Random constructions for translates of non-negative functions

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Abstract

Suppose Λ is a discrete infinite set of nonnegative real numbers. We say that Λ is type 2 if the series $s(x) = \sum_{\lambda \in \Lambda} f(x + \lambda)$ does not satisfy a zero-one law. This means that we can find a non-negative measurable "witness function" $f : \mathbb{R} \to [0, +\infty)$ such that both the convergence set $C(f, \Lambda) = \{x : s(x) < +\infty\}$ and its complement the divergence set $D(f, \Lambda) = \{x : s(x) = +\infty\}$ are of positive Lebesgue measure. If Λ is not type 2 we say that Λ is type 1.

The main result of our paper answers a question raised by Z. Buczolich, J-P. Kahane, and D. Mauldin. By a random construction we show that one can always choose a witness function which is the characteristic function of a measurable set.

We also consider the effect on the type of a set Λ if we randomly delete its elements.

Motivated by results concerning weighted sums $\sum c_n f(nx)$ and the Khinchin conjecture, we also discuss some results about weighted sums $\sum_{n=1}^{\infty} c_n f(x + \lambda_n)$.

1   Introduction

The original research leading to this paper started with questions concerning convergence properties of series of the type $\sum_{n=1}^{\infty} f(nx)$ for nonnegative measurable functions f.

First there were results about the periodic case. This is the case where $f : \mathbb{R} \to \mathbb{R}$ is a periodic measurable function and without limiting generality we can assume that its period $p = 1$.

Results of Mazur and Orlicz in [19] imply that if the periodic function f is the characteristic function of a set of positive (Lebesgue) measure, then for almost every x we have $\sum_n f(nx) = \infty$. Thus, in the periodic case we have a zero-one law: the series either converges or diverges almost everywhere.

In this case it is more interesting to consider the Cesàro 1 means of the partial sums of our series. A famous problem is the *Khinchin conjecture [16] (1923)*:

Assume that $E \subset (0, 1)$ is a measurable set and $f(x) = \chi_E(\{x\})$, where $\{x\}$ denotes the fractional part of x. Is it true that for almost every x

$$\frac{1}{k} \sum_{n=1}^{k} f(nx) \to \mu(E)?$$

(In our paper μ denotes the Lebesgue measure.)

Even at the time of the statement of the Khinchin conjecture it was a known result of H. Weyl [22], that there is a positive answer to the above question if f is Riemann integrable.

However in 1969 Marstrand [18] showed that the Khinchin conjecture is not true. Other counterexamples were given by J. Bourgain [6] by using his entropy method and by A. Quas and M. Wierdl [20]. For further results related to the Khinchin conjecture we also refer to [2] and [3].

In the non-periodic measurable case there was a question of Heinrich von Weizsäker [21] concerning a zero-one law:

Suppose $f : (0, +\infty) \to \mathbb{R}$ is a measurable function. Is it true that $\sum_{n=1}^{\infty} f(nx)$ either converges (Lebesgue) almost everywhere or diverges almost everywhere, i.e. is there a zero-one law for $\sum f(nx)$?

J. A. Haight in [14] also considered a similar question and his results implied that there exists a measurable set $H \subset (0, \infty)$ such that if $f(x) = \chi_H(x)$, the characteristic function of H then $\int_0^{\infty} f(x)dx = \infty$ and $\sum_{n=1}^{\infty} f(nx) < \infty$ everywhere.

In [9] Z. Buczolich and D. Mauldin answered the Haight–Weizsäker problem.

Theorem 1.1. *There exists a measurable function $f : (0, +\infty) \to \{0, 1\}$ and two nonempty intervals I_F, $I_\infty \subset [\frac{1}{2}, 1)$ such that for every $x \in I_\infty$ we have $\sum_{n=1}^{\infty} f(nx) = +\infty$ and for almost every $x \in I_F$ we have $\sum_{n=1}^{\infty} f(nx) < +\infty$. The function f is the characteristic function of an open set E.*

Later with Jean-Pierre Kahane in papers [7] and [8] we considered a more general, additive version of the Haight–Weizsäker problem. After a simple exponential/logarithmic substitution and change of variables one obtains almost everywhere convergence questions for the series $\sum_{\lambda \in \Lambda} f(x+\lambda)$ for non-negative functions defined on $\mathbb{R}$. Taking $\Lambda = \{\log n : n = 1, 2, ...\}$ we obtain an "additive" version of the question answered in Theorem 1.1. Of course, one can consider other infinite, unbounded sets Λ, different from $\{\log n : n = 1, 2, ...\}$.

In fact, in [15] Haight already considered this more general case in the original "multiplicative" setting. He proved, that for any countable set $\Lambda \subset [0, +\infty)$ such that the only accumulation point of Λ is $+\infty$ there exists a measurable set $E \subset (0, +\infty)$ such that choosing $f = \chi_E$ we have $\sum_{\lambda \in \Lambda} f(\lambda x) < \infty$, for any x but $\int_{\mathbb{R}^+} f(x)dx = \infty$.

In [7] we introduced the notion of type 1 and type 2 sets. Given Λ an unbounded, infinite discrete set of nonnegative numbers, and a measurable $f : \mathbb{R} \to [0, +\infty)$, we consider the sum

$$s(x) = s_f(x) = \sum_{\lambda \in \Lambda} f(x + \lambda),$$

and the complementary subsets of $\mathbb{R}$:

$$C = C(f, \Lambda) = \{x : s(x) < \infty\}, \qquad D = D(f, \Lambda) = \{x : s(x) = \infty\}.$$

Definition 1.2. The set Λ is type 1 if, for every f, either $C(f, \Lambda) = \mathbb{R}$ a.e. or $C(f, \Lambda) = \emptyset$ a.e. (or equivalently $D(f, \Lambda) = \emptyset$ a.e. or $D(f, \Lambda) = \mathbb{R}$ a.e.). Otherwise, Λ is type 2.

That is, for type 1 sets we have a "zero-one" law for the almost everywhere convergence properties of the series $\sum_{\lambda \in \Lambda} f(x + \lambda)$, while for type 2 sets the situation is more complicated.

Example 1.3. *Set $\Lambda = \cup_{k \in \mathbb{N}} \Lambda_k$, where $\Lambda_k = 2^{-k} \mathbb{N} \cap [k, k+1)$. In Theorem 1 of [7] it is proved that Λ is type 1. In fact, in a slightly more general version it is shown that if (n_k) is an increasing sequence of positive integers and $\Lambda = \cup_{k \in \mathbb{N}} \Lambda_k$ where $\Lambda_k = 2^{-k} \mathbb{N} \cap [n_k, n_{k+1})$ then Λ is type 1.*

Example 1.4. *Let (n_k) be a given increasing sequence of positive integers. By Theorem 3 of [7] there is an increasing sequence of integers $(m(k))$ such that the set $\Lambda = \cup_{k \in \mathbb{N}} \Lambda_k$ with $\Lambda_k = 2^{-m(k)} \mathbb{N} \cap [n_k, n_{k+1})$ is type 2.*

According to Theorem 6 of [7] type 2 sets form a dense open subset in the box topology of discrete sets while type 1 sets form a closed nowhere dense set. Therefore type 2 is typical in the Baire category sense in this topology. Indeed, this is in line with our experience that it is more difficult to find and verify that a Λ is type 1.

