Strong Approximation of the Anisotropic Random Walk Revisited

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Abstract
We study the path behavior of the anisotropic random walk on the two-dimensional lattice $\mathbb{Z}^2$. Strong approximation of its components with independent oscillating Brownian motions are proved.

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1 Introduction and main results

We consider random walks on the square lattice $\mathbb{Z}^2$ of the plane with possibly unequal symmetric horizontal and vertical step probabilities, so that these probabilities can only depend on the value of the vertical coordinate. In particular, if such a random walk is situated at a site on the horizontal line $y = j \in \mathbb{Z}$, then at the next step it moves with probability p_j to either vertical neighbor, and with probability $1/2 - p_j$ to either horizontal neighbor. A substantial motivation for studying such two-dimensional random walks on anisotropic lattice has originated from transport problems of statistical physics.

More formally, consider the random walk $\{\mathbf{C}(N) = (C_1(N), C_2(N))\,;\, N = 0, 1, 2, \ldots\}$ on $\mathbb{Z}^2$ with the transition probabilities

$$\mathbf{P}(\mathbf{C}(N+1) = (k+1, j)|\mathbf{C}(N) = (k, j)) = \mathbf{P}(\mathbf{C}(N+1) = (k-1, j)|\mathbf{C}(N) = (k, j)) = \frac{1}{2} - p_j,$$

$$\mathbf{P}(\mathbf{C}(N+1) = (k, j+1)|\mathbf{C}(N) = (k, j)) = \mathbf{P}(\mathbf{C}(N+1) = (k, j-1)|\mathbf{C}(N) = (k, j)) = p_j,$$

for $(k,j)\in\mathbb{Z}^2$, $N=0,1,2,\dots$ We assume throughout the paper that $0<p_j\le 1/2$ and $\min_{j\in\mathbb{Z}} p_j < 1/2$. Unless otherwise stated we assume also that $\mathbf{C}(0)=(0,0)$. In this paper we will have the following condition

$$n^{-1}\sum_{j=1}^{n} p_j^{-1} = 2\gamma_1 + o(n^{-\tau}), \qquad n^{-1}\sum_{j=1}^{n} p_{-j}^{-1} = 2\gamma_2 + o(n^{-\tau}) \tag{1.1}$$

as $n\to\infty$ for some constants $1 < max(\gamma_1,\gamma_2) < \infty$, and $1/2<\tau\le 1$. We will point out how this condition is different from the previous similar results. The case $p_j=1/4$, $j=0,\pm1,\pm2,\dots$ corresponds to simple symmetric random walk on the plane. For this case we refer to Erdős and Taylor [18], Dvoretzky and Erdős [17], and Révész [25]. The case $p_j=1/2$ for some j means that the horizontal line $y=j$ is missing. If all $p_j=1/2$, then the random walk takes place on the y axis, so it is only a one-dimensional random walk, and this case is excluded from the present investigations. The case however when $p_j=1/2$, $j=\pm1,\pm2,\dots$ but $p_0=1/4$ is an interesting one which is the so-called random walk on the two-dimensional comb. In this case $\gamma_1=\gamma_2=1$. For this model we may refer to Weiss and Havlin [30], Bertacchi and Zucca [2], Bertacchi [1], Csáki *et al.* [8]. In the comb model the scaling of the horizontal and vertical coordinates are different, namely for the first coordinate the scaling is of order $N^{1/4}$, so it is a so called sub-diffusion, and can be approximated with an iterated Wiener process while the second coordinate is of order $N^{1/2}$, and hence it can be approximated with a Wiener process.

In our paper Csáki *et al.* [10] we considered the case when both coordinates are of order $N^{1/2}$, hence can be approximated simultaneously with independent Wiener processes. More precisely we investigated the case when in (1.1) we have $\lambda_1=\lambda_2>1$. In Csáki *et al.* [10] we proved that

Theorem A *Under the condition* (1.1) *with* $\lambda=\lambda_1=\lambda_2>1$, *and* $1/2<\tau\le 1$, *on an appropriate probability space for the random walk*

$$\{\mathbf{C}(N)=(C_1(N),C_2(N));N=0,1,2,\dots\}$$

one can construct two independent standard Wiener processes $\{W_1(t);\ t\ge 0\}$, $\{W_2(t);\ t\ge 0\}$ *so that, as* $N\to\infty$, *we have with any* $\varepsilon>0$

$$\left|C_1(N)-W_1\left(\frac{\gamma-1}{\gamma}N\right)\right| + \left|C_2(N)-W_2\left(\frac{1}{\gamma}N\right)\right| = O(N^{5/8-\tau/4+\varepsilon}) \quad a.s. \tag{1.2}$$

The case $\lambda_1=\lambda_2>1$ has a considerable complex history, we refer to the interested reader to [10]. Here we just mention a few names; Silver *et al.* [28], Seshadri *et al.* [26], Shuler [27], Westcott [29]. Some of the most important contribution to this topic is due to Heyde [19], [20] and Heyde *et al.* [21]. Let $\{Y(t),t\ge 0\}$ be a diffusion process on the same probability space as $\{C_2(n)\}$ whose distribution is defined by

$$Y(t)=W(A^{-1}(t)), \quad t\ge 0,$$

where $\{W(t), t \geq 0\}$ is a standard Brownian motion, (or standard Wiener process) and

$$A(t) = \int_0^t \sigma^{-2}(W(s))\, ds$$

and

$$\sigma^2(y) = \begin{cases} \frac{1}{\gamma_1} & \text{for} \quad y \geq 0, \\ \frac{1}{\gamma_2} & \text{for} \quad y < 0. \end{cases}$$

Here $A^{-1}(\cdot)$ is the inverse of $A(\cdot)$. The process $Y(t)$ is called oscillating Brownian motion if $\gamma_1 \neq \gamma_2$, that is a diffusion with speed measure $m(dy) = 2\sigma^{-2}(y)dy$.

