# New maximum scattered linear sets of the projective line 

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#### Abstract

In [2] and [19] are presented the first two families of maximum scattered $\mathbb{F}_{q}$-linear sets of the projective line $\operatorname{PG}\left(1, q^{n}\right)$. More recently in [23] and in [5], new examples of maximum scattered $\mathbb{F}_{q}$-subspaces of $V\left(2, q^{n}\right)$ have been constructed, but the equivalence problem of the corresponding linear sets is left open.

Here we show that the $\mathbb{F}_{q}$-linear sets presented in [23] and in [5], for $n=6,8$, are new. Also, for $q$ odd, $q \equiv \pm 1,0(\bmod 5)$, we present new examples of maximum scattered $\mathbb{F}_{q}$-linear sets in $\operatorname{PG}\left(1, q^{6}\right)$, arising from trinomial polynomials, which define new $\mathbb{F}_{q}$-linear MRD-codes of $\mathbb{F}_{q}^{6 \times 6}$ with dimension 12 , minimum distance 5 and middle nucleus (or left idealiser) isomorphic to $\mathbb{F}_{q^{6}}$.


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## 1 Introduction

Linear sets are natural generalisations of subgeometries. Let $\Lambda=\mathrm{PG}\left(W, \mathbb{F}_{q^{n}}\right)$ $=\operatorname{PG}\left(r-1, q^{n}\right)$, where $W$ is a vector space of dimension $r$ over $\mathbb{F}_{q^{n}}$. A point set $L$ of $\Lambda$ is said to be an $\mathbb{F}_{q}$-linear set of $\Lambda$ of rank $k$ if it is defined by the non-zero vectors of a $k$-dimensional $\mathbb{F}_{q}$-vector subspace $U$ of $W$, i.e.

$$
L=L_{U}=\left\{\langle\mathbf{u}\rangle_{\mathbb{F}_{q^{n}}}: \mathbf{u} \in U \backslash\{\mathbf{0}\}\right\} .
$$

[^0]The maximum field of linearity of an $\mathbb{F}_{q^{-}}$linear set $L_{U}$ is $\mathbb{F}_{q^{t}}$ if $t \mid n$ is the largest integer such that $L_{U}$ is an $\mathbb{F}_{q^{t}}$-linear set. Two linear sets $L_{U}$ and $L_{W}$ of $\operatorname{PG}\left(r-1, q^{n}\right)$ are said to be PГL-equivalent (or simply equivalent) if there is an element $\phi$ in $\operatorname{P\Gamma L}\left(r, q^{n}\right)$ such that $L_{U}^{\phi}=L_{W}$. It may happen that two $\mathbb{F}_{q}$-linear sets $L_{U}$ and $L_{W}$ of $\mathrm{PG}\left(r-1, q^{n}\right)$ are equivalent even if the two $\mathbb{F}_{q}$-vector subspaces $U$ and $W$ are not in the same orbit of $\Gamma \mathrm{L}\left(r, q^{n}\right)$ (see [7] and [3] for further details). In the recent years, starting from the paper [18] by Lunardon, linear sets have been used to construct or characterise various objects in finite geometry, such as blocking sets and multiple blocking sets in finite projective spaces, two-intersection sets in finite projective spaces, translation spreads of the Cayley Generalized Hexagon, translation ovoids of polar spaces, semifield flocks and finite semifields. For a survey on linear sets we refer the reader to [22], see also [13]. It is clear that in the applications it is crucial to have methods to decide whether two linear sets are equivalent or not.

In this paper we focus on maximum scattered $\mathbb{F}_{q}$-linear sets of $\operatorname{PG}\left(1, q^{n}\right)$ with maximum field of linearity $\mathbb{F}_{q}$, that is, $\mathbb{F}_{q}$-linear sets of rank $n$ of $\operatorname{PG}\left(1, q^{n}\right)$ of size $\left(q^{n}-1\right) /(q-1)$. If $L_{U}$ is a maximum scattered $\mathbb{F}_{q}$-linear set, then $U$ is a maximum scattered $\mathbb{F}_{q}$-subspace.

If $\langle(0,1)\rangle_{\mathbb{F}_{q^{n}}}$ is not contained in the linear set $L_{U}$ of rank $n$ of $\operatorname{PG}\left(1, q^{n}\right)$ (which we can always assume after a suitable projectivity), then $U=U_{f}:=$ $\left\{(x, f(x)): x \in \mathbb{F}_{q^{n}}\right\}$ for some $q$-polynomial $f(x)=\sum_{i=0}^{n-1} a_{i} x^{q^{i}} \in \mathbb{F}_{q^{n}}[x]$. In this case we will denote the associated linear set by $L_{f}$. The known non-equivalent (under $\Gamma L\left(2, q^{n}\right)$ ) maximum scattered $\mathbb{F}_{q}$-subspaces are

1. $U_{s}^{1, n}:=\left\{\left(x, x^{q^{s}}\right): x \in \mathbb{F}_{q^{n}}\right\}, 1 \leq s \leq n-1, \operatorname{gcd}(s, n)=1([2, ~ 8])$,
2. $U_{s, \delta}^{2, n}:=\left\{\left(x, \delta x^{q^{s}}+x^{q^{n-s}}\right): x \in \mathbb{F}_{q^{n}}\right\}, n \geq 4, \mathrm{~N}_{q^{n} / q}(\delta) \notin\{0,1\}$ 1 , $\operatorname{gcd}(s, n)=1([19]$ for $s=1,[23,20]$ for $s \neq 1)$,
3. $U_{s, \delta}^{3, n}:=\left\{\left(x, \delta x^{q^{s}}+x^{q^{s+n / 2}}\right): x \in \mathbb{F}_{q^{n}}\right\}, n \in\{6,8\}, \operatorname{gcd}(s, n / 2)=1$, $\mathrm{N}_{q^{n} / q^{n / 2}}(\delta) \notin\{0,1\}$, for the precise conditions on $\delta$ and $q$ see [5, Theorems 7.1 and 7.2] 2.

The stabilisers of the $\mathbb{F}_{q}$-subspaces above in the group $\mathrm{GL}\left(2, q^{n}\right)$ were determined in [5, Sections 5 and 6]. They have the following orders:

1. for $U_{s}^{1, n}$ we have a group of order $q^{n}-1$,

[^1]2. for $U_{s, \delta}^{2, n}$ we have a group of order $q^{2}-1$,
3. for $U_{s, \delta}^{3, n}$ we have a group of order $q^{n / 2}-1$.

It is known, that for $n=3$ the maximum scattered $\mathbb{F}_{q}$-spaces of $V\left(2, q^{3}\right)$ are $\Gamma \mathrm{L}\left(2, q^{3}\right)$-equivalent to $U_{1}^{1,3}$ (cf. [15]), and for $n=4$ they are GL $\left(2, q^{4}\right)$ equivalent either to $U_{1}^{1,4}$ or to $U_{1, \delta}^{2,4}$ (cf. 9]).

To make notation easier, by $L_{s}^{i, n}$ and $L_{s, \delta}^{i, n}$ we will denote the $\mathbb{F}_{q}$-linear set defined by $U_{s}^{i, n}$ and $U_{s, \delta}^{i, n}$, respectively. The $\mathbb{F}_{q^{-}}$-linear sets equivalent to $L_{s}^{1, n}$ are called of pseudoregulus type. It is easy to see that $L_{1}^{1, n}=L_{s}^{1, n}$ for any $s$ with $\operatorname{gcd}(s, n)=1$ and that $U_{s, \delta}^{2, n}$ is $\operatorname{GL}\left(2, q^{n}\right)$-equivalent to $U_{n-s, \delta-1}^{2, n}$.

In [19, Theorem 3] Lunardon and Polverino proved that $L_{1, \delta}^{2, n}$ and $L_{1}^{1, n}$ are not $\operatorname{P\Gamma L}\left(2, q^{n}\right)$-equivalent when $q>3, n \geq 4$. For $n=5$, in [4] it is proved that $L_{2, \delta}^{2,5}$ is $\operatorname{P\Gamma L}\left(2, q^{5}\right)$-equivalent neither to $L_{1, \delta^{\prime}}^{2,5}$ nor to $L_{1}^{1,5}$.

In the first part of this paper we prove that for $n=6,8$ the linear sets $L_{1}^{1, n}, L_{s, \delta}^{2, n}$ and $L_{s^{\prime}, \delta^{\prime}}^{3, n}$ are pairwise non-equivalent for any choice of $s, s^{\prime}, \delta, \delta^{\prime}$.

