Linear Turán numbers of acyclic triple systems

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Abstract

The Turán number of hypergraphs has been studied extensively. Here we deal with a recent direction, the linear Turán number, and restrict ourselves to linear triple systems, a collection of triples on a set of points in which any two triples intersect in at most one point. For a fixed linear triple system F, the linear Turán number $\mathrm{ex}_L(n, F)$ is the maximum number of triples in a linear triple system with n points that does not contain F as a subsystem.

We initiate the study of the linear Turán number for an *acyclic* F. In this case $\mathrm{ex}_L(n, F)$ is linear in n and we aim for good bounds. Since the case of trees is already difficult for graphs (Erdős - Sós conjecture), we concentrate on matchings, paths and small trees.

In case of matchings, where M_k is the set of k pairwise disjoint triples, we prove that for fixed k and large enough n, $\mathrm{ex}_L(n, M_k) = f(n, k)$ where $f(n, k)$ is the maximum number of triples that can meet $k - 1$ points in a linear triple system on n points. This is an analogue of an old result of Erdős on hypergraph matchings. For the k-edge linear path P_k we show (extending some standard path increasing methods used for graphs) that $\mathrm{ex}_L(n, P_k) \le 1.5kn$ which is probably far from best possible.

Finding $\mathrm{ex}_L(n, F)$ relates to difficult problems on Steiner triple systems and interesting even for small trees. For example, for P_4, the path with four triples, $\mathrm{ex}_L(n, P_4) \le \frac{4n}{3}$ with equality only for disjoint union of affine planes of order 3. On the other hand, for E_4, the tree having three pairwise disjoint triples and a fourth one meeting all of them, we have bounds only: $6\lfloor \frac{n-3}{4} \rfloor \le \mathrm{ex}_L(n, E_4) \le 2n$.

1 Introduction, results

The Turán number of hypergraphs has been studied extensively, see for example surveys [9], [12], [16], and the book [13]. Here we concentrate on a recent

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direction, *the linear Turán number* and we restrict ourselves to the 3-uniform case. A *linear triple system* H is defined as a set of *triples* $E(H)$ on a set of *points* $V(H)$ with the property that any two triples intersect in at most one point.

For a fixed linear triple system F, the linear Turán number $\mathrm{ex}_L(n, F)$ is the maximum number of triples in a linear triple system on n points that does not contain F as a subsystem. This notion and notation is due to Collier-Cartaino, Graber, and Jiangin [2] 2018, although the famous result of Ruzsa and Szemerédi [18] 1976, can be also phrased as

$$n^2/e^{O(\sqrt{\log n})} < \mathrm{ex}_L(n, C) = o(n^2),$$

where C is the (linear) triangle. The $o(n^2)$ upper bound was extended recently to a wider family by two of the authors [14]. In [2] the order of magnitude of $\mathrm{ex}_L(n, F)$ was determined when F is a (linear) cycle of length at least four. A special case of a result of Füredi and Gyárfás [11] is that $\mathrm{ex}_L(n, F) \leq n^2/9$ (with equality when n is divisible by 3), where F, the fan, has 3 triples intersecting in a point v and an additional triple intersecting all of them in a point different from v.

In this paper we initiate studying $\mathrm{ex}_L(n, F)$ for an acyclic F. A linear triple system is *acyclic* if it can be built starting from one triple then at each step adding a new triple that intersects the union of the points of the previous triples in at most one point. If the new triple intersects the union of the points of the previous triples in *exactly one point* then we get the definition of a *tree*. A tree with k edges is denoted by T_k. Note that T_k has $2k+1$ points. Two special trees are S_k, P_k, the star and the path. The star S_k has k triples intersecting in the same point, the center. The path P_k has points $p_1, p_2, \ldots, p_{2k+1}$ and has triples $\{p_{2i-1}, p_{2i}, p_{2i+1}\}$ for $i = 1, \ldots, k$. Among disconnected acyclic triple systems the most important is M_k, the *matching of size* k, which has k pairwise disjoint triples.

An upper bound on $\mathrm{ex}_L(n, T_k)$ can be derived easily by adopting the well-known argument for graph trees.

Proposition 1.1. *Let T_k be a fixed tree with $k > 1$ edges. Then $\mathrm{ex}_L(n, T_k) \leq (2k-3)n$.*

Proof. In a minimal counterexample each point has degree at least $2k-2$ (otherwise we find a smaller counterexample by deleting a point with a smaller degree). Then we can build T_k with the greedy algorithm, adding in each step a triple that intersects the previous subtree in the required point. $\square$

Proposition 1.1 is a factor of 6 larger than the natural lower bound discussed later in Subsection 1.1. Since the case of trees is already notoriously difficult even for graphs (Erdős - Sós conjecture, [7] 1963), we concentrate on matchings, paths and small trees.

The Turán number of matchings and paths in graphs were determined by Erdős and Gallai [5], 1959. In the case of the matching, the extension of the

Turán problem to hypergraphs is also due to Erdős [4], 1965, who proved that for large enough n, the maximum number of edges in a uniform hypergraph without M_k occurs if all edges intersect a fixed set of $k-1$ vertices (see also Theorem 9.2 in [8]). It is worth noting that in the graph case $\mathrm{ex}(M_k, n)$ was determined for *every* n in [5] using tools of matching theory. These tools are not available for hypergraphs, that is the reason why Erdős' result is stated only for large enough n. We prove the following analogue of Erdős' result.