Remark 1.1 Observe that $A(t)$ above is equal to

$$A(t) = \gamma_1 \int_0^t I(W(s) \geq 0)\, ds + \gamma_2 \int_0^t I(W(s) < 0)\, ds. \tag{1.3}$$

Let

$$k^{-1} \sum_{j=1}^{k} p_j^{-1} = 2\gamma_1 + \varepsilon_k \quad k^{-1} \sum_{j=-k}^{-1} p_j^{-1} = 2\gamma_2 + \varepsilon_k^* \tag{1.4}$$

then the main result of Heyde *et al.* [21] is that

Theorem B ([21]) *Suppose that in* (1.4) ε_k *and* ε_k^* *are* $o(1)$ *as* $k \to \infty$. *Then*

$$\sup_{0 \leq t \leq N} |N^{-1/2} C_2([Nt]) - Y(t)| \to 0 \quad a.s.$$

Observe that here γ_1 and γ_2 might be different, and the convergence rates are much less restrictive, but the approximation is only for the second component.

Lets define an arbitrary set $B \subset \mathbb{Z}$ such that for

$$p_i = 1/4 \quad \text{if} \quad i \in B \quad \text{and} \quad p_i = 1/2 \quad i \in \mathbb{Z} \setminus B. \tag{1.5}$$

Thus we remove from the two-dimensional integer lattice all the horizontal edges which do not belong to the i-levels in B. In our paper [9] we investigated a simple random walk on the half-plane half-comb (HPHC) structure, which is another interesting special case where we define the set $B = \{i = 0, 1, 2, ...\}$, that is to say, all horizontal lines under the x-axis are deleted. Our main result there reads as follows.

Theorem C ([9]) *On an appropriate probability space for the HPHC random walk* $\{\mathbf{C}(N) = (C_1(N), C_2(N)); N = 0, 1, 2, \ldots\}$ *with* $p_j = 1/4$, $j = 0, 1, 2, \ldots$, $p_j = 1/2$, $j = -1, -2, \ldots$ *one can construct two independent standard Wiener processes* $\{W_1(t);\, t \geq 0\}$, $\{W_2(t);\, t \geq 0\}$ *such that, as* $N \to \infty$, *we have with any* $\varepsilon > 0$

$$|C_1(N) - W_1(N - A_2^{-1}(N))| + |C_2(N) - W_2((A_2^{-1}(N))| = O(N^{3/8+\varepsilon}) \quad a.s.,$$

where $A_2(t) = 2\int_0^t I(W_2(s) \geq 0)\, ds + \int_0^t I(W_2(s) < 0)\, ds.$

Clearly in this case $\lambda_1 = 2$ and $\lambda_2 = 1$, τ can be selected to be 1. In our paper Csáki and Földes [11] we considered the case when the set B is much more general than in Theorem C. Our main result in that paper can be formulated as follows:

Theorem D [11] *Let*

$$|B_n| := |B \cap \{-n, n\}| \sim cn \tag{1.6}$$

with some constant $c > 0$*, where* B *is defined in* (1.5) *and* $|B_n|$ *stands for the (finite) number of elements in the set* B_n*. Under the conditions* (1.1) *with* $\max(\gamma_1, \gamma_2) > 1$ *on an appropriate probability space for the random walk* $\{\mathbf{C}(N) = (C_1(N), C_2(N));\ N = 0, 1, 2, \ldots\}$ *one can construct two independent standard Wiener processes* $\{W_1(t);\ t \geq 0\}$*,* $\{W_2(t);\ t \geq 0\}$ *so that, as* $N \to \infty$*, we have with any* $\varepsilon > 0$

$$\left|C_1(N) - W_1\left(N - A_2^{-1}(N)\right)\right| + \left|C_2(N) - W_2\left(A_2^{-1}(N)\right)\right| = O(N^{5/8-\tau/4+\varepsilon}) \quad a.s. \tag{1.7}$$

where $A_2^{-1}(\cdot)$ *is the inverse of* $A_2(\cdot)$*.*

So in Theorem D we have exactly our condition (1.1), but all the p_i-s has to be 1/2 or 1/4. Our goal in this paper is to get a common generalization of Theorems A,B,C and D, namely we only need condition (1.1) and no other restrictions for the p_i-s.