In the second part of this paper we prove that the $\mathbb{F}_{q}$-linear set defined by

$$
U_{b}^{4}:=\left\{\left(x, x^{q}+x^{q^{3}}+b x^{q^{5}}\right): x \in \mathbb{F}_{q^{6}}\right\}
$$

with $b^{2}+b=1, q \equiv 0, \pm 1(\bmod 5)$ is maximum scattered in $\mathrm{PG}\left(1, q^{6}\right)$ and it is not $\operatorname{P\Gamma L}\left(2, q^{6}\right)$-equivalent to any previously known example. Connections between scattered $\mathbb{F}_{q}$-subspaces and MRD-codes have been investigated in [23, 6, 17]. Using the relation found in [23] we also present new examples of such codes.

## 2 Classes of $\mathbb{F}_{q}$-linear sets of rank $n$ of $\operatorname{PG}\left(1, q^{n}\right)$ and preliminary results

For $\alpha \in \mathbb{F}_{q^{n}}$ and a divisor $h$ of $n$ we will denote by $\mathrm{N}_{q^{n} / q^{h}}(\alpha)$ the norm of $\alpha$ over the subfield $\mathbb{F}_{q^{h}}$, that is, $\mathrm{N}_{q^{n} / q^{h}}(\alpha)=\alpha^{1+q^{h}+\ldots+q^{n-h}}$.

By [1, 3] for $f(x)=\sum_{i=0}^{n-1} a_{i} x^{q^{i}}$ and $\hat{f}(x)=\sum_{i=0}^{n-1} a_{i}^{q^{n-i}} x^{q^{n-i}}$, the $\mathbb{F}_{q^{-}}$ subspaces $U_{f}=\left\{(x, f(x)): x \in \mathbb{F}_{q^{n}}\right\}$ and $U_{\hat{f}}=\left\{(x, \hat{f}(x)): x \in \mathbb{F}_{q^{n}}\right\}$ define the same linear set of $\mathrm{PG}\left(1, q^{n}\right)$. On the other hand $U_{f}$ and $U_{\hat{f}}$ are not necessarily $\Gamma \mathrm{L}\left(2, q^{n}\right)$-equivalent (see [3, Section 3.2]) and this motivates the following definitions.

Definition 2.1. ([级) Let $L_{U}$ be an $\mathbb{F}_{q}$-linear set of $\mathrm{PG}\left(W, \mathbb{F}_{q^{n}}\right)=\mathrm{PG}\left(1, q^{n}\right)$ of rank $n$ with maximum field of linearity $\mathbb{F}_{q}$.

We say that $L_{U}$ is of $\Gamma \mathrm{L}$-class $s$ if $s$ is the greatest integer such that there exist $\mathbb{F}_{q}$-subspaces $U_{1}, \ldots, U_{s}$ of $W$ with $L_{U_{i}}=L_{U}$ for $i \in\{1, \ldots, s\}$ and there is no $f \in \Gamma \mathrm{~L}\left(2, q^{n}\right)$ such that $U_{i}=U_{j}^{f}$ for each $i \neq j, i, j \in$ $\{1,2, \ldots, s\}$. If $L_{U}$ has $\Gamma \mathrm{L}$-class one, then $L_{U}$ is said to be simple.

We say that $L_{U}$ is of $\mathcal{Z}(\Gamma \mathrm{L})$-class $r$ if $r$ is the greatest integer such that there exist $\mathbb{F}_{q}$-subspaces $U_{1}, U_{2}, \ldots, U_{r}$ of $W$ with $L_{U_{i}}=L_{U}$ for $i \in$ $\{1,2, \ldots, r\}$ and $U_{i} \neq \lambda U_{j}$ for each $\lambda \in \mathbb{F}_{q^{n}}^{*}$ and for each $i \neq j, i, j \in$ $\{1,2, \ldots, r\}$.

Result 2.2. ([3, Prop. 2.6]) Let $L_{U}$ be an $\mathbb{F}_{q}$-linear set of $\mathrm{PG}\left(1, q^{n}\right)$ of rank $n$ with maximum field of linearity $\mathbb{F}_{q}$ and let $\varphi$ be a collineation of $\operatorname{PG}\left(1, q^{n}\right)$. Then $L_{U}$ and $L_{U}^{\varphi}$ have the same $\mathcal{Z}(\Gamma \mathrm{L})$-class and $\Gamma \mathrm{L}$-class. Also, the $\Gamma \mathrm{L}$ class of an $\mathbb{F}_{q}$-linear set cannot be greater than its $\mathcal{Z}(\Gamma \mathrm{L})$-class.

For a $q$-polynomial $f(x)=\sum_{i=0}^{n-1} a_{i} x^{q^{i}}$ over $\mathbb{F}_{q^{n}}$ let $D_{f}$ denote the associated Dickson matrix (or $q$-circulant matrix)

$$
D_{f}:=\left(\begin{array}{cccc}
a_{0} & a_{1} & \ldots & a_{n-1} \\
a_{n-1}^{q} & a_{0}^{q} & \ldots & a_{n-2}^{q} \\
\vdots & \vdots & \vdots & \vdots \\
a_{1}^{q^{n-1}} & a_{2}^{q^{n-1}} & \ldots & a_{0}^{q^{n-1}}
\end{array}\right) .
$$

The rank of the matrix $D_{f}$ equals the rank of the $\mathbb{F}_{q}$-linear map $f$, see for example [24].

We will use the following results.
Proposition 2.3. Let $f$ and $g$ be two $q$-polynomials over $\mathbb{F}_{q^{n}}$. Then $L_{f} \subseteq L_{g}$ if and only if

$$
x^{q^{n}}-x \mid \operatorname{det} D_{F(Y)}(x) \in \mathbb{F}_{q^{n}}[x],
$$

where $F(Y)=f(x) Y-g(Y) x$. In particular, if $\operatorname{deg} \operatorname{det} D_{F(Y)}(x)<q^{n}$, then $L_{f} \subseteq L_{g}$ if and only if $\operatorname{det} D_{F(Y)}(x)$ is the zero polynomial.

Proof. $L_{f} \subseteq L_{g}$ if and only if

$$
\left\{\frac{f(x)}{x}: x \in \mathbb{F}_{q^{n}}^{*}\right\} \subseteq\left\{\frac{g(x)}{x}: x \in \mathbb{F}_{q^{n}}^{*}\right\},
$$

which means that $\frac{g(y)}{y}=\frac{f(x)}{x}$ can be solved in $y$ if we fix $x \in \mathbb{F}_{q^{n}}^{*}$. Fix $x \in \mathbb{F}_{q^{n}}^{*}$, then the $q$-polynomial $F(Y)=f(x) Y-g(Y) x$ has rank less than
$n$ since it has a non-zero solution. Since the Dickson matrix $D_{F(Y)}(x)$ of $F(Y)$ has the same rank as $F(Y)$, it follows that $\operatorname{det} D_{F(Y)}(x)=0$ for each $x$. It follows that $x^{q^{n}}-x \mid \operatorname{det} D_{F(Y)}(x)$.

Lemma 2.4. [3, Lemma 3.6] Let $f(x)=\sum_{i=0}^{n-1} a_{i} x^{q^{i}}$ and $g(x)=\sum_{i=0}^{n-1} b_{i} x^{q^{i}}$ be two q-polynomials over $\mathbb{F}_{q^{n}}$ such that $L_{f}=L_{g}$. Then

$$
\begin{equation*}
a_{0}=b_{0}, \tag{1}
\end{equation*}
$$

for $k=1,2, \ldots, n-1$ it holds that

$$
\begin{equation*}
a_{k} a_{n-k}^{q^{k}}=b_{k} b_{n-k}^{q^{k}}, \tag{2}
\end{equation*}
$$

for $k=2,3, \ldots, n-1$ it holds that

$$
\begin{equation*}
a_{1} a_{k-1}^{q} a_{n-k}^{q^{k}}+a_{k} a_{n-1}^{q} a_{n-k+1}^{q^{k}}=b_{1} b_{k-1}^{q} b_{n-k}^{q^{k}}+b_{k} b_{n-1}^{q} b_{n-k+1}^{q^{k}} . \tag{3}
\end{equation*}
$$

## 3 The $L_{s, \delta}^{2, n}$-linear sets in $\operatorname{PG}\left(1, q^{n}\right), n=6,8$

In this section we determine the $\mathcal{Z}(\Gamma L)$-class of the maximum scattered $\mathbb{F}_{q^{-}}$ linear sets of $\mathrm{PG}\left(1, q^{n}\right), n=6,8$, introduced by Lunardon and Polverino, and generalised by Sheekey. Recall that $U_{s, \delta}^{2, n}$ is $\mathrm{GL}\left(2, q^{n}\right)$-equivalent to $U_{n-s, \delta^{-1}}^{2, n}$, thus it is enough to study the linear sets $L_{s, \delta}^{2, n}$ with $s<n / 2$ and $\operatorname{gcd}(s, n)=1$.

