Some exact results for generalized Turán problems

Dániel Gerbner,[*] Cory Palmer[†]

June 9, 2020

Abstract

Fix a k-chromatic graph F. In this paper we consider the question to determine for which graphs H does the Turán graph $T_{k-1}(n)$ have the maximum number of copies of H among all n-vertex F-free graphs (for n large enough). We say that such a graph H is *F-Turán-good*. In addition to some general results, we give (among others) the following concrete results:

(i) For every complete multipartite graph H, there is k large enough such that H is K_k-Turán-good.

(ii) The path P_3 is F-Turán-good for F with $\chi(F) \geq 4$.

(iii) The path P_4 and cycle C_4 are C_5-Turán-good.

(iv) The cycle C_4 is F_2-Turán-good where F_2 is the graph of two triangles sharing exactly one vertex.

1 Introduction

Fix a graph F. We say that a graph G is *F-free* if it does not contain F as a subgraph. A cornerstone of extremal graph theory is Turán's theorem [25], which determines the maximum number of edges in an n-vertex K_k-free graph. The extremal construction is a complete $(k-1)$-partite graph on n vertices such that each class has cardinality either $\lceil n/(k-1) \rceil$ or $\lfloor n/(k-1) \rfloor$. Such a graph is called a *Turán graph* and is denoted $T_{k-1}(n)$.

Turán's theorem is the starting point of many avenues of research. The *Turán function* $\mathrm{ex}(n, F)$ is the maximum number of edges in an n-vertex F-free graph. In this notation, Turán's theorem states $\mathrm{ex}(n, K_k) = |E(T_{k-1}(n))|$. We call an n-vertex F-free graph with $\mathrm{ex}(n, F)$ edges an *extremal graph for F*. Thus, the Turán graph $T_{k-1}(n)$ is the extremal

graph for K_k. The fundamental Erdős-Stone-Simonovits theorem [7, 6] states that if the chromatic number of F is $k \geq 2$, then

$$\mathrm{ex}(n, F) = (1 + o(1))|E(T_{k-1}(n))| = \left(1 - \frac{1}{k-1} + o(1)\right) \binom{n}{2}.$$

Simonovits [24] characterized those graphs F that have the Turán graph as their unique extremal graph. We say that an edge e of a graph F is *color-critical* if deleting e from F results in a graph with smaller chromatic number.

Theorem 1 (Color-critical edge theorem, Simonovits [24]). *Let F be a k-chromatic graph. For n large enough, the unique extremal graph for F is the Turán graph $T_{k-1}(n)$ if and only if F has a color-critical edge.*

Here we consider a generalization of the results described above. Fix graphs H and G. Denote the number of copies of H in G by $\mathcal{N}(H, G)$. Now fix graphs F and H. Define

$$\mathrm{ex}(n, H, F) := \max\{\mathcal{N}(H, G) : G \text{ is an } n\text{-vertex } F\text{-free graph}\},$$

i.e., $\mathrm{ex}(n, H, F)$ is the maximum number of copies of the graph H in an n-vertex F-free graph. An early result of Zykov [26] (see also Erdős [4]) determines the maximum number of copies of K_r in a K_k-free graph.

Theorem 2 (Zykov [26]). *The Turán graph $T_{k-1}(n)$ is the unique n-vertex K_k-free graph with the maximum number of copies of K_r. Thus,*

$$\mathrm{ex}(n, K_r, K_k) = \mathcal{N}(K_r, T_{k-1}(n)) \leq \binom{k-1}{r} \left\lceil \frac{n}{k-1} \right\rceil^r.$$

After several other sporadic results (see, e.g., [2, 14, 16, 17, 18, 20]), the general investigation of this function was initiated by Alon and Shikhelman [1]. For several further recent results see, e.g., [8, 9, 10, 11, 12, 13, 19]. Despite these investigations, there are only few cases when $\mathrm{ex}(n, H, F)$ is determined exactly. One difficulty in determining $\mathrm{ex}(n, H, F)$ exactly is that there are few F-free graphs that are good candidates for being extremal constructions for maximizing copies of a graph H. Our understanding of F-free graphs is not deep enough to describe those graphs that are "largest" in some sense. An exception is the Turán graph. In this paper we examine when the Turán graph is the extremal construction for these so-called generalized Turán problems.

Definition 3. Fix a k-chromatic graph F and a graph H that does not contain F as a subgraph[1]. We say that H is *F-Turán-good* if $\mathrm{ex}(n, H, F) = \mathcal{N}(H, T_{k-1}(n))$ for every n large enough. If $F = K_k$, we use the briefer term *k-Turán-good*.

---

[1] We include the condition on H to avoid the degenerate case that $\mathrm{ex}(n, H, F) = \mathcal{N}(H, T_{k-1}(n)) = 0$ which would allow that H is F-Turán-good.

2

Using this notation, Turán's theorem states that K_2 is k-Turán-good for every $k > 2$, Theorem 1 states that K_2 is F-Turán-good for any F with a color-critical edge and Theorem 2 states that K_r is k-Turán-good for $r < k$.

Győri, Pach and Simonovits [18] considered the problem to determine which graphs H are k-Turán-good. In particular, they showed that a bipartite graph H on $m \geq 3$ vertices containing $\lfloor m/2 \rfloor$ independent edges is 3-Turán-good. This implies that the path P_l, the even cycle C_{2l} and the Turán graph $T_2(m)$ are all 3-Turán-good. They also gave the following general theorem.

Theorem 4 (Győri, Pach and Simonovits [18]). *Let $r \geq 3$ and let H be a $(k-1)$-partite graph with $m > k-1$ vertices containing $\lfloor m/(k-1) \rfloor$ vertex disjoint copies of K_{k-1}. Suppose further that for any two vertices u and v in the same component of H, there is a sequence $A_1, \ldots, A_s$ of $(k-1)$-cliques in H such that $u \in A_1$, $v \in A_s$, and for any $i < s$, A_i and A_{i+1} share $k-2$ vertices. Then H is k-Turán-good.*

They mention that this theorem implies that $T_{k-1}(m)$ is k-Turán-good and again that paths and even cycles are 3-Turán-good. When H is a complete multipartite graph they gave the following theorem.

