Induced Turán problems and traces of hypergraphs

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February 19, 2020

Abstract

Let F be a graph. We say that a hypergraph $\mathcal{H}$ contains an *induced Berge* F if the vertices of F can be embedded to $\mathcal{H}$ (e.g., $V(F) \subseteq V(\mathcal{H})$) and there exists an injective mapping f from the edges of F to the hyperedges of $\mathcal{H}$ such that $f(xy) \cap V(F) = \{x, y\}$ holds for each edge xy of F. In other words, $\mathcal{H}$ contains F as a trace.

Let $\mathrm{ex}_r(n, \mathrm{B}_{\mathrm{ind}}F)$ denote the maximum number of edges in an r-uniform hypergraph with no induced Berge F. Let $\mathrm{ex}(n, K_r, F)$ denote the maximum number of K_r's in an F-free graph on n vertices. We show that these two Turán type functions are strongly related.

Mathematics Subject Classification: 05D05, 05C65, 05C35.
Keywords: extremal hypergraph theory, Berge hypergraphs, traces.

1 Definitions, Berge F subhypergraphs

A hypergraph $\mathcal{H}$ is *r-uniform* or simply an *r-graph* if it is a family of r-element subsets of a finite set $V(\mathcal{H})$. If the *vertex set* $V(\mathcal{H})$ is clear from the text, then we associate an r-graph $\mathcal{H}$ with its edge set $E(\mathcal{H})$. Usually we take $V(\mathcal{H}) = [n]$, where $[n]$ is the set of first n integers, $[n] := \{1, 2, 3, \dots, n\}$. We also use the notation $\mathcal{H} \subseteq \binom{[n]}{r}$. For a set of vertices $S \subseteq V(\mathcal{H})$ define the *codegree* of S, denoted as $\deg(S)$, to be the number of edges of $\mathcal{H}$ containing S. The *s-shadow*, $\partial_s\mathcal{H}$, is the family of s-sets contained in the edges of $\mathcal{H}$. So $\partial_1\mathcal{H}$ is the set of non-isolated vertices, and $\partial_2\mathcal{H}$ is the graph whose edges are the pairs with positive co-degree in $\mathcal{H}$.

Definition 1.1. *For a graph F with vertex set $\{v_1, \dots, v_p\}$ and edge set $\{e_1, \dots, e_q\}$, a hypergraph $\mathcal{H}$ contains a* **Berge** *F if there exist distinct vertices $\{w_1, \dots, w_p\} \subseteq V(\mathcal{H})$ and distinct edges $\{f_1, \dots, f_q\} \subseteq E(\mathcal{H})$, such that if $e_i = v_\alpha v_\beta$, then $\{w_\alpha, w_\beta\} \subseteq f_i$. The vertices $\{w_1, \dots, w_p\}$ are called the* **base vertices** *of the Berge F.*

Definition 1.2. *For a graph F with vertex set $\{v_1, \dots, v_p\}$ and edge set $\{e_1, \dots, e_q\}$, a hypergraph $\mathcal{H}$ contains an* **induced Berge** *F if there exists a set of distinct vertices $W := \{w_1, \dots, w_p\} \subseteq V(\mathcal{H})$ and distinct edges $\{f_1, \dots, f_q\} \subseteq E(\mathcal{H})$, such that if $e_i = v_\alpha v_\beta$, then $\{w_\alpha, w_\beta\} = f_i \cap W$.*

In particular, in the case that $\mathcal{H}$ is a graph (2-uniform), an induced Berge F is just any copy of F in $\mathcal{H}$, not to be confused with the notion of induced subgraphs. If the two hypergraphs have the same number of edges, $e(\mathcal{H}) = e(\mathcal{F})$, then we say that $\mathcal{H}$ itself is a(n induced) Berge F hypergraph. The set of r-uniform (induced) Berge F hypergraphs is denoted by $\{\mathrm{B}(F)\}_r$ ($\{\mathrm{B}_{\mathrm{ind}}(F)\}_r$, resp.).

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For example, if F is a triangle, $E(F) = \{12, 13, 23\}$, then $\{B(F)\}_3$ contains four triple systems: $\{12a, 13a, 23a\}$, $\{12a, 13a, 23b\}$, $\{12a, 13b, 23c\}$, and $\{123, 13a, 23b\}$. The first three of them contains an induced C_3, the fourth does not. Parenthesis and indices are omitted when it does not cause ambiguities.

1.1 Three types of extremal numbers

Given a set of r-graphs $\mathcal{F}$ the hypergraph $\mathcal{H}$ is called *$\mathcal{F}$-free* if it does not have any subgraph isomorphic to any member of $\mathcal{F}$. The *Turán number* of $\mathcal{F}$, denoted by $\mathrm{ex}_r(n, \mathcal{F})$, is the maximum size of an $\mathcal{F}$-free $\mathcal{H} \subseteq \binom{[n]}{r}$. Usually it is assumed that $|\mathcal{F}|$ is finite, so the well-known fact $\mathrm{ex}_2(n, \{C_3, C_4, C_5, \dots\}) = n - 1$ usually is not considered a Turán type result because the set of forbidden graphs $\mathcal{F}$, the set of all cycles, is infinite. If $r = 2$ then the index is usually omitted. Also if $\mathcal{F}$ has only one member, $\mathcal{F} = \{F\}$, then we write $\mathrm{ex}_r(n, F)$ instead of $\mathrm{ex}_r(n, \{F\})$.