Definition 1.5. The unbounded, infinite discrete set $\Lambda = \{\lambda_1, \lambda_2, ...\}$, $\lambda_1 < \lambda_2 < ...$ is asymptotically dense if $d_n = \lambda_n - \lambda_{n-1} \to 0$, or equivalently:

$$\forall a > 0, \quad \lim_{x \to \infty} \#(\Lambda \cap [x, x+a]) = \infty.$$

If Λ is not asymptotically dense we say that it is asymptotically lacunary.

We recall Theorem 4 from of [7]

Theorem 1.6. *If Λ asymptotically lacunary, then Λ is type 2. Moreover, for some $f \in C_0^+(\mathbb{R})$, there exist intervals I and J, I to the left of J, such that $C(f, \Lambda)$ contains I and $D(f, \Lambda)$ contains J.*

We denote the non-negative continuous functions on $\mathbb{R}$ by $C^+(\mathbb{R})$, and if, in addition these functions tend to zero as $x \to +\infty$ they belong to $C_0^+(\mathbb{R})$.

In [7] we gave some necessary and some sufficient conditions for a set Λ to be type 2. A complete characterization of type 2 sets is still unknown. We recall here from [7] the theorem concerning the Haight–Weizsäker problem. This contains the additive version of the result of Theorem 1.1 with some additional information.

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Theorem 1.7. *The set $\Lambda = \{\log n : n = 1, 2, ...\}$ is type 2. Moreover, for some $f \in C_0^+(\mathbb{R})$, $C(f, \Lambda)$ has full measure on the half-line $(0, \infty)$ and $D(f, \Lambda)$ contains the half-line $(-\infty, 0)$. If for each c, $\int_c^{+\infty} e^y g(y) dy < +\infty$, then $C(g, \Lambda) = \mathbb{R}$ a.e. If $g \in C_0^+(\mathbb{R})$ and $C(g, \Lambda)$ is not of the first (Baire) category, then $C(g, \Lambda) = \mathbb{R}$ a.e. Finally, there is some $g \in C_0^+(\mathbb{R})$ such that $C(g, \Lambda) = \mathbb{R}$ a.e. and $\int_0^{+\infty} e^y g(y) dy = +\infty$.*

From the point of view of our current paper the following question (QUESTION 1 in [7]) is the most relevant:

Question 1.8. Is it true that Λ is type 2 if and only if there is a $\{0, 1\}$ valued measurable function f such that both $C(f, \Lambda)$ and $D(f, \Lambda)$ have positive Lebesgue measure?

In Section 3 we give a positive answer to this question. This result is very useful if one tries to study type 2 sets. In later sections of this paper and in another forthcoming paper [12] one can see applications of this result.

In Section 4 we take some type 1 sets from Example 1.3 and investigate the effect of random deletion of elements with probability q. We see in Theorem 4.3 that in the basic case of Example 1.3, that is, when $n_k = k$ after randomization Λ stays type 1. However in Theorem 4.5 we show that for some other n_ks one can turn a type 1 set into a type 2 set by random deletions.

In [7] two questions were stated. We have already mentioned Question 1, which is the main motivation for our paper. Question 2 was the following: *Given open sets G_1 and G_2 when is it possible to find Λ and f such that $C(f, \Lambda)$ contains G_1 and $D(f, \Lambda)$ contains G_2?* This question was essentially answered in our recent paper [11].

In the periodic case, corresponding to the Khinchin conjecture, several papers considered weighted averages $\sum c_k f(n_k x)$. See for example [1], [4], and [5]. This motivates the following definition:

Definition 1.9. We say that an asymptotically dense set Λ is $\boldsymbol{c}$-type 2 with respect to the positive sequence $\boldsymbol{c} = (c_n)_{n=1}^\infty$, if there exists a nonnegative measurable "witness" function f such that the series $s_{\boldsymbol{c}}(x) = s_{\boldsymbol{c},f}(x) = \sum_{n=1}^\infty c_n f(x + \lambda_n)$ does not converge almost everywhere and does not diverge almost everywhere either. Of course, those Λ which are not $\boldsymbol{c}$-type 2 will be called $\boldsymbol{c}$-type 1.

In the sense of our earlier definition, Λ is type 2 if it is $\boldsymbol{c}$-type 2 with respect to $c_n \equiv 1$. We also say in this case that Λ is $\boldsymbol{1}$-type 2. For the corresponding convergence and divergence sets we introduce the notation $C_{\boldsymbol{c}}(f, \Lambda)$ and $D_{\boldsymbol{c}}(f, \Lambda)$.

In Theorem 5.1 of Section 5 we see that if a set Λ is $\boldsymbol{1}$-type 2, then it is $\boldsymbol{c}$-type 2 with respect to any positive sequence $\boldsymbol{c}$. The key property behind this theorem is the fact that for $\boldsymbol{1}$-type 2 sets there is a always a witness function which is a

characteristic function according to the result of Theorem 3.1. This motivates the following definition.

Definition 1.10. A positive sequence $\boldsymbol{c}$ is a χ-sequence if for any $\boldsymbol{c}$-type 2 set Λ there is always a characteristic function to witness this property.

It would be interesting to see whether Theorem 5.1 holds for all χ-sequences.

It is also worthful to notice that there exist sequences which are not χ-sequences. Indeed, if $\sum c_n$ converges, then for any function f bounded by K we have $s_{\boldsymbol{c},f}(x) = \sum_{n=1}^{\infty} c_n f(x+\lambda_n) \leq \sum_{n=1}^{\infty} c_n K$, and hence $s_{\boldsymbol{c},f}$ converges everywhere. On the other hand, by Theorem 5.1 there are $\boldsymbol{c}$-type 2 sets Λ, in this case with unbounded witness functions.

Hence it is also a natural question for further research to characterize χ-sequences.

Finally, in Theorem 5.2 we prove that there are sequences $\boldsymbol{c}$ such that every discrete set Λ is $\boldsymbol{c}$-type 2.

2 Preliminaries

In the proof of Proposition 1 of [7] we used a simple argument based on the Borel–Cantelli lemma which we state here as the following lemma.

Lemma 2.1. *Suppose that Λ is type 2 and f is a bounded witness function for Λ. If we modify f on a set E such that $\mu(E \cap (x, \infty)) \leq \epsilon(x)$ where $\epsilon(x)$ is a positive decreasing function tending to 0 at infinity, and satisfying*

$$\sum_{l \in \mathbb{N}} \epsilon(l - K) \#(\Lambda \cap [l, l+1)) < \infty, \tag{1}$$

then the convergence and divergence sets in $[-K, K]$ for the modified function $\widetilde{f}$ do not change apart from a set of measure 0.

3 Characteristic functions are witness functions for type 2

Theorem 3.1. *Suppose that Λ is type 2, that is there exists a measurable witness function f such that both $D(f, \Lambda)$ and $C(f, \Lambda)$ have positive measure. Then there exists a witness function g which is the characteristic function of an open set and both $D(g, \Lambda)$ and $C(g, \Lambda)$ have positive measure.*

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Proof. First we observe that it is sufficient to find a suitable g which is the characteristic function of a measurable set: then we can modify it on a set of finite measure which does not change the measure of $D(g, \Lambda)$ and $C(g, \Lambda)$.

