NOTE ON THE SUM OF THE SMALLEST AND LARGEST EIGENVALUES OF A TRIANGLE-FREE GRAPH

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ABSTRACT. Let G be a triangle-free graph on n vertices with adjacency matrix eigenvalues $\mu_1(G) \ge \mu_2(G) \ge \cdots \ge \mu_n(G)$. In this paper we study the quantity

$$\mu_1(G) + \mu_n(G)$$
.

We prove that for any triangle-free graph G we have

$$\mu_1(G) + \mu_n(G) \le (3 - 2\sqrt{2})n.$$

This was proved for regular graphs by Brandt, we show that the condition on regularity is not necessary. We also prove that among triangle-free strongly regular graphs the Higman-Sims graph achieves the maximum of

$$\frac{\mu_1(G) + \mu_n(G)}{n}.$$

1. Introduction

In this paper every graph is simple. Motivated by the papers [2] and [4] we study the following problem. Let \mathcal{G}_3 be the family of triangle-free graphs, and for a graph G on v(G) = n vertices let $\mu_1(G) \ge \mu_2(G) \ge \cdots \ge \mu_n(G)$ be the eigenvalues of the adjacency matrix of G. The problem is to determine

$$c_3 = \sup_{G \in \mathcal{G}_2} \frac{\mu_1(G) + \mu_n(G)}{v(G)}.$$

Brandt [2] proved that for regular triangle-free graphs we have

$$\mu_1(G) + \mu_n(G) \le (3 - 2\sqrt{2})n.$$

Very recently Balogh, Clemen, Lidický, Norin and Volec proved that for regular triangle-free graphs we have $\mu_1(G) + \mu_n(G) \leq \frac{15}{94}n < 0.1596n$ and they mention in their paper that a similar but larger computation also gives the result 0.15467 instead of 0.1596. In fact, they study the smallest eigenvalue $q_n(G)$ of the so-called signless laplacian matrix $\overline{L} = D + A$, where D is is the diagonal matrix consisting of the degrees of the vertices and A is the adjacency matrix of the graph G. The quantity $q_n(G)$ coincides with $\mu_1(G) + \mu_n(G)$ if G is regular. Balogh, Clemen, Lidický, Norin and Volec mentions that in case of regular graphs they can further improve their result to prove $q_n(G) \leq 0.15442n$. Our first result is to prove that in Brandt's theorem one can drop the condition of regularity.

Theorem 1.1. Let G be a triangle-free graph on n vertices, and let $\mu_1(G) \ge \mu_2(G) \ge \cdots \ge \mu_n(G)$ be the eigenvalues of its adjacency matrix. Then

$$\mu_1(G) + \mu_n(G) \le (3 - 2\sqrt{2})n.$$

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The proof of Theorem 1.1 heavily relies on the following lemma which might be of independent interest.

Lemma 1.2. Let G be a triangle-free graph on n vertices, and let $\mu_1(G) \ge \mu_2(G) \ge \cdots \ge \mu_n(G)$ be the eigenvalues of its adjacency matrix. Then

$$\mu_1(G) \le \frac{-n\mu_n(G)}{\mu_1(G) - \mu_n(G)}.$$

Brandt [2] also realized that for the so-called Higman-Sims graph H we have

$$\frac{\mu_1(H) + \mu_n(H)}{v(H)} = \frac{22 + (-8)}{100} = 0.14$$

which gives a rather good lower bound for c_3 . Higman-Sims graph is the unique strongly regular graph with parameters (100, 22, 0, 6). Recall that a graph G is a strongly regular graph with paremeters (n, k, a, b) if it has n vertices, k-regular, any two adjacent vertices have exactly a common neighbors, and any two non-adjacent vertices have exactly b common neighbors. In this paper we show that among the strongly regular graphs, it is indeed the Higman-Sims graph which maximizes the quantity

$$\frac{\mu_1(G) + \mu_n(G)}{v(G)}.$$

Note that only finitely many triangle-free strongly regular graphs are known currently, but we do not rely on this fact.

Theorem 1.3. Let G be a triangle-free strongly regular graph on n vertices. Then

$$\mu_1(G) + \mu_n(G) \le 0.14n$$

with equality if and only if G is the Higman-Sims graph.