Proposition 3.1. If $n=6$, then the $\mathcal{Z}(\Gamma \mathrm{L})$-class of $L_{1, \delta}^{2,6}$ is two.
Proof. Since $g(x)=\delta x^{q}+x^{q^{5}}$ and $\hat{g}(x)=\delta^{q^{5}} x^{q^{5}}+x^{q}$ define the same linear set, we know $L_{1, \delta}^{2,6}=L_{5, \delta q^{5}}^{2,6}$. Suppose $L_{f}=L_{1, \delta}^{2,6}$ for some $f(x)=\sum_{i=0}^{5} a_{i} x^{q^{i}} \in$ $\mathbb{F}_{q^{6}}[x]$. We show that there exists $\lambda \in \mathbb{F}_{q^{6}}^{*}$ such that either $\lambda U_{f}=U_{1, \delta}^{2,6}$ or $\lambda U_{f}=U_{5, \delta q^{5}}^{2,6}$.

By (11) we obtain $a_{0}=0$, by (2) with $k=1,3$ we have

$$
\begin{equation*}
a_{1} a_{5}^{q}=\delta \tag{4}
\end{equation*}
$$

and $a_{3}=0$, respectively. Also, with $k=2$ in (2) and (3), taking (4) into account, we get $a_{2}=a_{4}=0$.

By Proposition (2.3) we get that the Dickson matrix associated to the $q$-polynomial

$$
F(Y)=\left(\frac{\delta}{a_{5}^{q}} x^{q}+a_{5} x^{q^{5}}\right) Y-x\left(\delta Y^{q}+Y^{q^{5}}\right)
$$

has zero determinant for each $x \in \mathbb{F}_{q^{6}}$. Direct computation shows that this determinant is

$$
\mathrm{N}_{q^{6} / q}\left(x / a_{5}\right)\left(\mathrm{N}_{q^{6} / q}\left(a_{5}\right)-1\right)\left(\mathrm{N}_{q^{6} / q}\left(a_{5}\right)-\mathrm{N}_{q^{6} / q}(\delta)\right),
$$

which has degree less than $q^{6}$, thus it is the zero polynomial. We have two possibilities:

1. If $\mathrm{N}_{q^{6} / q}\left(a_{5}\right)=1$, then putting $a_{5}=\lambda^{q^{5}-1}$ we obtain $\lambda U_{f}=U_{1, \delta}^{2,6}$.
2. If $\mathrm{N}_{q^{6} / q}\left(a_{5} / \delta\right)=1$, then choosing $a_{5}=\delta^{q^{5}} \lambda^{q^{5}-1}$ we get $\lambda U_{f}=U_{5, \delta q^{5}}^{2,6}$.

Because of the choice of $\delta$, that is $\mathrm{N}_{q^{6} / q}(\delta) \neq 1$, it follows that there is no $\mu \in \mathbb{F}_{q^{6}}$ such that $\mu U_{1, \delta}^{2,6}=U_{5, \delta^{5}}^{2,6}$ and this proves that the $\mathcal{Z}(\Gamma L)$-class of $L_{1, \delta}^{2,6}$ is exactly two.

Proposition 3.2. If $n=8$, then the $\mathcal{Z}(\Gamma \mathrm{L})$-class of $L_{1, \delta}^{2,8}$ is two.
Proof. Since $g(x)=\delta x^{q}+x^{q^{7}}$ and $\hat{g}(x)=\delta^{q^{7}} x^{q^{7}}+x^{q}$ define the same linear set, we have $L_{1, \delta}^{2,8}=L_{7, \delta q^{7}}^{2,8}$. Suppose $L_{f}=L_{1, \delta}^{2,8}$ for some $f(x)=\sum_{i=0}^{7} a_{i} x^{q^{i}} \in$ $\mathbb{F}_{q^{8}}[x]$. We show that there exists $\lambda \in \mathbb{F}_{q^{8}}^{*}$ such that either $\lambda U_{f}=U_{1, \delta}^{2,8}$ or $\lambda U_{f}=U_{7, \delta q^{7}}^{2,8}$.

By (1) we obtain $a_{0}=0$, by (2) with $k=1$ we have

$$
\begin{equation*}
a_{1} a_{7}^{q}=\delta \tag{5}
\end{equation*}
$$

and with $k=4$ we get $a_{4}=0$. Putting $k=2$ in (2) and (3), taking (5) into account, we get $a_{2}=a_{6}=0$. By (2) with $k=3$ we have $a_{3} a_{5}=0$.

If $a_{3}=0$, then $f(x)=a_{1} x^{q}+a_{5} x^{q^{5}}+a_{7} x^{q^{7}}$. Using Proposition 2.3, we get that the determinant of the Dickson matrix associated to the $q$-polynomial

$$
F(Y)=\left(a_{1} x^{q}+a_{5} x^{q^{5}}+a_{7} x^{q^{7}}\right) Y-x\left(a_{1} a_{7}^{q} Y^{q}+Y^{q^{7}}\right)
$$

is divisible by $x^{q^{8}}-x$. The coefficient of $x^{2\left(1+q+q^{2}+q^{3}\right)}$ after reducing the determinant modulo $x^{q^{8}}-x$ is $a_{1}^{1+q+q^{2}+q^{7}} a_{5}^{q^{3}+q^{4}+q^{5}+q^{6}}$, which is zero only when $a_{5}=0$ by (5).

On the other hand, if $a_{5}=0$, then $L_{f}=L_{\hat{f}}$ gives $a_{3}=0$.
Then $f(x)=\frac{\delta}{a_{7}^{q}} x^{q}+a_{7} x^{q^{7}}$. By Proposition 2.3, arguing as in the previous proof,

$$
\mathrm{N}_{q^{8} / q}\left(x / a_{7}\right)\left(\mathrm{N}_{q^{8} / q}\left(a_{7}\right)-1\right)\left(\mathrm{N}_{q^{8} / q}\left(a_{7}\right)-\mathrm{N}_{q^{8} / q}(\delta)\right)
$$

is the zero polynomial. We have two possibilities:

1. If $\mathrm{N}_{q^{8} / q}\left(a_{7}\right)=1$, then putting $a_{7}=\lambda^{q^{7}-1}$, we obtain $\lambda U_{f}=U_{1, \delta}^{2,8}$.
2. If $\mathrm{N}_{q^{8} / q}\left(a_{7} / \delta\right)=1$, then choosing $a_{7}=\delta^{q^{7}} \lambda^{q^{7}-1}$ we have $\lambda U_{f}=U_{7, \delta q^{7}}^{2,8}$. Because of the choice of $\delta$, that is $\mathrm{N}_{q^{8} / q}(\delta) \neq 1$, it follows that there is no $\mu \in \mathbb{F}_{q^{8}}$ such that $\mu U_{1, \delta}^{2,8}=U_{7, \delta q^{7}}^{2,8}$ and this proves that the $\mathcal{Z}(\Gamma \mathrm{L})$-class of $L_{1, \delta}^{2,8}$ is exactly two.
Proposition 3.3. If $n=8$, then the $\mathcal{Z}(\Gamma \mathrm{L})$-class of $L_{3, \delta}^{2,8}$ is two.
Proof. Since $g(x)=\delta x^{q^{3}}+x^{q^{5}}$ and $\hat{g}(x)=\delta^{q^{5}} x^{q^{5}}+x^{q^{3}}$ define the same linear set, we know $L_{3, \delta}^{2,8}=L_{5, \delta q^{5}}^{2,8}$. Suppose $L_{f}=L_{3, \delta}^{2,8}$ for some $f(x)=$ $\sum_{i=0}^{7} a_{i} x^{q^{i}} \in \mathbb{F}_{q^{8}}[x]$. We show that there exists $\lambda \in \mathbb{F}_{q^{8}}^{*}$ such that either $\lambda U_{f}=U_{3, \delta}^{2,8}$ or $\lambda U_{f}=U_{5, \delta^{5}}^{2,8}$.