Fix bounded sets $D \subset D(f, \Lambda)$ and $C \subset C(f, \Lambda)$ of positive measure, satisfying $C \cup D \subset [-K, K]$ for some $K \in \mathbb{N}$. We will suitably modify the function f by a sequence of steps such that the function obtained after each step satisfies the condition concerning the measures of D and C. Consider the intervals $I_1 = (-\infty, 1)$, $I_2 = [1, 2)$, $I_3 = [2, 3)$, ... For $n = 1, 2, ...$ we will choose sufficiently small real numbers $\varepsilon_n > 0$ and define f_0 such that $f_0(x) = f(x) + \varepsilon_k$ in I_k, $C \subset C(f_0, \Lambda)$ and $D \subset D(f_0, \Lambda)$. As $f_0 > f$, the second condition is obviously satisfied. Furthermore, as Λ is discrete and bounded from below, for fixed n there is a bounded number of λ_is with $\lambda_i \in I_n - [-K, K]$. Thus by choosing ε_n small enough, we can ensure that

$$\sum_{x + \lambda_i \in I_n} f_0(x + \lambda_i) < \frac{1}{2^n} + \sum_{x + \lambda_i \in I_n} f(x + \lambda_i) \text{ for any } x \in C \subset [-K, K].$$

As a consequence, for any $x \in C$ we have $s_{f_0}(x) < s_f(x) + 1 < \infty$, thus $C \subset C(f_0, \Lambda)$, as we stated. Hence f_0 is a function such that both $D(f_0, \Lambda)$ and $C(f_0, \Lambda)$ have positive measure, and f_0 is bounded away from zero on any interval of the form $(-\infty, t)$.

Now take $f_1(x) = \min(f_0(x), 1)$. For any $x \in D(f_0, \Lambda)$, if the sum $\sum_{\lambda \in \Lambda} f_0(x + \lambda)$ contains infinitely many terms which are at least 1, then these terms immediately guarantee that

$$\sum_{\lambda \in \Lambda} f_1(x + \lambda) = \infty. \tag{2}$$

On the other hand, if there are only finitely many such terms, then the sums associated to f_1 and f_0 differ only in these finitely many terms, which also yields (2). Then we have $D(f_1, \Lambda) = D(f_0, \Lambda)$, and consequently $C(f_1, \Lambda) = C(f_0, \Lambda)$. Thus we obtained a function f_1 which is bounded by 1. Moreover, both $D(f_1, \Lambda)$ and $C(f_1, \Lambda)$ have positive measure, and f_1 is bounded away from zero on any interval of the form $(-\infty, t)$.

Given f_1, we can construct a function f_2 with a rather simple range. Namely, for any x we choose $k_x \in \mathbb{N}$ such that $\frac{1}{2^{k_x}} < f_1(x) \leq \frac{1}{2^{k_x - 1}}$. As the range of f is contained by $(0, 1]$, there is such a k_x. Now take $f_2(x) = \frac{1}{2^{k_x}}$. Since we have $\frac{1}{2} f_1 \leq f_2 < f_1$, we may deduce $C(f_2, \Lambda) = C(f_1, \Lambda)$ and $D(f_2, \Lambda) = D(f_1, \Lambda)$. Moreover, as f_1 is bounded away from zero on any interval of the form $(-\infty, t)$, we have that f_2 has finite range in each such interval and vanishes nowhere. We can also assume $f_2 \equiv 1$ in $(-\infty, 0)$. As Λ is discrete and bounded from below the convergence and divergence sets remain the same.

Consider now the interval $[k-1, k)$ for $k \in \mathbb{N}$. The range of f_2 is finite in $[k-1, k)$, let it be $\{c_1, c_2, ..., c_l\}$. Now for any level set $\{f_2 = c_i\}$ we can define a relatively open set $U_i \subset [k-1, k)$ such that $\{f_2 = c_i\} \cap [k-1, k) \subset U_i$ and

$$\mu(U_i) < \mu(\{f_2 = c_i\} \cap [k-1, k)) + \frac{\delta_k}{2^i}$$

for some δ_k to be chosen later. Each U_i is a countable union of intervals. By choosing a sufficiently large finite subset of these intervals we can obtain a set V_i such that

$$\mu(U_i) < \mu(V_i) + \frac{\delta_k}{2^i}.$$

The intervals forming V_i are relatively open in $[k-1, k)$. Hence by adding finitely many points to each of them we can get sets V_i' which are finite unions of intervals of the form $[x, y)$. Finally, let $V_1^* = V_1'$, and for $i = 2, ..., l$ let

$$V_i^* = V_i' \setminus \left(\bigcup_{j=1}^{i-1} V_j^* \right).$$

Then the sets V_i^* are disjoint and each of them is a finite union of intervals of the form $[x, y)$ as such intervals form a semialgebra. Moreover, the complement V^* of their union in $[k-1, k)$ is also a set of this form. We define f_3 using these sets: on V_i^* let $f_3 = c_i$, and on V^* let $f_3 = c_1$.

When we redefine our function in V_i^* we modify it in a set of measure at most $\frac{\delta_k}{2^i}$, and when we redefine it in V^* we modify it in a set of measure at most $\sum_{i=1}^{l} \frac{\delta_k}{2^i}$. Hence f_2 and f_3 can differ only in a set of measure at most

$$2\delta_k \sum_{i=1}^{l} \frac{1}{2^i} < 2\delta_k.$$

Put $\epsilon(x) = \sum_{k \geq x} \delta_k$. If we choose a sufficiently rapidly decreasing sequence (δ_k) then we can ensure that

$$\sum_{l \in \mathbb{N}} \epsilon(l - K) \#(\Lambda \cap [l, l+1)) < \infty. \tag{3}$$

Since (3) is assumption (1) of Lemma 2.1, if we define f_3 in each of the intervals $[k-1, k)$ using the previous procedure, then the convergence and divergence sets are the same for f_2 and f_3 almost everywhere in $[-K, K]$. Moreover, the range of f_3 is a subset of the range of f_2 in any interval $(-\infty, t)$, and each bounded interval can be subdivided into finitely many subintervals of the form $[a, b)$ such that f_3 is constant on each of these subintervals. Denote the family of all these subintervals

in $\mathbb{R}$ by $\mathcal{I}$. We know that the sum s_{f_3} diverges in D apart from a null-set and converges in C apart from a null-set. For the sake of simplicity we assume that the sum s_{f_3} diverges in the entire set D and converges in the entire set C: if that does not hold, we can modify our initial sets. For ease of notation in the sequel we will denote f_3 by f, in fact it can be assumed that f was originally of this form.

In the following step we replace the family of intervals $\mathcal{I}$ by a "finer" family $\mathcal{J}$. Precisely, if $I \in \mathcal{I}$, first we subdivide it into sufficiently short subintervals $I_1, ..., I_m$ of equal length such that for any $x \in C \cup D$ we have that $x + \lambda \in I_i$ for at most one $\lambda \in \Lambda$ for any $i = 1, 2, ..., m$. In order to avoid technical complications, we define them to be closed from the left and open from the right, hence guaranteeing that they are disjoint. As $C \cup D$ is bounded, it is clear that this is possible. The family $\mathcal{J}$ will consist of all the previous short intervals for each $I \in \mathcal{I}$.

