(2r-1 \le U \le 2r)$$

$$= {\binom{2N}{N}} \frac{1}{4^{2N}} + \sum_{n=1}^{N} {\binom{2N-2n}{N-n}} \frac{1}{2^{2N}} \frac{1}{n+1} {\binom{2n}{n}} \sum_{r=1}^{n} {\binom{1+\frac{2r-n}{n}Q(r,n)}{P(2r-1 \le U \le 2r)}}$$

$$= {\binom{2N}{N}} \frac{1}{4^{2N}} + \sum_{n=1}^{N} {\binom{2N-2n}{N-n}} \frac{1}{2^{2N}} \frac{1}{n+1} {\binom{2n}{n}} \sum_{r=1}^{n} {\binom{1+\frac{2r-n}{n}Q(r,n)}{P(2r-1 \le U \le 2r)}}.$$

$$(3.1)$$

The first term in (3.1) is negligible, since

$$\binom{2N}{N}\frac{1}{4^{2N}} = O\left(\frac{1}{4^N}\right).$$

Thus

$$\begin{aligned} \mathbf{P}(\mathbf{C}(2N) &= (0,0)) \\ &\sim \sum_{n=1}^{N} \binom{2N-2n}{N-n} \frac{1}{2^{2N}} \binom{2n}{n} \frac{1}{n} \mathbf{P}(U \le 2n) \\ &+ \sum_{n=1}^{N} \binom{2N-2n}{N-n} \frac{1}{2^{2N}} \binom{2n}{n} \frac{1}{n} \sum_{r=1}^{n} \frac{2r-n}{n} Q(r,n) \mathbf{P}(2r-1 \le U \le 2r) = I + II. \end{aligned}$$

Observe that

$$\binom{2N-2n}{N-n}\frac{1}{2^{2N}}\binom{2n}{n} = c_{2N-2n}c_{2n} \sim \frac{1}{\pi}\frac{1}{\sqrt{n}}\frac{1}{\sqrt{N-n}}, \quad \text{when} \quad n \to \infty \quad \text{and} \quad N-n \to \infty.$$
(3.2)

Moreover, if only  $n \to \infty$  but N - n might be small, then

$$c_{2N-2n} c_{2n} \le \frac{c}{\sqrt{n}}.$$
 (3.3)

Here and in what follows c is a positive constant whose value can change from line to line. It is clear that for  $1 \le r \le n$ 

$$-1 \le \frac{2r-n}{n} \le 1.$$

We will show that term II is negligible compared to term I, so we use the above fact to give the following upper bound for II:

$$|II| \le II^* := \sum_{n=1}^N \frac{1}{n} \binom{2n}{n} \binom{2N-2n}{N-n} \frac{1}{2^{2N}} \sum_{r=1}^n Q(r,n) \mathbf{P}(2r-1 \le U \le 2r).$$

First we deal with the term I, dividing the sum for n into 5 parts:

$$\begin{array}{ll} (i) & 1 \leq n < \frac{N}{4} \\ (ii) & \frac{N}{4} \leq n < \frac{N}{2} - N^{1/2 + \alpha} \\ (iii) & \frac{N}{2} - N^{1/2 + \alpha} \leq n < \frac{N}{2} + N^{1/2 + \alpha} \\ (iv) & \frac{N}{2} + N^{1/2 + \alpha} \leq n < N - N^{1/2 - \alpha} \\ (v) & N - N^{1/2 - \alpha} \leq n \leq N, \end{array}$$

with some  $0 < \alpha < 1/2$ . Observe that

$$I = \sum_{n=1}^{N} \frac{1}{n} c_{2N-2n} c_{2n} \mathbf{P}(U \le 2n).$$
(3.4)