By (1) we obtain $a_{0}=0$, by (2) with $k=3$ we have

$$
a_{3} a_{5}^{q^{3}}=\delta
$$

and with $k=4$ we get $a_{4}=0$. Putting $k=1$ and $k=2$ in (2) we get

$$
\begin{equation*}
a_{1} a_{7}=0 \text { and } a_{2} a_{6}=0, \tag{6}
\end{equation*}
$$

respectively. With $k=2$ and $k=3$ in (3) we obtain

$$
\begin{equation*}
a_{1}^{q+1} a_{6}^{q^{2}}+a_{2} a_{7}^{q+q^{2}}=0 . \tag{7}
\end{equation*}
$$

and

$$
\begin{equation*}
a_{1} a_{2}^{q} a_{5}^{q^{3}}+a_{3} a_{7}^{q} a_{6}^{q^{3}}=0 \tag{8}
\end{equation*}
$$

By (77) and (8), taking (6) into account, at most one of $\left\{a_{1}, a_{2}, a_{6}, a_{7}\right\}$ is non-zero.

Hence $f(x)=a_{3} x^{q^{3}}+a_{5} x^{q^{5}}+a_{i} x^{q^{i}}$ with $i \in\{1,2,6,7\}$. For each $i \in\{1,2,6,7\}$, by Proposition [2.3, the determinant of the Dickson matrix $D_{F(Y)}(x)$ with $F(Y)=f(x) Y-x\left(a_{3} a_{5}^{q^{3}} Y^{q^{3}}+Y^{q^{5}}\right)$ is zero modulo $x^{q^{8}}-x$. Then the following hold:

- for $i=1$ the coefficient of $x^{3+3 q+q^{2}+q^{3}}$ in the reduced form of det $D_{F(Y)}(x)$ is $a_{1}^{1+q+q^{2}+q^{7}} a_{3}^{q^{5}+q^{6}} a_{5}^{q^{3}+q^{4}}$,
- for $i=2$ the coefficient of $x^{3+2 q+q^{2}+q^{3}+q^{4}}$ in the reduced form of $\operatorname{det} D_{F(Y)}(x)$ is $a_{2}^{1+q+q^{2}+q^{6}+q^{7}} a_{3}^{q^{5}} a_{5}^{q^{3}+q^{4}}$.
Thus $a_{i}=0$ for $i \in\{1,2\}$ and since $L_{f}=L_{\hat{f}}$, the same holds for $i \in\{6,7\}$. Then from (7) we get $f(x)=\frac{\delta}{a_{5}^{q^{3}}} x^{q^{3}}+a_{5} x^{q^{5}}$. By Proposition 2.3, arguing as in the previous proof,

$$
\mathrm{N}_{q^{8} / q}\left(x / a_{5}\right)\left(\mathrm{N}_{q^{8} / q}\left(a_{5}\right)-1\right)\left(\mathrm{N}_{q^{8} / q}\left(a_{5}\right)-\mathrm{N}_{q^{8} / q}(\delta)\right)
$$

is the zero polynomial. Then the following holds:

1. If $\mathrm{N}_{q^{8} / q}\left(a_{5}\right)=1$, then putting $a_{5}=\lambda^{q^{5}-1}$ gives $\lambda U_{f}=U_{3, \delta}^{2,8}$.
2. If $\mathrm{N}_{q^{8} / q}\left(a_{5} / \delta\right)=1$, then set $a_{5}=\delta^{q^{5}} \lambda^{q^{5}-1}$, and hence $\lambda U_{f}=U_{5, \delta^{5}}^{2,8}$.

As in the previous proof, it can be easily seen that the $\mathcal{Z}(\Gamma \mathrm{L})$-class is exactly two.

Theorem 3.4. The linear set $L_{s, \delta}^{2, n}$ is not of pseudoregulus type for each $n, s, \delta, q$. Also, the linear sets $L_{1, \delta}^{2,8}$ and $L_{3, \rho}^{2,8}$ are not $\mathrm{P} \Gamma \mathrm{L}\left(2, q^{8}\right)$-equivalent.
Proof. Suppose that $L_{s, \delta}^{2, n}$ is of pseudoregulus type. Then by [14] there exists an element $f$ of GL $\left(2, q^{n}\right)$ such that $\left(U_{s, \delta}^{2, n}\right)^{f}=U_{r}^{1, n}$ with $\operatorname{gcd}(r, n)=1$. Since the $\mathbb{F}_{q^{n}}$-linear automorphism groups of $U_{s, \delta}^{2, n}$ and $\left(U_{s, \delta}^{2, n}\right)^{f}$ are conjugated and since the groups of $U_{r}^{1, n}$ and $U_{s, \delta}^{2, n}$ have orders $q^{n}-1$ and $q^{2}-1$, respectively (cf. Introduction), we get a contradiction.

For the second part, suppose to the contrary that $L_{1, \delta}^{2,8}$ and $L_{3, \rho}^{2,8}$ are $\operatorname{P\Gamma L}\left(2, q^{8}\right)$-equivalent. Then by Proposition 3.3 there exists a field automorphism $\sigma$, an invertible matrix $\left(\begin{array}{ll}a & b \\ c & d\end{array}\right)$ and $\alpha, \beta \in \mathbb{F}_{q^{8}}^{*}$ such that for each $x \in \mathbb{F}_{q^{8}}$ there exists $z \in \mathbb{F}_{q^{8}}$ satisfying

$$
\left(\begin{array}{ll}
a & b \\
c & d
\end{array}\right)\binom{x^{\sigma}}{\delta^{\sigma} x^{\sigma q}+x^{\sigma q^{7}}}=\binom{z}{\alpha z^{q^{3}}+\beta z^{q^{5}}} .
$$

Equivalently, for each $x \in \mathbb{F}_{q^{8}}$

$$
c x^{\sigma}+d \delta^{\sigma} x^{\sigma q}+d x^{\sigma q^{7}}=\alpha\left(a^{q^{3}} x^{\sigma q^{3}}+\delta^{\sigma q^{3}} b^{q^{3}} x^{\sigma q^{4}}+b^{q^{3}} x^{\sigma q^{2}}\right)+
$$

$$
\beta\left(a^{q^{5}} x^{\sigma q^{5}}+\delta^{\sigma q^{5}} b^{q^{5}} x^{\sigma q^{6}}+b^{q^{5}} x^{\sigma q^{4}}\right)
$$

This is a polynomial identity in $x^{\sigma}$. Comparing the coefficients of $x^{q^{2}}$ and $x^{q^{3}}$ we get that $a=b=0$, which is a contradiction.

## 4 The $L_{s, \delta}^{3, n}$-linear sets in $\operatorname{PG}\left(1, q^{n}\right), n=6,8$

In this section we determine the $\mathcal{Z}(\Gamma L)$-class of the maximum scattered $\mathbb{F}_{q^{-}}$ linear sets of $\mathrm{PG}\left(1, q^{n}\right), n=6,8$, introduced in 5]. According to 5, Section 5 , pg. 7], $U_{s, \delta}^{3, n}$ is $\mathrm{GL}\left(2, q^{n}\right)$-equivalent to $U_{n-s, \delta^{n-s}}^{3, n}$ and to $U_{s+n / 2, \delta^{-1}}^{3, n}$, thus it is enough to study the linear sets $L_{s, \delta}^{3, n}$ with $s<n / 4, \operatorname{gcd}(s, n / 2)=1$ and hence only with $s=1$ for $n=6,8$.

Proposition 4.1. The $\mathcal{Z}(\Gamma \mathrm{L})$-class of $L_{g}$, with $g(x)=\delta x^{q}+x^{q^{4}}, \delta \neq 0$, is two and hence the $\mathcal{Z}(\Gamma \mathrm{L})$-class of $L_{1, \delta}^{3,6}$ is two as well. Moreover, $L_{1, \delta}^{3,6}$ is a simple linear set.

Proof. Since $g(x)$ and $\hat{g}(x)=\delta^{q^{5}} x^{q^{5}}+x^{q^{2}}$ define the same linear set, we know $L_{g}=L_{\hat{g}}$. Suppose $L_{f}=L_{g}$ for some $f(x)=\sum_{i=0}^{5} a_{i} x^{q^{i}} \in \mathbb{F}_{q^{6}}[x]$. We show that there exists $\lambda \in \mathbb{F}_{q^{6}}^{*}$ such that either $\lambda U_{f}=U_{g}$ or $\lambda U_{f}=U_{\hat{g}}$.

