New maximum scattered linear sets of the projective line

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Abstract

In [2] and [19] are presented the first two families of maximum scattered \mathbb{F}_q -linear sets of the projective line $\mathrm{PG}(1, q^n)$. More recently in [23] and in [5], new examples of maximum scattered \mathbb{F}_q -subspaces of $V(2, q^n)$ have been constructed, but the equivalence problem of the corresponding linear sets is left open.

Here we show that the \mathbb{F}_q -linear sets presented in [23] and in [5], for n = 6, 8, are new. Also, for q odd, $q \equiv \pm 1, 0 \pmod{5}$, we present new examples of maximum scattered \mathbb{F}_q -linear sets in PG(1, q^6), arising from trinomial polynomials, which define new \mathbb{F}_q -linear MRD-codes of $\mathbb{F}_q^{6\times 6}$ with dimension 12, minimum distance 5 and middle nucleus (or left idealiser) isomorphic to \mathbb{F}_{q^6} .

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1 Introduction

Linear sets are natural generalisations of subgeometries. Let $\Lambda = PG(W, \mathbb{F}_{q^n})$ = $PG(r-1, q^n)$, where W is a vector space of dimension r over \mathbb{F}_{q^n} . A point set L of Λ is said to be an \mathbb{F}_q -linear set of Λ of rank k if it is defined by the non-zero vectors of a k-dimensional \mathbb{F}_q -vector subspace U of W, i.e.

$$L = L_U = \{ \langle \mathbf{u} \rangle_{\mathbb{F}_{q^n}} : \mathbf{u} \in U \setminus \{\mathbf{0}\} \}.$$

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The maximum field of linearity of an \mathbb{F}_q -linear set L_U is \mathbb{F}_{q^t} if $t \mid n$ is the largest integer such that L_U is an \mathbb{F}_{q^t} -linear set. Two linear sets L_U and L_W of $\mathrm{PG}(r-1,q^n)$ are said to be PFL -equivalent (or simply equivalent) if there is an element ϕ in $\mathrm{PFL}(r,q^n)$ such that $L_U^{\phi} = L_W$. It may happen that two \mathbb{F}_q -linear sets L_U and L_W of $\mathrm{PG}(r-1,q^n)$ are equivalent even if the two \mathbb{F}_q -vector subspaces U and W are not in the same orbit of $\mathrm{FL}(r,q^n)$ (see [7] and [3] for further details). In the recent years, starting from the paper [18] by Lunardon, linear sets have been used to construct or characterise various objects in finite geometry, such as blocking sets and multiple blocking sets in finite projective spaces, two-intersection sets in finite projective spaces, translation spreads of the Cayley Generalized Hexagon, translation ovoids of polar spaces, semifield flocks and finite semifields. For a survey on linear sets we refer the reader to [22], see also [13]. It is clear that in the applications it is crucial to have methods to decide whether two linear sets are equivalent or not.

In this paper we focus on maximum scattered \mathbb{F}_q -linear sets of $\mathrm{PG}(1,q^n)$ with maximum field of linearity \mathbb{F}_q , that is, \mathbb{F}_q -linear sets of rank n of $\mathrm{PG}(1,q^n)$ of size $(q^n-1)/(q-1)$. If L_U is a maximum scattered \mathbb{F}_q -linear set, then U is a maximum scattered \mathbb{F}_q -subspace.

If $\langle (0,1) \rangle_{\mathbb{F}_{q^n}}$ is not contained in the linear set L_U of rank n of $\mathrm{PG}(1,q^n)$ (which we can always assume after a suitable projectivity), then $U = U_f :=$ $\{(x, f(x)): x \in \mathbb{F}_{q^n}\}$ for some q-polynomial $f(x) = \sum_{i=0}^{n-1} a_i x^{q^i} \in \mathbb{F}_{q^n}[x]$. In this case we will denote the associated linear set by L_f . The known non-equivalent (under $\mathrm{\GammaL}(2,q^n)$) maximum scattered \mathbb{F}_q -subspaces are

- 1. $U_s^{1,n} := \{(x, x^{q^s}) \colon x \in \mathbb{F}_{q^n}\}, \ 1 \le s \le n-1, \ \gcd(s, n) = 1 \ ([2, 8]), \ ([2, 8])$
- 2. $U_{s,\delta}^{2,n} := \{(x, \delta x^{q^s} + x^{q^{n-s}}) \colon x \in \mathbb{F}_{q^n}\}, n \ge 4, N_{q^n/q}(\delta) \notin \{0,1\}^{-1},$ gcd(s,n) = 1 ([19] for s = 1, [23, 20] for $s \ne 1$),
- 3. $U_{s,\delta}^{3,n} := \{(x, \delta x^{q^s} + x^{q^{s+n/2}}) : x \in \mathbb{F}_{q^n}\}, n \in \{6, 8\}, \text{gcd}(s, n/2) = 1,$ $N_{q^n/q^{n/2}}(\delta) \notin \{0, 1\}, \text{ for the precise conditions on } \delta \text{ and } q \text{ see } [5, \text{ Theorems 7.1 and 7.2}]^2.$

The stabilisers of the \mathbb{F}_q -subspaces above in the group $\operatorname{GL}(2, q^n)$ were determined in [5, Sections 5 and 6]. They have the following orders:

1. for $U_s^{1,n}$ we have a group of order $q^n - 1$,

¹This condition implies $q \neq 2$.

²Also here q > 2, otherwise $L^{3,n}_{s,\delta}$ is not scattered.

- 2. for $U_{s,\delta}^{2,n}$ we have a group of order $q^2 1$,
- 3. for $U_{s,\delta}^{3,n}$ we have a group of order $q^{n/2} 1$.

It is known, that for n = 3 the maximum scattered \mathbb{F}_q -spaces of $V(2, q^3)$ are $\Gamma L(2, q^3)$ -equivalent to $U_1^{1,3}$ (cf. [15]), and for n = 4 they are $GL(2, q^4)$ equivalent either to $U_1^{1,4}$ or to $U_{1,\delta}^{2,4}$ (cf. [9]).

To make notation easier, by $L_s^{i,n}$ and $L_{s,\delta}^{i,n}$ we will denote the \mathbb{F}_q -linear set defined by $U_s^{i,n}$ and $U_{s,\delta}^{i,n}$, respectively. The \mathbb{F}_q -linear sets equivalent to $L_s^{1,n}$ are called of pseudoregulus type. It is easy to see that $L_1^{1,n} = L_s^{1,n}$ for any s with gcd(s,n) = 1 and that $U_{s,\delta}^{2,n}$ is $GL(2,q^n)$ -equivalent to $U_{n-s,\delta^{-1}}^{2,n}$.

In [19, Theorem 3] Lunardon and Polverino proved that $L_{1,\delta}^{2,n}$ and $L_1^{1,n}$ are not $\mathrm{P\Gamma L}(2,q^n)$ -equivalent when q > 3, $n \ge 4$. For n = 5, in [4] it is proved that $L_{2,\delta}^{2,5}$ is $\mathrm{P\Gamma L}(2,q^5)$ -equivalent neither to $L_{1,\delta'}^{2,5}$ nor to $L_1^{1,5}$.