Theorem 5 (Győri, Pach and Simonovits [18]). *Let H be a complete r-partite graph and let n be large enough. If G is an n-vertex K_k-free graph with the maximum number of copies of H, then G is a complete $(k-1)$-partite graph.*

In [18], the authors remark that a graph G in the above theorem need not be a Turán graph and give an example where the ratio of the sizes of the largest and smallest class of G is not even bounded. They also gave an optimization argument to show that C_4 is k-Turán-good for any k. They state the same for $K_{2,3}$, but omit the details.

Less is known in the case when the forbidden graph F is not a clique. Ma and Qiu [21] proved that a $(k-1)$-partite graph H is k-Turán-good if it has $k-2$ classes of size s and one class of size t with $s \leq t < s + 1/2 + \sqrt{2s + 1/4}$. They also proved a common generalization of Theorems 1 and 2.

Theorem 6 (Ma and Qiu [21]). *Let F be graph with $\chi(F) = k > r$ and a color-critical edge. Then K_r is F-Turán-good. Moreover, for n large enough, $T_{k-1}(n)$ is the unique n-vertex F-free graph with the maximum number of copies of H.*

Let us discuss some simple conditions that force H to not be F-Turán-good. When $\chi(H) \geq \chi(F) = k$, then the Turán graph $T_{k-1}(n)$ contains no copies of H, so H cannot be F-Turán-good. Observe that if the sizes of color classes of H are very unbalanced, then it is possible that among the complete $(\chi(F) - 1)$-partite graphs on n vertices, the Turán graph does not have the maximum number of copies of H. When H is a complete multipartite graph, a straightforward optimization can determine which complete $(\chi(F) - 1)$-partite graphs contain the maximum number of copies of H. Some calculations of this type were performed in [18, 21, 3]. If F is a k-chromatic graph with no color-critical edge, then we can add an edge e to the Turán graph $T_{k-1}(n)$ and still have no copy of F. If $\chi(H) \leq \chi(F) - 2$,

then it is easy to see that in the resulting graph (for n large enough) there are copies of H that contain e. Therefore, in this case, H is not F-Turán-good. Thus, when F has no color-critical edge we can restrict our attention to the case when $\chi(H) = \chi(F) - 1$.

The rest of this paper is organized as follows. In Section 2 we consider k-Turán-good graphs and prove a theorem that is of a similar flavor to Theorem 4. We also show that for any complete multipartite graph H, there is a k_0 large enough such that if $k \geq k_0$, then H is k-Turán-good. In Section 3 we consider the case when F is not a clique. Among others, we prove that P_3 is F-Turán-good for F with $\chi(F) \geq 4$, that P_4 and C_4 are C_5-Turán-good, and that C_4 is F_2-Turán-good where F_2 is the graph of two triangles sharing exactly one vertex. We finish the paper with some concluding remarks and conjectures in Section 4.

2 Forbidding cliques

The main theorem of this section describes a method to construct k-Turán-good graphs.

Theorem 7. *Let H be a k-Turán-good graph. Let H' be any graph constructed from H in the following way. Choose a complete subgraph of H with vertex set X, add a vertex-disjoint copy of K_{k-1} to H and join the vertices in X to the vertices of K_{k-1} by edges arbitrarily. Then H' is k-Turán-good.*

Proof. By Theorem 2, the maximum number of copies of K_{k-1} in a K_k-free graph is achieved by the Turán graph $T_{k-1}(n)$. Since H is k-Turán-good, the Turán graph $T_{k-1}(n-k+1)$ has the maximum number of copies of H among K_k-free graphs on $n - k + 1$ vertices. We will show that $T_{k-1}(n)$ has the maximum number of copies of H'.

Let G be a K_k-free graph on n vertices with the maximum number of copies of H'. Since H' contains a copy of K_{k-1}, the graph G must contain a copy of K_{k-1}. Let K be a copy of K_{k-1} in G. Every other vertex of G is adjacent to at most $k - 2$ vertices of K. Let Y be a complete graph that is disjoint from K.

Consider an auxiliary bipartite graph with classes formed by the vertices of Y and K and join two vertices by an edge if they are non-adjacent in G.

Suppose this bipartite graph does not have a matching saturating the class Y, i.e., a matching that uses every vertex of Y. Then, by Hall's theorem, there exists a non-empty subset Y' of Y whose neighborhood in K has size less than $|Y'|$. In the original graph G this means that all of the vertices in Y' are connected to a fixed set of more than $|K| - |Y'|$ vertices in K. As Y' and K are complete graphs, this gives a copy of K_k in G, a contradiction. Therefore, this auxiliary bipartite graph has a matching saturating Y which implies that in G the edges between Y and K are a subgraph of a complete bipartite graph minus a matching saturating Y.

On the other hand, in a $(k-1)$-partite Turán graph the edges between K_{k-1} and a clique of size $|Y|$ form a complete bipartite graph minus a matching saturating the clique of size $|Y|$. This implies that there are at least as many ways to join the vertices of a copy of H with a copy of K_{k-1} in a Turán graph as in G.

4

The number of copies of H' is the product of the number of copies of K_{k-1}, the number of copies of H on the remaining $n-k+1$ vertices and the number of ways to join the vertices of K_{k-1} and H all divided by the number of times a copy of H' was counted. The first three quantities are maximized by the Turán graph, while the last quantity depends only on H'. This implies that the number of copies of H' is maximized by $T_{k-1}(n)$. □

We remark that Theorem 7 implies the same results mentioned after Theorem 4. However, neither Theorem 7 nor Theorem 4 imply the other. They both use copies of K_{k-1} as building blocks and connect them with additional edges, but Theorem 4 allows adding many edges. For example, when $k = 3$ the only assumptions on H are that H is bipartite and has a matching of size $\lfloor |V(H)|/2 \rfloor$. In Theorem 7, when $k = 3$, if we build H starting from a single edge, there is always an edge (the one added last) such that its vertices are incident to at most two other vertices. On the other hand, in Theorem 7 we do not need a sequence of $(k-1)$-cliques connecting any two vertices. For example we can take two copies of K_{k-1} and connect them with a single edge. The resulting graph is k-Turán-good because of Theorem 7.