Let us start with (i). In this case we can use the estimation

$$\mathbf{P}(U \le 2n) = \sum_{r=0}^{2n} \binom{2N-2n+r}{r} \frac{1}{2^{2N-2n+r+1}} \le 2n \binom{2N}{2n} \frac{1}{2^{2N}},\tag{3.5}$$

since the largest term in the previous sum corresponds to r = 2n. Thus

$$\sum_{(i)} \leq \sum_{1 \leq n < N/4} c_{2N-2n} c_{2n} \frac{1}{n} 2n \binom{2N}{2n} \frac{1}{2^{2N}}$$
  
$$\leq c \frac{1}{2^{2N}} \sum_{1 \leq n < N/4} \binom{2N}{2n} \leq c \frac{1}{2^{2N}} \frac{N}{4} \binom{2N}{N/2} \leq c \sqrt{N} \left(\frac{4}{3\sqrt{3}}\right)^N$$
(3.6)

with some constant c, by observing that the first two factor in our sum is the product of two probabilities. We used Stirling formula to get the last inequality.

In case (ii) we use normal approximation for negative binomial distribution, with Berry-Esseen bound as in (2.14) to get that for n belonging to the set (ii)

$$\mathbf{P}(U \le 2n) = \Phi\left(\frac{4n - 2N - 1}{\sqrt{2(2N - 2n + 1)}}\right) + O\left(\frac{1}{\sqrt{N - n}}\right) \le \Phi(-2N^{\alpha}) + \frac{c}{\sqrt{N}} \le \frac{c}{\sqrt{N}},$$

being the normal term exponentially small. Moreover, using (3.2)

$$\sum_{(ii)} \le c \sum_{N/4 < n \le N/2 - N^{1/2 + \alpha}} \frac{1}{n\sqrt{n(N-n)}} \frac{1}{\sqrt{N}} \le \frac{c}{N^{3/2}}.$$

Considering now term (*iii*), we can overestimate  $\mathbf{P}(U \leq 2n)$  by 1, and obtain, using (3.2) again,

$$\sum_{(iii)} \sim \frac{1}{\pi} \sum_{N/2 - N^{1/2 + \alpha} \le n < N/2 + N^{1/2 + \alpha}} \frac{1}{n^{3/2} (N - n)^{1/2}} \le \frac{c}{N^{3/2 - \alpha}}.$$

Skipping term (iv) to finish estimating the negligible terms, it is easy to see that

$$\sum_{(v)} \le \sum_{N-N^{1/2-\alpha} \le n < N} \frac{1}{n^{3/2}} \le c \frac{N^{1/2-\alpha}}{N^{3/2}} = \frac{c}{N^{1+\alpha}}.$$

using again only that  $\mathbf{P}(U \leq 2n) \leq 1$  and (3.3).

Now we want to show that part (iv) in sum I will give the order of magnitude claimed in the theorem. It is easy to see by normal approximation again that for  $n \in (iv)$  we obtain

$$\Phi(cN^{\alpha}) \le \mathbf{P}(U \le 2n) \le 1,$$

to conclude that for  $n \in (iv)$ 

$$\mathbf{P}(U \le 2n) = 1 - o(1), \quad as \quad N \to \infty.$$

So we need the asymptotic value of

$$\sum_{(iv)} \sim \frac{1}{\pi} \sum_{N/2 + N^{1/2 + \alpha} \le n < N - N^{1/2 - \alpha}} \frac{1}{n^{3/2} (N - n)^{1/2}}.$$
(3.7)

By showing that

$$\sum_{N/2 \le n \le N/2 + N^{1/2 + \alpha}} \frac{1}{n^{3/2} (N - n)^{1/2}} \le c \frac{N^{1/2 + \alpha}}{N^2} = \frac{c}{N^{3/2 - \alpha}}$$

and

$$\sum_{N-N^{1/2-\alpha} \le n \le N} \frac{1}{n^{3/2} (N-n)^{1/2}} \le c \frac{N^{1/2-\alpha}}{N^{3/2}} = \frac{c}{N^{1+\alpha}},$$

we can extend the interval of summation in (3.7) without changing the limit of the sum as follows:

$$I \sim \frac{1}{\pi N} \sum_{N/2 < n < N} \frac{1}{\left(\frac{n}{N}\right)^{3/2} \left(1 - \frac{n}{N}\right)^{1/2}} \frac{1}{N} \sim \frac{1}{\pi N} \int_{1/2}^{1} \frac{dv}{v^{3/2} (1 - v)^{1/2}} = \frac{2}{\pi N}.$$