Theorem 1.2. *For $n > 16(k-1)^2 + 1$, $\mathrm{ex}_L(n, M_k) = f(n,k)$, where $f(n,k)$ is the maximum number of triples that can intersect a fixed $(k-1)$-element set of points in a linear triple system with n points.*

Note that (as in Erdős theorem) Theorem 1.2 holds only for large enough n. For example $f(n,2) = \lfloor \frac{n-1}{2} \rfloor < \mathrm{ex}_L(n, M_2)$ for $n < 15$ as the Fano plane shows. In fact $n \geq 15$ is the sharp threshold in Theorem 1.2 when $k = 2$, because in a linear triple system pairwise intersecting triples either form a star or form a subsystem of the Fano plane. As noted by a referee, the coefficient 16 can be improved if k is not too small.

In fact $f(n,k)$ can be determined exactly, but to avoid a complicated formula (depending on the value of $k \pmod 6$), we just give a lower bound in Lemma 2.1 needed for the proof of Theorem 1.2 and provide a close upper bound in Lemma 2.2.

In the case of the path, the extension of the Turán problem to hypergraphs was studied for several different notions of paths. For loose paths Mubayi and Verstraëte [17], 2007, for linear paths Füredi, Jiang and Seiver [10], 2014, and for Berge paths Győri, Katona and Lemons [15], 2016, determined exactly (at least for large n and fixed k) the Turán number. It does not seem easy to find the asymptotic of $\mathrm{ex}_L(n, P_k)$. We slightly improve the general bound of Proposition 1.1, adopting the classical arguments of Dirac [3] and Erdős-Gallai [5].

Theorem 1.3. *For every $k, n \geq 1$, $\mathrm{ex}_L(n, P_k) \leq 1.5kn$.*

1.1 Lower bound for trees, Steiner systems

Lower bounds for linear Turán numbers of trees relate to analogues of complete graphs. The role of complete graphs in linear triple systems is played by Steiner triple systems.

A *Steiner triple system* $\mathrm{STS}(m)$ is a linear triple system on m points whose triples cover all pairs of points exactly once. They exist if and only if $m \equiv 1, 3 \pmod 6$, $\mathrm{STS}(7)$ is called the Fano plane, $\mathrm{STS}(9)$ is called the affine plane of order three. The notion can be extended to other values of m, *a maximal partial triple system*, $\mathrm{MPTS}(m)$, is a linear triple system on m points whose triples cover the maximum number of pairs of points. It is known that for $m \equiv 0, 2 \pmod 6$ any $\mathrm{MPTS}(m)$ is obtained from a $\mathrm{STS}(m+1)$ by deleting one point; for $m \equiv 5 \pmod 6$ an $\mathrm{MPTS}(m)$ is a linear triple system whose triples cover all pairs of m points except four pairs which form a four-cycle; for

$m \equiv 4 \pmod 6$ an MPTS(m) is a linear triple system whose triples cover all pairs of m points except three pairs which form a star. For more details see [1].

A natural lower bound for $\mathrm{ex}_L(n, T_k)$ is the maximum number of triples in a decomposition of n points into disjoint parts so that each part is an MPTS(m) with $m \leq 2k$. In particular, we get a natural lower bound of $\mathrm{ex}_L(n, T_k)$ when we can use Steiner systems STS($2k-1$) as components.

Proposition 1.4. *If n is divisible by $2k-1$ and $2k \equiv 2, 4 \pmod 6$ then $\mathrm{ex}_L(n, T_k) \geq \frac{n(k-1)}{3}$. This is sharp when T_k is the star S_k.*

However, the best construction is not always provided by taking the maximum number of parts with $2k$ points. For example, if $n \equiv 6 \pmod{10}$ and $k = 5$ then it is better to finish a chain of MPTS(10) with an MPTS(9) and an MPTS(7) than with an MPTS(10) and an MPTS(6). A more serious difficulty is that it is hard to decide whether all STS($2k+1$) contain a given tree T_k or not. This difficulty is shown convincingly by Elliott and Rödl [6], 2019, where they conjectured that for any $\nu > 0$ there exists $k \geq k_0(\nu)$ such that any STS(n) with $n \geq \nu(2k+1)$ contains every T_k (and gave affirmative answer for certain trees).

1.2 Results for small trees

It is obvious that $\mathrm{ex}_L(n, P_2) = \lfloor \frac{n}{3} \rfloor$. The case of P_3 is easy but worth mentioning.

Proposition 1.5. $\mathrm{ex}_L(n, P_3) \leq n$ *with equality if and only if H is the union of disjoint Fano planes.*

There are three trees with four triples apart from the star S_4 (treated in Proposition 1.4). We have sharp estimate for their linear Turán number in two cases. Let B_4 denote the tree obtained from S_3 by appending a triple at a point of degree one. It is easy to see that STS(9) does not contain B_4 or P_4 (see Figure 1, where straight lines with 3 points indicate triples).

Theorem 1.6. *Let $F \in \{B_4, P_4\}$. Then $\mathrm{ex}_L(n, F) \leq \frac{4n}{3}$. Equality holds if and only if the triple system is the union of disjoint affine planes of order* 3.

The fourth tree with four triples is E_4 (see Figure 2). It is obtained from three pairwise disjoint triples by adding one triple that intersects all of them. We construct an E_4-free triple system as follows. Consider a one-factorization of the graph mK_4 (m disjoint copies of K_4). Then extend each of the three factors into $2m$ triples with three distinct new points. This construction has $6m$ triples on $4m + 3$ points, and has no E_4. Adjusting this construction according to divisibility, let $\epsilon = 0$ if $n - 3 \equiv 0, 1 \pmod 4$, $\epsilon = 1$ if $n - 3 \equiv 2 \pmod 4$, $\epsilon = 3$ if $n - 3 \equiv 3 \pmod 4$ we get the lower bound of Theorem 1.7.