2. Proof of Theorem 1.1

We begin with proving Lemma 1.2. Before we actually start it let us mention that for regular graphs this lemma is a simple consequence of the Hoffman-Delsarte bound for independent sets. Indeed, let $\alpha(G)$ denote the size of the largest independent set of a d-regular graph. Then by the Hoffman-Delsarte bound we have

$$\alpha(G) \le \frac{-n\mu_n(G)}{d - \mu_n(G)}.$$

Since G is triangle-free, the neighbors of a vertex determine an independent set, whence $d \leq \alpha(G)$. Since $d = \mu_1(G)$ we get that

$$\mu_1(G) = d \le \alpha(G) \le \frac{-n\mu_n(G)}{d - \mu_n(G)} \le \frac{-n\mu_n(G)}{\mu_1(G) - \mu_n(G)}.$$

Based on this inequality Brandt proved that

$$\mu_1(G) + \mu_n(G) \le (3 - 2\sqrt{2})n.$$

So after proving Lemma 1.2 we practically copy the proof of Brandt.

Proof of Lemma 1.2. Let μ_s, \ldots, μ_n be the set of non-positive eigenvalues. Then

$$0 = 6 \cdot \text{number of triangles} = \sum_{i=1}^{n} \mu_i^3 \ge \mu_1^3 + \sum_{i=1}^{n} \mu_i^3.$$

Hence

$$\sum_{i=s}^{n} (-\mu_i)^3 \ge \mu_1^3.$$

On the other hand, we have

$$\sum_{i=s}^{n} (-\mu_i)^3 \le (-\mu_n) \sum_{i=s}^{n} (-\mu_i)^2 \le (-\mu_n) (2e(G) - \mu_1^2) \le (-\mu_n) (n\mu_1 - \mu_1^2).$$

Hence $\mu_1^3 \leq (-\mu_n)(n\mu_1 - \mu_1^2)$, thus $\mu_1^2 \leq (-\mu_n)(n - \mu_1)$, or in other words,

$$\mu_1 \le \frac{-n\mu_n}{\mu_1 - \mu_n}.$$

Proof of Theorem 1.1. As we mentioned earlier this proof practically follows the argument of [2].

We only need to solve the constrained maximization problem:

$$\max\left\{\frac{\mu_1 + \mu_n}{n} \mid \mu_1 \le \frac{-n\mu_n}{\mu_1 - \mu_n}\right\}.$$

Let $a = \mu_1$, $b = -\mu_n$ then we have $a \leq \frac{nb}{a-b}$ which is equivalent to $\frac{a^2}{n-a} \leq b$. Hence

$$\frac{a-b}{n} \le \frac{1}{n} \left(a - \frac{a^2}{n-a} \right) = \frac{an - 2a^2}{n(n-a)}.$$

So with the notation $\alpha = a/n$ we need to maximize $f(\alpha) := \frac{\alpha - 2\alpha^2}{1 - \alpha}$. Its derivative is $\frac{1 - 4\alpha + 2\alpha^2}{(1 - \alpha)^2}$ which is 0 at $\alpha = 1 \pm 1/\sqrt{2}$. Note that $\mu_1 \le \Delta \le n - 1$, where Δ is the largest degree, so $0 \le \alpha \le 1$. So we only need to consider $\alpha = 1 - 1/\sqrt{2}$ and the extreme points of the interval, $\alpha = 0$ and 1, to see that $f(\alpha)$ is indeed maximal at $1 - 1/\sqrt{2}$ and $f(\alpha) = 3 - 2\sqrt{2}$.

Hence $\mu_1 + \mu_n \le (3 - 2\sqrt{2})n$.

3. Proof of Theorem 1.3

In this secation we prove Theorem 1.3.