Theorem 1.1 *Under the conditions* (1.1)*, and* $1 \leq \gamma_2 < \gamma_1$ *on an appropriate probability space for the random walk* $\{\mathbf{C}(N) = (C_1(N), C_2(N));\ N = 0, 1, 2, \ldots\}$ *one can construct two independent standard Wiener processes* $\{W_1(t);\ t \geq 0\}$*,* $\{W_2(t);\ t \geq 0\}$ *so that, as* $N \to \infty$*, we have with any* $\varepsilon > 0$

$$\left|C_1(N) - W_1\left(N - A_2^{-1}(N)\right)\right| + \left|C_2(N) - W_2\left(A_2^{-1}(N)\right)\right| = O(N^{5/8-\tau/4+\varepsilon}) \quad a.s. \tag{1.8}$$

where $A_2(t) = \lambda_1 \int_0^t I(W_2(s) \geq 0)\, ds + \lambda_2 \int_0^t I(W_2(s) < 0)\, ds.$

and $A_2^{-1}(\cdot)$ *is the inverse of* $A_2(\cdot)$*.*

Remark 1.2 If $\gamma_1 = \gamma_2 > 1$ then $A_2(t) = \gamma_1 t$ and our theorem coincides with Theorem A. So we made the supposition that $1 \leq \gamma_2 < \gamma_1$, instead of $1 < max(\gamma_1, \gamma_2) < \infty$, even though it is not necessary but makes the flow of argument easier.

2 Preliminaries

First we are to redefine our random walk $\{\mathbf{C}(N);\ N = 0, 1, 2, \ldots\}$. It will be seen that the process described right below is equivalent to that given in the Introduction (cf. (2.2) below).

To begin with, on a suitable probability space consider two independent simple symmetric (one-dimensional) random walks $S_1(\cdot)$, and $S_2(\cdot)$. We may assume that on the same probability space we have a double array of independent geometric random variables $\{G_i^{(j)},\, i \ge 1, j \in \mathbb{Z}\}$ which are independent from $S_1(\cdot)$, and $S_2(\cdot)$, where $G_i^{(j)}$ has the following geometric distribution

$$\mathbf{P}(G_i^{(j)} = k) = 2p_j(1-2p_j)^k,\ \ k = 0, 1, 2, \dots \tag{2.1}$$

We now construct our walk $\mathbf{C}(N)$ as follows. We will take all the horizontal steps consecutively from $S_1(\cdot)$ and all the vertical steps consecutively from $S_2(\cdot)$. First we will take some horizontal steps from $S_1(\cdot)$, then exactly one vertical step from $S_2(\cdot)$, then again some horizontal steps from $S_1(\cdot)$ and exactly one vertical step from $S_2(\cdot)$, and so on. Now we explain how to get the number of horizontal steps on each occasion. Consider our walk starting from the origin proceeding first horizontally $G_1^{(0)}$ steps (note that $G_1^{(0)} = 0$ is possible with probability $2p_0$), after which it takes exactly one vertical step, arriving either to the level 1 or -1, where it takes $G_1^{(1)}$ or $G_1^{(-1)}$ horizontal steps (which might be no steps at all) before proceeding with another vertical step. If this step carries the walk to the level j, then it will take $G_1^{(j)}$ horizontal steps, if this is the first visit to level j, it takes $G_2^{(j)}$ horizontal steps, if this is its second visit at level j and so on. In general, if we finished the k -th vertical step and arrived to the level j for the i-th time, then it will take $G_i^{(j)}$ horizontal steps.

Let now H_N, V_N be the number of horizontal and vertical steps, respectively from the first N steps of the just described process. Consequently, $H_N + V_N = N$, and

$$\begin{aligned}\{\mathbf{C}(N);\, N = 0, 1, 2, \dots\} &= \{(C_1(N), C_2(N));\, N = 0, 1, 2, \dots\} \\ &\stackrel{d}{=} \{(S_1(H_N), S_2(V_N));\, N = 0, 1, 2, \dots\}\,, \end{aligned} \tag{2.2}$$

where $\stackrel{d}{=}$ stands for equality in distribution.

We will need the following two lemmas from Csáki at al. [10]

Lemma A *Let $\{G_i^{(j)},\, i = 1, 2, \dots, n_j,\, j = 1, 2, \dots, K\}$ be independent geometric random variables with distribution*

$$\mathbf{P}(G_i^{(j)} = k) = \alpha_j(1-\alpha_j)^k, \quad k = 0, 1, 2, \dots,$$

where $0 < \alpha_j \le 1$. Put

$$B_K = \sum_{j=1}^{K}\sum_{i=1}^{n_j} G_i^{(j)}, \quad \sigma^2 = Var B_K = \sum_{j=1}^{K} \frac{n_j(1-\alpha_j)}{\alpha_j^2}.$$

Then, for $\lambda < -\sigma^2 \log(1-\alpha_j)$ for each $j \in [1, K]$, we have

$$\mathbf{P}\left(\left|\sum_{j=1}^{K}\sum_{i=1}^{n_j}\left(G_i^{(j)} - \frac{1-\alpha_j}{\alpha_j}\right)\right| > \lambda\right) \le 2\exp\left(-\frac{\lambda^2}{2\sigma^2} + \sum_{\ell=3}^{\infty}\frac{\lambda^\ell}{\sigma^{2\ell}}\sum_{j=1}^{K}\frac{n_j}{\alpha_j^\ell}\right). \tag{2.3}$$