Nonetheless, both Theorem 4 and Theorem 7 require copies of K_{k-1} as building blocks. For example, we know that P_l is 3-Turán-good, and Theorem 5 implies that P_3 is k-Turán-good, but for longer paths neither Theorem 4 nor Theorem 7 can be applied. We conjecture that paths are k-Turán good (see Conjecture 19 in Section 4). Here we are able to show that the maximum number of copies of P_l in K_k-free graphs is asymptotic to the number of copies in the Turán graph. In fact, we can replace K_k with any k-chromatic graph F.

Proposition 8. *If F is k-chromatic with $k > 2$, then* $\mathrm{ex}(n, P_l, F) = (1+o(1))\mathcal{N}(P_l, T_{k-1}(n))$.

Proof. We will use spectral methods as they were used in [19] and [10]. We use the following simple facts: every path is a walk and a path with more than 2 vertices corresponds to two walks (one starting from each end-vertex of the path). Therefore, the number of walks of length $l - 1$ (i.e. having $l - 1$ edges) is at least twice the number of paths of length $l - 1$.

For a matrix M let $\mu(M)$ denote the largest eigenvalue of M. Now let $A(G)$ be the adjacency matrix of G. The number of walks of length $l - 1$ in G is at most $\mu(A(G))^{l-1}/n$. (This is well-known, see [19] and [10] for simple proofs.)

The largest eigenvalue of the adjacency matrix of graph is well-studied. Babai and Guidulli [15] and independently Nikiforov [22] proved that if F has chromatic number k and G is an n-vertex F-free graph, then $\mu(A(F)) = (1 - \frac{1}{k-1} + o(1))n$. Therefore, we obtain that

$$\mathrm{ex}(n, P_l, F) \leq \frac{1}{2}\left(1 - \frac{1}{k-1} + o(1)\right)^{l-1} n^l.$$

Now let us count the number of copies of P_l in the Turán graph $T_{k-1}(n)$. Counting greedily we have n choices for the first vertex. Each subsequent vertex must be in a different class of $T_{k-1}(n)$ from its predecessor and must be different from the previous vertices. Therefore, at each step (after the first) the number of choices for a vertex is at least

$$n - \left\lceil \frac{n}{k-1} \right\rceil - l + 1 = \left(1 - \frac{1}{k-1} - o(1)\right)n.$$

5

In this way, every path is counted twice. Therefore, the number of paths in $T_{k-1}(n)$ is

$$\mathcal{N}(P_l, T_{k-1}(n)) = \frac{1}{2}\left(1 - \frac{1}{k-1} - o(1)\right)^{l-1} n^l.$$

$\square$

We now turn our attention to the case when H is a complete multipartite graph. We begin with a lemma.

Lemma 9. *For any graph H there are integers k_0 and n_0 such that if $k \geq k_0$ and $n \geq n_0$, then for any complete $(k-1)$-partite n-vertex graph G we have $\mathcal{N}(H, G) \leq \mathcal{N}(H, T_{k-1}(n))$.*

Proof. Suppose G contains the maximum number of copies of H among all complete $(k-1)$-partite graphs on n vertices. Suppose, for the sake of a contradiction, that G is not the Turán graph $T_{k-1}(n)$.

Observe first that we can assume H is a complete multipartite graph. Indeed, if H has chromatic number r, then there is a constant number of ways to add edges to H to create a complete r-partite graph with $|V(H)|$ vertices. Each copy of H in G is contained in such a complete r-partite graph in G. Given such a complete r-partite graph, we can count the number of copies of H it contains. Therefore, if the number of copies of each such complete r-partite graph is maximized by the Turán graph $T_{k-1}(n)$, then the same holds for H.

We distinguish two cases.

Case 1: There are two vertex partition classes A and B of G such that $|A| \geq |V(H)||B|$.

In this case we will move a vertex from A to B to create a new complete $(k-1)$-partite graph. We will show that this new graph contains more copies of H than G. Observe that H intersects A and B in a bipartite graph H'. The number of ways to extend H' to H using other classes of G does not change when moving a vertex from A to B. Therefore, if the number of copies of each possible H' does not decrease by this change, then the number of copies of H does not decrease either. Moreover, if the number of copies of some H' increases, then the number of copies of H increases, which is a contradiction.

To show that the number of copies of H' increases, assume first that H' is connected. As H is complete multipartite, this implies that H' is a complete bipartite graph $K_{a,b}$ for some a, b with $a + b \leq |V(H)|$. We have $\binom{|A|}{a}\binom{|B|}{b} + \binom{|A|}{b}\binom{|B|}{a}$ copies of H' between A and B. It is easy to see that this number increases when we move a vertex from A to B.

If H' is disconnected, there may be multiple ways to embed it to the classes A and B. However, for every such embedding with a' and b' vertices in A and B, the same argument as above shows that the number of such embeddings increases when we move a vertex from A to B, thus the number of copies of H' increases.

Case 2: For every pair of partition classes A and B in G, we have $|A| < |V(H)||B|$.

Let us fix $\epsilon > 0$ and choose k_0 such that $k_0 - 1 > |V(H)|/\epsilon$. Now assume that $k \geq k_0$. Then the average size of the classes in G is

$$\frac{n}{k-1} \leq \frac{n}{k_0 - 1} < \frac{\epsilon n}{|V(H)|}.$$

6

Therefore, the size of each class X of G satisfies

$$\frac{1}{(k-1)|V(H)|}n \le |X| \le \frac{|V(H)|}{k-1}n < \epsilon n.$$

The graph G is not a Turán graph, so it has classes A and B such that $|A| > |B|+1$. Let us move a vertex from A to B to create a new complete $(k-1)$-partite graph G'. Let us count the number of copies of H destroyed and created when moving a vertex from A to B. It is well-known and easy to see that G' has more edges than G.

