Covering 2-colored complete digraphs by monochromatic d-dominating digraphs

Louis DeBiasio*, András Gyárfás†

February 26, 2021

Abstract

A digraph is *d-dominating* if every set of at most d vertices has a common out-neighbor. For all integers $d \geq 2$, let $f(d)$ be the smallest integer such that the vertices of every 2-edge-colored (finite or infinite) complete digraph (including loops) can be covered by the vertices of at most $f(d)$ monochromatic d-dominating subgraphs. Note that the existence of $f(d)$ is not obvious – indeed, the question which motivated this paper was simply to determine whether $f(d)$ is bounded, even for $d = 2$. We answer this question affirmatively for all $d \geq 2$, proving $4 \leq f(2) \leq 8$ and $2d \leq f(d) \leq 2d\left(\frac{d^d-1}{d-1}\right)$ for all $d \geq 3$. We also give an example to show that there is no analogous bound for more than two colors.

Our result provides a positive answer to a question regarding an infinite analogue of the Burr-Erdős conjecture on the Ramsey numbers of d-degenerate graphs. Moreover, a special case of our result is related to properties of d-paradoxical tournaments.

1 Introduction

Throughout this note a *directed graph* (or *digraph* for short) is a pair (V, E) where V can be finite or infinite and $E \subseteq V \times V$ (so in particular, loops are allowed). A digraph is *complete* if $E = V \times V$. For $v \in V$, we write $N^+(v) = \{u : (v, u) \in E\}$ and $N^-(v) = \{u : (u, v) \in E\}$. For a positive integer k, we define $[k] := \{1, \dots, k\}$. Note that regardless of whether $G = (V, E)$ is a graph or a digraph, if $H = (V', E')$ and $V' \subseteq V$ and $E' \subseteq E$, we will write $H \subseteq G$ and we will always refer to H as a *subgraph* of G rather than making a distinction between "subgraph" and "subdigraph."

Let $G = (V, E)$ be a digraph. For $X, Y \subseteq V$ we say that X *dominates* Y if $(x, y) \in E$ for all $x \in X, y \in Y$. We say that G is *d-dominating* if for all $S \subseteq V$ with $1 \leq |S| \leq d$, S dominates some $w \in V$. Note that it is possible for $w \in S$, in which case we must have $(w, w) \in E$. Reversing all edges of a *d-dominating* digraph gives a *d-dominated* digraph. These notions are well studied for tournaments (see Section 3).

A *cover* of a digraph $G = (V, E)$ is a set of subgraphs $\{H_1, \dots, H_t\}$ such that $V(G) = \bigcup_{i \in [t]} V(H_i)$. By a 2-*coloring* of $G = (V, E)$, we will always mean a 2-coloring of the edges of G; i.e. a function $c : E \to [2]$. Given a 2-coloring of G, we let E_i be the set of edges receiving color i (i.e. $E_i = c^{-1}(\{i\})$) and $G_i = (V, E_i)$ for $i \in [2]$. A *cover of G by monochromatic subgraphs* is a cover $\{H_1, \dots, H_t\}$ of G such that for all $i \in [t]$ there exists $j \in [2]$ such that $H_i \subseteq G_j$.

*Department of Mathematics, Miami University, debiasld@miamioh.edu. Research supported in part by NSF grant DMS-1954170.

†Alfréd Rényi Institute of Mathematics, Budapest, P.O. Box 127, Budapest, Hungary, H-1364. gyarfas.andras@renyi.hu. Research supported in part by NKFIH Grant No. K132696.

The following problem was raised in [4, Problem 6.6].

Problem 1.1. *Given a* 2*-colored complete digraph* K*, is it possible to cover* K *with at most four monochromatic* 2*-dominating subgraphs? (If not four, some other fixed number?)*

Our main result is a positive answer for the qualitative part of Problem 1.1 in a more general form.