Lemma B *Assume the conditions of* Lemma A *and put* $M = \sum_{j=1}^{K} n_j$*. For* $M \to \infty$ *and* $K \to \infty$ *assume moreover that*

$$K = K(M) = O(M^{1/2+\delta}), \quad \max_{1 \le j \le K} n_j = O(M^{1/2+\delta}), \tag{2.4}$$

for all $\delta > 0$*,*

$$\frac{1}{\alpha_j} \le c_1 |j|^{1-\tau}, \quad j = 0, 1, 2, \dots \tag{2.5}$$

for some $1/2 < \tau \le 1$ *and* $c_1 > 0$*,*

$$\sum_{j=1}^{K} \frac{1}{\alpha_j} = O(K), \qquad \frac{1}{\sigma} \le \frac{c_2}{M^{1/2}} \tag{2.6}$$

for some $c_2 > 0$*. Then we have as* $K, M \to \infty$*,*

$$\sum_{j=1}^{K} \sum_{i=1}^{n_j} G_i^{(j)} = \sum_{j=1}^{K} n_j \frac{1-\alpha_j}{\alpha_j} + O(M^{3/4-\tau/4+\varepsilon}) \quad a.s. \tag{2.7}$$

for some $\varepsilon > 0$*.*

Remark 2.1 Lemmas A and B in [10] are formulated for $-K \le j \le K$, but it is obvious that the present one can be stated and proved word by word as it is in [10].

Let $\{X_i\}$ be a sequence of independent i.i.d. random variable, with $\mathbf{P}(X_i = \pm 1) = 1/2$. Let $S(n) = \sum_{i=1}^{n} X_i$. Then $S(n)$ is a simple symmetric walk on the line. Its local time is defined by $\xi(j, n) = \#\{k : 0 < k < n, S(k) = j\}, \quad n = 1, 2, \ldots.$ for any integer j. Define $M_n = \max_{0 \le k \le n} |S(k)|$. Then we have the usual law of the iterated logarithm (LIL) and Chung's LIL [6].

Lemma C

$$\limsup_{n\to\infty} \frac{M_n}{(2n \log\log n)^{1/2}} = 1, \qquad \liminf_{n\to\infty} \left(\frac{\log\log n}{n}\right)^{1/2} M_n = \frac{\pi}{\sqrt{8}} \qquad a.s.$$

The following result is from Heyde [19], (see also in [12] Lemma 5)

Lemma D *For the simple symmetric random walk for any* $\varepsilon > 0$ *we have*

$$\lim_{n\to\infty} \frac{\sup_{x\in\mathbb{Z}} |\xi(x+1, n) - \xi(x, n)|}{n^{1/4+\varepsilon}} = 0 \quad a.s.$$

For the next Lemma see Kesten [22].

Lemma E *For the maximal local time*

$$\xi(n) = \sup_{x\in\mathbb{Z}} \xi(x,n)$$

we have

$$\limsup_{n\to\infty} \frac{\xi(n)}{(2n\log\log n)^{1/2}} = 1 \quad a.s.$$

We need a simple lemma about the properties of $A(t)$ from Csáki and Földes [11]

Lemma F Consider $A(t)$ defined by (1.3) and let $\alpha(t) = A(t) - t$. *Let* $\gamma_1 > \gamma_2 \geq 1$. *Then*

- $A(t)$ and $\alpha(t)$ are nondecreasing
- $\gamma_2 t \leq A(t) \leq \gamma_1 t$ and $\frac{t}{\gamma_1} \leq A^{-1}(t) \leq \frac{t}{\gamma_2}$.

We will need the famous KMT strong invariance principle (cf. Komlós *et al.* [23]).

Lemma G *On an appropriate probability space one can construct* $\{S(n),\, n = 1, 2, \ldots\}$, *a simple symmetric random walk on the line and a standard Wiener process* $\{W(t),\, t \geq 0\}$ *such that as* $n \to \infty$,

$$S(n) - W(n) = O(\log n) \qquad a.s.$$

The next lemma is a simultaneous strong approximation result of Révész [24]

Lemma H *On an appropriate probability space for a simple symmetric random walk* $\{S(n);\, n = 0, 1, 2, \ldots\}$ *with local time* $\{\xi(x,n);\, x = 0, \pm1, \pm2, \ldots;\, n = 0, 1, 2, \ldots\}$ *one can construct a standard Wiener process* $\{W(t);\, t \geq 0\}$ *with local time process* $\{\eta(x,t);\, x \in \mathbb{R};\, t \geq 0\}$ *such that, as* $n \to \infty$, *we have for any* $\varepsilon > 0$

- $|S(n) - W(n)| = O(n^{1/4+\varepsilon}) \quad a.s.$

  and

- $\sup_{x\in\mathbb{Z}} |\xi(x,n) - \eta(x,n)| = O(n^{1/4+\varepsilon}) \quad a.s.,$

simultaneously.

Concerning the increments of the Wiener process we quote the following result from Csörgő and Révész ([14], page 69).

Lemma I *Let* $0 < a_T \leq T$ *be a non-decreasing function of* T. *Then, as* $T \to \infty$, *we have*

$$\sup_{0\leq t\leq T-a_T} \sup_{s\leq a_T} |W(t+s) - W(t)| = O(a_T(\log(T/a_T) + \log\log T))^{1/2} \qquad a.s.$$

The above statement is also true if $W(\cdot)$ replaced by the simple symmetric random walk $S(\cdot)$.