In the first part of this paper we prove that for n = 6, 8 the linear sets $L_1^{1,n}$, $L_{s,\delta}^{2,n}$ and $L_{s',\delta'}^{3,n}$ are pairwise non-equivalent for any choice of s, s', δ, δ' .

In the second part of this paper we prove that the \mathbb{F}_q -linear set defined by

$$U_b^4 := \{ (x, x^q + x^{q^3} + bx^{q^5}) \colon x \in \mathbb{F}_{q^6} \}$$

with $b^2 + b = 1$, $q \equiv 0, \pm 1 \pmod{5}$ is maximum scattered in $\mathrm{PG}(1, q^6)$ and it is not $\mathrm{P\Gamma L}(2, q^6)$ -equivalent to any previously known example. Connections between scattered \mathbb{F}_q -subspaces and MRD-codes have been investigated in [23, 6, 17]. Using the relation found in [23] we also present new examples of such codes.

2 Classes of \mathbb{F}_q -linear sets of rank n of $PG(1, q^n)$ and preliminary results

For $\alpha \in \mathbb{F}_{q^n}$ and a divisor h of n we will denote by $N_{q^n/q^h}(\alpha)$ the norm of α over the subfield \mathbb{F}_{q^h} , that is, $N_{q^n/q^h}(\alpha) = \alpha^{1+q^h+\dots+q^{n-h}}$.

By [1, 3] for $f(x) = \sum_{i=0}^{n-1} a_i x^{q^i}$ and $\hat{f}(x) = \sum_{i=0}^{n-1} a_i^{q^{n-i}} x^{q^{n-i}}$, the \mathbb{F}_q subspaces $U_f = \{(x, f(x)) : x \in \mathbb{F}_{q^n}\}$ and $U_{\hat{f}} = \{(x, \hat{f}(x)) : x \in \mathbb{F}_{q^n}\}$ define the same linear set of PG(1, q^n). On the other hand U_f and $U_{\hat{f}}$ are not necessarily $\Gamma L(2, q^n)$ -equivalent (see [3, Section 3.2]) and this motivates the following definitions. **Definition 2.1.** ([3]) Let L_U be an \mathbb{F}_q -linear set of $PG(W, \mathbb{F}_{q^n}) = PG(1, q^n)$ of rank n with maximum field of linearity \mathbb{F}_q .

We say that L_U is of Γ L-class *s* if *s* is the greatest integer such that there exist \mathbb{F}_q -subspaces U_1, \ldots, U_s of *W* with $L_{U_i} = L_U$ for $i \in \{1, \ldots, s\}$ and there is no $f \in \Gamma L(2, q^n)$ such that $U_i = U_j^f$ for each $i \neq j$, $i, j \in \{1, 2, \ldots, s\}$. If L_U has Γ L-class one, then L_U is said to be simple.

We say that L_U is of $\mathcal{Z}(\Gamma L)$ -class r if r is the greatest integer such that there exist \mathbb{F}_q -subspaces U_1, U_2, \ldots, U_r of W with $L_{U_i} = L_U$ for $i \in \{1, 2, \ldots, r\}$ and $U_i \neq \lambda U_j$ for each $\lambda \in \mathbb{F}_{q^n}^*$ and for each $i \neq j, i, j \in \{1, 2, \ldots, r\}$.

Result 2.2. ([3, Prop. 2.6]) Let L_U be an \mathbb{F}_q -linear set of $\mathrm{PG}(1, q^n)$ of rank n with maximum field of linearity \mathbb{F}_q and let φ be a collineation of $\mathrm{PG}(1, q^n)$. Then L_U and L_U^{φ} have the same $\mathcal{Z}(\Gamma L)$ -class and ΓL -class. Also, the ΓL class of an \mathbb{F}_q -linear set cannot be greater than its $\mathcal{Z}(\Gamma L)$ -class.

For a q-polynomial $f(x) = \sum_{i=0}^{n-1} a_i x^{q^i}$ over \mathbb{F}_{q^n} let D_f denote the associated *Dickson matrix* (or q-circulant matrix)

$$D_f := \begin{pmatrix} a_0 & a_1 & \dots & a_{n-1} \\ a_{n-1}^q & a_0^q & \dots & a_{n-2}^q \\ \vdots & \vdots & \vdots & \vdots \\ a_1^{q^{n-1}} & a_2^{q^{n-1}} & \dots & a_0^{q^{n-1}} \end{pmatrix}$$

The rank of the matrix D_f equals the rank of the \mathbb{F}_q -linear map f, see for example [24].

We will use the following results.

Proposition 2.3. Let f and g be two q-polynomials over \mathbb{F}_{q^n} . Then $L_f \subseteq L_g$ if and only if

$$x^{q^n} - x \mid \det D_{F(Y)}(x) \in \mathbb{F}_{q^n}[x],$$

where F(Y) = f(x)Y - g(Y)x. In particular, if deg det $D_{F(Y)}(x) < q^n$, then $L_f \subseteq L_g$ if and only if det $D_{F(Y)}(x)$ is the zero polynomial.

Proof. $L_f \subseteq L_g$ if and only if

$$\left\{\frac{f(x)}{x} \colon x \in \mathbb{F}_{q^n}^*\right\} \subseteq \left\{\frac{g(x)}{x} \colon x \in \mathbb{F}_{q^n}^*\right\},\$$

which means that $\frac{g(y)}{y} = \frac{f(x)}{x}$ can be solved in y if we fix $x \in \mathbb{F}_{q^n}^*$. Fix $x \in \mathbb{F}_{q^n}^*$, then the q-polynomial F(Y) = f(x)Y - g(Y)x has rank less than

n since it has a non-zero solution. Since the Dickson matrix $D_{F(Y)}(x)$ of F(Y) has the same rank as F(Y), it follows that $\det D_{F(Y)}(x) = 0$ for each *x*. It follows that $x^{q^n} - x \mid \det D_{F(Y)}(x)$.

Lemma 2.4. [3, Lemma 3.6] Let $f(x) = \sum_{i=0}^{n-1} a_i x^{q^i}$ and $g(x) = \sum_{i=0}^{n-1} b_i x^{q^i}$ be two q-polynomials over \mathbb{F}_{q^n} such that $L_f = L_g$. Then

$$a_0 = b_0, \tag{1}$$

for $k = 1, 2, \ldots, n-1$ it holds that

$$a_k a_{n-k}^{q^k} = b_k b_{n-k}^{q^k}, (2)$$

for $k = 2, 3, \ldots, n-1$ it holds that

$$a_1 a_{k-1}^q a_{n-k}^{q^k} + a_k a_{n-1}^q a_{n-k+1}^{q^k} = b_1 b_{k-1}^q b_{n-k}^{q^k} + b_k b_{n-1}^q b_{n-k+1}^{q^k}.$$
 (3)

3 The $L_{s,\delta}^{2,n}$ -linear sets in $PG(1,q^n)$, n = 6, 8

In this section we determine the $\mathcal{Z}(\Gamma L)$ -class of the maximum scattered \mathbb{F}_{q} linear sets of PG(1, q^n), n = 6, 8, introduced by Lunardon and Polverino, and generalised by Sheekey. Recall that $U_{s,\delta}^{2,n}$ is GL(2, q^n)-equivalent to $U_{n-s,\delta^{-1}}^{2,n}$, thus it is enough to study the linear sets $L_{s,\delta}^{2,n}$ with s < n/2 and $\operatorname{gcd}(s,n) = 1$.