Theorem 1.7. $6\lfloor \frac{n-3}{4} \rfloor + \epsilon \leq \mathrm{ex}_L(n, E_4) \leq 2n.$

Our proof of the upper bound in Theorem 1.7 is involved and suggests that determining the asymptotic of $\mathrm{ex}_L(n, F)$ is difficult even for small trees.

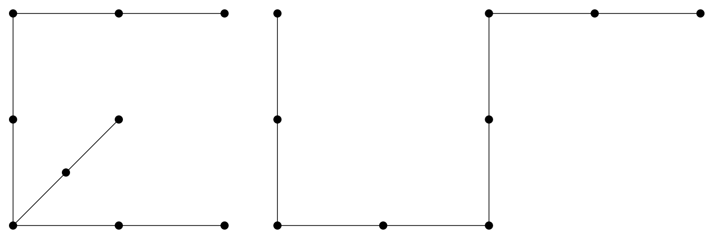

Figure 1: Configurations B_4 and P_4

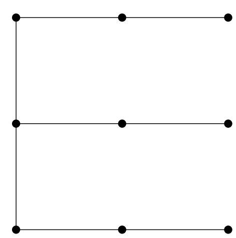

Figure 2: Configuration E_4

2 Matchings - Proof of Theorem 1.2

For the proof of Theorem 1.2 we need a lower bound on $f(n,k)$ for large enough n.

Lemma 2.1. $f(n,k) \geq (k-1)\lfloor\frac{(n-k+1)}{2}\rfloor + \frac{\binom{k-1}{2}}{3} - \frac{4}{3} - \frac{k-1}{6}$ *if* $n \geq 2k-2$.

Proof. Let A be a fixed $(k-1)$-element subset of n points. Place on A an MPTS$(k-1)$. This leaves $0, 3, 4$ or $\frac{k-1}{2}$ pairs of A uncovered (see Subsection 1.1). Then extend the points of A into triples using $k-1$ disjoint perfect or near perfect matchings of the complete graph on the $n-k+1$ points outside A. This can be done since $n-k+1 \geq k-1$. Thus we have at least

$$(k-1)\left\lfloor\frac{(n-k+1)}{2}\right\rfloor + \frac{\binom{k-1}{2} - \max\{0,3,4,\frac{k-1}{2}\}}{3}$$

triples, proving the lemma. $\square$

Although we do not need it for the proof of Theorem 1.2, we give a simple upper bound of $f(n,k)$ as well.

Lemma 2.2. $f(n,k) \le (k-1)\frac{(n-k+1)}{2} + \frac{\binom{k-1}{2}}{3}$.

Proof. Let A be a fixed $(k-1)$-element subset of points in a linear triple system H on n points where all triples of H intersects A. For $j = 1,2,3$ let e_j denote the number of edges intersecting A in j points. The triples intersecting A in two points define a graph G_A with vertex set A and degree sequence d_i $(i = 1,\dots,k-1)$. Using the linearity assumption we can easily get the following estimates for e_j.

- $e_1 \le \sum_{i=1}^{k-1} \frac{n-k+1-d_i}{2}$
- $e_2 = \sum_{i=1}^{k-1} \frac{d_i}{2}$
- $e_3 \le \frac{\binom{k-1}{2}-e_2}{3}$

which gives

$$e_1 + e_2 + e_3 \le (k-1)\frac{(n-k+1)}{2} + \frac{\binom{k-1}{2} - e_2}{3},$$

proving the lemma. □

We prove Theorem 1.2 by induction on k, it trivially holds for $k = 1$ and as we noted before, it also holds for $k = 2$.

To reach a contradiction, suppose that $k \ge 3$ and we have a linear triple system H on n points without M_k such that $|E(H)| > f(n,k)$ and $n > 16(k-1)^2+1$. By the inductive hypothesis H contains M_{k-1} with triples $X_i = \{a_i, b_i, c_i\}$ where $i = 1,\dots,k-1$. Let E_2 denote the set of triples of H intersecting $V(M_{k-1})$ in at least two points. Clearly $|E_2| \le \binom{3(k-1)}{2}$. Since we have no M_k in H, the set E_1 of triples of H not in E_2 intersect $V(M_{k-1})$ in one point.

A triple $\{a_i, b_i, c_i\}$ is called *good* if one of its points, say a_i has degree larger than $3(k-1)$ in E_1, otherwise the triple is called *bad*. W.l.o.g. assume we have j good triples (for some $j, 0 \le j \le k-1$), $X_1, X_2, \dots, X_j$ with points $a_1, a_2, \dots, a_j$ of degree larger than $3(k-1)$ in E_1.

Observe that if a point in X_i $(1 \le i \le k-1)$ has degree at least three in E_1 then the other two points of X_i have degree zero in E_1 - otherwise we have an M_k because the triple X_i in M_{k-1} can be replaced by two disjoint triples of E_1. This implies that the number triples of E_1 incident to a good triple X_i is the degree of a_i in E_1 and the number triples of E_1 incident to a bad triple X_i is at most $3(k-1)$ (using that $k \ge 3$). From this we get the following upper bound on the number of triples in H:

$$|E_2| + |E_1| \le \binom{3(k-1)}{2} + j\left\lfloor \frac{n-3(k-1)}{2} \right\rfloor + (k-1-j)3(k-1). \qquad (1)$$

We claim that this contradicts the assumption $|E(H)| > f(n,k)$ if $j < k-1$. It is enough to check that the RHS of (1) is smaller than the lower bound of

Lemma 2.1. Rewriting the second term of (1) as $j\left\lfloor\frac{n-k+1}{2}\right\rfloor - j(k-1)$ and rearranging we need that

$$\binom{3(k-1)}{2} - \frac{k^2-4k-5}{6} - j(k-1) + (k-1-j)3(k-1) < (k-1-j)\left\lfloor\frac{n-k+1}{2}\right\rfloor.$$