Theorem 1.2. *Let* d *be an integer with* $d \geq 2$*. In every* 2*-colored complete digraph* K*, there exists a cover of* K *with at most* $2 \times \sum_{i=1}^{d} d^i = 2d\left(\frac{d^d-1}{d-1}\right)$ *monochromatic* d*-dominating subgraphs. In case of* $d = 2$ *there exists a cover of* K *with at most eight monochromatic* 2*-dominating subgraphs.*

For all integers $d \geq 1$, let $f(d)$ be the minimum number of monochromatic d-dominating subgraphs needed to cover an arbitrarily 2-colored complete digraph. Note that obviously $f(1) = 2$ since the two sets of monochromatic loops provide an optimal cover. For $d \geq 2$, Theorem 1.2 shows that $f(d)$ is well-defined. Example 1.3 below (adapted from [4, Proposition 6.3]) combined with Theorem 1.2 gives

$$4 \leq f(2) \leq 8 \text{ and } 2d \leq f(d) \leq 2d\left(\frac{d^d-1}{d-1}\right) \text{ for all integers } d \geq 3. \tag{1}$$

Example 1.3. *Let* K *be a complete digraph on at least* $2d$ *vertices and partition* $V(K)$ *into non-empty sets* $R_1, \ldots, R_d$ *and* $B_1, \ldots, B_d$*, color all edges inside* R_i *red, all edges inside* B_i *blue, all edges from* R_i *to* B_j *red, all edges from* B_i *to* R_j *blue, all edges between* R_i *and* R_j *with* $i \neq j$ *blue, and all edges between* B_i *and* B_j *with* $i \neq j$ *red. One can check that every monochromatic* d*-dominating subgraph of* K *is entirely contained inside one of the sets* $R_1, \ldots, R_d, B_1, \ldots, B_d$*.*

Finally, the following example shows that for $d \geq 2$ there is no analogue of Theorem 1.2 for more than two colors (c.f. [4, Example 2.3]).

Example 1.4. *Let* V *be a totally ordered set and let* K *be the complete digraph on* V *where for all* $i \in V$*,* (i, i) *is green and for all* $i, j \in V$ *with* $i < j$*,* (i, j) *is red and* (j, i) *is blue. Note that for* $d \geq 2$ *the only monochromatic* d*-dominating subgraphs are the green loops and thus no bound can be put on the number of monochromatic* d*-dominating subgraphs needed to cover* V*.*

1.1 Motivation

A graph G is *d-degenerate* if there is an ordering of the vertices $v_1, v_2, \ldots$ such that for all $i \geq 1$, $|N(v_i) \cap \{v_1, \ldots, v_{i-1}\}| \leq d$ (equivalently, every subgraph has a vertex of degree at most d). Burr and Erdős conjectured [3] that for all positive integers d, there exists $c_d > 0$ such that every 2-coloring of K_n contains a monochromatic copy of every d-degenerate graph on at most $c_d n$ vertices. This conjecture was recently confirmed by Lee [8].

The motivation for Problem 1.1 relates to the following conjecture also raised in [4, Problem 1.5, Conjecture 10.2] which can be thought of as an infinite analogue of the Burr-Erdős conjecture.

Conjecture 1.5. *For all positive integers* d*, there exists a real number* $c_d > 0$ *such that if* G *is a countably infinite* d*-degenerate graph with no finite dominating set, then in every* 2*-coloring of the edges of* $K_{\mathbb{N}}$*, there exists a monochromatic copy of* G *with vertex set* $V \subseteq \mathbb{N}$ *such that the upper density of* V *is at least* c_d*.*

The case $d = 1$ was solved completely in [4] (regardless of whether G has a finite dominating set or not). For certain 2-colorings of $K_{\mathbb{N}}$, described below, Theorem 1.2 implies a positive solution to Conjecture 1.5 for $d \geq 2$.