Proposition 3.1. If n = 6, then the $\mathcal{Z}(\Gamma L)$ -class of $L_{1,\delta}^{2,6}$ is two.

Proof. Since $g(x) = \delta x^q + x^{q^5}$ and $\hat{g}(x) = \delta^{q^5} x^{q^5} + x^q$ define the same linear set, we know $L_{1,\delta}^{2,6} = L_{5,\delta q^5}^{2,6}$. Suppose $L_f = L_{1,\delta}^{2,6}$ for some $f(x) = \sum_{i=0}^5 a_i x^{q^i} \in \mathbb{F}_{q^6}[x]$. We show that there exists $\lambda \in \mathbb{F}_{q^6}^*$ such that either $\lambda U_f = U_{1,\delta}^{2,6}$ or $\lambda U_f = U_{5,\delta q^5}^{2,6}$.

By (1) we obtain $a_0 = 0$, by (2) with k = 1, 3 we have

$$a_1 a_5^q = \delta \tag{4}$$

and $a_3 = 0$, respectively. Also, with k = 2 in (2) and (3), taking (4) into account, we get $a_2 = a_4 = 0$.

By Proposition (2.3) we get that the Dickson matrix associated to the q-polynomial

$$F(Y) = \left(\frac{\delta}{a_5^q}x^q + a_5x^{q^5}\right)Y - x\left(\delta Y^q + Y^{q^5}\right)$$

has zero determinant for each $x \in \mathbb{F}_{q^6}$. Direct computation shows that this determinant is

$$N_{q^6/q}(x/a_5) \left(N_{q^6/q}(a_5) - 1 \right) \left(N_{q^6/q}(a_5) - N_{q^6/q}(\delta) \right),$$

which has degree less than q^6 , thus it is the zero polynomial. We have two possibilities:

- 1. If $N_{q^6/q}(a_5) = 1$, then putting $a_5 = \lambda^{q^5-1}$ we obtain $\lambda U_f = U_{1,\delta}^{2,6}$.
- 2. If $N_{q^6/q}(a_5/\delta) = 1$, then choosing $a_5 = \delta^{q^5} \lambda^{q^5-1}$ we get $\lambda U_f = U_{5,\delta^{q^5}}^{2,6}$.

Because of the choice of δ , that is $N_{q^6/q}(\delta) \neq 1$, it follows that there is no $\mu \in \mathbb{F}_{q^6}$ such that $\mu U_{1,\delta}^{2,6} = U_{5,\delta^{q^5}}^{2,6}$ and this proves that the $\mathcal{Z}(\Gamma L)$ -class of $L_{1,\delta}^{2,6}$ is exactly two.

Proposition 3.2. If n = 8, then the $\mathcal{Z}(\Gamma L)$ -class of $L_{1,\delta}^{2,8}$ is two.

Proof. Since $g(x) = \delta x^q + x^{q^7}$ and $\hat{g}(x) = \delta^{q^7} x^{q^7} + x^q$ define the same linear set, we have $L_{1,\delta}^{2,8} = L_{7,\delta^{q^7}}^{2,8}$. Suppose $L_f = L_{1,\delta}^{2,8}$ for some $f(x) = \sum_{i=0}^7 a_i x^{q^i} \in \mathbb{F}_{q^8}[x]$. We show that there exists $\lambda \in \mathbb{F}_{q^8}^*$ such that either $\lambda U_f = U_{1,\delta}^{2,8}$ or $\lambda U_f = U_{7,\delta^{q^7}}^{2,8}$.

By (1) we obtain $a_0 = 0$, by (2) with k = 1 we have

$$a_1 a_7^q = \delta \tag{5}$$

and with k = 4 we get $a_4 = 0$. Putting k = 2 in (2) and (3), taking (5) into account, we get $a_2 = a_6 = 0$. By (2) with k = 3 we have $a_3a_5 = 0$.

If $a_3 = 0$, then $f(x) = a_1 x^q + a_5 x^{q^5} + a_7 x^{q^7}$. Using Proposition 2.3, we get that the determinant of the Dickson matrix associated to the q-polynomial

$$F(Y) = (a_1 x^q + a_5 x^{q^5} + a_7 x^{q^7}) Y - x(a_1 a_7^q Y^q + Y^{q^7})$$

is divisible by $x^{q^8} - x$. The coefficient of $x^{2(1+q+q^2+q^3)}$ after reducing the determinant modulo $x^{q^8} - x$ is $a_1^{1+q+q^2+q^7}a_5^{q^3+q^4+q^5+q^6}$, which is zero only when $a_5 = 0$ by (5).

On the other hand, if $a_5 = 0$, then $L_f = L_{\hat{f}}$ gives $a_3 = 0$.

Then $f(x) = \frac{\delta}{a_7^q} x^q + a_7 x^{q^7}$. By Proposition 2.3, arguing as in the previous coof

proof,

$$N_{q^{8}/q}(x/a_{7}) \left(N_{q^{8}/q}(a_{7}) - 1 \right) \left(N_{q^{8}/q}(a_{7}) - N_{q^{8}/q}(\delta) \right)$$

is the zero polynomial. We have two possibilities:

- 1. If $N_{q^8/q}(a_7) = 1$, then putting $a_7 = \lambda^{q^7-1}$, we obtain $\lambda U_f = U_{1,\delta}^{2,8}$.
- 2. If $N_{q^8/q}(a_7/\delta) = 1$, then choosing $a_7 = \delta^{q^7} \lambda^{q^7-1}$ we have $\lambda U_f = U_{7,\delta^{q^7}}^{2,8}$.

Because of the choice of δ , that is $N_{q^8/q}(\delta) \neq 1$, it follows that there is no $\mu \in \mathbb{F}_{q^8}$ such that $\mu U_{1,\delta}^{2,8} = U_{7,\delta^{q^7}}^{2,8}$ and this proves that the $\mathcal{Z}(\Gamma L)$ -class of $L_{1,\delta}^{2,8}$ is exactly two.

Proposition 3.3. If n = 8, then the $\mathcal{Z}(\Gamma L)$ -class of $L^{2,8}_{3,\delta}$ is two.

Proof. Since $g(x) = \delta x^{q^3} + x^{q^5}$ and $\hat{g}(x) = \delta^{q^5} x^{q^5} + x^{q^3}$ define the same linear set, we know $L_{3,\delta}^{2,8} = L_{5,\delta^{q^5}}^{2,8}$. Suppose $L_f = L_{3,\delta}^{2,8}$ for some $f(x) = \sum_{i=0}^{7} a_i x^{q^i} \in \mathbb{F}_{q^8}[x]$. We show that there exists $\lambda \in \mathbb{F}_{q^8}^*$ such that either $\lambda U_f = U_{3,\delta}^{2,8}$ or $\lambda U_f = U_{5,\delta^{q^5}}^{2,8}$.