Concerning the term  $II^*$ , it is clear that Q(r, n) being a probability, the four negligible terms which we investigated as terms of I are also negligible compared to the main term. The only problem is to estimate the sum  $II^*$  for  $n \in (iv)$ . This however is a delicate calculation. We split the sum for r into 3 parts:

(1) 
$$0 \le r \le n/4$$
,

(2) 
$$n/4 < r \le n - n^{\beta}$$
,

$$(3) \quad n - n^{\beta} < r \le n.$$

with some 
$$0 < \beta < 1/2 + \alpha < 1$$
.

For (1) we use that Q(r, n) is a probability, obtaining just as in (i) in (3.5) that

$$\sum_{r \le n/4} Q(r,n) \mathbf{P}(2r-1 \le U \le 2r) \le c \mathbf{P}(U \le N/2) < \frac{N}{2} \binom{2N}{N/2} \frac{1}{2^{2N}},$$

 $\operatorname{So}$ 

$$\sum_{n \in (iv)} \frac{1}{n} {2n \choose n} {2N-2n \choose N-n} \frac{1}{2^{2N}} \sum_{r \in (1)} Q(r,n) \mathbf{P}(2r-1 < U \le 2r)$$
$$\le \sum_{n \in (iv)} \frac{cN}{n} {2N \choose N/2} \frac{1}{2^{2N}} \le c {2N \choose N/2} \frac{1}{2^{2N}} \le c \left(\frac{4}{3\sqrt{3}}\right)^N,$$

where the last inequality is coming from Stirling formula as in (3.6).

In case (2), using Stirling formula, we have

$$\sum_{r \in (2)} Q(r,n) \mathbf{P}(2r-1 \le U \le 2r) \le \sum_{r \in (2)} \frac{c\sqrt{n}}{\sqrt{r}\sqrt{n-r}} \mathbf{P}(2r-1 \le U \le 2r)$$
$$\le \frac{c}{n^{\beta/2}} \sum_{r \in (2)} \mathbf{P}(2r-1 \le U \le 2r) \le \frac{c}{n^{\beta/2}} \le \frac{c}{N^{\beta/2}},$$

where the last inequality holds as  $n \in (iv)$ . Consequently, similarly to (iv) in calculating I, we have  $\frac{c}{N^{\beta/2}}$  times the sum in (3.7) implying that

$$\sum_{n \in (iv)} \sum_{r \in (2)} \le \frac{c}{N^{1+\beta/2}}.$$

For the case  $r \in (3)$  we have

$$\sum_{r \in (3)} Q(r,n) \mathbf{P}(2r-1 \le U \le 2r) \le c \mathbf{P}(2n-2n^{\beta}-1 \le U \le 2n) \le c \mathbf{P}(U \ge 2n-2n^{\beta}-1).$$

Recall now that  $U = U_{2N-2n+1}$  with E(U) = 2N - 2n + 1 and Var(U) = 2(2N - 2n + 1). Applying now Chebyshev inequality in the form

$$\mathbf{P}(X - \mu \ge x\sigma) \le \mathbf{P}(|X - \mu| \ge x\sigma) \le \frac{1}{x^2}$$

we arrive to

$$\mathbf{P}(U-2N+2n-1 \ge 4n-2N-2n^{\beta}-2) = \mathbf{P}\left(U-E(U) \ge \frac{4n-2N-2n^{\beta}-2}{(4N-4n+2)^{1/2}}\sigma\right)$$
$$\le \frac{4N-4n+2}{(4n-2N-2n^{\beta}-2)^2} \sim \frac{N-n}{(2n-N-n^{\beta})^2}.$$