Suppose that for some finite subset $F \subseteq \mathbb{N}$, we have a partition of $\mathbb{N} \setminus F$ into (finitely or infinitely many) infinite sets $\mathcal{X} = \{X_1, \dots, X_n, \dots\}$. Also suppose that we have ultrafilters $\mathscr{U}_1, \mathscr{U}_2, \dots, \mathscr{U}_n, \dots$ on $\mathbb{N}$ such that for all $i \geq 1$, $X_i \in \mathscr{U}_i$. Finally, suppose that for all $i, j \geq 1$ there exists $c_{i,j} \in [2]$ such that for all $v \in X_i$, $\{u : \{u, v\}$ has color $c_{i,j}\} \cap X_j \in \mathscr{U}_j$. This last condition ensures that if there exists $X_{i_1}, \dots, X_{i_n}$ and X_j such that $c_{i_1,j} = \cdots = c_{i_n,j} =: c$, then every finite collection of vertices in $X_{i_1} \cup \cdots \cup X_{i_n}$ has infinitely many common neighbors of color c in X_j. Note that such a scenario can be realized as follows: For all i, j, let $c_{i,j} \in [2]$ and color the edges from X_i to X_j so that every vertex in X_i is incident with cofinitely many edges of color $c_{i,j}$ (by using the half graph coloring[1] when $c_{i,j} \neq c_{j,i}$ for instance).

The above coloring of $K_{\mathbb{N}}$ naturally corresponds to a 2-colored complete digraph in the following way: Let K be a 2-colored complete digraph on $\mathcal{X}$ where we color (X_i, X_j) with color c if for all $v \in X_i$, $\{u : \{u, v\}$ has color $c\} \cap X_j \in \mathscr{U}_j$. Now by Theorem 1.2, K can be covered by $t \leq f(d+1)$ monochromatic $(d+1)$-dominating subgraphs $G_1, \dots, G_t$. Since $\mathbb{N} \setminus F = \bigcup_{i \in [t]} \left(\bigcup_{X \in V(G_i)} X \right)$, there exists $i \in [t]$ such that $V_i := \bigcup_{X \in V(G_i)} X$ has upper density at least $1/f(d+1)$. Without loss of generality, suppose the edges of G_i are red. By the construction, V_i has the property that for all $S \subseteq V_i$ with $1 \leq |S| \leq d+1$, there is an infinite subset $W \subseteq V_i$ such that every edge in $E(S, W)$ is red. As shown in [4, Proposition 6.1], if G is a graph satisfying the hypotheses of Conjecture 1.5, then there exists a red copy of G which spans V_i and thus has upper density at least $1/f(d+1)$.

2 Covering digraphs, proof of Theorem 1.2

For a graph G, we denote the order of a largest clique (pairwise adjacent vertices) in G by $\omega(G)$. Given a 2-colored complete digraph K and a set $U \subseteq V(K)$, define $G[U]_{\text{blue}}$ to be the graph on U where $\{u, v\} \in G[U]_{\text{blue}}$ if and only if (u, v) and (v, u) are blue in K; define $G[U]_{\text{red}}$ analogously.

Given positive integers ω and d, let $f(\omega, d)$ be the smallest positive integer D such that if K is a 2-colored complete digraph on vertex set V where every loop has the same color, say red, and $\omega(G[V]_{\text{blue}}) = \omega$, then V can be covered by at most D monochromatic d-dominating subgraphs. Also define $f(0, d) = 0$.

Lemma 2.1.

(1) $f(1, 2) = 1$

(2) $f(\omega, d) \leq d(f(\omega - 1, d) + 1)$ *for all* $1 \leq \omega \leq d$ *(in particular,* $f(1, d) \leq d$*). In fact, all d-dominating subgraphs in the covering have the same color as the loops.*

Note that the upper bound $\omega \leq d$ is not strictly necessary, but we include it here for clarity since in the next lemma, we will prove a stronger result when $\omega \geq d + 1$.

Proof. Let K be a 2-colored complete digraph on vertex set V where all loops have the same color, say red.

(1) is trivial since for all distinct $u, v \in V$ both (u, u) and (v, v) are red and $\omega(G[V]_{\text{blue}}) = 1$ implies that either (u, v) or (v, u) is red.