By (1) we obtain $a_0 = 0$, by (2) with k = 3 we have

$$a_3 a_5^{q^3} = \delta$$

and with k = 4 we get $a_4 = 0$. Putting k = 1 and k = 2 in (2) we get

$$a_1 a_7 = 0 \text{ and } a_2 a_6 = 0,$$
 (6)

respectively. With k = 2 and k = 3 in (3) we obtain

$$a_1^{q+1}a_6^{q^2} + a_2a_7^{q+q^2} = 0. (7)$$

and

$$a_1 a_2^q a_5^{q^3} + a_3 a_7^q a_6^{q^3} = 0. ag{8}$$

By (7) and (8), taking (6) into account, at most one of $\{a_1, a_2, a_6, a_7\}$ is non-zero.

Hence $f(x) = a_3 x^{q^3} + a_5 x^{q^5} + a_i x^{q^i}$ with $i \in \{1, 2, 6, 7\}$. For each $i \in \{1, 2, 6, 7\}$, by Proposition 2.3, the determinant of the Dickson matrix $D_{F(Y)}(x)$ with $F(Y) = f(x)Y - x(a_3a_5^{q^3}Y^{q^3} + Y^{q^5})$ is zero modulo $x^{q^8} - x$. Then the following hold:

- for i = 1 the coefficient of $x^{3+3q+q^2+q^3}$ in the reduced form of det $D_{F(Y)}(x)$ is $a_1^{1+q+q^2+q^7}a_3^{q^5+q^6}a_5^{q^3+q^4}$,
- for i = 2 the coefficient of $x^{3+2q+q^2+q^3+q^4}$ in the reduced form of det $D_{F(Y)}(x)$ is $a_2^{1+q+q^2+q^6+q^7}a_3^{q^5}a_5^{q^3+q^4}$.

Thus $a_i = 0$ for $i \in \{1, 2\}$ and since $L_f = L_{\hat{f}}$, the same holds for $i \in \{6, 7\}$. Then from (7) we get $f(x) = \frac{\delta}{a_5^{q^3}} x^{q^3} + a_5 x^{q^5}$. By Proposition 2.3, arguing as in the previous proof,

$$N_{q^8/q}(x/a_5) (N_{q^8/q}(a_5) - 1) (N_{q^8/q}(a_5) - N_{q^8/q}(\delta))$$

is the zero polynomial. Then the following holds:

- 1. If $N_{q^8/q}(a_5) = 1$, then putting $a_5 = \lambda^{q^5-1}$ gives $\lambda U_f = U_{3,\delta}^{2,8}$.
- 2. If $N_{q^8/q}(a_5/\delta) = 1$, then set $a_5 = \delta^{q^5} \lambda^{q^5-1}$, and hence $\lambda U_f = U_{5,\delta^{q^5}}^{2,8}$.

As in the previous proof, it can be easily seen that the $\mathcal{Z}(\Gamma L)$ -class is exactly two.

Theorem 3.4. The linear set $L_{s,\delta}^{2,n}$ is not of pseudoregulus type for each n, s, δ, q . Also, the linear sets $L_{1,\delta}^{2,8}$ and $L_{3,\rho}^{2,8}$ are not $P\Gamma L(2, q^8)$ -equivalent.

Proof. Suppose that $L_{s,\delta}^{2,n}$ is of pseudoregulus type. Then by [14] there exists an element f of $\operatorname{GL}(2,q^n)$ such that $(U_{s,\delta}^{2,n})^f = U_r^{1,n}$ with $\operatorname{gcd}(r,n) = 1$. Since the \mathbb{F}_{q^n} -linear automorphism groups of $U_{s,\delta}^{2,n}$ and $(U_{s,\delta}^{2,n})^f$ are conjugated and since the groups of $U_r^{1,n}$ and $U_{s,\delta}^{2,n}$ have orders $q^n - 1$ and $q^2 - 1$, respectively (cf. Introduction), we get a contradiction.

For the second part, suppose to the contrary that $L_{1,\delta}^{2,8}$ and $L_{3,\rho}^{2,8}$ are $\mathrm{P}\Gamma\mathrm{L}(2,q^8)$ -equivalent. Then by Proposition 3.3 there exists a field automorphism σ , an invertible matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\alpha, \beta \in \mathbb{F}_{q^8}^*$ such that for each $x \in \mathbb{F}_{q^8}$ there exists $z \in \mathbb{F}_{q^8}$ satisfying

$$\left(\begin{array}{cc}a&b\\c&d\end{array}\right)\left(\begin{array}{c}x^{\sigma}\\\delta^{\sigma}x^{\sigma q}+x^{\sigma q^{7}}\end{array}\right)=\left(\begin{array}{c}z\\\alpha z^{q^{3}}+\beta z^{q^{5}}\end{array}\right).$$

Equivalently, for each $x \in \mathbb{F}_{q^8}$

$$cx^{\sigma} + d\delta^{\sigma} x^{\sigma q} + dx^{\sigma q^{7}} = \alpha (a^{q^{3}} x^{\sigma q^{3}} + \delta^{\sigma q^{3}} b^{q^{3}} x^{\sigma q^{4}} + b^{q^{3}} x^{\sigma q^{2}}) +$$

$$\beta(a^{q^5}x^{\sigma q^5} + \delta^{\sigma q^5}b^{q^5}x^{\sigma q^6} + b^{q^5}x^{\sigma q^4}).$$

This is a polynomial identity in x^{σ} . Comparing the coefficients of x^{q^2} and x^{q^3} we get that a = b = 0, which is a contradiction.

4 The $L_{s,\delta}^{3,n}$ -linear sets in $PG(1,q^n)$, n = 6, 8

In this section we determine the $\mathcal{Z}(\Gamma L)$ -class of the maximum scattered \mathbb{F}_{q} linear sets of PG(1, q^n), n = 6, 8, introduced in [5]. According to [5, Section 5, pg. 7], $U_{s,\delta}^{3,n}$ is GL(2, q^n)-equivalent to $U_{n-s,\delta^{q^{n-s}}}^{3,n}$ and to $U_{s+n/2,\delta^{-1}}^{3,n}$, thus it is enough to study the linear sets $L_{s,\delta}^{3,n}$ with s < n/4, $\gcd(s, n/2) = 1$ and hence only with s = 1 for n = 6, 8.

Proposition 4.1. The $\mathcal{Z}(\Gamma L)$ -class of L_g , with $g(x) = \delta x^q + x^{q^4}$, $\delta \neq 0$, is two and hence the $\mathcal{Z}(\Gamma L)$ -class of $L_{1,\delta}^{3,6}$ is two as well. Moreover, $L_{1,\delta}^{3,6}$ is a simple linear set.