Now we define a sequence of random variables. Consider the intervals in $\mathcal{J}$ in increasing order: $J_1, J_2, ...$ Let $J_n \in \mathcal{J}$. Then we have that $f = 2^{-\kappa_n}$ on J_n for some $\kappa_n \in \mathbb{N}$. We define the sequence (X_n) of random variables such that they are independent and $X_n = 1$ with probability $2^{-\kappa_n}$, otherwise X_n is 0. By Kolmogorov's consistency theorem such random variables can be defined on a suitable probability measure space Ω. Given these random variables, we can define a random characteristic function: for any $\omega \in \Omega$ and $x \in \mathbb{R}$ let $g(\omega, x) = X_n(\omega)$ if $x \in J_n$.

We claim that almost surely, that is for $\mathbb{P}$ almost every ω

$$s_g(\omega, x) = \sum_{\lambda \in \Lambda} g(\omega, x + \lambda)$$

converges in C apart from a μ null-set, and diverges in D apart from a μ null-set. Proving the claim finishes the proof of the theorem as we can define $g = g(\omega)$ for one of the ωs of these almost sure events.

First let us consider the behaviour of $s_g(\omega, x)$ in D. Fix $x \in D$. Also fix $\lambda \in \Lambda$. Let us observe that

$$\mathbb{P}(g(\omega, x + \lambda) = 1) = f(x + \lambda).$$

Indeed if $f(x + \lambda) = 2^{-\kappa}$ for some $\kappa \in \mathbb{N}$, we have that $x + \lambda$ lies in an interval J_n where $f = 2^{-\kappa}$, thus

$$\mathbb{P}(g(\omega, x + \lambda) = 1) = \mathbb{P}(X_n(\omega) = 1) = 2^{-\kappa} = f(x + \lambda),$$

as we claimed. As a consequence, by the definition of D for $x \in D$ we clearly have that

$$\sum_{\lambda \in \Lambda} \mathbb{P}(g(\omega, x + \lambda) = 1) = \sum_{\lambda \in \Lambda} f(x + \lambda) = \infty. \tag{4}$$

Observe the events appearing in the leftmost expression. For fixed λ, the value $g(\omega, x+\lambda)$ depends on at most one of the independent random variables $X_1, X_2, ...$

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Moreover, by the procedure by which we replaced $\mathcal{I}$ by $\mathcal{J}$, for fixed n and x the random variable X_n affects at most one of the values $g(\omega, x + \lambda)$, $\lambda \in \Lambda$. Thus by the independence of (X_n), for fixed $x \in D$ the events

$$A_{\lambda, x} = \{\omega : g(\omega, x + \lambda) = 1\} \tag{5}$$

are also independent. As the series of their probabilites diverges, by the second Borel–Cantelli lemma we have that with probability one infinitely many of them occur, which is equivalent to the fact that $s_g(\omega, x) = \infty$. Thus for any fixed $x \in D$ we obtain $s_g(\omega, x) = \infty$ almost surely.

Now let us define $\Omega_D = \{(\omega, x) : x \in D \, , \, s_g(\omega, x) = \infty\} \subset \Omega \times D$. We claim that it is measurable. Indeed, let

$$\Omega_{\lambda_k} = \{(\omega, x) : g(\omega, x + \lambda_k) = 1\} \, .$$

It would be sufficient to verify that such a set is measurable as

$$\Omega_D = \limsup_{k \to \infty} \Omega_{\lambda_k} = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \Omega_{\lambda_k}$$

clearly holds. Now we simply observe that for fixed k the set D can be subdivided using finitely many intervals in each of which $g(\omega, x + \lambda_k)$ depends only on one of the random variables in the sequence (X_n). Consequently, Ω_{λ_k} can be written as a finite union of rectangles, hence it is measurable in the product space, which verifies our claim: Ω_D is measurable. By the earlier observations we obtain for its measure

$$\mu_{\Omega \times D}(\Omega_D) = \int_{\Omega \times D} \mathbf{1}_{\Omega_D} d\omega dx = \int_D \left(\int_\Omega \mathbf{1}_{\Omega_D} d\omega \right) dx = \int_D 1 dx = \mu(D).$$

Hence Ω_D is of full measure in the product space $\Omega \times D$. Thus almost surely $s_g(\omega, x)$ diverges in D apart from a null-set, that is

$$\mathbb{P}\left(\omega : s_g(\omega, x) = \infty \text{ for a.e. } x \in D\right) = \mathbb{P}(\Omega'_D) = 1.$$

The behaviour of $s_g(\omega, x)$ in C can be treated similarly. More precisely, the beginning of the argument up to (4) can be repeated and in place of (4) we obtain

$$\sum_{\lambda \in \Lambda} \mathbb{P}(g(\omega, x + \lambda) = 1) = \sum_{\lambda \in \Lambda} f(x + \lambda) < \infty \tag{6}$$

for fixed $x \in C$. We do not even have to check independence in this case; we can simply apply the first Borel–Cantelli lemma which tells us that with probability 1 only finitely many of the events $A_{\lambda, x}$ in (5) occur for $x \in C$, hence for fixed

$x \in C$ almost surely $s_g(\omega, x) < \infty$. The conclusion is also similar: the measure of $\Omega_C = \{(\omega, x) : x \in C$, $s_g(\omega, x) < \infty\} \subset \Omega \times C$ equals

$$\mu_{\Omega \times C}(\Omega_C) = \int_{\Omega \times C} \mathbf{1}_{\Omega_C} d\omega dx = \int_C \left(\int_\Omega \mathbf{1}_{\Omega_C} d\omega \right) dx = \int_C 1 dx = \mu(C).$$

(The measurability of Ω_C can be verified analogously to that of Ω_D.) Hence Ω_C is of full measure in the product space $\Omega \times C$. Thus $s_g(\omega, x)$ converges in C apart from a null-set almost surely, as we stated, that is

$$\mathbb{P} \left(\omega : s_g(\omega, x) < \infty \text{ for a.e. } x \in C \right) = \mathbb{P}(\Omega'_C) = 1.$$

This concludes the proof: the choice $g = g(\omega)$ for any $\omega \in \Omega'_D \cap \Omega'_C$ satisfies the claims of the theorem, thus there exists a satisfactory characteristic function. $\quad\square$

4   Randomly deleted points from Λ

Lemma 4.1. *Assume that $C \subset [0, 1)$ is Lebesgue measurable, that is $C \in \mathcal{L}[0, 1)$. Then for almost every $x \in [0, 1)$ we have*

$$\lim_{n \to -\infty} \frac{\#((x + 2^n \mathbb{Z}) \cap C)}{2^{-n}} = \mu(C). \tag{7}$$