Those copies of H in G that do not have any edge from A to B remain in the graph. For each edge uv between A and B consider the copies of H where u and v are the only vertices of H in $A \cup B$. Observe that their number does not depend which vertices u and v we choose from A and B. We can greedily pick a vertex from each of the other $|V(H)| - 2$ distinct classes to extend uv to a unique such copy of H. At each step we can choose from at least $n - |V(H)|\epsilon n$ vertices. Therefore, the number of such copies of H is at least

$$((1 - |V(H)|\epsilon)n)^{|V(H)|-2}.$$

As there are more edges between A and B in G' than in G, we have created at least $((1 - |V(H)|\epsilon)n)^{|V(H)|-2}$ new copies of H.

Now consider a copy of H that has at least two edges between A and B. Such copy of H has $p \ge 3$ vertices in $A \cup B$. These p vertices can be extended to a copy of H in at most $n^{|V(H)|-p}$ ways. Now pick an arbitrary bipartite subgraph H' of H with $p \ge 3$ vertices. We claim that the number of copies of H' in $A \cup B$ decreases by at most $\epsilon c n^{p-2}$ when we move a vertex v from A to B for some constant c that depends only on H.

Indeed, consider a proper 2-coloring of H' with a vertices of color red and b vertices of color blue. We may suppose that v is in our copy of H' otherwise H is unchanged. Then we have to pick $a - 1$ vertices from A and b vertices from B (or vice versa) to form a copy of H'. Therefore, we start with $\binom{|A|}{a-1}\binom{|B|}{b} + \binom{|A|}{b}\binom{|B|}{a-1}$ copies of H' and, after moving v, we end up with $\binom{|A|-1}{a-1}\binom{|B|+1}{b} + \binom{|A|-1}{b}\binom{|B|+1}{a-1}$ copies of H'. Recall that only $|A|$ and $|B|$ depend on n and ϵ. Simple expansion gives

$$\binom{|A|-1}{a-1}\binom{|B|+1}{b} = \frac{(|A|-a+1)(|B|+1)}{|A|(|B|+1-b)}\binom{|A|}{a-1}\binom{|B|}{b}.$$

Therefore, the difference between the first terms of these counts of H' is

$$\frac{(|A|-a+1)(|B|+1) - |A|(|B|+1-b)}{|A|(|B|+1-b)}\binom{|A|}{a-1}\binom{|B|}{b}$$

$$\le \frac{|A|b - |B|(a-1)}{|A|(|B|+1-b)}|A|^{a-1}|B|^b \le \frac{|A|b - |B|(a-1)}{|B|+1-b}(\epsilon n)^{a+b-2} = c_0(\epsilon n)^{a+b-2},$$

where $c_0 \le |V(H)|^2(b+1)$. A similar bound can be obtained for the difference of the second terms, proving our claim.

This shows that the number of copies of H that have more than one edge between A and B decreases by at most $c\epsilon n^{|V(H)|-2}$, thus the total number of copies of H increases, a contradiction. $\qquad\square$

When H is a star S_t, Cutler, Nir and Radcliffe [3] state that numerical evidence suggests for small t (i.e., large k) that S_t is k-Turán-good. We can confirm this statement for every multipartite H. Indeed, Theorem 5 and Lemma 9 together imply the following theorem.

Theorem 10. *For every complete multipartite graph H there is an integer k_0 such that if $k \geq k_0$, then H is k-Turán-good.*

We believe that Theorem 10 should hold for any graph H. See Conjecture 20 in Section 4 for details.

3 Forbidding non-cliques

We begin this section with a simple proposition.

Proposition 11. *If F is a graph with chromatic number $\chi(F) = k \geq 4$ and a color-critical edge, then P_3 is F-Turán-good.*

Proof. Fix an F-free n-vertex graph G and let $p(G)$ be the number of induced copies of P_3. Let us count the number of pairs (uv, w) where uv is an edge in G and w is a vertex in G that is distinct from u and v. Clearly, there are $|E(G)|(n-2)$ such pairs. On the other hand, on any three vertices there is at most one triangle or one induced P_3. Moreover, each triangle consists of three such pairs (uv, w) and every induced P_3 consists of two such pairs (uv, w). Thus

$$2p(G) + 3\mathcal{N}(K_3, G) \leq |E(G)|(n-2). \tag{1}$$

For the graph $G = T_{k-1}(n)$ we have equality in (1). By Theorems 1 and 6, we have that $|E(G)| \leq |E(T_{k-1}(n))|$ and $\mathcal{N}(K_3, G) \leq \mathcal{N}(K_3, T_{k-1}(n))$. Counting copies of P_3 in G gives

$$
\begin{aligned}
\mathrm{ex}(n, P_3, F) = \mathcal{N}(P_3, G) = p(G) + 3\mathcal{N}(K_3, G) &= \left(p(G) + \frac{3}{2}\mathcal{N}(K_3, G)\right) + \frac{3}{2}\mathcal{N}(K_3, G) \\
&\leq \left(p(G) + \frac{3}{2}\mathcal{N}(K_3, G)\right) + \frac{3}{2}\mathcal{N}(K_3, T_{k-1}(n)) \\
&\leq \frac{1}{2}|E(G)|(n-2) + \frac{3}{2}\mathcal{N}(K_3, T_{k-1}(n)) \\
&\leq \frac{1}{2}|E(T_{k-1}(n))|(n-2) + \frac{3}{2}\mathcal{N}(K_3, T_{k-1}(n)) = \mathcal{N}(P_3, T_{k-1}(n)).
\end{aligned}
$$

$\square$

We believe that the condition on the chromatic number of F can be reduced to 3 in Proposition 11.