To see (2), note first that we may assume that K itself is not spanned by a red d-dominating subgraph, otherwise we are done. This is witnessed by a set $U = \{u_1, \dots, u_d\} \subseteq V$, such that there is no $w \in V$ with (u_i, w) red for all $i \in [d]$.

[1] Given a totally ordered set Z and disjoint $X, Y \subseteq Z$ the *half graph coloring* of the complete bipartite graph $K_{X,Y}$ is a 2-coloring of the edges of $K_{X,Y}$ where for all $i \in X$, $j \in Y$, $\{i, j\}$ is red if and only if $i \leq j$.

For all $i \in [d]$ we define

$$W_i = \{v \in V : (v, u_i) \text{ is red}\}.$$

Note that $u_i \in W_i$ and $K[W_i]$ is spanned by a red d-dominating subgraph for all $i \in [d]$.

Set $V' = V \setminus (\cup_{i\in[d]} W_i)$ and define

$$T_i = \{v \in V' : (u_i, v) \text{ is blue}\}.$$

Note, that by the definition of V', (v, u_i) is also blue for all $v \in T_i$ and $i \in [d]$. Moreover, from the selection of U, every vertex in V' receives a blue edge from some vertex in U and therefore $V' = \cup_{i=1}^{d} T_i$.

Note that if $\omega = 1$, then $T_i = \emptyset$ for all $i \in [d]$ and thus $\cup_{i\in[d]} W_i$ is a cover of K with d red d-dominating subgraphs; i.e. $f(1, d) \le d = d(f(0, d) + 1)$.

Otherwise, we have that $\omega(K[T_i]_{\text{blue}}) \le \omega - 1$ and thus K is covered by at most

$$d + d \cdot f(\omega - 1, d) = d(f(\omega - 1, d) + 1)$$

red d-dominating subgraphs. □

Lemma 2.2. *Let K be a 2-colored complete digraph K where R is the set of red loops and B is the set of blue loops. If $\omega(G[R]_{\text{blue}}) \ge d + 1$, then $V(K)$ can be covered by at most d red d-dominating subgraphs and at most one blue d-dominating subgraph. Likewise, if $\omega(G[B]_{\text{red}}) \ge d + 1$. In particular, this implies $f(\omega, d) \le d + 1$ for $\omega \ge d + 1$.*

Proof. Suppose $\omega(G[R]_{\text{blue}}) \ge d + 1$ and let $X = \{x_1, \dots, x_d, x_{d+1}\} \subseteq R$ be a set of order $d + 1$ which witnesses this fact. For $i \in [d]$ we define

$$W_i = \{v \in V(K) : (v, x_i) \text{ is red}\}.$$

Note that $x_i \in W_i$ and $K[W_i]$ is spanned by a red d-dominating subgraph for all $i \in [d]$.

Set $V' = X \cup (V(K) \setminus (\cup_{i\in[d]} W_i))$ and note that for all $v \in V'$, $[v, X]$ is blue. Now let $S \subseteq V'$ such that $1 \le |S| \le d$. If $S \subseteq X$, then since $|S| < |X|$, there exists $x_i \in X \setminus S$ such that every edge in $[S, x_i]$ is blue; otherwise $|S \cap X| \le d - 1$ and there exists $i \in [d]$ such that $x_i \notin S$ and every edge in $[S, x_i]$ is blue. So there is one blue d-dominating subgraph which covers V', which together with the red d-dominating subgraphs $K[W_1], \dots, K[W_d]$ gives the result.

When $\omega(G[B]_{\text{red}}) \ge d + 1$, the proof is the same by switching the colors. □

Now we are ready to prove our main result.