Proof. Since g(x) and $\hat{g}(x) = \delta^{q^5} x^{q^5} + x^{q^2}$ define the same linear set, we know $L_g = L_{\hat{g}}$. Suppose $L_f = L_g$ for some $f(x) = \sum_{i=0}^5 a_i x^{q^i} \in \mathbb{F}_{q^6}[x]$. We show that there exists $\lambda \in \mathbb{F}_{q^6}^*$ such that either $\lambda U_f = U_g$ or $\lambda U_f = U_{\hat{g}}$.

By (1), we obtain $a_0 = 0$ and by (2) with k = 2 we get $a_3 = 0$. Also, by (2) with k = 1 and k = 2, we have

$$a_1 a_5 = 0 \tag{9}$$

and

$$a_2 a_4 = 0, (10)$$

respectively. By (3) with k = 2 we get

$$a_1^{q+1}a_4^{q^2} + a_2a_5^{q+q^2} = \delta^{q+1}.$$
 (11)

From (9), (10) and (11) it follows that either

$$f(x) = \frac{\delta^{q+1}}{a_5^{q+q^2}} x^{q^2} + a_5 x^{q^5}$$

or

$$f(x) = a_1 x^q + \left(\frac{\delta}{a_1}\right)^{q^5 + q^4} x^{q^4}$$

In both cases, the determinant of the Dickson matrix associated with $F(Y) = f(x)Y - x(\delta Y^q + Y^{q^4})$ is the zero-polynomial after reducing modulo $x^{q^6} - x$

and hence in the first case we obtain $N_{q^6/q}(a_5/\delta) = 1$, in the second case $N_{q^6/q}(a_1/\delta) = 1$. In the former case $a_5 = \delta^{q^5} \lambda^{q^5-1}$ and hence $\lambda U_f = U_{\hat{g}}$. In the latter case $a_1 = \delta \lambda^{q-1}$ implying $\lambda U_f = U_g$.

This means that the $\mathcal{Z}(\Gamma L)$ -class of U_g is at most two. Straightforward computation shows that it is exactly two. In case of $L^{3,6}_{1,\delta}$ (and hence with $N_{q^6/q^3}(\delta) \neq 1$) it follows from [5, Section 5] that $U^{3,6}_{1,\delta}$ and $U^{3,6}_{5,\delta^{q^5}}$ are $\Gamma L(2,q^6)$ -equivalent and hence $L^{3,6}_{1,\delta}$ is simple.

Proposition 4.2. The $\mathcal{Z}(\Gamma L)$ -class of L_g , with $g(x) = \delta x^q + x^{q^5}$, $\delta \neq 0$, is two and hence the $\mathcal{Z}(\Gamma L)$ -class of $L_{1,\delta}^{3,8}$ is two as well. Moreover, $L_{1,\delta}^{3,8}$ is a simple linear set.

Proof. Since $g(x) = \delta x^q + x^{q^5}$ and $\hat{g}(x) = \delta^{q^7} x^{q^7} + x^{q^3}$ define the same linear set, we have $L_g = L_{\hat{g}}$. Suppose $L_f = L_g$ for some $f(x) = \sum_{i=0}^7 a_i x^{q^i} \in \mathbb{F}_{q^8}[x]$. We show that there exists $\lambda \in \mathbb{F}_{q^8}^*$ such that either $\lambda U_f = U_g$ or $\lambda U_f = U_{\hat{g}}$.

By (1), we obtain $a_0 = 0$ and by (2) with k = 4 we get $a_4 = 0$. Also, by (2) with k = 1, k = 2 and k = 3 we get

$$a_1 a_7 = a_2 a_6 = a_3 a_5 = 0. \tag{12}$$

By (3), with k = 2 we obtain

$$a_1^{q+1}a_6^{q^2} + a_2a_7^{q+q^2} = 0, (13)$$

and with k = 3 we get

$$a_1 a_2^q a_5^{q^3} + a_3 a_7^q a_6^{q^3} = 0. (14)$$

By (12), first suppose $a_1 = a_2 = a_3 = 0$. Proposition 2.3 yields that the determinant of the Dickson matrix associated with

$$F(Y) = (a_5 x^{q^5} + a_6 x^{q^6} + a_7 x^{q^7}) Y - x(\delta Y^q + Y^{q^5}),$$

has to be the zero polynomial after reducing modulo $x^8 - x$. The coefficient of $x^{1+2q+2q^2+2q^3+q^4}$ is $-a_5^{q^4+q^5+q^6+q^7}\delta^{1+q+q^2}$, hence $a_5 = 0$. The coefficient of $x^{1+q+2q^2+2q^3+q^4+q^5}$ is $-a_6^{q^4+q^5+q^6+q^7}\delta^{1+q+q^2}$, hence $a_6 = 0$. The coefficient of $x^{1+q+q^2+2q^3+q^4+q^5+q^6}$ is $-a_7^{q^4+q^5+q^6+q^7}\delta^{1+q+q^2}$, hence $a_7 = 0$, a contradiction. Now suppose $a_1 = a_2 = a_5 = a_7 = 0$. Again, Proposition 2.3 yields that the determinant of the Dickson matrix associated with

$$F(Y) = (a_3 x^{q^3} + a_6 x^{q^6}) Y - x(\delta Y^q + Y^{q^5}),$$

has to be the zero polynomial after reducing modulo $x^8 - x$. The coefficient of $x^{2+2q+3q^2+q^3}$ is $-a_3^{q^5+q^6+q^7}a_6^{q^4}\delta^{1+q+q^2}$, hence $a_3a_6 = 0$. We cannot have $a_3 = 0$ because of the previous paragraph, hence $a_6 = 0$, but then the coefficient of $x^{1+2q+2q^2+2q^3+q^4}$ is $-a_3^{1+q^5+q^6+q^7}\delta^{q+q^2+q^3}$. Then again $a_3 = 0$ follows, a contradiction.

Taking into account $L_f = L_{\hat{f}}$ and (12), (13), (14), two cases remain: $f(x) = a_3 x^{q^3} + a_7 x^{q^7}$ and $f(x) = a_1 x^q + a_5 x^{q^5}$.

In the former case Proposition 2.3 yields that the determinant of the Dickson matrix associated with

$$F(Y) = (a_3 x^{q^3} + a_7 x^{q^7}) Y - x(\delta Y^q + Y^{q^5}),$$

has to the zero polynomial after reducing modulo $x^8 - x$. The coefficient of $x^{2+2q+2q^2+2q^3}$ is $a_3^{q^5+q^6+q^7}a_7^{q^4}(a_3a_7^{q+q^2+q^3}-\delta^{1+q+q^2})$, hence

$$a_3 = \delta^{1+q+q^2} / a_7^{q+q^2+q^3}.$$

Since the coefficient of $x^{2+q+2q^2+2q^3+q^5}$ is

$$(N_{q^8/q}(\delta) - N_{q^8/q}(a_7))\delta^{2+q+q^2+q^6+2q^7}/a_7^{3+2q+2q^2+q^3+q^5+q^6+2q^7},$$

which has to be zero and hence it follows that $N_{q^8/q}(a_7/\delta) = 1$. Then there exists $\lambda \in \mathbb{F}_{q^8}^*$ such that $a_7 = \delta^{q^7} \lambda^{q^7-1}$ and hence $a_3 = \lambda^{q^3-1}$, i.e. $\lambda U_f = U_{\hat{g}}$.