Replacing $\left\lfloor\frac{n-k+1}{2}\right\rfloor$ by the smaller $\frac{n-k-1}{2}$, rearranging and multiplying by 2 we arrive to

$$9(k-1)^2 - 3(k-1) - \frac{k^2-4k-5}{3} - 2j(k-1) + (k-1-j)(6(k-1)+k+1) < (k-1-j)n.$$

The last term on the left hand side is largest for $j=0$ thus it is enough to show that

$$9(k-1)^2 - (k-1) - \frac{k^2-4k-5}{3} - 2j(k-1) + 7(k-1)^2 < (k-1-j)n,$$

and, since the sum of the three terms with a negative sign is less than one for $k \geq 3$, we arrive to

$$16(k-1)^2 + 1 < (k-1-j)n,$$

which is true by the assumption $n > 16(k-1)^2+1$, unless $j = k-1$.

However, $j = k-1$ means that all triples X_i are good. We claim that the set $A = \{a_1, a_2, \ldots, a_{k-1}\}$ meets all triples of H. Indeed, otherwise there exists a triple B such that $B \cap A = \emptyset$. Using that the degrees of the points a_i are larger than $3(k-1)$, we can find by the greedy algorithm $k-1$ pairwise disjoint triples that are disjoint from B as well. This is a contradiction.

We conclude that A intersects all triples of H implying that we have at most $f(n,k)$ triples, contradicting $|E(H)| > f(n,k)$. $\square$

3 Paths - Proof of Theorem 1.3

Starting with standard preparations, let H be a counterexample, a linear triple system on n points with more than $1.5kn$ triples containing no P_k. We may assume that H is a minimal counterexample (neither k nor n can be decreased). Clearly $n > 3$ and $k \geq 5$, since for $k = 2, 3, 4$ we have sharp results (with bounds smaller than $1.5kn$, see Proposition 1.5 and Theorem 1.6). From the choice of k, H contains a path $P = P_{k-1}$ with triples $e_i = \{x_{2i-1}, x_{2i}, x_{2i+1}\}$ for $i = 1, 2, \ldots, k-1$. Also, every point of H has degree at least $1.5k$, otherwise deleting a point with a smaller degree we would get a smaller counterexample.

The points x_1, x_2 are the *left endpoints*, the points x_{2k-2}, x_{2k-1} are the *right endpoints*, and the other points of P are the *interior points* of P. The points of H not on P are called *exterior* points.

Let $A_1(x_1), A_1(x_2)$ denote the set of triples in H containing the left endpoint x_1, x_2, respectively, one interior point, and one exterior point. Similarly, $B_1(x_{2k-1}), B_1(x_{2k-2})$ is the set of triples in H containing the right endpoint x_{2k-1}, x_{2k-2}, respectively, one interior point, and one exterior point. Since a left endpoint and x_3 (a right endpoint and x_{2k-3}) cannot be in a triple of A_1 (B_1) by the linearity of H, we have

$$|A_1(x_1)|, |A_1(x_2)|, |B_1(x_{2k-1})|, |B_1(x_{2k-2})| \leq 2(k-3). \tag{2}$$

Set

$$A_1 = A_1(x_1) \cup A_1(x_2), B_1 = B_1(x_{2k-1}) \cup B_1(x_{2k-2}).$$

We use four claims to capture the maximality of the path P to decrease the trivial bound in (2) for a suitable pair of a left endpoint and a right endpoint. The first claim is obvious.

Claim 3.1. *If $e \in E(H)$ intersects an endpoint of P then $|e \cap P| \geq 2$.*

A *touching pair* is a pair of triples $f_1, f_2 \in E(H)$ such that $f_1 \in A_1, f_2 \in B_1$ and their interior points are the same x_{2i}.

Claim 3.2. *There are no touching pairs in $E(H)$.*

Proof. Let w.l.o.g. $f_1 = \{x_1, p, x_{2i}\}, f_2 = \{x_{2k-1}, q, x_{2i}\}$ be a touching pair and note that $p \neq q$. Thus the path

$$e_{i+1}, \dots, e_{k-1}, f_2, f_1, e_1, \dots, e_{i-1}$$

has k triples, leading to a contradiction. □

Two triples $f_1, f_2 \in E(H)$ are *crossing* over two consecutive interior points x_i, x_{i+1} if $f_1 \in A_1, f_2 \in B_1$ and $x_{i+1} \in f_1, x_i \in f_2$.

Claim 3.3. *If $f_1, f_2 \in E(H)$ are crossing then their exterior points are the same.*

Proof. Suppose w.l.o.g. that $f_1 = \{x_1, p, x_{i+1}\}$ and $f_2 = \{x_{2k-1}, q, x_i\}$ are crossing and $p \neq q$. If i is even, then

$$f_2, e_{k-1}, e_{k-2}, \dots, e_{\frac{i}{2}+1}, f_1, e_1, \dots, e_{\frac{i}{2}-1},$$

and if i is odd, then

$$f_1, e_1, e_2, \dots, \dots, e_{\frac{i-1}{2}}, f_2, e_{k-1}, \dots, e_{\frac{i+3}{2}}$$

defines a path with k triples, a contradiction. □

For a left endpoint x_j (right endpoint x_m) we denote by $\overline{x_j}$ ($\overline{x_m}$) the other left endpoint (other right endpoint).