Proof of Theorem 1.2. Let $V(K) = R \cup B$ where R, B are the vertex sets of the red and blue loops, respectively. If $\omega(G[R]_{\text{blue}}) \ge d + 1$ or $\omega(G[B]_{\text{red}}) \ge d + 1$, then by Lemma 2.2, $R \cup B$ can be covered by at most $d + 1$ monochromatic d-dominating subgraphs. So suppose $\omega(G[R]_{\text{blue}}) \le d$ and $\omega(G[B]_{\text{red}}) \le d$. Now by Lemma 2.1, each of $K[R]$ and $K[B]$ can be covered by at most 4 monochromatic d-dominating subgraphs when $d = 2$, and by at most $\sum_{i=1}^{\omega} d^i \le \sum_{i=1}^{d} d^i$ monochromatic d-dominating subgraphs when $d \ge 3$. □

3 Paradoxical tournaments

In the above section, we proved that $f(1, 2) = 1$ and $f(1, d) \le d$ for all $d \ge 3$. Naturally, we wondered if the upper bound on $f(1, d)$ could be improved when $d \ge 3$ (since any improvement on $f(1, d)$ would improve the general upper bound on $f(d)$). In this section we show that it cannot; that is, $f(1, d) = d$ for all $d \ge 3$.

A *tournament* is a digraph (V, E) such that for all distinct $x, y \in V$ exactly one of $(x, y), (y, x)$ is in E and $(x, x) \notin E$. Given a digraph $G = (V, E)$, we say that $S \subseteq V$ is an out-dominating set if for all $v \in V \setminus S$, there exists $u \in S$ such that $(u, v) \in E$, and we say that S is an in-dominating set if for all $v \in V \setminus S$, there exists $u \in S$ such that $(v, u) \in E$. Note that a tournament T is d-dominating (d-dominated) if and only if T has no in-dominating (out-dominating) set of order d.

We call a d-dominating (d-dominated) tournament *critical* if its proper subtournaments are not d-dominating (d-dominated). For a tournament T, let T^* be the digraph obtained from T by adding a loop at every vertex.

Our main result of this section is the following.

Theorem 3.1. *For all integers $d \geq 2$, if T is a critical d-dominated tournament with no $(d+1)$-dominating subtournaments, then $f(1, d+1) = d+1$.*

However, before proving Theorem 3.1, we show that such a tournament exists for all $d \geq 2$ from which we obtain the following corollary.

Corollary 3.2. *For all $d \geq 3$, $f(1, d) = d$.*

Note that the absence of loops and two-way oriented edges make the existence of d-dominated tournaments a nontrivial problem. This existence problem for d-dominated tournaments was proposed by Schütte (see [5]) and was first proved by Erdős [5] with the probabilistic method, then Graham and Spencer [6] gave an explicit construction using sufficiently large Paley tournaments[2].

Note that Babai [1] coined the term *d-paradoxical tournament* for what we refer to as d-dominated tournament. In this spirit, we say that a tournament is *perfectly d-paradoxical* if it is d-dominating, d-dominated, has no $(d+1)$-dominating subtournaments, and has no $(d+1)$-dominated subtournaments. A result of Esther and George Szekeres [7] combined with the fact that Paley tournaments are self-complementary implies that QT_7 is perfectly 2-paradoxical and QT_{19} is perfectly 3-paradoxical. It is an open question (which to the best of our knowledge we are raising here for the first time) whether every Paley tournament is perfectly d-paradoxical for some d. While we can't settle that question, the following beautiful example of Bukh [2] shows that perfectly d-paradoxical tournaments exist for all $d \geq 2$. We repeat his proof here (tailored to the terminology of this paper) for completeness.

Example 3.3 (Bukh [2])**.** *For all integers $d \geq 2$, there exists a perfectly d-paradoxical tournament. In particular, there exists a critical d-dominated tournament which has no $(d+1)$-dominating subtournaments.*

Proof. Let d be an integer with $d \geq 2$ and let $n = m(d+1)$ where $m = 2^{3d}$. Let $V = \{0, 1, \ldots, n-1\}$ and let G be the oriented graph on V where $(i, j) \in E(G)$ if and only if $1 \leq j - i \leq m-1$ (with addition modulo n). In other words G is the oriented $(m-1)$st power of a cycle on n vertices. Now we define a tournament T by starting with the oriented graph G and for all distinct $i, j \in V$, if $(i, j), (j, i) \notin E(G)$, then independently and uniformly at random let $(i, j) \in E(T)$ or $(j, i) \in E(T)$.