For a simple random walk with local time $\xi(\cdot,\cdot)$, let

$$\widehat{A}(n)=\gamma_1\sum_{j=0}^{\infty}\xi(j,n)+\gamma_2\sum_{j=1}^{\infty}\xi(-j,n). \tag{2.8}$$

We will need the following lemma from Csáki and Földes [11]

Lemma J *On a probability space as in Lemma H*

$$|\widehat{A}(n)-A(n)|=O(n^{3/4+\varepsilon}) \qquad a.s.$$

where $A(.)$ is defined in (1.3). The next lemma is using some ideas from Heyde [19].

Lemma 2.1 *Let $\{S(i)\}_{i=1}^{\infty}$ a simple symmetric random walk with local time $\xi(i,n)$. Let $\beta_j>0 \quad j=1,2,\ldots$ Suppose that*

$$n^{-1}\sum_{j=1}^{n}\beta_j^{-1}=\rho+o(n^{-\tau}), \tag{2.9}$$

as $n\to\infty$ for some constants $0<\rho<\infty$, and $1/2<\tau\le 1$. Then

$$\sum_{j=1}^{N}\xi(j,N)\frac{1}{\beta_j}=\rho\sum_{j=1}^{\infty}\xi(j,N)+O(N^{5/4-\tau/2+\varepsilon}) \quad a.s. \tag{2.10}$$

Proof: Introduce the notation:

$$\frac{1}{j}\sum_{k=1}^{j}\frac{1}{\beta_k}=\kappa_j \quad j=1,2,\ldots.$$

$$\sum_j\xi(j,N)\frac{1}{\beta_j}=\sum_{j=1}^{\infty}\xi(j,N)(j\kappa_j-(j-1)\kappa_{j-1})$$

$$=\sum_{j=1}^{\infty}j\kappa_j(\xi(j,N)-\xi(j+1,N))$$

$$=\sum_{j=1}^{\infty}j(\kappa_j-\rho)(\xi(j,N)-\xi(j+1,N))+\rho\sum_{j=1}^{\infty}j(\xi(j,N)-\xi(j+1,N))$$

$$=\rho\sum_{j=1}^{\infty}\xi(j,N)+\sum_{j=1}^{\infty}j(\kappa_j-\rho)(\xi(j,N)-\xi(j+1,N))$$

Observe that from (2.9) we have that

$$|j(\kappa_j - \rho)| \le c j^{1-\tau}$$

for some $c > 0$. Now applying Lemma C for $S(\cdot)$, and Lemma D, we get that

$$\sum_{j=1}^{\infty} j(\kappa_j - \rho)(\xi(j, N) - \xi(j+1, N))$$

$$= O(N^{1/4+\epsilon}) \sum_{j=1}^{\max_{k\le N}|S(k)|} j^{1-\tau} = O(N^{1/4+\epsilon}) O(N^{1-\tau/2+\varepsilon}) = O(N^{5/4-\tau/2+\varepsilon}) \qquad a.s.,$$

where here and throughout the paper the value of ε might change from line to line. □

3 Proofs

Proof of Theorem 1.1 Recall that H_N and V_N are the number of horizontal and vertical steps respectively of the first N steps of $\{\mathbf{C}(.)\}$. First we would like to approximate H_N almost surely as $N \to \infty$.

Consider the sum

$$G_1^{(j)} + G_2^{(j)} + ... + G_{\xi_2(j, V_N)}^{(j)}$$

which is the total number of horizontal steps on the level j, where $\xi_2(\cdot, \cdot)$ is the local time of the walk $S_2(\cdot)$. This statement is slightly incorrect if j happens to be the level where the last vertical step (up to the total of N steps) takes the walk. In this case the last geometric random variable might be truncated. However the error which might occur from this simplification will be part of the $O(\cdot)$ term. This can be seen as follows. Let

$$H_N^* = \sum_j \sum_{i=1}^{\xi_2(j, V_N)} G_i^{(j)},$$

where $G_i^{(j)}$ has distribution (2.1). Clearly

$$H_N^* - H_N \le \max_j G_{\xi_2(j, V_N)}^{(j)}.$$

Here and in the sequel

$$\sum_j = \sum_{\min_{0\le k\le V_N} S_2(k) \le j \le \max_{0\le k\le V_N} S_2(k)}$$

and

$$\max_j = \max_{\min_{0\le k\le V_N} S_2(k)\le j\le \max_{0\le k\le V_N} S_2(k)}$$

Note that from (1.1) we have

$$\frac{1}{\alpha_j} = \frac{1}{2p_j} \le c_1|j|^{1-\tau} \quad j = \pm 1, \pm 2, ...$$

$$\begin{aligned} P(\max_j G^{(j)}_{\xi_2(j,V_N)} > N^{1/2+\delta}) &\le \sum_j P(G^j_1 > N^{1/2+\delta}) \le \sum_j (1-\alpha_j)^{N^{1/2+\delta}} \\ &\le \sum_j \exp(-\alpha_j N^{1/2+\delta}) \le \sum_j \exp(-cj^{\tau-1}N^{1/2+\delta}) \\ &\le N^{1/2+\delta^*}\exp(-cN^{\tau-1}N^{1/2+\delta}) \\ &\le N^{1/2+\delta^*}\exp(-cN^{\tau-1/2+\delta}) \le \exp(-cN^{\varepsilon}) \end{aligned} \tag{3.1}$$

with some small $\varepsilon > 0, \delta > 0, \delta^* > 0$. In the last line we used that $1/2 < \tau \le 1$ and Lemma C. Here and in what follows the value of c can be different from line to line. By Borel Cantelli we have now that for $N \to \infty$

$$H^*_N - H_N \le N^{1/2+\delta} \tag{3.2}$$

almost surely for any $\delta > 0$.