Proof. Consider the measurable function $\mathbf{1}_C$ and the negatively indexed increasing sequence of σ-algebras $\mathcal{F}_n = \{(A + 2^n \mathbb{Z}) \cap [0, 1) : A \in \mathcal{L}([0, 1))\}$, $n \in -\mathbb{N}$. Moreover, denote by $\mathcal{F}_{-\infty}$ their intersection. By Lebesgue's density theorem one can easily see that $\mathcal{F}_{-\infty}$ contains only full measure sets and null-sets. Hence the conditional expectation $\mathbb{E}(\mathbf{1}_C | \mathcal{F}_{-\infty})$ is almost everywhere constant, therefore it equals $\mathbb{E}(\mathbf{1}_C) = \mu(C)$. On the other hand, by Theorem 5.6.3 in [13] about backwards martingales we know that

$$\mathbb{E}(\mathbf{1}_C | \mathcal{F}_n) \to \mathbb{E}(\mathbf{1}_C | \mathcal{F}_{-\infty}) \tag{8}$$

almost surely as $n \to -\infty$. Next we show that

$$\mathbb{E}(\mathbf{1}_C | \mathcal{F}_n)(x) = \frac{\#((x + 2^n \mathbb{Z}) \cap C)}{2^{-n}}, \text{ for } \mu \text{ a.e. } x \in [0, 1) \text{ for all } n \in -\mathbb{N}. \tag{9}$$

The function on the right-handside of (9) is defined for any $x \in \mathbb{R}$. It is Lebesgue measurable and invariant under translations by values in $2^n \mathbb{Z}$, hence its restriction onto $[0, 1)$ is clearly $\mathcal{F}_n$ measurable. Suppose that $A' \in \mathcal{F}_n$. We denote by A the one periodic set obtained from A', that is $A = A' + \mathbb{Z}$. Then $A + 2^n k = A$ for any $k \in \mathbb{Z}$. Moreover, for any $n \in -\mathbb{N}$

$$\int_{A'} \frac{\#((x + 2^n \mathbb{Z}) \cap C)}{2^{-n}} d\mu(x) = 2^n \int_0^1 \left(\sum_{k \in \mathbb{Z}} \mathbf{1}_{C + 2^n k}(x) \right) \mathbf{1}_A(x) d\mu(x) \tag{10}$$

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$$= 2^n \int_0^1 \left(\sum_{k \in \mathbb{Z}} \mathbf{1}_{C+2^n k}(x) \mathbf{1}_{A+2^n k}(x) \right) d\mu(x)$$

$$= 2^n \sum_{m=0}^{2^{-n}-1} \left(\sum_{k \in \mathbb{Z}} \int_{m2^n}^{(m+1)2^n} \mathbf{1}_{C+2^n k}(x) \mathbf{1}_{A+2^n k}(x) d\mu(x) \right).$$

However,

$$\sum_{k \in \mathbb{Z}} \int_{m2^n}^{(m+1)2^n} \mathbf{1}_{C+2^n k}(x) \mathbf{1}_{A+2^n k}(x) d\mu(x) = \sum_{k \in \mathbb{Z}} \int_0^{2^n} \mathbf{1}_{C+2^n k}(x) \mathbf{1}_{A+2^n k}(x) d\mu(x)$$

$$= \sum_{k \in \mathbb{Z}} \mu(A \cap C \cap [-k2^n, -(k+1)2^n)) = \mu(A \cap C) = \mu(A' \cap C).$$

Hence the left-hand side of (10) equals $\mu(A' \cap C)$. Since we have this property for any $A' \in \mathcal{F}_n$ we proved (9). Using this result in (8) and taking limit in (8) we obtain (7). $\qquad \square$

Let

$$\widetilde{\Lambda} = \bigcup_{k=1}^{\infty} (2^{-k} \mathbb{N} \cap [k, k+1)). \tag{11}$$

We know from Example 1.3 that $\widetilde{\Lambda}$ is type 1.

Definition 4.2. Let $0 < p < 1$. Then we say that $\Lambda \subset \widetilde{\Lambda}$ is chosen with probability p from $\widetilde{\Lambda}$ if for each $\lambda \in \widetilde{\Lambda}$ the probability that $\lambda \in \Lambda$ is p. That is, we consider $\Omega = \{0, 1\}^{\mathbb{N}}$ with the product measure $\mathbb{P}$ which is obtained as the product of the measures which assign probability p to $\{1\}$ and $q = 1 - p$ to $\{0\}$. We order the elements of $\widetilde{\Lambda}$ in increasing order, that is $\widetilde{\Lambda} = \{\lambda_1 < \lambda_2 < ...\}$ and for an element ω of our probability space Ω we assign the random set Λ_ω which is obtained from $\widetilde{\Lambda}$ by keeping λ_k if ω_k, the kth entry of ω is 1 and deleting it otherwise. To make this a little more precise we consider independent identically distributed random variables $X_k(\omega)$ with values in $\{0, 1\}$ with $X_k(\omega) = \omega_k$. Then $\mathbb{P}(X_k = 1) = p$, $\mathbb{P}(X_k = 0) = q = 1 - p$ and we keep λ_k in Λ_ω if $X_k(\omega) = 1$.

We say that a property holds almost surely if the $\mathbb{P}$ measure of those ωs for which Λ_ω has this property equals 1.

For ease of notation often we omit the subscript ω and we just speak about almost sure subsets $\Lambda \subset \widetilde{\Lambda}$.

It is clear that almost surely if $\Lambda \subset \widetilde{\Lambda}$ is chosen with probability p from $\widetilde{\Lambda}$ then Λ is an infinite discrete set.

Theorem 4.3. *Suppose that $0 < p < 1$ and Λ is chosen with probability p from $\widetilde{\Lambda} = \cup_{k=1}^{\infty}(2^{-k}\mathbb{N} \cap [k, k+1))$. Then almost surely Λ is type 1.*

Lemma 4.4. *Suppose that $0 < p < 1$ and Λ is chosen with probability p from $\widetilde{\Lambda}$. Then almost surely Λ satisfies the following:*

For every $L \in \mathbb{N}$ there exists $N \in \mathbb{N}$ such that for all $x \geq N$ we have

$$\#\left(\Lambda \cap \left[x, x + \frac{1}{2^L}\right)\right) > p \cdot 2^{J-L-2}, \tag{12}$$

where $J = \lfloor x \rfloor$.

Proof. We will use the notation from Definition 4.2. We consider $\Lambda = \Lambda_\omega$ obtained from $\widetilde{\Lambda}$ by using the i.i.d. random variables $X_k = X_k(\omega) = \omega_k$.

We recall from the standard Chebyshev's inequality proof of the Weak Law of Large Numbers (see for example [13, Ch. 2.2]) that there exists a constant C_p depending only on p such that for any $K, n \in \mathbb{N}$

$$\mathbb{P}\left\{\left|\frac{X_{K+1} + ... + X_{K+n}}{n} - p\right| > p(1 - 2^{-1.9})\right\} < \frac{C_p}{n}. \tag{13}$$

Observe that $\widetilde{\Lambda} \cap [J, J+1) = \{\lambda_{2^J-1}, \lambda_{2^J-1+1}, ..., \lambda_{2^J-1+2^J-1}\}$ for any $J \in \mathbb{N}$. We say that ω is J-L'-*good* if

$$\left|\frac{\sum_{j=0}^{2^{J-L'}-1} X_{2^J-1+l \cdot 2^{J-L'}+j}(\omega)}{2^{J-L'}} - p\right| \leq p(1 - 2^{-1.9}) \tag{14}$$

holds for every $l = 0, ..., 2^{L'} - 1$.