First note that every induced subgraph of G has an in-dominating set of order at most $d+1$ and an out-dominating set of order at most $d+1$ and thus the same is true of every subtournament of T. This implies that T has no $(d+1)$-dominating subtournaments and no $(d+1)$-dominated subtournaments.

Now we claim that with positive probability, T has no out-dominating sets of order d and no in-dominating sets of order d and thus T is d-dominated and d-dominating. Let

[2] For a prime power p, $p \equiv -1 \pmod 4$, the Paley tournament QT_p is defined on vertex set $V = [0, p-1]$ and (a, b) is a directed edge if and only if $a - b$ is a non-zero square in the finite field $\mathbb{F}_p$.

$S \subseteq V$ with $|S| = d$ and let

$$N_G^+[S] = \{v \in V : v \in S \text{ or there exists } u \in S \text{ such that } (u, v) \in E(G)\}.$$

Let $V' := V \setminus N_G^+[S]$ and note that $|N_G^+[S]| \leq dm$ and thus $|V'| \geq m$. The probability that $v \in V'$ is dominated by S in T is $1 - 2^{-d}$ and thus the probability that every vertex in V' is dominated by S is $(1 - 2^{-d})^{|V'|} \leq (1 - 2^{-d})^m \leq e^{-2^{-d}m} = e^{-4^d}$. Likewise for every vertex of V' dominating S. So the expected number of out-dominating or in-dominating sets of order d is at most

$$2\binom{n}{d}e^{-4^d} < 2(em)^d e^{-4d} < 2(e^{3d+1})^d e^{-4^d} < 1$$

(where the last inequality holds since $(3d + 1)d < 4^d$ for all $d \geq 2$), which establishes the claim.

Starting with a perfectly d-paradoxical tournament T, let T' be a minimal subtournament of T which is d-dominated. So T' is critical d-dominated and has no $(d + 1)$-dominating subtournaments. □

The proof of Theorem 3.1 will follow from two more general lemmas.

Lemma 3.4. *Let T be a tournament and let $d \geq 2$. If T is 2-dominating and there exists a set $W \subseteq V(T)$ with $|W| = d$ such that W dominates exactly one vertex v, then T^* is not $(d + 1)$-dominating. In particular, if T is critical d-dominating, then T^* is not $(d + 1)$-dominating.*

Proof. Let $W = \{w_1, \ldots, w_d\}$ and v be as in the statement. To see that T^* is not $(d + 1)$-dominating, it is enough to prove that for some $u \in N^+(v)$ the set $W \cup \{u\}$ does not dominate any vertex in T^* (note that since T is 2-dominating, $N^+(v) \neq \emptyset$). Suppose for contradiction that this is not the case; that is, for all $u \in N^+(v)$ the set $W \cup \{u\}$ dominates some vertex x in T^*. Note that by the definition of W and the fact that $u \in N^+(v)$, it must be the case that $x \in W$; without loss of generality, suppose $x = w_1$. This implies that for all $i \in [d]$, $(w_i, w_1) \in E(T)$. But now this implies that for all $u \in N^+(v)$, $W \cup \{u\}$ dominates w_1. On the other hand since T is 2-dominating, it must be the case that there exists a vertex which is dominated by $\{w_1, v\}$ in T, but every outneighbor of v is an inneighbor of w_1 and thus we have a contradiction.

To get the second part of the lemma, first note that if T is critical d-dominating, then T is 2-dominating. Moreover, for all $v \in V$, since $T - v$ is not d-dominating there exists $W = \{w_1, \ldots w_d\} \subseteq V(T) \setminus \{v\}$ which does not dominate any vertex in $V(T) \setminus \{v\}$, but since T is d-dominating, W must dominate v. □

If $G = (V, E)$ is a digraph such that there exists $w \in V$ such that $(v, w) \in E$ for all $v \in V$ (including $v = w$), then note that G is d-dominating for all $d \leq |V|$. In this case we call G *trivially d-dominating.*

Lemma 3.5. *Let T be a tournament. If T is critical d-dominating, then T^* cannot be covered by less than $d + 1$ $(d + 1)$-dominating subgraphs.*

Proof. Suppose for contradiction that for some $t \leq d$ there are $(d+1)$-dominating subgraphs $H_1, \ldots, H_t$ which cover T^*. Since T is critical d-dominating we have by Lemma 3.4 that T^* is not $(d + 1)$-dominating, and thus all $V(H_i)$ are proper subsets of $V(T^*)$.