On the other hand, if $f(x) = a_1 x^q + a_5 x^{q^5}$, then the previous paragraph yields that there exists $\lambda \in \mathbb{F}_{q^8}^*$ such that $\lambda U_{\hat{f}} = U_{\hat{g}}$ and hence $\lambda^{-1} U_f = U_g$.

Since there is no $\mu \in \mathbb{F}_{q^8}^*$ such that $U_g = \mu U_{\hat{g}}$, it follows that the $\mathcal{Z}(\Gamma L)$ class of U_g is exactly two. In case of $L_{1,\delta}^{3,8}$ (and hence with $N_{q^8/q^4}(\delta) \neq 1$) it follows from [5, Section 5] that $U_{1,\delta}^{3,8}$ and $U_{7,\delta^{q^7}}^{3,8}$ are $\Gamma L(2,q^8)$ -equivalent and hence $L_{1,\delta}^{3,8}$ is simple.

Theorem 4.3. The linear set $L_{1,\delta}^{3,n}$, n = 6, 8, is not of pseudoregulus type and not $P\Gamma L(2, q^n)$ -equivalent to $L_{s,\rho}^{2,n}$.

Proof. Since the \mathbb{F}_{q^n} -linear automorphism group of $U_{1,\delta}^{3,n}$ has order $q^{n/2} - 1$ (cf. [5, Corollary 5.2]), the same arguments as in the proof of Theorem 3.4 can be applied to show that $L_{1,\delta}^{3,n}$ is not of pseudoregulus type.

Suppose that $L_{1,\delta}^{3,n}$ is equivalent to $L_{s,\rho}^{2,n}$ for some $n \in \{6,8\}$. Then by Propositions 3.1, 3.2, 3.3, there exists $f \in \Gamma L(2,q^n)$ such that either $(U_{1,\delta}^{3,n})^f = U_{s,\rho}^{2,n}$ or $(U_{1,\delta}^{3,n})^f = U_{n-s,\rho q^{n-s}}^{2,n}$. This gives a contradiction, since the sizes of the corresponding automorphism groups are different.

5 New maximum scattered linear sets in $PG(1, q^6)$

In this section we show that L_g with $g(x) = x^q + x^{q^3} + bx^{q^5} \in \mathbb{F}_{q^6}[x]$, q odd, $q \equiv 0, \pm 1 \pmod{5}$, $b^2 + b = 1$ is a maximum scattered \mathbb{F}_q -linear set of $\mathrm{PG}(1, q^6)$ which is not equivalent to any other previously known example. Note that, under these assumptions we have $b \in \mathbb{F}_q$.

The \mathbb{F}_q -subspace $U_g = \{(x, x^q + x^{q^3} + bx^{q^5}) : x \in \mathbb{F}_{q^6}\}$ is scattered if and only if for each $m \in \mathbb{F}_{q^6}$

$$\frac{x^q + x^{q^3} + bx^{q^5}}{x} = -m$$

has at most q solutions. Those m which admit exactly q solutions correspond to points $\langle (1, -m) \rangle_{\mathbb{F}_{q^6}}$ of L_g with weight one. It follows that U_g is scattered if and only if for each $m \in \mathbb{F}_{q^6}$ the kernel of

$$r_{m,b}(x) := mx + x^q + x^{q^3} + bx^{q^5}$$

has dimension less than two, or, equivalently, the Dickson matrix associated with $r_{m,b}(x)$, that is,

$$D_{m,b} = \begin{pmatrix} m & 1 & 0 & 1 & 0 & b \\ b & m^{q} & 1 & 0 & 1 & 0 \\ 0 & b & m^{q^{2}} & 1 & 0 & 1 \\ 1 & 0 & b & m^{q^{3}} & 1 & 0 \\ 0 & 1 & 0 & b & m^{q^{4}} & 1 \\ 1 & 0 & 1 & 0 & b & m^{q^{5}} \end{pmatrix}$$

has rank at least 5 for each $m \in \mathbb{F}_{q^6}$.

Denote by $M_{i,j}$ the determinant of the matrix obtained from $D_{m,b}$ by deleting the *i*-th row and *j*-th column and consider the two minors:

$$M_{6,3} = 2 - 3b + (b-1)(\mathcal{N}_{q^6/q^3}(m) + \mathcal{N}_{q^6/q^3}(m^q)) + \mathcal{N}_{q^6/q^3}(m)^{q+1} + (1-b)(m^{1+q} - m^{q^3+q^4}) + \mathcal{N}_{q^6/q^3}(m^q) + \mathcal{N}_$$

$$M_{6,4} = 2m - 3bm + 2m^{q^2} - 3bm^{q^2} + m^{q^4} - bm^{q^4} + m^{1+q+q^2} + bm^{1+q+q^4} + bm^{q+q^2+q^4} + bm^{q+q^2+q^$$

Theorem 5.1. If $q \equiv 0, \pm 1 \pmod{5}$, q odd and $b^2 + b = 1$ (hence $b \in \mathbb{F}_q$), then U_g is a maximum scattered \mathbb{F}_q -subspace for $g(x) = x^q + x^{q^3} + bx^{q^5}$.

Proof. It is sufficient to show that $M_{6,3}$ and $M_{6,4}$ cannot be both zeros for the same value of $m \in \mathbb{F}_{q^6}$. If m = 0, then $M_{6,3} = 2 - 3b \neq 0$ since b = 2/3does not satisfy our condition. First suppose that $M_{6,3}$ vanishes for some $m \in \mathbb{F}_{q^6}^*$. Then

$$m^{1+q} - m^{q^3+q^4} = \frac{2 - 3b + (b-1)(N_{q^6/q^3}(m) + N_{q^6/q^3}(m^q)) + N_{q^6/q^3}(m)^{q+1}}{b-1}$$

and since the righ-hand side is in \mathbb{F}_{q^3} , the same follows for the left-hand side, and hence $m^{1+q} - m^{q^3+q^4} = m^{q^3+q^4} - m^{1+q}$, from which $m^{1+q} \in \mathbb{F}_{q^3}^*$ follows. So, if $m^{1+q} \notin \mathbb{F}_{q^3}^*$, then $rk(D_{m,b}) \geq 5$. Now, suppose $m^{1+q} \in \mathbb{F}_{q^3}^*$, then $M_{6,3}$ can be written as

$$M_{6,3} = 2 - 3b + (1 - b)(-m^{1+q^3} - m^{q+q^4}) + m^{2(1+q)} = ((1 - b) - m^{1+q^3})^{q+1}$$

Since $M_{6,3} = 0$, we have

$$1 - b = m^{1+q^3} \in \mathbb{F}_q. \tag{15}$$

Then $m^{(q^3+1)(q+1)} = m^{2(q+1)} = (1-b)^2$ and hence either $m^{q+1} = 1-b$, or $m^{q+1} = b-1$. In both cases $m^{q+1} \in \mathbb{F}_q$ follows. The latter case cannot hold. Indeed by (15) we would get $m^{q^3+1} = -m^{q+1}$, so $m^{q^2} = -m$, which holds only if $m \in \mathbb{F}_{q^4} \cap \mathbb{F}_{q^6} = \mathbb{F}_{q^2}$, a contradiction. In the former case, by (15) we obtain $m^{q^3+1} = m^{q+1}$, so $m \in \mathbb{F}_{q^2}$. It follows that, taking $m^{1+q} = 1-b=b^2$ into account, $M_{6,4} = 4m(1-b)$, which cannot be zero.