Proof of Theorem 1.3. Suppose for contradiction that G is a strongly regular graph with eigenvalues $(k, \mu_2^{(m_2)}, \mu_n^{(m_n)})$ such that $\frac{k+\mu_n}{n} > 0.14$. Let $-\mu_n = r$ and r/k = x. Again we use that $k \leq \alpha(G) \leq \frac{-n\mu_n}{k-\mu_n}$. Hence $\frac{r}{k} \geq \frac{k+r}{n}$. Then

$$0.14 < \frac{k-r}{n} = \frac{k-r}{k+r} \cdot \frac{k+r}{n} \le \frac{k-r}{k+r} \cdot \frac{r}{k} = \frac{x(1-x)}{1+x}.$$

From which we get that x > 1/5. Secondly, $m_n \ge \alpha(G) \ge k$ since we can assume that $\mu_2 > 0$. (Note that $\mu_2 > 0$ if G is not a blow-up of a complete graph.) Hence $kn = 2e(G) \ge m_n \mu_n^2 \ge kr^2$. So we have $n > r^2$. So we have two inequalities: $0.14 < \frac{k-r}{n} = \frac{k(1-x)}{n}$ and $n > r^2$. Then $n > r^2 = (kx)^2 > x^2 \frac{0.14^2}{(1-x)^2} n^2$. Thus $\frac{1}{0.14^2} \frac{(1-x)^2}{x^2} > n$. Since x > 1/5 we have $\frac{(1-x)^2}{x^2} < 16$. Hence $n < \frac{16}{0.14^2} \approx 816.33$. Now we can finish the proof since we know all possible strongly regular graph parameters up to 816, see Brouwer's website [3] and for the triangle-free strongly regular graph parameters the table on the next page. One can check that indeed the Higman-Sims graph achieves the maximum of $(\mu_1(G) + \mu_n(G))/n$.

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Remark 3.1. An interesting thing arises from the table on Andries Brouwer's website. If there were a strongly regular graph G with parameters (28, 9, 0, 4) then for this graph G we would have

$$\frac{\mu_1(G) + \mu_n(G)}{v(G)} = \frac{9 + (-5)}{28} = \frac{1}{7} > 0.14.$$

It is known that there is no such strongly regular graph just as there is no strongly regular graph with parameters (64, 21, 0, 10). For this graph we would have

$$\frac{\mu_1(G) + \mu_n(G)}{v(G)} = \frac{21 + (-11)}{64} = \frac{10}{64} > \frac{1}{7} > 0.14.$$

n	k	a	b	ϑ_1	ϑ_2	m_1	m_2	$\frac{k+\vartheta_2}{n}$	Appr.	Existence
5	2	0	1	$\frac{\sqrt{5}-1}{2}$	$\frac{-\sqrt{5}-1}{2}$	2	2	$\frac{3-\sqrt{5}}{10}$	0.076	Yes
10	3	0	1	1	-2	4	5	$\frac{1}{10}$	0.1	Yes
16	5	0	2	1	-3	10	5	$\frac{2}{16}$	0.125	Yes
28	9	0	4	1	-5	21	6	$\frac{4}{28}$	0.142	No
50	7	0	1	2	-3	28	21	$\frac{4}{50}$	0.08	Yes
56	10	0	2	2	-4	35	20	$\frac{6}{56}$	0.106	Yes
64	21	0	10	1	-11	56	7	$\frac{10}{64}$	0.156	No
77	16	0	4	2	-6	55	21	$\frac{10}{77}$	0.129	Yes
100	22	0	6	2	-8	77	22	$\frac{14}{100}$	0.14	Yes
162	21	0	3	3	-6	105	56	$\frac{15}{162}$	0.092	?
176	25	0	4	3	-7	120	55	$\frac{18}{176}$	0.102	?
210	33	0	6	3	-9	154	55	$\frac{24}{210}$	0.114	?
266	45	0	9	3	-12	209	56	33 266	0.124	?
324	57	0	12	3	-15	266	57	$\frac{42}{324}$	0.129	No
352	26	0	2	4	-6	208	143	$\frac{20}{352}$	0.056	?
352	36	0	4	4	-8	231	120	$\frac{28}{352}$	0.079	?
392	46	0	6	4	-10	276	115	$\frac{36}{392}$	0.091	?
552	76	0	12	4	-16	437	114	$\frac{60}{552}$	0.108	?
638	49	0	4	5	-9	406	231	$\frac{40}{638}$	0.062	?
650	55	0	5	5	-10	429	220	50 650	0.076	?
667	96	0	16	4	-20	551	115	$\frac{76}{667}$	0.113	?
704	37	0	2	5	-7	407	296	$\frac{30}{704}$	0.042	?
784	116	0	20	4	-24	667	116	$\frac{92}{784}$	0.117	?
800	85	0	10	5	-15	595	204	$\frac{70}{800}$	0.087	?

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