The next step is to show that $\sum_j \xi_2(j, V_N)\frac{1}{2p_j}$ is close to $A(V_N)$.

To see this we apply Lemma 2.1 for the vertical walk $S_2(.)$ two times (separately for positive and negative j indices) with $\beta_j = 2p_j = \alpha_j$ for $j = \pm 1, \pm 2, \dots$ and $\rho = \gamma_1$ and γ_2 respectively for positive and negative indices and with V_N instead of N to conclude, that

$$\begin{aligned} \sum_j \frac{\xi_2(j, V_N)}{\alpha_j} &= \gamma_1 \sum_{j=1}^{\infty} \xi_2(j, V_N) + \gamma_2 \sum_{j=1}^{\infty} \xi_2(-j, V_N) + \xi_2(0, V_N)\frac{1}{2p_0} + O(N^{5/4-\tau/2+\varepsilon}) \\ &= \widehat{A}_2(V_N) + O(N^{5/4-\tau/2+\varepsilon}) = A_2(V_N) + O(N^{5/4-\tau/2+\varepsilon}). \end{aligned} \tag{3.3}$$

where in the last line we used Lemmas E and J and that $V_n \le N$.

The rest of proof will be different for $\gamma_2 > 1$ and for $\gamma_2 = 1$.

Consider first the case $\gamma_2 > 1$. In this case we will apply Lemma B twice for

$$H^*_N = \sum_j \left(G^{(j)}_1 + G^{(j)}_2 + ... + G^{(j)}_{\xi_2(j,V_N)} \right).$$

Introduce $V_N(+) = \sum_{j=0}^{\infty} \xi_2(j, V_N)$ and $V_N(-) = \sum_{j=1}^{+\infty} \xi_2(-j, V_N)$, and and let $K = \max_{0\le k\le V_N} |S_2(k)|$. Define

$$\sigma_1^2 = \sum_{j=0}^{K} \frac{\xi_2(j, V_N)(1-\alpha_j)}{\alpha_j^2}, \quad \text{and} \quad \sigma_2^2 = \sum_{j=1}^{K} \frac{\xi_2(-j, V_N)(1-\alpha_{-j})}{\alpha_{-j}^2}.$$

Let $M = V_N(+)$ and $M = V_N(-)$ for the first and second application respectively and $n_j = \xi_2(j, V_N)$, $\alpha_j = 2p_j$, $j = 0, 1, 2, \ldots$ for the first and $n_j = \xi_2(j, V_N)$, $\alpha_j = 2p_j$, $j = -1, -2, \ldots - K$ indices for the second application.

We need to check all the assumptions of the lemma in both cases . (2.4) follows from Lemma C and Lemma E, and (2.5) follows from(1.1). The first part of (2.6) follows from (2.9) with ρ equal γ_1 and γ_2 respectively. It remains to verify the second part of (2.6). Using Lemma 2.1 we have almost surely as $N \to \infty$,

$$\begin{aligned} \frac{\sigma_1^2}{V_N(+)} &= \frac{1}{V_N(+)} \sum_{j\ge 0} \frac{\xi_2(j, V_N)(1-2p_j)}{(2p_j)^2} = \frac{1}{V_N)(+)} \sum_{j\ge 0} \frac{\xi_2(j, V_N(+))(1-2p_j)}{(2p_j)^2} \\ &\ge \frac{1}{V_N(+)} \sum_{j\ge 0} \frac{\xi_2(j, V_N(+))(1-2p_j)}{2p_j} = \frac{1}{V_N(+)} \sum_{j\ge 0} \frac{\xi_2(j, V_N(+))}{2p_j} - 1 \to \gamma_1 - 1 > 0. \end{aligned} \tag{3.4}$$

where the last inequality follows from our supposition of $\gamma_1 > 1$. The corresponding argument for σ_2^2 goes the same way using now that $\gamma_2 > 1$ as well. So we checked all the conditions of Lemma B and we can conclude that we have almost surely, as $N \to \infty$,

$$H_N^*(+) := \sum_{j\ge 0} \left(G_1^{(j)} + G_2^{(j)} + \ldots + G_{\xi_2(j,V_N)}^{(j)} \right) = \sum_{j\ge 0} \xi_2(j, V_N) \frac{1-2p_j}{2p_j} + O(N^{3/4-\tau/4+\varepsilon}) \tag{3.5}$$

and a similarly

$$H_N^*(-) := \sum_{j<0} \left(G_1^{(j)} + G_2^{(j)} + \ldots + G_{\xi_2(j,V_N)}^{(j)} \right) = \sum_{j<0} \xi_2(j, V_N) \frac{1-2p_j}{2p_j} + O(N^{3/4-\tau/4+\varepsilon}) \tag{3.6}$$

As for the case $\gamma_2 = 1$ we don't have (3.6), we need a different argument, as follows.