If ω is J-L'-good for every $J \geq J_0$ then we say that ω is J_0-L'-∞-*good*.

By (13) one can see that

$$\mathbb{P}\{\omega : \omega \text{ is } J\text{-}L'\text{-good}\} \geq 1 - 2^{L'} \cdot \frac{C_p}{2^{J-L'}} \tag{15}$$

and hence

$$\mathbb{P}\{\omega : \omega \text{ is } J_0\text{-}L'\text{-}\infty\text{-good}\} \geq 1 - \sum_{J \geq J_0} 2^{L'} \cdot \frac{C_p}{2^{J-L'}}. \tag{16}$$

It is easy to see that if L' is sufficiently large, say $L' = L + 100$ and $\Lambda = \Lambda_\omega$ for a J_0-L'-∞-good ω, then (12) holds with $N = J_0$.

Using (16) it is also clear that for $\mathbb{P}$ a.e. ω there is a J_0 such that ω is J_0-L'-∞-good. This completes the proof of the lemma. $\square$

Proof of Theorem 4.3. Let $0 < p < 1$ and assume that Λ has been chosen with probability p from $\widetilde{\Lambda}$.

Pursuing a contradiction, we assume that Λ is type 2.

By Theorem 3.1 we can choose a measurable set $S \subset \mathbb{R}$ such that $f = \mathbf{1}_S$ witnesses that Λ is type 2. Thus $\mu(D(f, \Lambda)) > 0$ and $\mu(C(f, \Lambda)) > 0$ and therefore we can choose $R \in \mathbb{N}$ and an interval I of length $R-1$ such that $\mu(D(f, \Lambda) \cap I) > 0$ and $\mu(C(f, \Lambda) \cap I) > 0$. Then using the Lebesgue Density Theorem we choose intervals $I_{\mathbf{D}}$ and $I_{\mathbf{C}}$ subsets of I of length 2^{-L} where $L \in \mathbb{N}$ such that

$$\mu(I_{\mathbf{C}} \cap C(f, \Lambda)) > \left(1 - \frac{p}{2^{R+7}}\right) \cdot 2^{-L} \tag{17}$$

and

$$\mu(I_{\mathbf{D}} \cap D(f, \Lambda)) > 0. \tag{18}$$

We assume without loss of generality that $I_{\mathbf{C}} = [0, \frac{1}{2^L})$ and $I_{\mathbf{D}} = [\frac{-N}{2^L}, \frac{-(N-1)}{2^L})$ for some $N \in \mathbb{Z}$. Since the cases $N \leq 0$ are easier than the ones when $N > 0$ we provide details only for the case $N \in \mathbb{N}$.

Note that we have

$$N \leq R \cdot 2^L. \tag{19}$$

For each $n \in \mathbb{N}$ we define $C_n^* = \{x \in C(f, \Lambda) \cap I_{\mathbf{C}} : (x + \Lambda) \cap [n, \infty) \cap S = \emptyset\}$. Since f is a characteristic function, it follows that $\cup_{n=1}^{\infty} C_n^* = C(f, \Lambda) \cap I_{\mathbf{C}}$ and therefore we can choose $C \subset C(f, \Lambda) \cap I_{\mathbf{C}}$ and $M \in \mathbb{N}$ such that

$$\mu(C) \geq \left(1 - \frac{p}{2^{R+6}}\right) \cdot 2^{-L} \tag{20}$$

and

$$(C + \Lambda) \cap [M \cdot 2^{-L}, \infty) \cap S = \emptyset. \tag{21}$$

For each $n \geq L$ define $n^* = \lfloor \frac{n}{2^L} \rfloor$ and let

$$C_n = \left\{x \in I_{\mathbf{C}} : \#((x + 2^{-k}\mathbb{Z}) \cap C) > \left(1 - \frac{p}{2^{R+5}}\right) 2^{k-L} \text{ for all } k \geq (n + N)^*\right\}$$

and $E_n = C_n - \frac{N}{2^L} \subset I_{\mathbf{D}}$. Note that for all $n \geq L$ we have $C_n \subset C_{n+1} \subset I_{\mathbf{C}}$ and by a rescaled version of Lemma 4.1 we know that

$$\mu(C_n) \to 2^{-L} = \mu(I_{\mathbf{C}}) \text{ and hence } \mu(I_{\mathbf{D}} \backslash E_n) \to 0. \tag{22}$$

Note also that C_n is $\frac{1}{2^{(n+N)^*}}$ periodic on $I_{\mathbf{C}}$ and E_n is $\frac{1}{2^{(n+N)^*}}$ periodic on $I_{\mathbf{D}}$ for all $n \geq L$.

For each $n \in \mathbb{N}$ define

$$S_n = \left\{y \in \left[\frac{n}{2^L}, \frac{n+1}{2^L}\right) : (y - \Lambda) \cap C_n = \emptyset\right\},$$

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and
$$S'_n = \left\{ y \in \left[\frac{n}{2^L}, \frac{n+1}{2^L}\right) : (y - \Lambda) \cap C = \emptyset \right\}.$$

Using Lemma 4.4, we may assume that we can choose $P \in \mathbb{N}$ such that

$$\#\left(\Lambda \cap \left[x, x + \frac{1}{2^L}\right)\right) > p \cdot 2^{J-L-2} \text{ for all } x \geq P, \tag{23}$$

where $J = \lfloor x \rfloor$.

Next we show that

if $n > B := \max\{M, (P+L) \cdot 2^L + 1\}$, then $S \cap \left[\frac{n}{2^L}, \frac{n+1}{2^L}\right) \subset S'_n \subset S_n$. $\quad$ (24)

Assume that $n > B$. Since $n > M$, by (21) we have $(C + \Lambda) \cap S \cap [\frac{n}{2^L}, \frac{n+1}{2^L}) = \emptyset$ and therefore $S \cap [\frac{n}{2^L}, \frac{n+1}{2^L}) \subset S'_n$.

Now suppose that $y \in [\frac{n}{2^L}, \frac{n+1}{2^L}) \backslash S_n$. Then we can choose $x \in C_n$ and $\lambda \in \Lambda$ such that $x + \lambda = y$. Since $\lambda = y - x < \frac{n+1}{2^L} \leq n^* + 1$ by (11) and $\Lambda \subset \widetilde{\Lambda}$ we have $\lambda \in 2^{-n^*}\mathbb{Z}$ which implies

$$y + 2^{-n^*}\mathbb{Z} = x + 2^{-n^*}\mathbb{Z} \text{ and } (y - \Lambda) \cap \left[0, \frac{1}{2^L}\right) \subset (y + 2^{-n^*}\mathbb{Z}) \cap \left[0, \frac{1}{2^L}\right). \tag{25}$$

From the definition of C_n we have that

$$\#((x + 2^{-(n+N)^*}\mathbb{Z}) \cap C) > \left(1 - \frac{p}{2^{R+5}}\right) \cdot 2^{(n+N)^*-L}, \text{ that is}$$

$$\frac{p}{2^{R+5}} \cdot 2^{(n+N)^*-L} > \#\left(\left((x + 2^{-(n+N)^*}\mathbb{Z}) \cap \left[0, \frac{1}{2^L}\right)\right) \backslash C\right)$$

$$\geq \#\left(\left((x + 2^{-n^*}\mathbb{Z}) \cap \left[0, \frac{1}{2^L}\right)\right) \backslash C\right).$$