Claim 3.4. *Assume that $f_1, f_2 \in E(H)$ are crossing over the interior points x_i, x_{i+1} and x_j is the left endpoint of f_1, x_m is the right endpoint of f_2. Then $\{\overline{x_j}, x_{i+1}\}$ is not covered by any triple of A_1 and $\{\overline{x_m}, x_i\}$ is not covered by any triple of B_1.*

Proof. From Claim 3.3 we know that $f_1 = \{x_j, p, x_{i+1}\}, f_2 = \{x_m, p, x_i\}$. If $\{\overline{x_j}, x_{i+1}\}$ is covered by a triple $g = \{\overline{x_j}, s, x_{i+1}\} \in A_1$ then from the linearity of H, $s \neq p$. Now g, f_2 is a crossing pair over x_i, x_{i+1} with different exterior points, contradicting Claim 3.3. The proof of the second statement is similar. □

Lemma 3.5. *For a suitable choice of $x_j \in \{x_1, x_2\}$, and $x_m \in \{x_{2k-2}, x_{2k-1}\}$, we have*

$$|A_1(x_j)| + |B_1(x_m)| \leq 2(k-3.)$$

Proof. From (2), $|A_1| + |B_1| \leq 8(k-3)$. We will give a lower bound for the number of "missing triples" from $A_1 \cup B_1$ as follows. For every fixed $i \in [2, k-2]$ we consider two cases.

Case 1. No triple of $A_1 \cup B_1$ contains x_{2i}. This means that for any endpoint v the pair (x_{2i}, v) is not covered by any triple of $A_1 \cup B_1$ thus we have exactly four missing triples. (See Figure 3, where dashed lines indicate "missing pairs", i.e. pairs which are not covered by any triple of $A_1 \cup B_1$.)

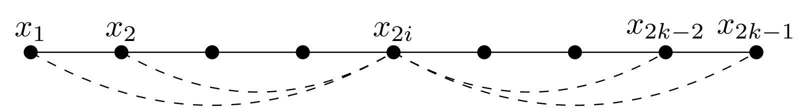

Figure 3: Case 1: No triple of $A_1 \cup B_1$ contains x_{2i}.

Case 2. There exists a triple $e \in A_1 \cup B_1$ containing x_{2i}. The *mate* x^* of x_{2i} is x_{2i-1} if $e \in A_1$ and it is x_{2i+1} if $e \in B_1$. Note that x^* is always an interior point.

Subcase 2.1. There exists $e' \in A_1 \cup B_1$ such that e, e' is a crossing pair over x_{2i}, x^*. By Claim 3.3 e, e' have a common exterior point p. W.l.o.g. $e = (x_j, p, x_{2i}) \in A_1$, $e' = (x_m, p, x^*) \in B_1$. By Claim 3.4, applied to the interior points x^*, x_{2i}, the pair $\{\overline{x_j}, x_{2i}\}$ is not covered by any triple of A_1 and the pair $\{\overline{x_m}, x^*\}$ is not covered by any triple of B_1. Moreover, by Claim 3.2, neither $\{x_{2i}, x_m\}$ nor $\{x_{2i}, \overline{x_m}\}$ is covered by any triple of B_1. Thus we have at least four missing triples. (See Figure 4, where $x^* = x_{2i-1}, x_j = x_1, \overline{x_j} = x_2, x_m = x_{2k-1}, \overline{x_m} = x_{2k-2}$, solid lines indicate covered pairs and dashed lines indicate "missing pairs".)

Subcase 2.2. There is no $e' \in A_1 \cup B_1$ such that e, e' is a crossing pair over x_{2i}, x^*. W.l.o.g. $e = (x_j, p, x_{2i}) \in A_1$ and (from the definition of the subcase) neither $\{x^*, x_m\}$ nor $\{x^*, \overline{x_m}\}$ is covered by any triple of B_1. As in the previous subcase, neither $\{x_{2i}, x_m\}$ nor $\{x_{2i}, \overline{x_m}\}$ are covered by any triple of B_1. Again, we have at least four missing triples. (See Figure 5, where $x^* = x_{2i-1}, x_j = x_1$.)

We conclude that in all cases we have at least four missing triples. Missing triples are not doubly counted, because missing triples on x^* are in A_1 if $x^* = x_{2i+1}$ and in B_1 if $x^* = x_{2i-1}$. Thus altogether we have at least $4(k-3)$

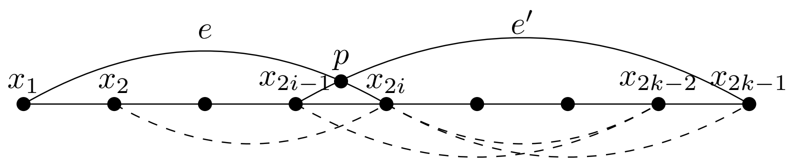

Figure 4: Subcase 2.1: There is a crossing pair.

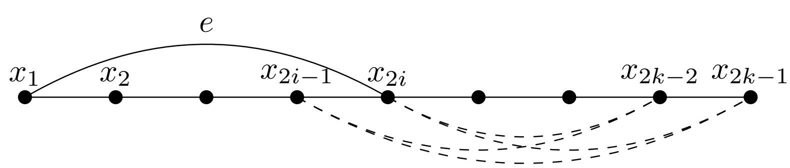

Figure 5: Subcase 2.2.

missing triples in $A_1 \cup B_1$. This implies that we have either at least $2(k-3)$ missing triples with endpoints x_1, x_{2k-1} or at least $2(k-3)$ missing triples with endpoints x_2, x_{2k-2}. This proves the lemma. $\square$