3.1   Forbidding a book

Recall that a *book* B_k is the graph of k triangles all sharing exactly one common edge. Note that book B_2 is simply the graph resulting from removing an edge from K_4. We will show that both C_4 and P_4 are B_2-Turán-good.

Let $\overline{M_k}$ be the complement of the graph of k independent edges, i.e., $\overline{M_k}$ is a clique on $2k$ vertices with the edges of a perfect matching removed. Let $\overline{M_k}^+$ be the graph resulting from adding an edge to $\overline{M_k}$, i.e, $\overline{M_k}^+$ is the graph of a clique on $2k$ vertices with all but one of the edges of a perfect matching removed. Note that $\overline{M_k}$ and $\overline{M_k}^+$ differ by a single edge and that $\chi(\overline{M_k}) = k$ and $\chi(\overline{M_k}^+) = k+1$, i.e., the graph $\overline{M_k}^+$ has a color-critical edge. Also note that when $k = 2$, we have that $\overline{M_k}$ is the cycle C_4 and $\overline{M_k}^+$ is the book B_2 (i.e., K_4 minus an edge).

Lemma 12. *Let H be a $2k$-vertex graph consisting of two vertex-disjoint copies of K_k joined together with edges arbitrarily. If $\overline{M_k}$ has the maximum number of copies of H among all $2k$-vertex $\overline{M_k}^+$-free graphs, then H is $\overline{M_k}^+$-Turán-good. In particular, $\overline{M_k}$ is $\overline{M_k}^+$-Turán-good.*

Proof. We can count the copies of $\overline{M_k}$ by counting the number of ways to choose a pair of disjoint copies of K_k and then counting the number of copies of $\overline{M_k}$ spanned by these two copies of K_k. We show that each of these quantities is maximized among n-vertex $\overline{M_k}^+$-free graphs by the Turán graph $T_k(n)$.

By Theorem 6, for n large enough, the Turán graphs $T_k(n)$ and $T_k(n-k)$ contain the maximum number of copies of K_k among all n-vertex and $(n-k)$-vertex $\overline{M_k}^+$-free graphs. Therefore, $T_k(n)$ maximizes the number of pairs of disjoint copies of K_k. In an $\overline{M_k}^+$-free graph, if two disjoint copies of K_k span a copy of $\overline{M_k}$, then there can be no further edges between them as otherwise we have a copy of $\overline{M_k}^+$. Thus, any two disjoint copies of K_k span at most one copy of $\overline{M_k}$. Observe that in $T_k(n)$ any pair of disjoint copies of K_k span exactly one copy of $\overline{M_k}$. As the number of copies of H on $2k$-vertices is maximized by $\overline{M_k}$ we have that the number of copies of H is maximized in $T_k(n)$. $\qquad\square$

When $k = 4$ the graphs P_4 and C_4 are both candidates for the graph H in Lemma 12. This gives the following proposition.

Proposition 13. *The cycle C_4 and path P_4 are B_2-Turán-good.*

We remark that one can also obtain that C_4 is B_2-Turán-good from a result of Pippenger and Golumbic [23]. They showed that $T_2(n)$ contains the largest number of *induced* copies of C_4 among n-vertex graphs. As every copy of a C_4 is induced in a B_2-free graph, this implies that C_4 is B_2-Turán-good.

3.2   Forbidding odd cycles

Gerbner, Győri, Methuku and Vizer [10] counted paths and even cycles when forbidding an odd cycle. In particular, they proved that for any $k \geq 1$ and $l \geq 2$,

$$\mathrm{ex}(n, P_l, C_{2k+1}) = (1 + o(1)) \left(\frac{n}{2}\right)^l = (1 + o(1))\mathcal{N}(P_l, T_2(n))$$

$$\mathrm{ex}(n, C_{2l}, C_{2k+1}) = (1 + o(1))\frac{1}{2l}\left(\frac{n^2}{4}\right)^l = (1 + o(1))\mathcal{N}(C_{2l}, T_2(n)).$$

In this subsection we show that both P_4 and C_4 are C_5-Turán-good, i.e, the results above are exact for $k = 2$ and P_4 and C_4, respectively. In the case of P_4 we also obtain a stability result.

Theorem 14. *The path P_4 is C_5-Turán-good. Moreover, if G is a C_5-free graph on n vertices and G has α edges contained in triangles, then the number of copies of P_4 in G is at most*

$$\mathcal{N}(P_4, T_2(n)) - (1 + o(1))\alpha\frac{n^2}{12}.$$

Proof. Let G be an n-vertex C_5-free graph. We will show that every edge of G is contained in at most $3\lfloor n/2 - 1\rfloor\lceil n/2 - 1\rceil$ copies of P_4. As the number of edges is maximized in the Turán graph $T_2(n)$ and in the Turán graph every edge is contained in $3\lfloor n/2 - 1\rfloor\lceil n/2 - 1\rceil$ copies of P_4, this will show that P_4 is C_5-Turán-good. In order to prove the second part of the theorem we will examine the number of copies of P_4 containing a fixed edge e where e is contained in a triangle in G.

Consider an edge uv and let G' be obtained from G by deleting u and v. As G' is a C_5-free graph on $n - 2$ vertices and $n - 2$ is large enough, Theorem 1 implies that G' satisfies

$$|E(G')| \leq |E(T_2(n - 2))| = \lfloor n/2 - 1\rfloor\lceil n/2 - 1\rceil.$$

Now partition $V(G')$ into a set A of vertices adjacent to both u and v, a set B of vertices adjacent to u but not v, a set C of vertices adjacent to v but not u, and a set D of the remaining vertices (not adjacent to u nor v). As G is C_5-free, no vertex in $V(G')$ is adjacent to a vertex in $A \cup B$ and a distinct vertex in $A \cup C$.

Observe that if $|A| \geq 1$, then there is no edge between B and C. If $|A| \geq 2$, then there is no edge between A and $B \cup C$. If $|A| \geq 3$, then there is no edge in A.