By (1), we obtain $a_{0}=0$ and by (2) with $k=2$ we get $a_{3}=0$. Also, by (2) with $k=1$ and $k=2$, we have

$$
\begin{equation*}
a_{1} a_{5}=0 \tag{9}
\end{equation*}
$$

and

$$
\begin{equation*}
a_{2} a_{4}=0 \tag{10}
\end{equation*}
$$

respectively. By (3) with $k=2$ we get

$$
\begin{equation*}
a_{1}^{q+1} a_{4}^{q^{2}}+a_{2} a_{5}^{q+q^{2}}=\delta^{q+1} \tag{11}
\end{equation*}
$$

From (9), (10) and (11) it follows that either

$$
f(x)=\frac{\delta^{q+1}}{a_{5}^{q+q^{2}}} x^{q^{2}}+a_{5} x^{q^{5}}
$$

or

$$
f(x)=a_{1} x^{q}+\left(\frac{\delta}{a_{1}}\right)^{q^{5}+q^{4}} x^{q^{4}}
$$

In both cases, the determinant of the Dickson matrix associated with $F(Y)=$ $f(x) Y-x\left(\delta Y^{q}+Y^{q^{4}}\right)$ is the zero-polynomial after reducing modulo $x^{q^{6}}-x$
and hence in the first case we obtain $\mathrm{N}_{q^{6} / q}\left(a_{5} / \delta\right)=1$, in the second case $\mathrm{N}_{q^{6} / q}\left(a_{1} / \delta\right)=1$. In the former case $a_{5}=\delta^{q^{5}} \lambda^{q^{5}-1}$ and hence $\lambda U_{f}=U_{\hat{g}}$. In the latter case $a_{1}=\delta \lambda^{q-1}$ implying $\lambda U_{f}=U_{g}$.

This means that the $\mathcal{Z}(\Gamma \mathrm{L})$-class of $U_{g}$ is at most two. Straightforward computation shows that it is exactly two. In case of $L_{1, \delta}^{3,6}$ (and hence with $\left.\mathrm{N}_{q^{6} / q^{3}}(\delta) \neq 1\right)$ it follows from [5, Section 5] that $U_{1, \delta}^{3,6}$ and $U_{5, \delta^{5}}^{3,6}$ are $\Gamma \mathrm{L}\left(2, q^{6}\right)$ equivalent and hence $L_{1, \delta}^{3,6}$ is simple.

Proposition 4.2. The $\mathcal{Z}(\Gamma \mathrm{L})$-class of $L_{g}$, with $g(x)=\delta x^{q}+x^{q^{5}}, \delta \neq 0$, is two and hence the $\mathcal{Z}(\Gamma \mathrm{L})$-class of $L_{1, \delta}^{3,8}$ is two as well. Moreover, $L_{1, \delta}^{3,8}$ is a simple linear set.
Proof. Since $g(x)=\delta x^{q}+x^{q^{5}}$ and $\hat{g}(x)=\delta^{q^{7}} x^{q^{7}}+x^{q^{3}}$ define the same linear set, we have $L_{g}=L_{\hat{g}}$. Suppose $L_{f}=L_{g}$ for some $f(x)=\sum_{i=0}^{7} a_{i} x^{q^{i}} \in$ $\mathbb{F}_{q^{8}}[x]$. We show that there exists $\lambda \in \mathbb{F}_{q^{8}}^{*}$ such that either $\lambda U_{f}=U_{g}$ or $\lambda U_{f}=U_{\hat{g}}$.

By (1), we obtain $a_{0}=0$ and by (2) with $k=4$ we get $a_{4}=0$. Also, by (2) with $k=1, k=2$ and $k=3$ we get

$$
\begin{equation*}
a_{1} a_{7}=a_{2} a_{6}=a_{3} a_{5}=0 \tag{12}
\end{equation*}
$$

By (3), with $k=2$ we obtain

$$
\begin{equation*}
a_{1}^{q+1} a_{6}^{q^{2}}+a_{2} a_{7}^{q+q^{2}}=0 \tag{13}
\end{equation*}
$$

and with $k=3$ we get

$$
\begin{equation*}
a_{1} a_{2}^{q} a_{5}^{q^{3}}+a_{3} a_{7}^{q} a_{6}^{q^{3}}=0 \tag{14}
\end{equation*}
$$

By (12), first suppose $a_{1}=a_{2}=a_{3}=0$. Proposition 2.3 yields that the determinant of the Dickson matrix associated with

$$
F(Y)=\left(a_{5} x^{q^{5}}+a_{6} x^{q^{6}}+a_{7} x^{q^{7}}\right) Y-x\left(\delta Y^{q}+Y^{q^{5}}\right)
$$

has to be the zero polynomial after reducing modulo $x^{8}-x$. The coefficient of $x^{1+2 q+2 q^{2}+2 q^{3}+q^{4}}$ is $-a_{5}^{q^{4}+q^{5}+q^{6}+q^{7}} \delta^{1+q+q^{2}}$, hence $a_{5}=0$. The coefficient of $x^{1+q+2 q^{2}+2 q^{3}+q^{4}+q^{5}}$ is $-a_{6}^{q^{4}+q^{5}+q^{6}+q^{7}} \delta^{1+q+q^{2}}$, hence $a_{6}=0$. The coefficient of $x^{1+q+q^{2}+2 q^{3}+q^{4}+q^{5}+q^{6}}$ is $-a_{7}^{q^{4}+q^{5}+q^{6}+q^{7}} \delta^{1+q+q^{2}}$, hence $a_{7}=0$, a contradiction.

Now suppose $a_{1}=a_{2}=a_{5}=a_{7}=0$. Again, Proposition 2.3 yields that the determinant of the Dickson matrix associated with

$$
F(Y)=\left(a_{3} x^{q^{3}}+a_{6} x^{q^{6}}\right) Y-x\left(\delta Y^{q}+Y^{q^{5}}\right),
$$

has to be the zero polynomial after reducing modulo $x^{8}-x$. The coefficient of $x^{2+2 q+3 q^{2}+q^{3}}$ is $-a_{3}^{q^{5}+q^{6}+q^{7}} a_{6}^{q^{4}} \delta^{1+q+q^{2}}$, hence $a_{3} a_{6}=0$. We cannot have $a_{3}=0$ because of the previous paragraph, hence $a_{6}=0$, but then the coefficient of $x^{1+2 q+2 q^{2}+2 q^{3}+q^{4}}$ is $-a_{3}^{1+q^{5}+q^{6}+q^{7}} \delta^{q+q^{2}+q^{3}}$. Then again $a_{3}=0$ follows, a contradiction.

Taking into account $L_{f}=L_{\hat{f}}$ and (12), (13), (14), two cases remain: $f(x)=a_{3} x^{q^{3}}+a_{7} x^{q^{7}}$ and $f(x)=a_{1} x^{q}+a_{5} x^{q^{5}}$.

In the former case Proposition 2.3 yields that the determinant of the Dickson matrix associated with

$$
F(Y)=\left(a_{3} x^{q^{3}}+a_{7} x^{q^{7}}\right) Y-x\left(\delta Y^{q}+Y^{q^{5}}\right),
$$

has to the zero polynomial after reducing modulo $x^{8}-x$. The coefficient of $x^{2+2 q+2 q^{2}+2 q^{3}}$ is $a_{3}^{q^{5}+q^{6}+q^{7}} a_{7}^{q^{4}}\left(a_{3} a_{7}^{q+q^{2}+q^{3}}-\delta^{1+q+q^{2}}\right)$, hence

$$
a_{3}=\delta^{1+q+q^{2}} / a_{7}^{q+q^{2}+q^{3}} .
$$

Since the coefficient of $x^{2+q+2 q^{2}+2 q^{3}+q^{5}}$ is

$$
\left(\mathrm{N}_{q^{8} / q}(\delta)-\mathrm{N}_{q^{8} / q}\left(a_{7}\right)\right) \delta^{2+q+q^{2}+q^{6}+2 q^{7}} / a_{7}^{3+2 q+2 q^{2}+q^{3}+q^{5}+q^{6}+2 q^{7}},
$$

which has to be zero and hence it follows that $\mathrm{N}_{q^{8} / q}\left(a_{7} / \delta\right)=1$. Then there exists $\lambda \in \mathbb{F}_{q^{8}}^{*}$ such that $a_{7}=\delta^{q^{7}} \lambda^{q^{7}-1}$ and hence $a_{3}=\lambda^{q^{3}-1}$, i.e. $\lambda U_{f}=U_{\hat{g}}$.