Similarly to the proof of [5, Proposition 5.2] it is easy to prove the following result.

Proposition 5.2. The linear automorphism group of U_g (defined as in Theorem 5.1) is

$$\mathcal{G} = \left\{ \left(egin{array}{cc} \lambda & 0 \ 0 & \lambda^q \end{array}
ight) : \lambda \in \mathbb{F}_{q^2}^*
ight\}.$$

and

Proposition 5.3. The maximum scattered \mathbb{F}_q -subspace U_g defined in Theorem 5.1 is not $\Gamma L(2, q^6)$ -equivalent to the \mathbb{F}_q -subspaces $U_s^{1,6}$, $U_{t,\rho}^{2,6}$ and $U_{h,\xi}^{3,6}$.

Proof. As in the proof of Theorem 3.4, the size of the linear automorphism group of U_g is different from the size of the group of $U_s^{1,6}$ and of $U_{h,\xi}^{3,6}$ (cf. Introduction), hence it remains to show that U_g is not $\Gamma L(2, q^6)$ -equivalent to $U_{t,\rho}^{2,6}$.

Since any \mathbb{F}_q -subspace of the form $U_{5,\eta}^{2,6}$ is $\operatorname{GL}(2,q^6)$ -equivalent to $U_{1,\rho}^{2,6}$ for some ρ , it is enough to show that U_g and $U_{t,\rho}^{2,6}$ lie on different orbits of $\Gamma L(2,q^6)$. Suppose the contrary, then there exist $\sigma \in \operatorname{Aut}(\mathbb{F}_{q^6})$ and an invertible matrix $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ such that for each $x \in \mathbb{F}_{q^6}$ there exists $z \in \mathbb{F}_{q^6}$ satisfying

$$\left(\begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array}\right) \left(\begin{array}{c} x^{\sigma} \\ \rho^{\sigma} x^{\sigma q} + x^{\sigma q^5} \end{array}\right) = \left(\begin{array}{c} z \\ z^{q} + z^{q^3} + b z^{q^5} \end{array}\right).$$

Equivalently, for each $x \in \mathbb{F}_{q^6}$ we have

$$\gamma x^{\sigma} + \delta(\rho^{\sigma} x^{\sigma q} + x^{\sigma q^5}) = \alpha^q x^{\sigma q} + \beta^q (\rho^{\sigma q} x^{\sigma q^2} + x^{\sigma}) + \alpha^{q^3} x^{\sigma q^3} + \beta^{q^3} (\rho^{\sigma q^3} x^{\sigma q^4} + x^{\sigma q^2}) + b(\alpha^{q^5} x^{\sigma q^5} + \beta^{q^5} (\rho^{\sigma q^5} x^{\sigma} + x^{\sigma q^4}))$$

This is a polynomial identity in x^{σ} . Comparing coefficients we get $\alpha = \delta = 0$ and

$$\begin{cases} \beta^q \rho^{\sigma q} + \beta^{q^3} = 0, \\ \beta^{q^3} \rho^{\sigma q^3} + b\beta^{q^5} = 0. \end{cases}$$

Subtracting the second equation from the q^2 -th power of the first gives $\beta^{q^5}(1-b) = 0$, and hence $\beta = 0$, a contradiction.

Theorem 5.4. The maximum scattered \mathbb{F}_q -linear set L_g of $PG(1, q^6)$, where g is defined in Theorem 5.1, is not $P\Gamma L(2, q^6)$ -equivalent to any any other previously known maximum scattered \mathbb{F}_q -linear set.

Proof. We have to confront L_g with $L_s^{1,6}$, $L_{t,\rho}^{2,6}$ and $L_{h,\xi}^{3,6}$. Suppose that L_g is equivalent to one of these linear sets, then by [14] and by Propositions 3.1 and 4.1, respectively, there exists $\varphi \in \Gamma L(2, q^n)$ such that U_g^{φ} equals one of $U_s^{1,6}$, $U_{t,\rho}^{2,6}$ and $U_{h,\xi}^{3,6}$, a contradiction by Proposition 5.3.

For the sake of completeness we show that the $\mathcal{Z}(\Gamma L)$ -class of L_g , defined as in Theorem 5.1, is one. **Proposition 5.5.** The $\mathcal{Z}(\Gamma L)$ -class of L_g of $PG(1, q^6)$, where $g(x) = x^q + x^{q^3} + bx^{q^5}$, is at most two if $b \neq 0$.

Proof. Since g(x) and $\hat{g}(x) = b^q x^q + x^{q^3} + x^{q^5}$ define the same linear set, we know $L_g = L_{\hat{g}}$. Suppose $L_f = L_g$ for some $f(x) = \sum_{i=0}^5 a_i x^{q^i} \in \mathbb{F}_{q^6}[x]$. We show that there exists $\lambda \in \mathbb{F}_{q^6}^*$ such that either $\lambda U_f = U_g$ or $\lambda U_f = U_{\hat{g}}$.

By (1) we obtain $a_0 = 0$, by (2) with k = 1, 3 we have

$$a_1 a_5^q = b^q$$

and

$$a_3^{q^3+1} = 1. (16)$$

By (3), with k = 2 we have

$$a_1^{q+1}a_4^{q^2} + a_2a_5^{q+q^2} = 0$$

and taking this into account, together with (2) applied for k = 2 we obtain $a_2 = a_4 = 0$.

Using Proposition 2.3, we get that the determinant of the Dickson matrix associated to the q-polynomial

$$F(Y) = (a_1 x^q + a_3 x^{q^3} + a_5 x^{q^5}) Y - x(Y^q + Y^{q^3} + bY^{q^5})$$

is the zero-polynomial modulo $x^{q^6} - x$. Substituting $a_1 = (b/a_5)^q$ it turns out that the coefficient of $x^{1+q+2q^4+2q^5}$ in the reduced form of this determinant is

$$a_3^{q^2}a_5^{-1-q-q^4-q^5}(a_3^{q^3}a_5^{2+q+q^4+2q^5}(a_3^{q+q^4}-1)-(a_5^{1+q+q^5}-a_3^q)b^{1+q+q^5}+a_3^{q^4}a_5^{2+2q+2q^5}-a_5^{1+q+q^5}).$$

Applying $a_3^{q^3+1} = 1$, it follows that

$$(a_3^q - a_5^{1+q+q^5})b^{1+q+q^5} = a_5^{1+q+q^5} - a_3^{q^4}a_5^{2+2q+2q^5} = (a_3^q - a_5^{1+q+q^5})a_3^{q^4}a_5^{1+q+q^5}$$

If $a_3^q = a_5^{1+q+q^5}$, then (16) yields $N_{q^6/q}(a_5) = 1$, and hence there exists $\lambda \in \mathbb{F}_{q^6}^*$ such that $a_5 = \lambda^{q^5-1}$. It is easy to see that in this case $\lambda U_f = U_{\hat{g}}$.