Consider two vertices x and y of G' (necessarily distinct from u and v). Let $f(x, y)$ denote the number of copies of P_4 in G containing the edge uv and vertices x and y. If $x, y \in A$ and xy is an edge, then $f(x, y) = 6$ and if xy is not an edge, then $f(x, y) = 2$. If $x \in A$ and $y \in B \cup C$ and xy is an edge, then $f(x, y) = 4$ and if xy is not an edge, then $f(x, y) = 1$. If $x \in B$ and $y \in C$ and xy is an edge, then $f(x, y) = 3$ and if xy is not an edge, then $f(x, y) = 1$. If $x, y \in B$ and xy is an edge, then $f(x, y) = 2$ and if xy is not an edge, then $f(x, y) = 0$. The same argument holds when $x, y \in C$. If $x \in D$ is not adjacent

to y, then $f(x, y) = 0$. If x is adjacent to y and $y \in A$, then $f(x, y) = 2$ and if $y \in B \cup C$ then $f(x, y) = 1$ and if $y \in D$, then $f(x, y) = 0$. Let

$$q(u, v) := \sum_{x,y \in V(G')} f(x, y),$$

i.e., $q(u, v)$ is the number of copies of P_4 containing the edge uv. We determine an upper-bound on $q(u, v)$ in four cases based on the size of A.

Case 1: $A = \emptyset$, i.e., uv is not contained in any triangles.

For every pair x, y of vertices, $f(x, y)$ depends on which of the three sets B, C and D they belong to and whether x and y are adjacent in G'. Observe that when $x \in B$ and $y \in C$, then $f(x, y) \leq 3$ if xy is an edge and $f(x, y) \leq 1$ otherwise. In all other cases $f(x, y) \leq 2$ if xy is an edge and $f(x, y) \leq 0$ otherwise. Therefore, $q(u, v) \leq 2|E(G')| + |C||B|$. Both terms of this sum are maximized by the Turán graph, $T_2(n-2)$, so $q(u, v) \leq 3\lfloor n/2 - 1 \rfloor \lceil n/2 - 1 \rceil$.

Case 2: $A = \{w\}$.

Then $\sum_{y \in V(G')} f(w, y) \leq 4(n - 3)$. If $x \neq w \neq y$, then $f(x, y) \leq 2$. Indeed, if x and y are adjacent, then it is impossible that one of them is in B and the other is in C. Moreover, if x and y are not adjacent, then $f(x, y) = 0$. Therefore, we have $q(u, v) \leq 4(n-3) + 2|E(G')| \leq (1 + o(1))n^2/2$.

Case 3: $A = \{w, w'\}$.

If $\{x, y\} = \{w, w'\}$, then $f(x, y) \leq 6$. By the same reasoning as in Case 2, if $|\{x, y\} \cap \{w, w'\}| = 1$, then $f(x, y) \leq 4$, and if $|\{x, y\} \cap \{w, w'\}| = 0$, then $f(x, y) \leq 2$. Moreover, in this latter case, if x and y are not adjacent, then $f(x, y) = 0$. Therefore, we obtain $q(u, v) \leq 6 + 8(n - 3) + 2|E(G')| \leq (1 + o(1))n^2/2$.

Case 4: $|A| = m \geq 3$.

Then we know $f(x, y) \leq 2$ if $x, y \in A$, since xy is not an edge of G'. Furthermore, the only other case when $f(x, y) \geq 2$ is when $x \in A$ and $y \in D$ are adjacent and we have $f(x, y) = 2$. Observe that $y \in D$ can be adjacent to at most one $x \in A$. Therefore this situation can occur at most once for each element of D, i.e., at most $n - m - 2$ total times. All other pairs x, y have $f(x, y) \leq 1$ and therefore $q(u, v) \leq \binom{n}{2} + \binom{m}{2} + n - m - 2$.

Let us call a subgraph of G a *large book* if it consists of the book spanned by the edge uv and all the $m \geq 3$ common neighbors $w_1, \ldots, w_m$ of u and v. Observe that each edge of the form uw_i or vw_i in G is contained in the single triangle uvw_i as otherwise we can form a C_5 with w_j for some $j \neq i$ (as $m \geq 3$), a contradiction.

This implies that large books are pairwise edge-disjoint. Therefore, we can calculate the sum of $q(u, v)$ for all edges uv contained in a large book by summing them for each large book. In a large book H with $m + 2$ vertices, we have $2m$ edges each contained in exactly one triangle (thus we can use Case 2) and one edge contained in exactly m triangles (where we use Case 4). Therefore,

$$\sum_{uv \in E(H)} q(u, v) \leq 2m(1 + o(1))n^2/2 + \binom{n}{2} + \binom{m}{2} + n - m - 2 \leq (2m + 1)(1 + o(1))n^2/2 + \binom{m}{2}$$

11

This implies that for edges in large books, the average of $q(u, v)$ is

$$\frac{1}{2m+1} \sum_{uv \in E(H)} q(u, v) \leq (1 + o(1))n^2/2 + \frac{1}{2m+1} \binom{m}{2} = (1 + o(1))n^2/2.$$

For the at most $\lfloor n^2/4 \rfloor - \alpha$ edges not in triangles, we have $q(u, v) \leq 3 \lfloor n/2 - 1 \rfloor \lceil n/2 - 1 \rceil$ by Case 1. For edges in triangles but not in large books, we have $q(u, v) \leq (1 + o(1))n^2/2$ by Cases 2 and 3. Therefore,

$$3\mathcal{N}(P_4, G) = \sum_{uv \in E(G)} q(u, v)$$
$$\leq 3 \lfloor n/2 - 1 \rfloor \lceil n/2 - 1 \rceil (\lfloor n^2/4 \rfloor - \alpha) + (1 + o(1))\alpha n^2/2 =$$
$$= 3\mathcal{N}(P_4, T_2(n)) - (1 + o(1))\alpha n^2/4,$$

completing the proof. $\qquad \square$

Lemma 15. *Fix a graph F and let G_0 be a complete bipartite graph with $\mathcal{N}(P_{2k}, G_0) = \mathrm{ex}(n, P_{2k}, F)$. Then G_0 satisfies $\mathcal{N}(C_{2k}, G_0) = \mathrm{ex}(n, C_{2k}, F)$.*