Being  $n \in (iv)$  we have  $\frac{N}{2} + N^{1/2+\alpha} \le n < N - N^{1/2-\alpha}$ , implying that

$$N-n \le N/2$$
 and  $2n-N \ge 2N^{1/2+\alpha}$ 

Knowing also that  $1>1/2+\alpha>\beta>0$  we can conclude that

$$\mathbf{P}(U \ge 2n - 2n^{\beta} - 1) \le \frac{c}{N^{2\alpha}}$$

which goes to zero as  $N \to +\infty$ , so the term  $\sum_{n \in (iv)} \sum_{r \in (3)}$  is negligible compared to  $\sum_{n \in (iv)}$  in the main term.

This completes the proof of Theorem 3.1.  $\Box$ 

## 4 Laws of the iterated logarithm for the local time

Define the local time of the random walk on the HPHC lattice as

$$\Xi((k,j),N) = \sum_{r=0}^{N} I\{\mathbf{C}(r) = (k,j)\}, \ (k,j) \in \mathbb{Z}^2$$

From Theorem 3.1 we can calculate the truncated Green function  $g(\cdot)$ :

$$g(N) = \sum_{k=0}^{\lfloor N/2 \rfloor} \mathbf{P}(\mathbf{C}(2k) = 0) \sim \frac{2}{\pi} \log N \qquad as \ N \to \infty.$$

Our random walk being Harris recurrent, we can infer (e.g. Chen [5]) that

$$\lim_{N \to \infty} \frac{\Xi((k_1, j_1), N)}{\Xi((k_2, j_2), N)} = \frac{\mu(k_1, j_1)}{\mu(k_2, j_2)} \quad a.s.$$

where  $\mu(\cdot)$  is an invariant measure. Here the invariant measure is defined as the solution of the equation

$$\mu(A) = \sum_{(k,j) \in \mathbb{Z}^2} \mu(k,j) \mathbf{P}(\mathbf{C}(N+1) \in A | \mathbf{C}(N) = (k,j)).$$

For  $(k, j) \in \mathbb{Z}^2$ , in our case we have

$$\mu(k,j) = \mu(k+1,j)\left(\frac{1}{2} - p_j\right) + \mu(k-1,j)\left(\frac{1}{2} - p_j\right) + \mu(k,j+1)p_{j+1} + \mu(k,j-1)p_{j-1}$$

where

$$p_j = \frac{1}{4}$$
 if  $j \ge 0$  and  $p_j = \frac{1}{2}$  if  $j < 0$ .

It is easy to see that

$$\mu(k,j) = \frac{1}{p_j}, \quad (k,j) \in \mathbb{Z}^2$$

is one possible invariant measure. So from Theorem 17.3.2 of Meyn and Tweedie [15] we get the following result

**Corollary 4.1** For all integers  $k_1, k_2$  we have

$$\lim_{N \to \infty} \frac{\Xi((k_1, j_1), N)}{\Xi((k_2, j_2), N)} = 1 \quad a.s. \quad \text{if} \quad j_1 \ge 0, \ j_2 \ge 0 \quad \text{or} \quad j_1 < 0, \ j_2 < 0,$$

and

$$\lim_{N \to \infty} \frac{\Xi((k_1, j_1), N)}{\Xi((k_2, j_2), N)} = 2 \quad a.s. \quad \text{if} \quad j_1 \ge 0, \ j_2 < 0.$$

Using g(N) again, we get from Darling and Kac [10] the following result.

#### Corollary 4.2

$$\lim_{N \to \infty} \mathbf{P}\left(\frac{\Xi((0,0),N)}{g(N)} \ge x\right) = \lim_{N \to \infty} \mathbf{P}\left(\frac{\pi \, \Xi((0,0),N)}{2 \log N} \ge x\right) = e^{-x}.$$

As to the law of the iterated logarithm, we get from Theorem 2.4 of Chen [5] that it reads as follows.