W.l.o.g. let x_1, x_{2k-1} be the pair of endpoints ensured by Lemma 3.5. Denote by $A_2(x_1)$ the set of triples in H containing x_1 and intersecting $P - \{x_1\}$ in two points. Similarly, $B_2(x_{2k-1})$ is the set of triples in H containing x_{2k-1} and intersecting $P - \{x_{2k-1}\}$ in two points. Since H is linear, $2|A_2(x_1)| + |A_1(x_1)| \leq 2k-2$ and $2|B_2(x_{2k-1})| + |B_1(x_{2k-1})| \leq 2k-2$. Adding these inequalities to the inequality $|A_1(x_1)| + |B_1(x_{2k-1})| \leq 2(k-3)$ (from Lemma 3.5) we obtain

$$2(|A_1(x_1)| + |A_2(x_1)| + |B_1(x_{2k-1})| + |B_2(x_{2k-1})|) \leq 6k - 10,$$

from which either $d_H(x_1) = |A_1(x_1)| + |A_2(x_1)| \leq 1.5k - 2.5$ or $d_H(x_{2k-1}) = |B_1(x_{2k-1})| + |B_2(x_{2k-1})| \leq 1.5k - 2.5$, contradicting the minimum degree condition in a minimal counterexample. $\square$

4 Proofs for small trees

Proof of Proposition 1.5. First we prove that $\mathrm{ex}_L(n, P_3) \leq n$. Let H be a minimal counterexample, a P_3-free linear triple system with n points, more than n triples and n is as small as possible. From the minimality H has just one connected component. All points of H have degree at least two, otherwise we find a smaller counterexample by deleting a point with a smaller degree. We may also assume that there exists a point p with degree at least 4, otherwise we have at most n edges in H. Select a star S_k with center p with $k \geq 4$. Then select a point q in S_k different from p. There is a triple e containing q and not containing p. Then e with two triples of S_k form a P_3, leading to a contradiction, thus $\mathrm{ex}_L(n, P_3) \leq n$.

The above argument also shows that in case of $|E(H)| = n$, each connected component of H is 3-regular. Selecting any $A = S_3$, we can observe that any triple intersecting it must be completely inside A, otherwise we have a P_3 leading to a contradiction. Thus each connected component of H is 3-regular on seven points, i.e. an STS(7). $\square$

Proof of Theorem 1.6. Let $F \in \{B_4, P_4\}$, first we prove that $\mathrm{ex}_L(n, F) \leq \frac{4n}{3}$. Let H be a minimal counterexample, an F-free linear triple system with n points, more than $\frac{4n}{3}$ triples and n is as small as possible. From the minimality, H has just one connected component. As in the proof of Proposition 1.5, all points of H have degree at least two, and there exists a point p with degree at least 5. Select the largest star S_m in H with center p (clearly $m \geq 5$). Let $e_i = (p, x_i, y_i), i = 1, \ldots, m$ be the triples of S_m.

Case 1: $F = B_4$. Select a point $q \neq p$ in S_m. Clearly, there must be a triple e through q which does not belong to S_m. Then e and three suitable triples of S_m define a B_4 leading to contradiction. This argument also implies that in case of $|E(H)| = \frac{4n}{3}$, H is a 4-regular triple system. Selecting any $A = S_4$, we can observe that any triple intersecting it must be completely inside A otherwise we have a B_4, leading to contradiction. Thus each component of H is an STS(9).

Case 2: $F = P_4$. We first claim that S_m covers all points of H. Indeed, if w is any point of H not in S_m then the shortest path P from w to $V(S_m)$ has just one triple, otherwise with two suitable triples of S_m we can extend P to a P_4, a contradiction. Thus, since w has degree at least two in H, we have two triples f_1, f_2 containing w such that both of them intersect S_m in points different from the center of S_m. Since $m \geq 5$, we find a triple of S_m (say e_i) disjoint from $(f_1 \cap S_m) \cup (f_2 \cap S_m)$. Then with an appropriate triple (say e_j) of S_m containing a point from either $f_1 \cap S_m$ or $f_2 \cap S_m$, we have the P_4 defined by the triples e_i, e_j, f_1, f_2 in this order unless f_1 and f_2 both intersect the same two triples of S_m, say e_1 and e_2. In this case consider an arbitrary point v of e_i with $i \geq 3$. This point v also has degree at least two, so there must be a triple containing v different from e_i. This must also intersect both f_1 and f_2, otherwise we are done again similarly as above. But there are only two remaining pairs like that between f_1 and f_2, but we have at least 6 vertices that may play the role of v, a contradiction. This proves the claim, i.e. $2m + 1 = n$.

There is a pair f_1, f_2 of intersecting triples in H not containing the center of S_m. Indeed, otherwise we have at most $\frac{n-1}{3} + \frac{n-1}{2} < \frac{4n}{3}$ triples in H, contradicting the assumption. We have two possibilities for f_1 and f_2: either f_1 and f_2 intersect exactly the same three triples of S_m, say e_1, e_2 and e_3, or $m = 5$ and $f_1 \cup f_2$ intersects all triples of S_5, otherwise f_1 and f_2 with two suitable triples of S_m form a P_4, a contradiction again. Then if $m > 5$, we always must have the first possibility; i.e. all triples intersecting f_1 intersect e_1, e_2 and e_3. Then again consider a point v different from p in e_4. There is a triple g containing v different from e_4. By the above, g cannot intersect f_1, but then f_1 and g with two suitable triples of S_m form a P_4.

Thus we may assume that $m = 5, n = 11$, and $|E(H)| \geq 15$. We may also assume that for any pair of intersecting triples in H not containing the center of S_5 we must have the above two possibilities.