Proof. Let G be an n-vertex F-free graph with $\mathrm{ex}(n, C_{2k}, F)$ copies of C_{2k}. Observe that $\mathcal{N}(P_{2k}, G) \leq \mathrm{ex}(n, P_{2k}, F)$. Every copy of C_{2k} contains $2k$ copies of P_{2k} and each copy of P_{2k} is contained in at most one C_{2k}. Thus,

$$2k \cdot \mathcal{N}(C_{2k}, G) \leq \mathcal{N}(P_{2k}, G).$$

Note that copies of P_{2k} not contained in a C_{2k} are not counted here. As G_0 is a complete bipartite graph, every copy of P_{2k} in G_0 is contained in a C_{2k}. Therefore,

$$2k \cdot \mathcal{N}(C_{2k}, G_0) = \mathcal{N}(P_{2k}, G_0).$$

Thus,

$$\mathrm{ex}(n, C_{2k}, F) = \mathcal{N}(C_{2k}, G) \leq \mathcal{N}(P_{2k}, G)/2k \leq \mathrm{ex}(n, P_{2k}, F)/2k$$
$$= \mathcal{N}(P_{2k}, G_0)/2k = \mathcal{N}(C_{2k}, G_0).$$

$\qquad \square$

Theorem 14 and Lemma 15 imply the following corollary.

Corollary 16. Let F be a 3-chromatic graph. If P_{2k} is F-Turán-good, then C_{2k} is F-Turán-good. In particular, C_4 is C_5-Turán-good.

3.3 Forbidding fans

Until this point we only considered forbidden graphs that have a color-critical edge. By Theorem 1, an extremal graph for a k-chromatic graph F without a color-critical edge has more edges than $T_{k-1}(n)$. This suggests that there may not be graphs H that are F-Turán-good in this case. Proposition 18 below shows that this is false in general.

For $k \geq 2$, the *k-fan* F_k is the graph of k triangles all sharing exactly one common vertex. Note that the fan F_k does not contain a color-critical edge. Erdős, Füredi, Gould and Gunderson [5] determined the exact Turán number of F_2 for n large enough.

Theorem 17 (Erdős, Füredi, Gould and Gunderson [5]). *Let F_2 be the graph of two triangles sharing exactly one vertex. Then, for n large enough, the unique extremal graph for F_2 is the graph resulting from adding a single edge to one class of the Turán graph $T_2(n)$. Thus, for n large enough,*

$$\mathrm{ex}(n, F_2) = \left\lfloor \frac{n^2}{4} \right\rfloor + 1.$$

We use Theorem 17 to show that C_4 is F_2-Turán-good.

Proposition 18. *The cycle C_4 is F_2-Turán-good.*

Proof. Let n be large enough and G be an n-vertex F_2-free graph. If G has more than $\lfloor n^2/4 \rfloor$ edges, then Theorem 17 gives the exact structure of G. In particular, G has $\mathcal{N}(C_4, T_2(n))$ copies of C_4 (observe that the edge added to the $T_2(n)$ is not in any copy of C_4).

Therefore, we may assume that G has at most $\lfloor n^2/4 \rfloor$ edges. Let uv be an arbitrary edge of G. We claim that uv is in at most $\lfloor (n-2)^2/4 \rfloor$ copies of C_4, which will complete the proof as it implies $\mathcal{N}(C_4, G) \leq \frac{1}{4} \lfloor n^2/4 \rfloor \lfloor (n-2)^2/4 \rfloor = \mathcal{N}(C_4, T_2(n))$. We distinguish two cases.

Case 1: uv is not contained in a K_4.

Delete u and v from G and let G' be the resulting graph. Every edge of G' forms at most one C_4 with uv and we count each such C_4 exactly once this way. The number of edges in G' is at most $\lfloor (n-2)^2/4 \rfloor + 1$ by Theorem 17. We are done unless $|E(G')| = \lfloor (n-2)^2/4 \rfloor + 1$ and every edge of G' forms a C_4 with uv in G. So, by Theorem 17, we may assume that G' is a $T_2(n-2)$ with classes A and B and an extra edge xy in class A. Without loss of generality, we may assume $xyuvx$ is a C_4 in G. Thus vx and uy are edges of G.

Now let z be an arbitrary vertex of B. Observe that if ux (or vy) is an edge of G, then we have a copy of F_2 spanned by the two triangles $xyzx$ and $uvxu$ (or $uvyu$), a contradiction. On the other hand, the edges xz and yz are each in a C_4 with uv. This implies that zu and zv are both edges of G. But then the two triangles $uvzu$ and $xyzx$ span a copy of F_2, a contradiction.

Case 2: uv is contained in a K_4.

Let u, v, x and y be the vertices of a K_4. Delete these four vertices and let G' be the resulting F_2-free graph on $n-4$ vertices. Observe that each vertex of G' is adjacent to at most one of u, v, x, y as otherwise we have an F_2 in G. Therefore, each edge of G' forms at most one C_4 with uv. By Theorem 17 we have $|E(G')| \leq \lfloor (n-4)^2/4 \rfloor + 1$. Therefore,

the number of copies of C_4 containing uv is at most $2 + \lfloor (n-4)^2/4 \rfloor + 1 \leq \lfloor (n-2)^2/4 \rfloor$, completing the proof. $\square$

4 Concluding remarks

Theorem 7 and a weaker version of Proposition 11 previously appeared in the authors' first and second arXiv version of [12], but were ultimately not included in the published version.