Claim 3.6. *Each H_i is trivially $(d + 1)$-dominating.*

Proof. The claim is obvious if $|V(H_i)| \leq d$; so suppose that $|V(H_i)| \geq d+1$. Since T is critical d-dominating, the subtournament T_i of T spanned by $V(H_i)$ is not d-dominating. This is witnessed by a set $W = \{w_1, \ldots, w_d\} \subseteq V(T_i)$ such that W does not dominate any vertex in $U = V(T_i) \setminus W$. Let $u \in U$. Since H_i is $(d+1)$-dominating, $W \cup u$ dominates some vertex $x \in V(H_i)$ which must be in W from the definition of W. Without loss of generality, let $x = w_1$. This implies that $(u, x_1) \in E(T)$ and for all $i \in [d]$, $(w_i, w_1) \in E(T)$. But now this implies that for all $u \in U$, $W \cup \{u\}$ dominates w_1 and thus all vertices of $V(H_i)$ (including w_1) are oriented to w_1 proving the claim. □

Claim 3.6 implies that for all $i \in [t]$ there is a vertex $v_i \in V(H_i)$ which is dominated by all vertices of H_i. But since $\cup_{i=1}^{t} V(H_i) = V(T)$, the set $\{v_1, \ldots, v_t\}$ does not dominate any vertex in T, contradicting the fact that T is d-dominating. □

Proof of Theorem 3.1. First note that $f(1, d+1) \leq d+1$ by Lemma 2.1.

Let T_B be a tournament on vertex set V such that T_B is critical d-dominated and has no $(d+1)$-dominating subtournaments. Define the 2-colored complete digraph K on V by coloring all edges of T_B blue, and all edges of $(V \times V) \setminus E(T_B)$ red. Let T_R be the tournament with $E(T_R) = \{(y, x) : (x, y) \in E(T_B)\}$ and note that every edge of T_R is red and T_R has no loops. Since T_B is critical d-dominated, this implies that T_R is critical d-dominating (since T_R is obtained by reversing all the edges of T_B).

Note that by the assumption on T_B, every monochromatic $(d+1)$-dominating subgraph in K must be red. However, since T_R is crtical d-dominating, we get that $f(1, d+1) \geq d+1$ from Lemma 3.5. □

Acknowledgements. We thank Boris Bukh for Example 3.3 and for his comments on the paper.

References

[1] L. Babai, Character sums, Weil's estimate and paradoxical tournaments, http://people.cs.uchicago.edu/ laci/reu02/paley.pdf (2002).

[2] B. Bukh. *Dominating sets in subtournaments of the Paley tournament*, https://mathoverflow.net/q/383245

[3] S. A. Burr, P. Erdős. *On the magnitude of generalized Ramsey numbers for graphs*, Colloq. Math. Soc. János Bolyai **10** (1975), 215–240.

[4] J. Corsten, L. DeBiasio, P. McKenney. *Density of monochromatic infinite subgraphs II*, arXiv preprint arXiv:2007.14277 (2020).

[5] P. Erdős. On a problem in Graph Theory, *Math. Gazette* **47** (1963) 220-223.

[6] R. L. Graham, J. H. Spencer, A constructive solution to a tournament problem, *Canadian Math. Bulletin* **14** (1971) 45-48.

[7] E. Szekeres, G. Szekeres, On a problem of Schütte and Erdős, *Math. Gazette* **49** (1965) 290-293.

[8] C. Lee. *Ramsey numbers of degenerate graphs*, Ann. of Math. **185** (2017), no. 3, 791-829.