On the other hand, there must be two disjoint triples g, h in H not containing the center of S_5. Indeed, otherwise the at least 10 triples that do not contain p pairwise intersect thus they form either a star or must be part of a Fano plane (as noted already before). But within 10 points there is no room for an S_{10} and part of a Fano plane cannot have 10 triples either. Thus we have g, h as required. We get a contradiction again by finding a $P_4 = g, e_i, e_j, h$ where the triples e_i, e_j are in S_5 - unless $g \cup h$ covers exactly three triples of S_5, say e_1, e_2, e_3 and $g = (x_1, x_2, x_3)$ and $h = (y_1, y_2, y_3)$. This argument holds for any two disjoint triples in the role of g, h thus any further triple ℓ (different from g, h and triples of S_5) must intersect both g and h, so in particular ℓ must intersect two of the triples e_1, e_2 and e_3. Applying this to a triple ℓ containing a point v different from p of e_4, ℓ must intersect both g and h. However, we cannot have the first possibility for the intersecting pair g and ℓ, so we must have the second, however, $g \cup \ell$ now cannot intersect all triples of S_5 (since they both intersect the same two triples of S_5, together they may intersect at most 4 triples of S_5), contradicting the assumption. This finishes the proof of $\mathrm{ex}_L(n, P_4) \leq \frac{4n}{3}$.

In case of $|E(H)| = \frac{4n}{3}$ we may assume that H is 4-regular, otherwise H has a point of degree at least 5 and the above argument leads to a contradiction. We prove that the first claim in Case 2 is still valid, any connected component C of H containing an S_4 must contain only the points of S_4. Indeed, otherwise all the four triples f_1, f_2, f_3, f_4 on a point w not on S_4 must intersect S_4 (in a point different from the center of S_4). However, if any f_i intersects S_4 in one point then with f_j $(j \neq i)$ the triples f_i, f_j and two suitable triples of S_4 create a P_4. Thus all f_i intersect S_4 in two points, moreover the union of any two intersections must either cover all triples of S_4 or just two of them. W.l.o.g. we may assume that the four triples on w are $(w, x_1, x_2), (w, y_1, y_2), (w, x_3, x_4), (w, y_3, y_4)$. Since n must be divisible by 3, there exist two further points q, r in C different from w outside S_4. Repeating the argument for q, it is easy to check that the four triples through q must be $(q, x_1, y_2), (q, x_2, y_1), (q, x_3, y_4), (w, x_4, y_3)$ otherwise there would be a triple on w and a triple on q that define a P_4 with two suitable triples of S_4. However, there is no room to place *any* triple on r without creating a P_4. This proves that C is 4-regular on 9 points, thus it is an affine plane of order 3. $\square$

Proof of Theorem 1.7. Let H be a linear triple system with n points containing no E_4. We prove by induction on n, it is trivial to launch it from $n = 1$. If there is a point v with degree $d(v) \leq 2$ we are done by induction on n because removing v we have at most $2 + 2(n-1) = 2n$ triples. Thus $d(v) \geq 3$ holds for all $v \in V(H)$. We define the partition $V(H) = Y \cup Z$ where Z contains the points of degree at most four, Y is the set of remaining points (of degree at least five).

For any triple $e = \{a, b, c\} \in E(H)$ let $D(e)$ denote the vector

$$(d(a), d(b), d(c)),$$

where the degrees are in decreasing order. We always assume that the points a, b, c are ordered the same way as their degrees. We consider these vectors partially ordered by $D(e) \geq D(f)$ if at all positions the coordinate of e is larger than or equal to the coordinate of f. In the proof we consider two cases.

Case I. There is no $e \in E(H)$ such that $D(e) \geq (5, 5, 3)$.

In this case all triples of H intersect Z in at least two points. Therefore $\sum_{v \in Z} d(v)$ is at least a double count of $|E(H)|$ implying

$$|E(H)| \leq \frac{1}{2} \sum_{v \in Z} d(v) \leq \frac{1}{2}(4|Z|) \leq 2n,$$

finishing the proof of Theorem 1.7 in Case I.

Case II. There exists $e = \{a, b, c\} \in E(H)$ with $D(e) \geq (5, 5, 3)$. Select e such that $D(e) = (d_1, d_2, d_3)$ is a minimal element in the up-set of $(5, 5, 3)$. The next proposition restricts d_1.

Proposition 4.1. *If $d_1 > 5$ then H contains E_4.*

Proof. Since $d_3 > 1$, we can select a triple f containing c and different from e, then by $d_2 > 3$ we can select a triple g containing b and disjoint from f. Finally, using that $d_1 > 5$ we can select a triple h containing a and disjoint from both f and g. Thus e, f, g, h form an E_4. $\square$

Proposition 4.1 implies that

$$D(e) \in \{(5, 5, 5), (5, 5, 4), (5, 5, 3)\}. \tag{3}$$

We show in Claim 2 that the assumption $D(e) \geq (5, 5, 3)$ is very restrictive. Let $F \subset (H \setminus e)$ be the triple system spanned by the 4 triples containing a, the 4 triples containing b and 2 triples containing c. These ten triples can be denoted as $a \cup \alpha_i, b \cup \beta_j, c \cup \gamma_k$ where $i, j = 1, 2, 3, 4; k = 1, 2$. Note that by (3) there may be at most two more *extra triples* (not in F) containing c. We define the graph G with edge set $\alpha_i, \beta_j, \gamma_k$ on vertex set $V(H) \setminus \{a, b, c\}$. We ignore isolated points and consider G with the vertex set spanned by its (ten) edges.

A set of three pairwise disjoint edges of G, $\alpha_i, \beta_j, \gamma_k$ is called a *rainbow matching*. Since a rainbow matching in G together with e form a copy of an E_4, there is no rainbow matching in G.