It follows that

$$\#((x + 2^{-n^*}\mathbb{Z}) \cap C) > 2^{n^*-L} - \frac{p}{2^{R+5}}(2^{(n+N)^*-L}) = 2^{n^*-L}\left(1 - \frac{p}{2^{R+5}}2^{(n+N)^*-n^*}\right).$$

Using $(n + N)^* - n^* \leq N^* + 1$ and by (19), $R \geq N^*$, we conclude that

$$\#((y + 2^{-n^*}\mathbb{Z}) \cap C) = \#((x + 2^{-n^*}\mathbb{Z}) \cap C) > \left(1 - \frac{p}{8}\right) \cdot 2^{n^*-L}. \tag{26}$$

Now using (23) with $y - \frac{1}{2^L}$ in place of x we find that since $y - \frac{1}{2^L} \geq \frac{n-1}{2^L} \geq P$

$$\#\left((y - \Lambda) \cap \left[0, \frac{1}{2^L}\right)\right) = \#\left(\Lambda \cap \left[y - \frac{1}{2^L}, y\right)\right) > p \cdot 2^{n^*-1-L-2} = \frac{p}{8} \cdot 2^{n^*-L}.$$

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Using (25), (26), the last inequality and the pigeon-hole principle, we conclude that there must exist $x' \in C$ and $\lambda' \in \Lambda$ such that $x' + \lambda' = y$ and therefore $y \notin S'_n$. It follows that $S'_n \subset S_n$ and we are done with the proof of (24).

Next we continue with some definitions. For each $n \in \mathbb{N}$ we define $D_n = (S_n - \Lambda) \cap I_{\boldsymbol{D}}$ and let $D'_n = D_n \cap E_n$ and $D''_n = D_n \backslash D'_n$. Note that if $x \in I_{\boldsymbol{D}} \backslash D_n$ and $n > B$, then $(x + \Lambda) \cap S'_n = \emptyset$ so $f(x + \lambda) = 0$ for all $\lambda \in [\frac{n+N}{2^L} - x, \frac{n+N+1}{2^L} - x)$. From these considerations it follows that

$$D(f, \Lambda) \cap I_{\boldsymbol{D}} \subset \cap_{k=1}^\infty \cup_{n=k}^\infty D_n = (\cap_{k=1}^\infty \cup_{n=k}^\infty D'_n) \cup (\cap_{k=1}^\infty \cup_{n=k}^\infty D''_n).$$

Furthermore, for all $n \in \mathbb{N}$ we have $D''_n \subset I_{\boldsymbol{D}} \backslash E_n$ where $I_{\boldsymbol{D}} \backslash E_{n+1} \subset I_{\boldsymbol{D}} \backslash E_n$ and by (22), $\mu(I_{\boldsymbol{D}} \backslash E_n) \to 0$ and therefore

$$\mu(\cap_{k=1}^\infty \cup_{n=k}^\infty D''_n) = 0. \tag{27}$$

Thus, if we can prove that $\mu(\cap_{k=1}^\infty \cup_{n=k}^\infty D'_n) = 0$ we can conclude that $\mu(D(f, \Lambda) \cap I_{\boldsymbol{D}}) = 0$, which contradicts (18) and finishes the proof of the theorem. Actually we prove that $D'_n = \emptyset$ for large n.

Suppose that $n > B$. Then $n^* \geq L$ and $\lambda'' = \frac{n}{2^L} \in 2^{-n^*}\mathbb{Z}$. We show that

$$\left(C_n + \frac{n}{2^L}\right) \cap S_n = (C_n + \lambda'') \cap S_n = \emptyset. \tag{28}$$

Indeed, suppose that $y = x + \lambda''$ with $x \in C_n$ then we show that one can find $x' \in C_n$ and $\lambda' \in \Lambda$ such that $y = x' + \lambda'$ and hence $y \notin S_n$. This follows easily, since C_n is $\frac{1}{2^{n^*}}$ periodic on $I_{\boldsymbol{C}}$ and one can apply (23) for $x'' = x + \frac{n-1}{2^L}$ and observe that there are points of Λ in $(x + \frac{n-1}{2^L}, x + \frac{n}{2^L})$. Select such a point λ'. Then $\lambda'' - \lambda' \in 2^{-n^*}\mathbb{Z}$ and hence if we let $x' = x + (\lambda'' - \lambda')$ then $y = x' + \lambda'$ and $x' \in [0, \frac{1}{2^L})$. Since C_n is 2^{-n^*} periodic in $[0, \frac{1}{2^L})$ we obtained that $x' \in C_n$, proving (28).

Now recall that C_n is $\frac{1}{2^{(n+N)^*}}$ periodic on $I_{\boldsymbol{C}}$. This implies that $C_n + \lambda'' = C_n + \frac{n}{2^L}$ is $\frac{1}{2^{(n+N)^*}}$ periodic on $[\frac{n}{2^L}, \frac{n+1}{2^L})$. By (28)

$$S_n \subset \widetilde{C}_n := \left[\frac{n}{2^L}, \frac{n+1}{2^L}\right) \backslash \left(C_n + \frac{n}{2^L}\right).$$

Obviously $\widetilde{C}_n$ is also $\frac{1}{2^{(n+N)^*}}$ periodic on $[\frac{n}{2^L}, \frac{n+1}{2^L})$. Since we also know that $E_n = C_n - \frac{N}{2^L}$ is $\frac{1}{2^{(n+N)^*}}$ periodic on $[\frac{-N}{2^L}, \frac{-(N-1)}{2^L})$, it follows that

$$(\widetilde{C}_n - 2^{-(n+N)^*}\mathbb{N}) \cap E_n = \emptyset. \tag{29}$$

Now observe that if $y \in \widetilde{C}_n$ and $y - \lambda \in E_n$, then we have $\lambda \in [\frac{N+n-1}{2^L}, \frac{N+n+1}{2^L})$. Moreover, we also have that $\lambda \in [\frac{N+n-1}{2^L}, \frac{N+n+1}{2^L}) \cap \Lambda$ implies that $\lambda \in 2^{-(n+N)^*}\mathbb{N}$. Since $S_n \subset \widetilde{C}_n$, it follows from (29) that $(S_n - \Lambda) \cap E_n = \emptyset$, which implies that $D'_n = \emptyset$ for $n > B$. This concludes the proof of Theorem 4.3. $\qquad\square$

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Theorem 4.5. *Suppose that (m_k) and (n_k) are strictly increasing sequences of positive integers. For each $k \in \mathbb{N}$, define $\Lambda_k = 2^{-m_k}\mathbb{N} \cap [n_k, n_{k+1})$ and let $\widetilde{\Lambda} = \bigcup_{k=1}^{\infty} \Lambda_k$. Moreover, fix $0 < p < 1$ and suppose that Λ is chosen with probability p from $\widetilde{\Lambda}$. Set $q = 1 - p$. For fixed (m_k), if (n_k) tends to infinity sufficiently fast then almost surely Λ is type 2. Notably, if the series $\sum_{k=1}^{\infty} 1 - \left(1 - q^{2^{m_k}}\right)^{n_{k+1}-n_k}$ diverges then almost surely Λ is type 2.*

Remark 4.6. If $m_k = k$ then by Example 1.3, Λ is type 1 for any n_k and hence it may happen that a type 1 set is turned into a type 2 set by random deletion of its elements.