Recall (1.1) and that in this condition $1/2 < \tau \le 1$. Select $\delta > 0$ such that $\tau = 1/2 + 2\delta$ should hold. We will show that for N big enough

$$H_N^*(-) \le N^{1-\tau/2+\delta} \quad \text{a.s.} \tag{3.7}$$

To this end observe that for N big enough by Lemma C and Lemma E

$$M_N \le N^{1/2} \log N \quad \text{a.s.} \quad \text{and} \quad \xi(j, V_N) \le \xi(N) \le N^{1/2} \log N \quad \text{a.s.}$$

where $M_N = max_{0\le k\le N} |S_2(k)|$. Introduce the notation $r_N = N^{1/2} \log N$

implying that

$$H_N^*(-) \le \sum_{j=1}^{r_N} \sum_{i=1}^{r_N} G_i^{(-j)}. \quad a.s. \tag{3.8}$$

Let $N_k = k^k, \quad \lambda_N = N^{1-\tau/2+\delta}$ with some $\delta > 0$. Then $\frac{N_{k+1}}{N_k} \sim e(k+1)$. From Markov inequality and (1.1) we have that

$$\begin{aligned}
\mathbf{P}\left(\sum_{j=1}^{r_{N_{K+1}}} \sum_{i=1}^{r_{N_{K+1}}} G_i^{(-j)} > \lambda_{N_K}\right) &\le \frac{\mathbf{E}\left(\sum_{j=1}^{r_{N_{K+1}}} \sum_{i=1}^{r_{N_{K+1}}} G_i^{(-j)}\right)}{\lambda_{N_K}} = \frac{\left(\sum_{i=1}^{r_{N_{K+1}}} \sum_{j=1}^{r_{N_{K+1}}} \mathbf{E}G_i^{(-j)}\right)}{\lambda_{N_K}} \\
&= \frac{\sum_{i=1}^{r_{N_{K+1}}} \sum_{j=1}^{r_{N_{K+1}}} \frac{1-2p_{-j}}{2p_{-j}}}{\lambda_{N_K}} = \frac{\sum_{i=1}^{r_{N_{K+1}}} \left(\sum_{j=1}^{r_{N_{K+1}}} \frac{1}{2p_{-j}} - r_{N_{K+1}}\right)}{\lambda_{N_K}} \\
&= \frac{r_{N_{K+1}} o(r_{N_{K+1}})^{1-\tau}}{\lambda_{N_K}} \le \frac{(N_{K+1})^{1-\tau/2+\delta/2}}{(N_K)^{1-\tau/2+\delta}} \\
&\sim (e(K+1))^{1-\tau/2+\delta} \frac{1}{(K+1)^{\frac{(K+1)\delta}{2}}}.
\end{aligned} \tag{3.9}$$

So we got the $(K+1)$-th term of a convergent series. By Borel-Cantelli and the monotonicity of $H_N^*(-)$ we conclude that for $N_K \le N \le N_{K+1}$

$$H_N^*(-) \le H_{N_{K+1}}^*(-) \le \lambda_{N_K} \le \lambda_N = N^{1-\tau/2+\delta} \quad a.s. \tag{3.10}$$

for N big enough, proving (3.7). Applying now Lemma 2.1 with $\rho = 1$ and $\beta_j = 2p_{-j}$, $j = 1, 2, \ldots$ imply that

$$\sum_{j<0} \xi_2(j, V_N) \frac{1}{2p_j} = V_N(-) + O(N^{5/4-\tau/2+\varepsilon})$$

or equivalently

$$\sum_{j<0} \xi_2(j, V_N) \frac{1-2p_j}{2p_j} = O(N^{5/4-\tau/2+\varepsilon})$$

Consequently

$$\begin{aligned}
H_N^*(-) &= \sum_{j<0} \xi_2(j, V_N) \frac{1-2p_j}{2p_j} + O(N^{5/4-\tau/2+\varepsilon}) + N^{1-\tau/2+\delta} \\
&= \sum_{j<0} \xi_2(j, V_N) \frac{1-2p_j}{2p_j} + O(N^{5/4-\tau/2+\varepsilon}).
\end{aligned} \tag{3.11}$$

as we can select δ to be arbitrary small. Consequently, we have by (3.5), (3.6) and (3.11) that for $1 \leq \lambda_1 < \lambda_2$

$$\begin{aligned}
H_N^* &= \sum_j \left(G_1^{(j)} + G_2^{(j)} + \ldots + G_{\xi_2(j,V_N)}^{(j)} \right) \\
&= \sum_j \xi_2(j, V_N) \frac{1-2p_j}{2p_j} + O(N^{3/4-\tau/4+\varepsilon}) + O(N^{5/4-\tau/2+\varepsilon}). \\
&= -V_N + \sum_j \xi_2(j, V_N) \frac{1}{2p_j} + O(N^{3/4-\tau/4+\varepsilon}) + +O(N^{5/4-\tau/2+\varepsilon}). \\
&= -V_N + \widehat{A}_2(V_N) + O(N^{3/4-\tau/4+\varepsilon}) + O(N^{5/4-\tau/2+\varepsilon}). \\
&= -V_N + A_2(V_N) + O(N^{5/4-\tau/2+\varepsilon}) + O(N^{3/4+\varepsilon}) \\
&= -V_N + A_2(V_N) + O(N^{5/4-\tau/2+\varepsilon}) \quad a.s. \qquad (3.12)
\end{aligned}$$

where we used the definition of $\widehat{A}_2(.)$ and Lemma J.