Győri, Pach and Simonovits [18] also studied when the Turán graph is the unique extremal graph. We have so far avoided this for simplicity. Let us say that a graph H is *strictly F-Turán-good* for a k-chromatic graph F if for every n large enough, $T_{k-1}(n)$ is the unique F-free graph with $\text{ex}(n, H, F)$ copies of H. By Theorem 6, if F does not have a color-critical edge, then there is no strictly F-Turán-good graph. However, it is not hard to show that our results when F has a color-critical edge hold for the strict version as well.

Let us conclude with several natural conjectures supported by the results in this paper.

Conjecture 19. *For every pair of integers l and k, the path P_l is k-Turán-good.*

Theorem 4 and Corollary 11 imply that the conjecture holds for $l = 3$ and $k \geq 3$. Proposition 8 shows that the conjecture holds asymptotically.

Conjecture 20. *For every graph H there is an integer k_0 such that if $k \geq k_0$, then H is k-Turán-good.*

Theorem 10 implies that the conjecture is true for H a complete multipartite graph. As noted in the introduction, this conjecture cannot in general be extended to hold for small k. Observe that Conjecture 20 would imply that if we increase k, sooner or later every graph becomes k-Turán-good. In some of our examples if a graph was k-Turán-good, then it was also $(k+1)$-Turán-good. We do not know if this behavior holds for every graph H.

Conjecture 21. *The path P_k and the even cycle C_{2k} are C_{2l+1}-Turán-good.*

The asymptotic version of this statement was proved in [10]. By Theorem 4, the conjecture holds when $l = 1$. In this paper, we proved that it holds for P_3 when $l \geq 1$ (Corollary 11) and for P_4 and C_4 when $l = 2$ (Theorem 14 and Corollary 16).

References

[1] N. Alon and C. Shikhelman. Many T copies in H-free graphs. *Journal of Combinatorial Theory, Series B* **121** (2016) 146–172.

[2] B. Bollobás and E. Győri. Pentagons vs. triangles. *Discrete Mathematics* **308** (2008) 4332–4336.

[3] J. Cutler, J. Nir and J. Radcliffe. Supersaturation for subgraph counts. arXiv:1903.08059 (2019)

[4] P. Erdős. On the number of complete subgraphs contained in certain graphs. *Magyar Tud. Akad. Mat. KutatóInt. Közl.* **7** (1962) 459–464.

[5] P. Erdős, Z. Füredi, R.J. Gould and D. S. Gunderson. Extremal graphs for intersecting triangles. *Journal of Combinatorial Theory, Series B* **64** (1995) 89–100.

[6] P. Erdős and M. Simonovits. A limit theorem in graph theory. *Studia Sci. Math. Hungar.* **1** (1966) 51–57.

[7] P. Erdős and A. H. Stone. On the structure of linear graphs. *Bulletin of the American Mathematical Society.* **52** (1946) 1087–1091.

[8] B. Ergemlidze, E. Győri, A. Methuku and N. Salia. A note on the maximum number of triangles in a C_5-free graph. *Journal of Graph Theory* **90** (2019) 227-230.

[9] B. Ergemlidze and A. Methuku. Triangles in C_5-free graphs and Hypergraphs of Girth Six. arXiv:1811.11873 (2018)

[10] D. Gerbner, E. Győri, A. Methuku and M. Vizer. Generalized Turán problems for even cycles. *Journal of Combinatorial Theory, Series B* **145** (2019) 169–213.

[11] D. Gerbner, A. Methuku and M. Vizer. Generalized Turán problems for disjoint copies of graphs. *Discrete Mathematics* **342** (2019) 3130–3141.

[12] D. Gerbner and C. Palmer. Counting copies of a fixed subgraph in F-free graphs. *European Journal of Combinatorics* **82** (2019) 103001.

[13] L. Gishboliner and A. Shapira. A Generalized Turán Problem and its Applications. *International Math Research Notices (IMRN)* **11** (2020) 3417–3452.

[14] A. Grzesik. On the maximum number of five-cycles in a triangle-free graph. *Journal of Combinatorial Theory, Series B* **102** (2012) 1061–1066.

[15] B. Guiduli. Spectral Extrema for Graphs. Ph.D. Thesis, University of Chicago, Dec. 1996.

[16] E. Győri. On the number of C_5's in a triangle-free graph. *Combinatorica* **9** (1989) 101–102.

[17] E. Győri and H. Li. The maximum number of triangles in C_{2k+1}-free graphs. *Combinatorics, Probability and Computing* **21** (2011) 187–191.

[18] E. Győri, J. Pach and M. Simonovits. On the maximal number of certain subgraphs in K_r-free graphs. *Graphs and Combinatorics* **7** (1991) 31–37.

[19] E. Győri, N. Salia, C. Tompkins and O. Zamora. The maximum number of P_l copies in P_k-free graphs. *Acta Mathematica Universitatis Comenianae* **88** (2019) 773–778.

[20] H. Hatami, J. Hladký, D. Král', D. Norine and A. Razborov. On the number of pentagons in triangle-free graphs. *Journal of Combinatorial Theory, Series A* **120** (2012) 722–732.

[21] Jie Ma and Yu Qiu. Some sharp results on the generalized Turán numbers. *European Journal of Combinatorics* **84** (2020) 103026.

[22] V. Nikiforov. A spectral Erdős-Stone-Bollobás theorem. *Combinatorics, Probability and Computing* **18** (2009) 455–458.

[23] N. Pippenger and M. C. Golumbic. The inducibility of graphs. *Journal of Combinatorial Theory, Series B* **19** (1975) 189–203.

[24] M. Simonovits. A method for solving extremal problems in graph theory, stability problems, in: Theory of Graphs, Proc. Colloq., Tihany, 1966, Academic Press, New York, (1968), pp. 279319.

[25] P. Turán. On an extremal problem in graph theory (in Hungarian). *Matematikai es Fizikai Lapok* **48** (1941) 436–452.

[26] A. A. Zykov. On some properties of linear complexes. *Matematicheskii Sbornik* **66** (1949) 163–188.