Proof. Let A_k denote the event in which there exists $a \in \mathbb{N}$ such that $[a, a + 1) \subset [n_k, n_{k+1})$ and $[a, a + 1) \cap \Lambda = \emptyset$. We can quickly deduce that the probability of the complement is
$$\mathbb{P}(A_k^c) = \left(1 - q^{2^{m_k}}\right)^{n_{k+1}-n_k}.$$

Consequently,
$$\mathbb{P}(A_k) = 1 - \left(1 - q^{2^{m_k}}\right)^{n_{k+1}-n_k}.$$

By assumption, the series of these probabilites diverges. Consider now the sequence of events $(A_k)_{k=1}^{\infty}$. They are clearly independent, hence by the second Borel–Cantelli lemma the aforementioned divergence implies that almost surely infinitely many of the events A_k occurs. However, this immediately yields that almost surely the set Λ is asymptotically lacunary and hence by Theorem 1.6 it is type 2. $\qquad\square$

5 c-type 1 and 2 sets

The following theorem is a nice consequence of Theorem 3.1.

Theorem 5.1. *If a set Λ is $\mathbf{1}$-type 2, then it is $\mathbf{c}$-type 2 with respect to any positive sequence $\mathbf{c} = (c_n)_{n=1}^{\infty}$.*

Proof. By Theorem 3.1, choose an open set such that its characteristic function f witnesses that Λ is $\mathbf{1}$-type 2. Then both $D(f, \Lambda)$ and $C(f, \Lambda)$ have positive measure. Choose a bounded $D \subset D(f, \Lambda)$ and a bounded $C \subset C(f, \Lambda)$ of positive measure. Then the set $\{f \neq 0\}$ equals a countable union of intervals $I_1, I_2, \ldots$. We will construct g verifying the statement such that for any $x \in I_k$, $k = 1, 2, \ldots$ we have $g(x) = \alpha_k f(x)$ for some $\alpha_k > 0$. We define α_k as follows: since D is bounded, for any $k = 1, 2, \ldots$ there are finitely many $\lambda_{k_1}, \ldots, \lambda_{k_m}$ such that $x + \lambda_{k_i} \in I_k$ for some $x \in D$ and $i = 1, \ldots, m$. As $c_n > 0$ for each n, we have that the finite set $\{c_{k_1}, \ldots, c_{k_m}\}$ is bounded away from 0. Thus α_k can be chosen sufficiently large to

guarantee $\alpha_k c_{k_i} \geq 1$ for $i = 1, ..., m$. By this choice for any $x \in D$ we have that

$$\sum_{\lambda_j \,:\, x+\lambda_j \in I_k} c_j g(x + \lambda_j) \geq \sum_{\lambda_j \,:\, x+\lambda_j \in I_k} f(x + \lambda_j).$$

However, if we add these latter sums for all the intervals I_k, we find that our sum diverges. As a consequence, $\sum_{n=1}^{\infty} c_n g(x + \lambda_n)$ diverges for any $x \in D$, which guarantees the positive measure of the divergence set.

Concerning the convergence set, we have an easy task: for any $x \in C$ we have that $x + \lambda \in \{f \neq 0\}$ only for finitely many λs since otherwise $\sum_{n=1}^{\infty} f(x + \lambda_n)$ would diverge as $\{f \neq 0\} = \{f = 1\}$. Thus we also have $x + \lambda \in \{g \neq 0\}$ only for finitely many λs. This guarantees that $\sum_{n=1}^{\infty} c_n g(x + \lambda_n)$ converges for any $x \in C$, which guarantees the positive measure of the convergence set. □

The previous theorem displays that the sequence $c_n \equiv 1$ is minimal in some sense: the family of type 2 sets is as small as possible. It is natural to ask whether all χ-sequences have this property.

Theorem 5.2 shows that not all sequences have this property by showing the other extreme: sequences for which every Λ is $\boldsymbol{c}$-type 2.

Theorem 5.2. *Suppose that $\boldsymbol{c} = (c_n)$ is a sequence of positive numbers satisfying the following condition:*

$$\sum_{j=n+1}^{\infty} c_j < 2^{-n} c_n \text{ for every } n \in \mathbb{N}. \tag{30}$$

Then every discrete set Λ is $\boldsymbol{c}$-type 2.

Proof. Let $\Lambda = \{\lambda_1, \lambda_2, \dots\}$ with $\lambda_1 < \lambda_2 < \dots$ and $\lambda_n \to \infty$. Choose $y_n \nearrow \infty$ such that $y_{n+1} - y_n > 1$ and $\Lambda \cap [y_n, y_n + \frac{1}{2}] \neq \emptyset$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ let

$$T_n = \left\{ j \,:\, \lambda_j \in \left[y_n, y_n + \frac{1}{2} \right] \right\},$$

and define

$$d_n = \frac{1}{\sum_{j \in T_n} c_j} \text{ and } f = \sum_{n=1}^{\infty} d_n \mathbf{1}_{[y_n, y_n+1]}. \tag{31}$$

Claim 5.3. $[0, \frac{1}{2}] \subset D_{\boldsymbol{c}}(f, \Lambda)$.

Proof. Let $x \in [0, \frac{1}{2}]$. Then for every $j \in T_n$ we have $x + \lambda_j \in [y_n, y_n + 1]$. Hence $f(x + \lambda_j) = d_n$. Thus we obtain

$$\sum_{j=1}^{\infty} c_j f(x + \lambda_j) \geq \sum_{n=1}^{\infty} \sum_{j \in T_n} c_j f(x + \lambda_j) = \sum_{n=1}^{\infty} \frac{1}{d_n} d_n = \infty,$$

finishing the proof of Claim 5.3. □

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Claim 5.4. $(-\infty, -\frac{1}{2}) \subset C_{\boldsymbol{c}}(f, \Lambda)$.

Proof. Let $x \in (-\infty, -\frac{1}{2})$. For each $n \in \mathbb{N}$ define

$$R_n(x) = \{j : \; x + \lambda_j \in [y_n, y_n + 1]\}.$$

Note that if $j \in R_n(x)$, then $\lambda_j > y_n + \frac{1}{2}$ and it follows that $j > i$ for all $i \in T_n$. Therefore, using (30) we obtain

$$\sum_{j \in R_n(x)} c_j < 2^{-n} \sum_{j \in T_n} c_j.$$

Therefore using (31) we deduce

$$\sum_{j=1}^{\infty} c_j f(x + \lambda_j) = \sum_{n=1}^{\infty} \sum_{j \in R_n(x)} c_j f(x + \lambda_j) = \sum_{n=1}^{\infty} \left(d_n \sum_{j \in R_n(x)} c_j \right)$$

$$< \sum_{n=1}^{\infty} d_n 2^{-n} \sum_{j \in T_n} c_j = \sum_{n=1}^{\infty} 2^{-n} = 1.$$

$\square$

Clearly the proof of Claim 5.4 also concludes the proof of Theorem 5.2. $\square$

6 Acknowledgements

References

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