On the density theorem of Halász and Turán

by
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Abstract

Gábor Halász and Paul Turán were the first who proved unconditionally the Density Hypothesis for Riemann's zeta function in a fixed horizontal strip $c_0 < \text{Re } s < 1$. They also showed that the Lindelöf Hypothesis implies a surprisingly strong bound on the number of zeros with $\text{Re } s \geq c_1 > 3/4$. In the present work we use an alternative approach to prove their result which does not use either Turán's power sum method or the large sieve.

1 Introduction

Density theorems play a central role in the study of Riemann's zeta and allied functions, in particular for Dirichlet $\mathcal{L}$-functions. In case of Riemann's zeta function the most important consequence of them is an upper bound for the difference of consecutive primes. Denoting the non-trivial zeros of $\zeta(s)$ by $\varrho = \beta + i\gamma$, the famous Density Hypothesis (DH) asserts for any $\varepsilon > 0$

$$(1.1) \qquad N(\sigma, T) := \sum_{\substack{\varrho \\ \beta \geq \sigma,\, 0<\gamma<T}} \ll_\varepsilon T^{2(1-\sigma)+\varepsilon} \qquad (1/2 \leq \sigma < 1).$$

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This would imply $p_{n+1} - p_n \ll_\varepsilon p_n^{1/2+\varepsilon}$ (p_n denotes the n^{th} prime). On the other hand, Ingham [Ing] showed that the DH would follow from the Lindelöf Hypothesis (LH), stating for any $\varepsilon > 0$

$$M(\alpha, T) = \max_{|t| \le T} |\zeta(\alpha + it)| \ll_{\alpha,\varepsilon} T^\varepsilon \quad \text{for} \quad 1/2 \le \alpha \le 1, \tag{1.2}$$

which is by convexity arguments equivalent with

$$M(1/2, T) = \max_{|t| \le T} |\zeta(1/2 + it)| \ll_\varepsilon T^\varepsilon \quad (\text{for any } \varepsilon > 0). \tag{1.3}$$

Carlson [Car] was the first to prove a density theorem. In 1920 he showed (1.1) with $A(\sigma) = 4\sigma$ in place of 2.

In the following half century a long series of works improved his estimate but they never reached (1.1) for any $\sigma < 1$.

It was Gábor Halász and Paul Turán who first succeeded to show (1.1) for a fixed range $c < \sigma < 1$ with an explicitly calculable constant $c < 1$. Their original result,

$$N(\sigma, T) \le C_1 T^{(1-\sigma)^{3/2} \log^3(1/1-\sigma)} \quad \text{for } T > C_2 \tag{1.4}$$

was shown in a sharper form in Turán's book [Tur2], Theorem 38.2:

$$N(\sigma, T) < T^{1.2 \cdot 10^5 (1-\sigma)^{3/2}} \log^c T. \tag{1.5}$$

Their results were further sharpened and extended by Bombieri [Bom] and Montgomery [Mon]. Halász and Turán [HT2] proved also a q-analogue of the result using the large sieve too.

They also established a very important conditional result in connection with the mentioned result of Ingham [Ing]. Turán [Tur1] conjectured already in 1954 that (LH) has a much stronger effect on the zeros of $\zeta(s)$, namely that (LH) implies beyond the (DH) the inequality $N(\sigma, T) \ll_\varepsilon T^\varepsilon$ for any $\sigma > 1/2$.

The other pioneering result of Halász and Turán was that in their mentioned work [HT1] they showed that supposing the Lindelöf Hypothesis one has

$$N(\sigma, T) \ll_{\sigma,\varepsilon} T^\varepsilon \quad \text{for any} \quad \varepsilon > 0 \quad \text{and} \quad \sigma > 3/4. \tag{1.6}$$

The two main new ideas behind the proof of (1.4) were

(i) an ingenious idea of Halász [Hal] which was first used by him in the investigations of mean values of general multiplicative function;

(ii) the second main theorem of Turán's power sum method (Theorem 8.1 of [Tur2]).

The proof of (1.4)–(1.5) needed additionally another deep result about the growth $M(\alpha, T)$ of the zeta function for α near to 1. They used Richert's bound [Ric]

$$M(\alpha, T) \leq AT^{B(1-\alpha)^{3/2}} \log^{2/3} t \quad \text{for} \quad 1/2 \leq \alpha \leq 1, \ |t| \geq 3, \tag{1.7}$$

with $B = 100$, A an absolute constant. The proof was based on ideas of I. M. Vinogradov and A. Korobov.