Now suppose $a_3^q \neq a_5^{1+q+q^5}$ and hence $b^{1+q+q^5} = a_3^{q^4} a_5^{1+q+q^5}$. Taking $(q^3 + 1)$ -th powers yields $N(b/a_5) = 1$ and hence there exists $\lambda \in \mathbb{F}_{q^6}^*$ such that $a_5 = b\lambda^{q^5-1}$. It is easy to see that in this case $\lambda U_f = U_g$.

Corollary 5.6. The $\mathcal{Z}(\Gamma L)$ -class of L_g of $PG(1, q^6)$, where $g(x) = x^q + x^{q^3} + bx^{q^5}$, is two if $b^2 + b = 1$. In particular, it is two if g is defined as in Theorem 5.1.

Proof. If $\lambda U_g = U_{\hat{g}}$ for some $\lambda \in \mathbb{F}_{q^6}$, then $\lambda g(x) = \hat{g}(\lambda x)$ for each $x \in \mathbb{F}_{q^6}^*$ and hence comparing coefficients gives $b^q \lambda^{q-1} = 1$ and $\lambda^{q^3-1} = 1$. Then $b = \lambda^{q^2-1}$ and hence $N_{q^6/q^2}(b) = 1$. Also, $b \in \mathbb{F}_{q^2}$ from which $b^3 = 1$ follows, contradicting $b^2 + b = 1$.

6 New MRD-codes

The set of $m \times n$ matrices $\mathbb{F}_q^{m \times n}$ over \mathbb{F}_q is a rank metric \mathbb{F}_q -space with rank metric distance defined by d(A, B) = rk (A - B) for $A, B \in \mathbb{F}_q^{m \times n}$. A subset $\mathcal{C} \subseteq \mathbb{F}_q^{m \times n}$ is called a *rank distance code* (RD-code for short). The minimum distance of \mathcal{C} is

$$d(C) = \min_{A,B\in\mathcal{C},\ A\neq B} \{d(A,B)\}.$$

In [11] the Singleton bound for an $m \times n$ rank metric code C with minimum rank distance d was proved:

$$\#\mathcal{C} < q^{\max\{m,n\}(\min\{m,n\}-d+1)}.$$

If this bound is achieved, then C is an *MRD-code*.

When \mathcal{C} is an \mathbb{F}_q -linear subspace of $\mathbb{F}_q^{m \times n}$, we say that \mathcal{C} is an \mathbb{F}_q -linear code and the dimension $\dim_q(\mathcal{C})$ is defined to be the dimension of \mathcal{C} as a subspace over \mathbb{F}_q . If d is the minimum distance of \mathcal{C} we say that \mathcal{C} has parameters (m, n, q; d).

We will use the following equivalence definition for codes of $\mathbb{F}_q^{m \times m}$. If \mathcal{C} and \mathcal{C}' are two codes then they are *equivalent* if and only if there exist two invertible matrices $A, B \in \mathbb{F}_q^{m \times m}$ and a field automorphism σ such that $\{AC^{\sigma}B \colon C \in \mathcal{C}\} = \mathcal{C}'$, or $\{AC^{T\sigma}B \colon C \in \mathcal{C}\} = \mathcal{C}'$, where T denotes transposition. The code \mathcal{C}^T is also called the *adjoint* of \mathcal{C} .

In [23, Section 5] Sheekey showed that scattered \mathbb{F}_q -linear sets of $\mathrm{PG}(1,q^n)$ of rank n yield \mathbb{F}_q -linear MRD-codes with parameters (n, n, q; n - 1). We briefly recall here the construction from [23]. Let $U_f = \{(x, f(x)) : x \in \mathbb{F}_{q^n}\}$ for some q-polynomial f(x). Then, after fixing an \mathbb{F}_q -bases $\{b_1, \ldots, b_n\}$ for \mathbb{F}_{q^n} we can define an isomorphism between the rings $\mathrm{End}(\mathbb{F}_{q^n}, \mathbb{F}_q)$ and $\mathbb{F}_q^{n \times n}$. More precisely, to $f \in \mathrm{End}(\mathbb{F}_{q^n}, \mathbb{F}_q)$ we associate the matrix M_f of $\mathbb{F}_q^{n \times n}$ with *i*-th column $(a_{1,i},\ldots,a_{n,i})^T$, where $f(b_i) = \sum_{j=1}^n a_{j,i} b_j$.³ In this way the set

$$\mathcal{C}_f := \{ x \mapsto af(x) + bx \colon a, b \in \mathbb{F}_{q^n} \}$$

corresponds to a set of $n \times n$ matrices over \mathbb{F}_q forming an \mathbb{F}_q -linear MRDcode with parameters (n, n, q; n - 1). Also, since \mathcal{C}_f is an \mathbb{F}_{q^n} -subspace of End $(\mathbb{F}_{q^n}, \mathbb{F}_q)$, its *middle nucleus* $\mathcal{N}(\mathcal{C})$ (cf. [21], or [16] where the term *left idealiser* was used) is the set of scalar maps $\mathcal{F}_n := \{x \in \mathbb{F}_{q^n} \mapsto \alpha x \in \mathbb{F}_{q^n}\}$, i.e. $\mathcal{N}(\mathcal{C}_f) \cong \mathbb{F}_{q^n}$. Note that equivalent codes have isomorphic middle nuclei. For further details see [5, Section 6].

Let C_f and C_h be two MRD-codes arising from maximum scattered subspaces U_f and U_h of $\mathbb{F}_{q^n} \times \mathbb{F}_{q^n}$. In [23, Theorem 8] the author showed that there exist invertible matrices A, B and $\sigma \in \operatorname{Aut}(\mathbb{F}_q)$ such that $AC_f^{\sigma}B = C_h$ if and only if U_f and U_h are $\Gamma L(2, q^n)$ -equivalent.

Theorem 6.1. The \mathbb{F}_q -linear MRD-code \mathcal{C}_g arising from the maximum scattered \mathbb{F}_q -subspace U_g , g as in Theorem 5.1, with parameters (6, 6, q; 5) and with middle nucleus isomorphic to \mathbb{F}_{q^6} is not equivalent to any previously known MRD-code.

Proof. From [5, Section 6], the previously known \mathbb{F}_q -linear MRD-codes with parameters (6, 6, q; 5) and with middle nucleus isomorphic to \mathbb{F}_{q^6} , up to equivalence, arise from one of the following maximum scattered subspaces of $\mathbb{F}_{q^6} \times \mathbb{F}_{q^6}$: $U_s^{1,6}$, $U_{s,\delta}^{2,6}$, $U_{s,\delta}^{3,6}$. From Proposition 5.3 the result follows. \Box

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³In the paper [21] the anti-isomorphism $f \mapsto M_f^T$ is considered.

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