On the other hand, if $f(x)=a_{1} x^{q}+a_{5} x^{q^{5}}$, then the previous paragraph yields that there exists $\lambda \in \mathbb{F}_{q^{8}}^{*}$ such that $\lambda U_{\hat{f}}=U_{\hat{g}}$ and hence $\lambda^{-1} U_{f}=U_{g}$.

Since there is no $\mu \in \mathbb{F}_{q^{8}}^{*}$ such that $U_{g}=\mu U_{\hat{g}}$, it follows that the $\mathcal{Z}(\Gamma L)$ class of $U_{g}$ is exactly two. In case of $L_{1, \delta}^{3,8}$ (and hence with $\mathrm{N}_{q^{8} / q^{4}}(\delta) \neq 1$ ) it follows from [5, Section 5] that $U_{1, \delta}^{3,8}$ and $U_{7, \delta q^{7}}^{3,8}$ are $\Gamma \mathrm{L}\left(2, q^{8}\right)$-equivalent and hence $L_{1, \delta}^{3,8}$ is simple.

Theorem 4.3. The linear set $L_{1, \delta}^{3, n}, n=6,8$, is not of pseudoregulus type and not $\mathrm{P} \Gamma \mathrm{L}\left(2, q^{n}\right)$-equivalent to $L_{s, \rho}^{2, n}$.

Proof. Since the $\mathbb{F}_{q^{n}}$-linear automorphism group of $U_{1, \delta}^{3, n}$ has order $q^{n / 2}-1$ (cf. [5, Corollary 5.2]), the same arguments as in the proof of Theorem 3.4 can be applied to show that $L_{1, \delta}^{3, n}$ is not of pseudoregulus type.

Suppose that $L_{1, \delta}^{3, n}$ is equivalent to $L_{s, \rho}^{2, n}$ for some $n \in\{6,8\}$. Then by Propositions 3.1, 3.2 , 3.3, there exists $f \in \Gamma L\left(2, q^{n}\right)$ such that either $\left(U_{1, \delta}^{3, n}\right)^{f}=U_{s, \rho}^{2, n}$ or $\left(U_{1, \delta}^{3, n}\right)^{f}=U_{n-s, \rho^{q^{n-s}}}^{2, n}$. This gives a contradiction, since the sizes of the corresponding automorphism groups are different.

## 5 New maximum scattered linear sets in $\operatorname{PG}\left(1, q^{6}\right)$

In this section we show that $L_{g}$ with $g(x)=x^{q}+x^{q^{3}}+b x^{q^{5}} \in \mathbb{F}_{q^{6}}[x], q$ odd, $q \equiv 0, \pm 1(\bmod 5), b^{2}+b=1$ is a maximum scattered $\mathbb{F}_{q}$-linear set of $\operatorname{PG}\left(1, q^{6}\right)$ which is not equivalent to any other previously known example. Note that, under these assumptions we have $b \in \mathbb{F}_{q}$.
 only if for each $m \in \mathbb{F}_{q^{6}}$

$$
\frac{x^{q}+x^{q^{3}}+b x^{q^{5}}}{x}=-m
$$

has at most $q$ solutions. Those $m$ which admit exactly $q$ solutions correspond to points $\langle(1,-m)\rangle_{\mathbb{F}_{q^{6}}}$ of $L_{g}$ with weight one. It follows that $U_{g}$ is scattered if and only if for each $m \in \mathbb{F}_{q^{6}}$ the kernel of

$$
r_{m, b}(x):=m x+x^{q}+x^{q^{3}}+b x^{q^{5}}
$$

has dimension less than two, or, equivalently, the Dickson matrix associated with $r_{m, b}(x)$, that is,

$$
D_{m, b}=\left(\begin{array}{llllll}
m & 1 & 0 & 1 & 0 & b \\
b & m^{q} & 1 & 0 & 1 & 0 \\
0 & b & m^{q^{2}} & 1 & 0 & 1 \\
1 & 0 & b & m^{q^{3}} & 1 & 0 \\
0 & 1 & 0 & b & m^{q^{4}} & 1 \\
1 & 0 & 1 & 0 & b & m^{q^{5}}
\end{array}\right)
$$

has rank at least 5 for each $m \in \mathbb{F}_{q^{6}}$.
Denote by $M_{i, j}$ the determinant of the matrix obtained from $D_{m, b}$ by deleting the $i$-th row and $j$-th column and consider the two minors:
$M_{6,3}=2-3 b+(b-1)\left(\mathrm{N}_{q^{6} / q^{3}}(m)+\mathrm{N}_{q^{6} / q^{3}}\left(m^{q}\right)\right)+\mathrm{N}_{q^{6} / q^{3}}(m)^{q+1}+(1-b)\left(m^{1+q}-m^{q^{3}+q^{4}}\right)$,
and
$M_{6,4}=2 m-3 b m+2 m^{q^{2}}-3 b m^{q^{2}}+m^{q^{4}}-b m^{q^{4}}+m^{1+q+q^{2}}+b m^{1+q+q^{4}}+b m^{q+q^{2}+q^{4}}$.

Theorem 5.1. If $q \equiv 0, \pm 1(\bmod 5), q$ odd and $b^{2}+b=1\left(\right.$ hence $\left.b \in \mathbb{F}_{q}\right)$, then $U_{g}$ is a maximum scattered $\mathbb{F}_{q}$-subspace for $g(x)=x^{q}+x^{q^{3}}+b x^{q^{5}}$.

Proof. It is sufficient to show that $M_{6,3}$ and $M_{6,4}$ cannot be both zeros for the same value of $m \in \mathbb{F}_{q^{6}}$. If $m=0$, then $M_{6,3}=2-3 b \neq 0$ since $b=2 / 3$ does not satisfy our condition. First suppose that $M_{6,3}$ vanishes for some $m \in \mathbb{F}_{q^{6}}^{*}$. Then
$m^{1+q}-m^{q^{3}+q^{4}}=\frac{2-3 b+(b-1)\left(\mathrm{N}_{q^{6} / q^{3}}(m)+\mathrm{N}_{q^{6} / q^{3}}\left(m^{q}\right)\right)+\mathrm{N}_{q^{6} / q^{3}}(m)^{q+1}}{b-1}$,
and since the righ-hand side is in $\mathbb{F}_{q^{3}}$, the same follows for the left-hand side, and hence $m^{1+q}-m^{q^{3}+q^{4}}=m^{q^{3}+q^{4}}-m^{1+q}$, from which $m^{1+q} \in \mathbb{F}_{q^{3}}^{*}$ follows. So, if $m^{1+q} \notin \mathbb{F}_{q^{3}}^{*}$, then $r k\left(D_{m, b}\right) \geq 5$. Now, suppose $m^{1+q} \in \mathbb{F}_{q^{3}}^{*}$, then $M_{6,3}$ can be written as
$M_{6,3}=2-3 b+(1-b)\left(-m^{1+q^{3}}-m^{q+q^{4}}\right)+m^{2(1+q)}=\left((1-b)-m^{1+q^{3}}\right)^{q+1}$.
Since $M_{6,3}=0$, we have

$$
\begin{equation*}
1-b=m^{1+q^{3}} \in \mathbb{F}_{q} . \tag{15}
\end{equation*}
$$

Then $m^{\left(q^{3}+1\right)(q+1)}=m^{2(q+1)}=(1-b)^{2}$ and hence either $m^{q+1}=1-b$, or $m^{q+1}=b-1$. In both cases $m^{q+1} \in \mathbb{F}_{q}$ follows. The latter case cannot hold. Indeed by (15) we would get $m^{q^{3}+1}=-m^{q+1}$, so $m^{q^{2}}=-m$, which holds only if $m \in \mathbb{F}_{q^{4}} \cap \mathbb{F}_{q^{6}}=\mathbb{F}_{q^{2}}$, a contradiction. In the former case, by (15) we obtain $m^{q^{3}+1}=m^{q+1}$, so $m \in \mathbb{F}_{q^{2}}$. It follows that, taking $m^{1+q}=1-b=b^{2}$ into account, $M_{6,4}=4 m(1-b)$, which cannot be zero.