In an important work, using a refinement of Vinogradov's method K. Ford [For] proved an explicit sharpening of (1.7) as

(1.8)

$$M_1(\alpha, T) := \max_{\sigma \geq \alpha, 3 < |t| \leq T} \max \left(|\zeta(s)|, \max_{x \leq T} \left| \sum_{n \leq x} n^{-s} \right| \right) \leq AT^{B(1-\alpha)^{3/2}} \log^{1/3} T$$

with

$$A = 76.2, \quad B = 4.45. \tag{1.9}$$

It should be noted that Turán in his earlier works supposed a weaker form of the Lindelöf Hypothesis which according to Turán's view "does not seem to be hopeless" (see p. 359 of Turán's book [Tur2]). However, no new results were proved in connection with this weaker form (in fact, a consequence of LH) in the past half century after the works [HT1] and [HT2].

By a slight refinement of Theorem 12.3 of Montgomery [Mon] the estimate (1.8) of Ford led to his improvement of (1.5):

$$N(\sigma, T) \leq CT^{58.05(1-\sigma)^{3/2}} \log^{15} T. \tag{1.10}$$

2 Results, methods

The goal of the present work is to give an alternative, relatively simple proof of Ingham's theorem and the results (1.5) and (1.6) of Halász and Turán. The proof is based

(i) on a method of the author [Pin1], developed for investigation of the oscillations of the sum $\sum_{n\le x}\mu(n)$;

(ii) on the simple but ingenious idea of Halász (see (4.5)–(4.6)) appearing in the works [HT1], [HT2] (in Theorems 1–2).

Additionally in the proofs of the unconditional Theorem 1 we use the estimates (1.8)–(1.9) of Ford. In the course of proof we can avoid both Turán's method and large sieve type results used in an ingenious way by Bombieri [Bom] and Montgomery [Mon]. Further we need a restricted knowledge of the theory of complex functions (actually only Cauchy's residue theorem) and in particular, of Riemann's zeta function (essentially only the estimates (1.7)–(1.9) in case of the unconditional Theorem 1).

Finally I note that the application of my method [Pin] was inspired by the recent work of S. M. Gonek, S. W. Graham and Y. Lee [GGL] who applied this method in connection with the Lindelöf Hypothesis.

We shall prove the following results.

Theorem 1. $N(1-\eta,T)\ll T^{86\eta^{3/2}}(\log T)^8$.

Theorem 2. *The Lindelöf Hypothesis* (1.3) *implies*

$$N(\sigma,T)\ll_{\varepsilon,\sigma}T^{\varepsilon}\quad\text{for any}\quad\varepsilon>0\quad\text{and}\quad 3/4<\sigma\le 1. \tag{2.1}$$

Theorem 3. *The Lindelöf Hypothesis* (1.3) *implies the Density Hypothesis* (1.1).

Remark. In view of the zero-free region ($t>C$)

$$\zeta(\sigma,t)\ne 0\quad\text{for}\quad\sigma>1-\frac{c_2}{(\log t)^{2/3}(\log\log t)^{1/3}} \tag{2.2}$$

our results are clearly true if

$$(1-\sigma)^{3/2}\le\frac{c_3}{\log T(\log_2 T)^{1/2}}\qquad(T=|t|+2) \tag{2.3}$$

We also note that in the range

$$\frac{c_4}{(\log T)^{2/3}(\log\log T)^{1/3}}<1-\sigma<c_5\frac{(\log\log T)^{2/3}}{(\log T)^{2/3}} \tag{2.4}$$

the log-power is the dominant factor in the estimates of Theorem 1.

A further remark is that Theorem **??** is weaker than the density hypothesis if $\sigma > 0.9994\ldots$. The Density Hypothesis was proved already for a relatively wide range more than forty years ago (Heath–Brown [Hea] showed it for $\sigma \geq 11/14 = 0.7857\ldots$), however the best known result of type

$$N(\sigma, T) \leq T^{b(1-\sigma)+\varepsilon} \quad \text{for any} \quad \varepsilon > 0,\ 1/2 \leq \sigma \leq 1 \tag{2.5}$$

is still today $b = 12/5$ (the hardest case being the point $\sigma = 3/4$).

The above result was proved by Huxley [Hux] in 1972 and it implied the Prime Number Theorem for short intervals of type $\left[x, x + x^{a+\varepsilon}\right]$ with $a = 1 - 1/b = 7/12$.