#### Corollary 4.3

$$\limsup_{N \to \infty} \frac{\Xi((0,0), N)}{\log N \log \log \log N} = \frac{2}{\pi} \qquad a.s.$$

To conclude we would like to discuss how these results relate to the corresponding ones for other anisotropic planar walks. In the anisotropic walk in general, everything is determined by the return probability to zero, which allow us to calculate the Green function, which leads to the results about the local time. As we will see below the return probability, which we got for the HPHC walk is only differ in a constant from the return probability of the simple symmetric walk of the plane, and much smaller than the return probability to zero of the two dimensional comb.

As far as we know the return probability to zero for the anisotropic random walk is not known. However for the periodic anisotropic random walk  $\mathbf{C}^{\mathbf{P}}(\cdot)$  which is defined by  $p_j = p_{j+L}$  for each  $j \in \mathbb{Z}$ , where L is a positive integer, we proved in [9] that

$$\mathbf{P}(\mathbf{C}^{\mathbf{P}}(2N) = (0,0)) \sim \frac{1}{4\pi N p_0 \sqrt{\gamma - 1}} \quad as \quad N \to \infty,$$
$$\gamma = \frac{\sum_{j=0}^{L-1} p_j^{-1}}{2L}.$$

where

This leads to the following local time results

$$\lim_{N \to \infty} \frac{\Xi^P((0,0),N)}{\Xi^P((k,j),N)} = \frac{p_j}{p_0} \qquad a.s.$$
$$\lim_{N \to \infty} \sup_{N \to \infty} \frac{\Xi^P((0,0),N)}{\log \log \log \log N} = \frac{1}{4p_0\pi\sqrt{\gamma-1}} \qquad a.s.$$

In case of the simple symmetric walk on the plane is, when  $p_j = p_0 = 1/4$  for  $j = \pm 1, \pm 2, ...$ and L = 1 this reduces to the well known asymptotic formula

$$\mathbf{P}(\mathbf{C}(2N) = (0,0)) \sim \frac{1}{\pi N} \quad as \quad N \to \infty.$$

This leads to the famous Erdős -Taylor integral test [12] (see e.g in [18]) containing e.g. that

$$\limsup_{N \to \infty} \frac{\Xi((0,0),N)}{\log N \log \log \log N} = \frac{1}{\pi} \qquad a.s.$$

On the other hand in the case of the 2-dimensional comb, when  $p_0 = 1/4$ , and  $p_j = 1/2$  for  $j = \pm 1, \pm 2, \ldots$  we have from Bertacchi and Zucca [3] that

$$\mathbf{P}(\mathbf{C}(2N) = (0,0)) \sim \frac{1}{2^{9/2} \Gamma(1/4) N^{3/4}} \quad as \quad N \to \infty.$$

For the local time of the two-dimensional comb [7], we have for any fix k

$$\limsup_{N \to \infty} \frac{\Xi((k,0),N)}{N^{1/4} (\log \log N)^{3/4}} = \frac{2^{9/4}}{3^{3/4}} \qquad a.s$$

and for any fix k and any fix  $j \neq 0$ 

$$\limsup_{N \to \infty} \frac{\Xi((k,j),N)}{N^{1/4} (\log \log N)^{3/4}} = \frac{2^{5/4}}{3^{3/4}} \qquad a.s.$$

As we mentioned above, the local time behavior is determined by the return probability to zero. The order of magnitude (apart from a constant factor) of the return probability for the simple symmetric random walk, for the periodic walk on the plane discussed above, and for the HPHC walk are the same. So their local time behavior are the same as well. However as the order of the return probability of the comb is different, its local time behavior is very different from the other three cases above. It would be interesting to find examples which shed some light upon this transition between these two types of behavior.

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