Similarly to the proof of [5, Proposition 5.2] it is easy to prove the following result.

Proposition 5.2. The linear automorphism group of $U_{g}$ (defined as in Theorem (5.1) is

$$
\mathcal{G}=\left\{\left(\begin{array}{cc}
\lambda & 0 \\
0 & \lambda^{q}
\end{array}\right): \lambda \in \mathbb{F}_{q^{2}}^{*}\right\} .
$$

Proposition 5.3. The maximum scattered $\mathbb{F}_{q}$-subspace $U_{g}$ defined in Theorem 5.1 is not $\Gamma \mathrm{L}\left(2, q^{6}\right)$-equivalent to the $\mathbb{F}_{q}$-subspaces $U_{s}^{1,6}, U_{t, \rho}^{2,6}$ and $U_{h, \xi}^{3,6}$.

Proof. As in the proof of Theorem 3.4, the size of the linear automorphism group of $U_{g}$ is different from the size of the group of $U_{s}^{1,6}$ and of $U_{h, \xi}^{3,6}$ (cf. Introduction), hence it remains to show that $U_{g}$ is not $\Gamma \mathrm{L}\left(2, q^{6}\right)$-equivalent to $U_{t, \rho}^{2,6}$.

Since any $\mathbb{F}_{q^{-}}$-subspace of the form $U_{5, \eta}^{2,6}$ is $\operatorname{GL}\left(2, q^{6}\right)$-equivalent to $U_{1, \rho}^{2,6}$ for some $\rho$, it is enough to show that $U_{g}$ and $U_{t, \rho}^{2,6}$ lie on different orbits of $\Gamma \mathrm{L}\left(2, q^{6}\right)$. Suppose the contrary, then there exist $\sigma \in \operatorname{Aut}\left(\mathbb{F}_{q^{6}}\right)$ and an invertible matrix $\left(\begin{array}{ll}\alpha & \beta \\ \gamma & \delta\end{array}\right)$ such that for each $x \in \mathbb{F}_{q^{6}}$ there exists $z \in \mathbb{F}_{q^{6}}$ satisfying

$$
\left(\begin{array}{cc}
\alpha & \beta \\
\gamma & \delta
\end{array}\right)\binom{x^{\sigma}}{\rho^{\sigma} x^{\sigma q}+x^{\sigma q^{5}}}=\binom{z}{z^{q}+z^{q^{3}}+b z^{q^{5}}} .
$$

Equivalently, for each $x \in \mathbb{F}_{q^{6}}$ we have

$$
\begin{gathered}
\gamma x^{\sigma}+\delta\left(\rho^{\sigma} x^{\sigma q}+x^{\sigma q^{5}}\right)=\alpha^{q} x^{\sigma q}+\beta^{q}\left(\rho^{\sigma q} x^{\sigma q^{2}}+x^{\sigma}\right)+ \\
+\alpha^{q^{3}} x^{\sigma q^{3}}+\beta^{q^{3}}\left(\rho^{\sigma q^{3}} x^{\sigma q^{4}}+x^{\sigma q^{2}}\right)+b\left(\alpha^{q^{5}} x^{\sigma q^{5}}+\beta^{q^{5}}\left(\rho^{\sigma q^{5}} x^{\sigma}+x^{\sigma q^{4}}\right)\right) .
\end{gathered}
$$

This is a polynomial identity in $x^{\sigma}$. Comparing coefficients we get $\alpha=\delta=0$ and

$$
\left\{\begin{array}{l}
\beta^{q} \rho^{\sigma q}+\beta^{q^{3}}=0, \\
\beta^{q^{3}} \rho^{\sigma q^{3}}+b \beta^{q^{5}}=0
\end{array}\right.
$$

Subtracting the second equation from the $q^{2}$-th power of the first gives $\beta^{q^{5}}(1-b)=0$, and hence $\beta=0$, a contradiction.

Theorem 5.4. The maximum scattered $\mathbb{F}_{q}$-linear set $L_{g}$ of $\mathrm{PG}\left(1, q^{6}\right)$, where $g$ is defined in Theorem 5.1, is not $\mathrm{P} \Gamma \mathrm{L}\left(2, q^{6}\right)$-equivalent to any any other previously known maximum scattered $\mathbb{F}_{q}$-linear set.

Proof. We have to confront $L_{g}$ with $L_{s}^{1,6}, L_{t, \rho}^{2,6}$ and $L_{h, \xi}^{3,6}$. Suppose that $L_{g}$ is equivalent to one of these linear sets, then by [14] and by Propositions 3.1 and 4.1, respectively, there exists $\varphi \in \Gamma \mathrm{L}\left(2, q^{n}\right)$ such that $U_{g}^{\varphi}$ equals one of $U_{s}^{1,6}, U_{t, \rho}^{2,6}$ and $U_{h, \xi}^{3,6}$, a contradiction by Proposition 5.3.

For the sake of completeness we show that the $\mathcal{Z}(\Gamma \mathrm{L})$-class of $L_{g}$, defined as in Theorem 5.1, is one.

Proposition 5.5. The $\mathcal{Z}(\Gamma \mathrm{L})$-class of $L_{g}$ of $\mathrm{PG}\left(1, q^{6}\right)$, where $g(x)=x^{q}+$ $x^{q^{3}}+b x^{q^{5}}$, is at most two if $b \neq 0$.
Proof. Since $g(x)$ and $\hat{g}(x)=b^{q} x^{q}+x^{q^{3}}+x^{q^{5}}$ define the same linear set, we know $L_{g}=L_{\hat{g}}$. Suppose $L_{f}=L_{g}$ for some $f(x)=\sum_{i=0}^{5} a_{i} x^{q^{i}} \in \mathbb{F}_{q^{6}}[x]$. We show that there exists $\lambda \in \mathbb{F}_{q^{6}}^{*}$ such that either $\lambda U_{f}=U_{g}$ or $\lambda U_{f}=U_{\hat{g}}$.

By (1) we obtain $a_{0}=0$, by (2) with $k=1,3$ we have

$$
a_{1} a_{5}^{q}=b^{q}
$$

and

$$
\begin{equation*}
a_{3}^{q^{3}+1}=1 . \tag{16}
\end{equation*}
$$

By (3), with $k=2$ we have

$$
a_{1}^{q+1} a_{4}^{q^{2}}+a_{2} a_{5}^{q+q^{2}}=0
$$

and taking this into account, together with (2) applied for $k=2$ we obtain $a_{2}=a_{4}=0$.

Using Proposition 2.3, we get that the determinant of the Dickson matrix associated to the $q$-polynomial

$$
F(Y)=\left(a_{1} x^{q}+a_{3} x^{q^{3}}+a_{5} x^{q^{5}}\right) Y-x\left(Y^{q}+Y^{q^{3}}+b Y^{q^{5}}\right)
$$

is the zero-polynomial modulo $x^{q^{6}}-x$. Substituting $a_{1}=\left(b / a_{5}\right)^{q}$ it turns out that the coefficient of $x^{1+q+2 q^{4}+2 q^{5}}$ in the reduced form of this determinant is

$$
\begin{gathered}
a_{3}^{q^{2}} a_{5}^{-1-q-q^{4}-q^{5}}\left(a_{3}^{q^{3}} a_{5}^{2+q+q^{4}+2 q^{5}}\left(a_{3}^{q+q^{4}}-1\right)-\right. \\
\left.\left(a_{5}^{1+q+q^{5}}-a_{3}^{q}\right) b^{1+q+q^{5}}+a_{3}^{q^{4}} a_{5}^{2+2 q+2 q^{5}}-a_{5}^{1+q+q^{5}}\right) .
\end{gathered}
$$

Applying $a_{3}^{q^{3}+1}=1$, it follows that
$\left(a_{3}^{q}-a_{5}^{1+q+q^{5}}\right) b^{1+q+q^{5}}=a_{5}^{1+q+q^{5}}-a_{3}^{q^{4}} a_{5}^{2+2 q+2 q^{5}}=\left(a_{3}^{q}-a_{5}^{1+q+q^{5}}\right) a_{3}^{q^{4}} a_{5}^{1+q+q^{5}}$.
If $a_{3}^{q}=a_{5}^{1+q+q^{5}}$, then (16) yields $\mathrm{N}_{q^{6} / q}\left(a_{5}\right)=1$, and hence there exists $\lambda \in \mathbb{F}_{q^{6}}^{*}$ such that $a_{5}=\lambda^{q^{5}-1}$. It is easy to see that in this case $\lambda U_{f}=U_{\hat{g}}$.