3 Notation

In our work, T, Y will denote sufficiently large reals, B as in (1.8)–(1.9), $\sigma = 1 - \eta$, $Y_1 = Ye^3$, $L = \log T$, $\lambda = \log Y$. $\mu(n)$ will denote the Möbius function. Zeros of the zeta function will be denoted by $\varrho_j = \beta_j + i\gamma_j = 1 - \eta_j + i\gamma_j$ if $\eta_j \leq \eta$, $0 \leq \gamma_j \leq T$ $(1 \leq j \leq K)$, $\delta_j = \eta - \eta_j \geq 0$. Δ will be a parameter depending on η, $\xi = \Delta + \eta$ (actually we will choose $\Delta = \eta$, $\xi = 2\eta$ in Theorem 1 and $\Delta = 1/4 - \eta$, $\xi = 1/4$ in Theorem 2). ε will be a generic positive constant not necessarily the same at each occurrence. The signs $\ll$ and O substitute absolute constants unless the dependence is specified.

4 Proof of Theorems 1 and 2

We shall use some ideas from the proof of [Pin], where we showed $\left|\sum_{n\leq x} \mu(n)\right| \geq \frac{x^{\beta_0}}{6|\gamma_0|^3}$ for any ϱ, in particular $\left|\sum_{n\leq x} \mu(n)\right| \geq c_7\sqrt{x}$. Let

$$\begin{aligned} I := \frac{1}{2\pi i} \int\limits_{(3)} \sum_{n=1}^{\infty} \frac{\mu(n)}{n^{s+\varrho_j}} \frac{\zeta(s+\varrho_j)}{s} e^{s^2/\lambda+\lambda s} ds \\ = \frac{1}{2\pi i} \int\limits_{(3)} \frac{e^{s^2/\lambda+\lambda s}}{s} ds = 1 + \int\limits_{(-3)} \frac{e^{s^2/\lambda+\lambda s}}{s} ds \end{aligned} \tag{4.1}$$

$$= 1 + O\left(\frac{1}{Y^2}\right).$$

The integral I is a weighted sum of the terms $\mu(n)n^{-\varrho_j}$

$$I_j := I = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^{\varrho_j}} w_j(\lambda - \log n), \tag{4.2}$$

$$w_j(h) := \frac{1}{2\pi i} \int_{(3)} \frac{\zeta(s + \varrho_j)}{s} e^{s^2/\lambda + hs} ds. \tag{4.3}$$

For $h \le -3 \iff n > Ye^3$ we shift the integral to the line $\operatorname{Re} s = \lambda$:

$$w_j(h) \ll e^{-(|h|-1)\lambda}, \tag{4.4}$$

consequently,

$$I_j = \sum_{n \le Y_1} \frac{\mu(n)}{n^{\varrho_j}} w_j(\lambda - \log n) + O\left(\frac{1}{Y^2}\right). \tag{4.5}$$

Shifting the integral in (4.1) to $\operatorname{Re} s = -\Delta - \delta_j$ ($\eta_j + \delta_j + \Delta = \xi$):

$$1 + O\left(\frac{1}{Y^2}\right) = I'_j := \frac{1}{2\pi i} \int_{-2\lambda}^{2\lambda} \sum_{n \le Y_1} \frac{\mu(n)}{n^{1-\xi+i(\gamma_j+t)}} f_j(t) dt, \tag{4.6}$$

where

$$f_j(t) = \frac{\zeta(1 - \xi + i(\gamma_j + t))}{-\Delta - \delta_j + it} Y^{-\Delta-\delta_j+it} e^{(\Delta+\delta_j-it)^2/\lambda}, \tag{4.7}$$

$$\int_{-2\lambda}^{2\lambda} |f_j(t)| dt \ll Y^{-\Delta} M_1(1 - \xi, 2T) \log \frac{\lambda}{\Delta}, \tag{4.8}$$

so, following the idea of Halász,

$$\exists\, \widetilde{\gamma}_j \in [\gamma_j - 2\lambda, \gamma_j + 2\lambda], \quad \alpha_j,\ |\alpha_j| = 1 \ \text{ s.t.} \tag{4.9}$$

(4.10)
$$\alpha_j \sum_{n\le Y_1} \frac{\mu(n)}{n^{1-\xi+i\widetilde{\gamma}_j}} = \left|\sum_{n\le Y_1} \frac{\mu(n)}{n^{1-\xi+i\widetilde{\gamma}_j}}\right| \gg \left(\log\frac{\lambda}{\Delta}\right)^{-1} Y^{\Delta} M_1^{-1}(1-\xi,2T).$$

Since $N(T+1)-N(T) \ll L$ we can choose from $\widetilde{\gamma}_j$ $(j=1,2,\dots,K)$ a subset γ_ν^* $(\nu=1,\dots,K^*)$ with

(4.11)
$$|\gamma_\nu^* - \gamma_\kappa^*| > 5\lambda \quad \text{and cardinality} \quad K^* \gg K/\lambda L.$$

Summing over $\nu \le K^*$, squaring and using the Cauchy–Schwarz inequality:

(4.12)
$$\frac{c_6(K^*)^2 Y^{2\Delta}}{\log^2\lambda M_1^2(1-\xi,2T)} \le \left(\sum_{n\le Y_1}\frac{\mu(n)}{\sqrt{n}}\cdot\sum_{\nu=1}^{K^*}\frac{\alpha_\nu}{n^{1/2-\xi+i\gamma_\nu^*}}\right)^2$$
$$\le \left(\sum_{n\le Y_1}\frac{\mu^2(n)}{n}\right)\left(\sum_{\nu,\kappa=1}^{K^*}\alpha_\nu\overline{\alpha}_\kappa\sum_{n\le Y_1}\frac{1}{n^{1-2\xi+i(\gamma_\nu^*-\gamma_\kappa^*)}}\right)$$
$$\le 2\lambda\left\{K^*(K^*-1)M_1(1-2\xi,2T)+K^*Y_1^{2\xi}\,/\,\xi\right\}.$$

If the first sum on the RHS of (4.12) is less than half of the LHS, i.e. if

(4.13)
$$Y^{2\Delta} \ge c_7\lambda\log^2\lambda M_1^2(1-\xi,2T)M_1(1-2\xi,2T),$$

then by $\xi = \Delta+\eta$ we get

(4.14)
$$K^* \ll \frac{\lambda}{\eta}\log^2\lambda M^2(1-\xi,2T)Y^{2\eta}.$$

Choosing $\Delta=\eta \;\Leftrightarrow\; \xi=2\eta$, Y with equality in (4.13) we obtain $\lambda \asymp \sqrt{\eta}L$. Hence,

(4.15)
$$K \ll \lambda L K^* \ll \frac{\lambda^2}{\eta}L\log^2\lambda M_1^4(1-2\eta,2T)M_1(1-4\eta,2T)$$
$$\ll L^{14/3}\log^2 L\, T^{8(1+\sqrt{2})B\eta^{3/2}} \ll T^{86\eta^{3/2}}\log^5 T.$$

Theorem 1 is proved.

In case of Theorem 2 let $\Delta = 1/4-\eta \Leftrightarrow \xi = 1/4$. Then

(4.16)
$$K \ll L^{C(\eta)}M_1\left(\frac{3}{4},2T\right)^{2(\eta/\Delta+1)}M_1\left(\frac{1}{2},2T\right)^{\eta/\Delta} \ll_{\varepsilon,\eta} T^{\varepsilon}.$$

□

5 Proof of Theorem 3

We begin as in (4.1)–(4.5) but choose $Y_1 = T$ $(\lambda = L - 3)$ and shift the line of integration to $\mathrm{Re}\, s = \eta_j - 1/2$. By LH we obtain for every j by $\int\limits_x^{x+y} |g(u)|du \ll \left(y \int\limits_x^{x+y} \left|g^2(u)\right|du\right)^{1/2}$

(5.1)
$$\int\limits_{\gamma_j-2\lambda}^{\gamma_j+2\lambda} \left|\sum_{n\le T} \frac{\mu(n)}{n^{1/2+it}}\right| dt \gg T^{1/2-\eta-\varepsilon},$$

(5.2)
$$\int\limits_{\gamma_j-2\lambda}^{\gamma_j+2\lambda} \left|\sum_{n\le T} \frac{\mu(n)}{n^{1/2+it}}\right|^2 dt \gg T^{1-2\eta-3\varepsilon}.$$

Selecting the largest subset of K^* zeros with $\left|\gamma_\nu^* - \gamma_\kappa^*\right| > 5\lambda$ we have by $\log\left(1 + \frac{n-m}{m}\right) \gg \frac{n-m}{m}$ for $n \le 2m$

(5.3)
$$\begin{aligned} K^* T^{1-2\eta-3\varepsilon} &\ll \int\limits_0^{2T} \left|\sum_{n\le T} \frac{\mu(n)}{n^{1/2+it}}\right|^2 dt \\ &\ll T \sum_{n\le T} \frac{\mu^2(n)}{n} + \\ &\quad + \sum_{n\le T} \frac{1}{n^{1/2}} \left\{ \sum_{n/2<m<n} \frac{1}{\sqrt{m}\log(n/m)} + \sum_{m\le n/2} \frac{1}{\sqrt{m}\log(n/m)} \right\} \\ &\ll TL + \sum_{n\le T} \frac{1}{\sqrt{n}} \left\{ \sum_{\ell\le n/2} \frac{\sqrt{n}}{\ell} + \sqrt{n} \right\} \ll TL. \end{aligned}$$

Consequently

(5.4)
$$K \ll \lambda L K^* \ll L^3 T^{2\eta+3\varepsilon} \ll T^{2\eta+4\varepsilon}. \qquad \square$$

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