Clearly, using (3.2) and (3.12)

$$N = H_N + V_N = H_N^* + V_N + O(N^{1/2+\delta}) = A_2(V_N) + O(N^{5/4-\tau/2+\varepsilon}) \quad a.s.$$

and

$$V_N = A_2^{-1}(N) + O(N^{5/4-\tau/2+\varepsilon}) \qquad a.s.$$

Remark 3.1 In the previous line we used the fact that $A_2^{-1}(u+v) - A_2^{-1}(u) \leq v$. To see this, first recall from Lemma 3.1 that $A_2(t)$, $A_2^{-1}(t)$ and $\alpha(t) = A_2(t) - t$ are all nondecreasing. Then

$$\begin{aligned}
v = A_2(A_2^{-1}(u+v)) - A_2(A_2^{-1}(u)) &= \alpha(A_2^{-1}(u+v)) + A_2^{-1}(u+v) - \alpha(A_2^{-1}(u)) - A_2^{-1}(u) \\
&\geq A_2^{-1}(u+v) - A_2^{-1}(u).
\end{aligned}$$

So we can conclude, using Lemmas H and I that

$$\begin{aligned}
C_2(N) = S_2(V_N) = W_2(V_N) + O(N^{1/4+\varepsilon}) &= W_2((A_2^{-1}(N) + O(N^{5/4-\tau/2+\varepsilon})) + O(N^{1/4+\varepsilon}) \\
&= W_2((A_2^{-1}(N)) + O(N^{5/8-\tau/4+\varepsilon}) \qquad a.s.
\end{aligned}$$

$$C_1(N) = S_1(H_N) = S_1(N-V_N) = W_1(N-V_N) + O(N^{1/4+\varepsilon}) = W_1(N-A_2^{-1}(N)) + O(N^{5/8-\tau/4+\varepsilon})) \qquad a.s.$$

proving our theorem. □

Remark 3.2 We could extend our result for the case $\gamma_1 = \gamma_2 = 1$, by using the argument of the case $\gamma_2 = 1$ for both of the positive and the negative side. Then we would get the following result:

Under the conditions (1.1), *and* $\gamma_1 = \gamma_2 = 1$ *on an appropriate probability space for the random walk* $\{\mathbf{C}(N) = (C_1(N), C_2(N));\ N = 0, 1, 2, \ldots\}$ *one can construct two independent standard Wiener processes* $\{W_1(t);\ t \geq 0\}$, $\{W_2(t);\ t \geq 0\}$ *so that, as* $N \to \infty$, *we have with any* $\varepsilon > 0$

$$|C_1(N)| + |C_2(N) - W_2(N)| = O(N^{5/8-\tau/4+\varepsilon}) \quad a.s. \tag{3.13}$$

Remark 3.3 In our paper ([11] Lemma 4.1) we calculated the density function of $A^{-1}(t)$ and $t - A^{-1}(t)$. Our result was the following:

Suppose that $\gamma_1 > \gamma_2 \geq 1$.

$$P(A^{-1}(t) \in dv) = \frac{t}{\pi v} \frac{1}{\sqrt{(v\gamma_1 - t)(t - \gamma_2 v)}} dv \quad \text{for} \quad \frac{t}{\gamma_1} < v < \frac{t}{\gamma_2},$$

$$P(t - A^{-1}(t) \in dv) = \frac{t}{\pi(t-v)} \frac{1}{\sqrt{((\gamma_1 - 1)t - \gamma_1 v)(t(1-\gamma_2) + \gamma_2 v)}} dv$$

$$\text{for} \quad t\left(1 - \frac{1}{\gamma_2}\right) < v < t\left(1 - \frac{1}{\gamma_1}\right).$$

As in our Theorem 1.1 our random walk $\{\mathbf{C}(N) = (C_1(N), C_2(N)$ is approximated with the same pair of oscillating Wiener processes as in Theorem D, we get the same consequences as in case of the p_i-s were restricted to be $1/2$ or $1/4$. We proved the following laws of the iterated logarithm ([11]).

Corollary A ([11] Corollary 4.1). *Under condition* (1.1) *with* $\gamma_1 > \gamma_2 \geq 1$ *the following laws of the iterated logarithm hold.*

$$\limsup_{t\to\infty} \frac{W_1(t - A^{-1}(t))}{\sqrt{t \log\log t}} = \limsup_{N\to\infty} \frac{C_1(N)}{\sqrt{N \log\log N}} = \sqrt{2\left(1 - \frac{1}{\gamma_1}\right)} \quad a.s.,$$

$$\liminf_{t\to\infty} \frac{W_1(t - A^{-1}(t))}{\sqrt{t \log\log t}} = \liminf_{N\to\infty} \frac{C_1(N)}{\sqrt{N \log\log N}} = -\sqrt{2\left(1 - \frac{1}{\gamma_1}\right)} \quad a.s.,$$

$$\limsup_{t\to\infty} \frac{W_2(A^{-1}(t))}{\sqrt{t \log\log t}} = \limsup_{N\to\infty} \frac{C_2(N)}{\sqrt{N \log\log N}} = \sqrt{\frac{2}{\gamma_1}} \quad a.s.,$$

$$\liminf_{t\to\infty} \frac{W_2(A^{-1}(t))}{\sqrt{t \log\log t}} = \liminf_{N\to\infty} \frac{C_2(N)}{\sqrt{N \log\log N}} = -\sqrt{\frac{2}{\gamma_2}} \quad a.s.$$

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