Two among the many possible graphs G are G_1, G_2 defined as follows. Both have two point-disjoint alternating four-cycles with edges $\alpha_1, \beta_1, \alpha_2, \beta_2$ and $\alpha_3, \beta_3, \alpha_4, \beta_4$. In G_1 the edges γ_1, γ_2 are the diagonals of the first four-cycle. In G_2 the edges γ_1, γ_2 are diagonals in the first and the second four-cycle, respectively (see Figures 6 and 7).

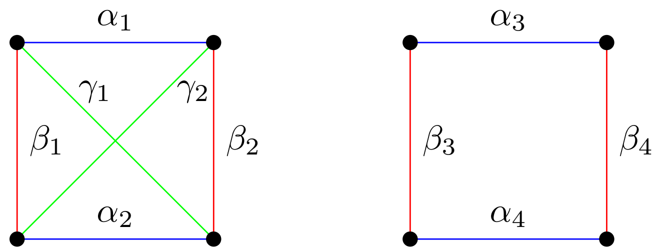

Figure 6: The graph G_1

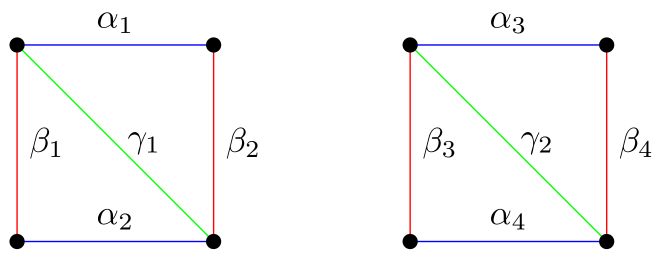

Figure 7: The graph G_2

Claim 2. Apart from permutations of the α_i's, β_j's, γ_k's, either $G = G_1$ or $G = G_2$.

Proof. Clearly, there exist at least two α_i's and two β_j's, say $\alpha_3, \alpha_4, \beta_3, \beta_4$ not incident to γ_1. For any $i, j \in \{3, 4\}$ we have $\alpha_i \cap \beta_j \neq \emptyset$ otherwise there is a rainbow matching together with γ_1. Therefore $\alpha_3, \beta_3, \alpha_4, \beta_4$ form an alternating four-cycle with vertex set X_2. Note that none of the edges $\alpha_1, \beta_1, \alpha_2, \beta_2$ can be incident to X_2 but all of them have to be incident to γ_1, otherwise we have a rainbow matching. Therefore w.l.o.g α_1, β_1 and α_2, β_2 are incident to different endpoints of γ_1 and all of them are inside $V(G) \setminus X_2$.

Assume that either $\alpha_1 \cap \beta_2 = \emptyset$ or $\alpha_2 \cap \beta_1 = \emptyset$, w.l.o.g the first. Then, to avoid a rainbow matching, γ_2 must intersect both of them. However, then $\gamma_2, \beta_1, \alpha_3$ is a rainbow matching, contradiction. Thus $\alpha_i \cap \beta_j \neq \emptyset$ for $i, j \in \{1, 2\}$ implying that $\alpha_1, \beta_1, \alpha_2, \beta_2$ form an alternating four-cycle with vertex set X_1 (with diagonal γ_1). Finally, γ_2 can be the second diagonal of this four-cycle or a diagonal in the four-cycle in X_2, proving the claim. $\square$

Claim 3. At most four triples of H have nonempty intersection with both $V(F)$ and $V(H) \setminus V(F)$.

Proof. Suppose that the claim is not true and consider any $f \notin E(F)$ that intersects both $V(F)$ and $V(H) \setminus V(F)$. From Claim 2 we have two alternating four-cycles, say $\alpha_1, \beta_1, \alpha_2, \beta_2$ and $\alpha_3, \beta_3, \alpha_4, \beta_4$ in G with disjoint vertex sets, let X_1, X_2 denote these. We look at possibilities for $f \cap V(F)$. Observe first that only at most two extra triples may intersect $e = \{a, b, c\}$. Assume that f is not an extra triple. Then $|f \cap V(G)| \neq 0$. We have the following two possibilities:

- 1. $|f \cap V(G)| = 1$. Assume w.l.o.g that $\{x\} = f \cap V(G) \in X_1$ and $x = \alpha_1 \cap \beta_1$. Then $f, \beta_2 \cup b, \alpha_3 \cup a$ are three pairwise disjoint triples and the triple $\alpha_1 \cup a$ intersects all of them, thus we have an E_4, a contradiction.
- 2. $|f \cap V(G)| = 2$. Assume $\{x, y\} = f \cap V(G)$. If $x \in X_1, y \in X_2$, we may assume again by symmetry that $x = \alpha_1 \cap \beta_1$. Select $\alpha_j \in \{\alpha_3, \alpha_4\}$ so that $y \notin \alpha_j$. Then $f, \beta_2 \cup b, \alpha_j \cup a$ are three pairwise disjoint triples and the triple $\alpha_1 \cup a$ intersects all of them, thus we have an E_4, a contradiction. Thus x, y are on the same four-cycle. Therefore, by the linearity of H, $f \cap V(G)$ is one of the two missing diagonals of the two four cycles.

We conclude that only the extra triples and the ones from case 2 have nonempty intersection with both $V(F)$ and $V(H) \setminus V(F)$, proving Claim 3. $\square$

Observe that the total number of triples incident to $V(F)$ is at most 15: the eleven triples defined by the vector $(5, 5, 3)$, the at most two extra triples on c, and the at most two triples that cover the two missing edges in the components of G_1 or G_2. Therefore, applying the inductive hypothesis for $H[V(H) \setminus V(F)]$, $|E(H)| \leq 15 + 2(n - 11) < 2n$, finishing the proof of Theorem 1.7 in Case II. $\square$

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