Now suppose $a_{3}^{q} \neq a_{5}^{1+q+q^{5}}$ and hence $b^{1+q+q^{5}}=a_{3}^{q^{4}} a_{5}^{1+q+q^{5}}$. Taking $\left(q^{3}+1\right)$-th powers yields $\mathrm{N}\left(b / a_{5}\right)=1$ and hence there exists $\lambda \in \mathbb{F}_{q^{6}}^{*}$ such that $a_{5}=b \lambda^{q^{5}-1}$. It is easy to see that in this case $\lambda U_{f}=U_{g}$.

Corollary 5.6. The $\mathcal{Z}(\Gamma \mathrm{L})$-class of $L_{g}$ of $\mathrm{PG}\left(1, q^{6}\right)$, where $g(x)=x^{q}+$ $x^{q^{3}}+b x^{q^{5}}$, is two if $b^{2}+b=1$. In particular, it is two if $g$ is defined as in Theorem 5.1.

Proof. If $\lambda U_{g}=U_{\hat{g}}$ for some $\lambda \in \mathbb{F}_{q^{6}}$, then $\lambda g(x)=\hat{g}(\lambda x)$ for each $x \in \mathbb{F}_{q^{6}}^{*}$ and hence comparing coefficients gives $b^{q} \lambda^{q-1}=1$ and $\lambda^{q^{3}-1}=1$. Then $b=\lambda^{q^{2}-1}$ and hence $\mathrm{N}_{q^{6} / q^{2}}(b)=1$. Also, $b \in \mathbb{F}_{q^{2}}$ from which $b^{3}=1$ follows, contradicting $b^{2}+b=1$.

## 6 New MRD-codes

The set of $m \times n$ matrices $\mathbb{F}_{q}^{m \times n}$ over $\mathbb{F}_{q}$ is a rank metric $\mathbb{F}_{q}$-space with rank metric distance defined by $d(A, B)=r k(A-B)$ for $A, B \in \mathbb{F}_{q}^{m \times n}$. A subset $\mathcal{C} \subseteq \mathbb{F}_{q}^{m \times n}$ is called a rank distance code (RD-code for short). The minimum distance of $\mathcal{C}$ is

$$
d(C)=\min _{A, B \in \mathcal{C}, A \neq B}\{d(A, B)\}
$$

In [11] the Singleton bound for an $m \times n$ rank metric code $\mathcal{C}$ with minimum rank distance $d$ was proved:

$$
\# \mathcal{C} \leq q^{\max \{m, n\}(\min \{m, n\}-d+1)}
$$

If this bound is achieved, then $\mathcal{C}$ is an MRD-code.
When $\mathcal{C}$ is an $\mathbb{F}_{q}$-linear subspace of $\mathbb{F}_{q}^{m \times n}$, we say that $\mathcal{C}$ is an $\mathbb{F}_{q}$-linear code and the dimension $\operatorname{dim}_{q}(\mathcal{C})$ is defined to be the dimension of $\mathcal{C}$ as a subspace over $\mathbb{F}_{q}$. If $d$ is the minimum distance of $\mathcal{C}$ we say that $\mathcal{C}$ has parameters ( $m, n, q ; d$ ).

We will use the following equivalence definition for codes of $\mathbb{F}_{q}^{m \times m}$. If $\mathcal{C}$ and $\mathcal{C}^{\prime}$ are two codes then they are equivalent if and only if there exist two invertible matrices $A, B \in \mathbb{F}_{q}^{m \times m}$ and a field automorphism $\sigma$ such that $\left\{A C^{\sigma} B: C \in \mathcal{C}\right\}=\mathcal{C}^{\prime}$, or $\left\{A C^{T \sigma} B: C \in \mathcal{C}\right\}=\mathcal{C}^{\prime}$, where $T$ denotes transposition. The code $\mathcal{C}^{T}$ is also called the adjoint of $\mathcal{C}$.

In [23, Section 5] Sheekey showed that scattered $\mathbb{F}_{q}$-linear sets of $\mathrm{PG}\left(1, q^{n}\right)$ of rank $n$ yield $\mathbb{F}_{q}$-linear MRD-codes with parameters $(n, n, q ; n-1)$. We briefly recall here the construction from [23]. Let $U_{f}=\left\{(x, f(x)): x \in \mathbb{F}_{q^{n}}\right\}$ for some $q$-polynomial $f(x)$. Then, after fixing an $\mathbb{F}_{q}$-bases $\left\{b_{1}, \ldots, b_{n}\right\}$ for $\mathbb{F}_{q^{n}}$ we can define an isomorphism between the rings $\operatorname{End}\left(\mathbb{F}_{q^{n}}, \mathbb{F}_{q}\right)$ and $\mathbb{F}_{q}^{n \times n}$. More precisely, to $f \in \operatorname{End}\left(\mathbb{F}_{q^{n}}, \mathbb{F}_{q}\right)$ we associate the matrix $M_{f}$ of $\mathbb{F}_{q}^{n \times n}$ with
$i$-th column $\left(a_{1, i}, \ldots, a_{n, i}\right)^{T}$, where $f\left(b_{i}\right)=\sum_{j=1}^{n} a_{j, i} b_{j} \sqrt[3]{ }$ In this way the set

$$
\mathcal{C}_{f}:=\left\{x \mapsto a f(x)+b x: a, b \in \mathbb{F}_{q^{n}}\right\}
$$

corresponds to a set of $n \times n$ matrices over $\mathbb{F}_{q}$ forming an $\mathbb{F}_{q}$-linear MRDcode with parameters $(n, n, q ; n-1)$. Also, since $\mathcal{C}_{f}$ is an $\mathbb{F}_{q^{n}}$-subspace of $\operatorname{End}\left(\mathbb{F}_{q^{n}}, \mathbb{F}_{q}\right)$, its middle nucleus $\mathcal{N}(\mathcal{C})$ (cf. [21], or [16] where the term left idealiser was used) is the set of scalar maps $\mathcal{F}_{n}:=\left\{x \in \mathbb{F}_{q^{n}} \mapsto \alpha x \in\right.$ $\left.\mathbb{F}_{q^{n}}: \alpha \in \mathbb{F}_{q^{n}}\right\}$, i.e. $\mathcal{N}\left(\mathcal{C}_{f}\right) \cong \mathbb{F}_{q^{n}}$. Note that equivalent codes have isomorphic middle nuclei. For further details see [5, Section 6].

Let $\mathcal{C}_{f}$ and $\mathcal{C}_{h}$ be two MRD-codes arising from maximum scattered subspaces $U_{f}$ and $U_{h}$ of $\mathbb{F}_{q^{n}} \times \mathbb{F}_{q^{n}}$. In [23, Theorem 8] the author showed that there exist invertible matrices $A, B$ and $\sigma \in \operatorname{Aut}\left(\mathbb{F}_{q}\right)$ such that $A \mathcal{C}_{f}^{\sigma} B=\mathcal{C}_{h}$ if and only if $U_{f}$ and $U_{h}$ are $\Gamma \mathrm{L}\left(2, q^{n}\right)$-equivalent.

Theorem 6.1. The $\mathbb{F}_{q}$-linear $M R D$-code $\mathcal{C}_{g}$ arising from the maximum scattered $\mathbb{F}_{q}$-subspace $U_{g}, g$ as in Theorem 5.1, with parameters $(6,6, q ; 5)$ and with middle nucleus isomorphic to $\mathbb{F}_{q^{6}}$ is not equivalent to any previously known MRD-code.

Proof. From [5, Section 6], the previously known $\mathbb{F}_{q}$-linear MRD-codes with parameters $(6,6, q ; 5)$ and with middle nucleus isomorphic to $\mathbb{F}_{q^{6}}$, up to equivalence, arise from one of the following maximum scattered subspaces of $\mathbb{F}_{q^{6}} \times \mathbb{F}_{q^{6}}: U_{s}^{1,6}, U_{s, \delta}^{2,6}, U_{s, \delta}^{3,6}$. From Proposition 5.3 the result follows.

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[^1]:    ${ }^{1}$ This condition implies $q \neq 2$.
    ${ }^{2}$ Also here $q>2$, otherwise $L_{s, \delta}^{3, n}$ is not scattered.

[^2]:    ${ }^{3}$ In the paper 21 the anti-isomorphism $f \mapsto M_{f}^{T}$ is considered.