The *generalized Turán number* for graphs, pioneered by Erdős [3] and recently systematically investigated by Alon and Shikhelman [1], is the following extremal problem. We only formulate the case relevant to this paper. Given a graph F, let $\mathrm{ex}(n, K_r, F)$ denote the maximum possible number of copies of K_r's in an F-free, n-vertex graph, i.e.,

$$\mathrm{ex}(n, K_r, F) := \max\left\{|\mathcal{N}_r(H)| : H \text{ is } F\text{-free }, H \subseteq \binom{[n]}{2}\right\},$$

where $\mathcal{N}_r(H) \subseteq \binom{[n]}{r}$ is the family of r-element vertex sets that span a K_r in H. In particular $\mathcal{N}_2(H) = E(H)$ and $\mathrm{ex}(n, K_2, F) = \mathrm{ex}(n, F)$ is the regular Turán number of F.

For a graph F and positive integer r, let

$$\mathrm{ex}_r(n, \mathrm{B}F) := \max\{e(\mathcal{H}) : \mathcal{H} \subseteq \binom{[n]}{r} \text{ and } \mathcal{H} \text{ is Berge } F\text{-free}\}.$$

Ever since Győri, G. Y. Katona, and Lemons [8] investigated hypergraphs without long Berge paths there is a renewed interest concerning extremal Berge type problems. Here we define a related function, the *induced Berge Turán number* of F. Special cases were studied earlier, especially the 3-uniform case (e.g., Maherani and Shahsiah [13], Gyárfás [7], Sali and Spiro [18]).

$$\mathrm{ex}_r(n, \mathrm{B}_{\mathrm{ind}}F) := \max\{e(\mathcal{H}) : \mathcal{H} \subseteq \binom{[n]}{r} \text{ and } \mathcal{H} \text{ is induced Berge } F\text{-free}\}.$$

We consider the relationship between these three functions. Obviously,

$$\mathrm{ex}(n, K_r, F) \le \mathrm{ex}_r(n, \mathrm{B}F) \le \mathrm{ex}_r(n, \mathrm{B}_{\mathrm{ind}}F). \tag{1}$$

Indeed, consider a graph G with $|\mathcal{N}_r(G)| = \mathrm{ex}(n, K_r, F)$. Since G is F-free, the r-graph $\mathcal{N}_r(G)$ is Berge F-free, implying $|\mathcal{N}_r(G)| \le \mathrm{ex}_r(n, \mathrm{B}F)$. The second inequality holds because if a hypergraph contains no Berge F then it also contains no induced Berge F.

The induced Berge F problem is motivated by the forbidden configuration problem for matrices (see Anstee [2] for a survey). It can also be reformulated as a hypergraph trace problem (see, e.g., Mubayi and Zhao [15]). Few results are known for the induced Berge Turán problem. In [15], the value of $\mathrm{ex}_r(n, \mathrm{B}_{\mathrm{ind}}K_t)$ is determined asymptotically for K_3 and K_4, as well as K_t when t is close to the uniformity r.

A special case of induced Berge hypergraphs, so called *expansions* were intensively studied, see, e.g., Pikhurko [16], Kostochka, Mubayi, and Verstraëte [11], and the survey by Mubayi and Verstraëte [14].

There are also other areas of research in extremal graph theory which are called 'induced' Turán type results. E.g., Prömel and Steger [17] investigated the extremal properties of graphs not containing an induced copy of a given graph F. A more recent version is by Loh, Tait, Timmons, and Zhou [12]. But most of these are only distant relatives of our induced Berge question.

2 Main results, bounds for $\mathrm{ex}_r(n, \mathrm{B}_{\mathrm{ind}}F)$

2.1 The order of magnitude

Let F be a graph, $r \geq 2$. Our aim is to determine the order of magnitude of the induced Berge Turán number of F as $n \to \infty$, or to reduce it to known problems. Then in the next subsection we define a large class of 3-chromatic graphs $\mathcal{G}_{\mathrm{tri}}$ which contains, e.g., all outerplanar graphs, and apply our results and methods to determine their induced Berge Turán number more precisely.

Theorem 2.1. *Let $r \geq 2$, and fix a graph F such that $E(F) \neq \emptyset$. Then, as $n \to \infty$*

$$\mathrm{ex}_r(n, \mathrm{B}_{\mathrm{ind}}F) = \Theta(\max_{2 \leq s \leq r}\{\mathrm{ex}(n, K_s, F)\}).$$

This theorem shows that the order of magnitudes of the three functions in (1) behave differently as r changes. For small r, in the range $r \leq \chi(F) - 1$, all the three, $\mathrm{ex}_r(n, F)$, $\mathrm{ex}_r(n, \mathrm{B}F)$, and $\mathrm{ex}_r(n, \mathrm{B}_{\mathrm{ind}}F)$, are of order $\Theta(n^r)$ because the balanced complete $(\chi(F)-1)$-partite r-graph contains no Berge F (so its 2-shadow, the r-partite Turán graph is r-chromatic).

If $r \geq |V(F)|$ then $\mathrm{ex}(n, K_r, F) = 0$ (since a K_r contains a copy of F). For general graphs F, the behavior of the three functions in the range $\chi(F) \leq r \leq |V(F)| - 1$ is still unknown. Determining the order of $\mathrm{ex}(n, K_r, F)$ for r in this range would give an answer for the growth of $\mathrm{ex}_r(n, \mathrm{B}_{\mathrm{ind}}F)$.

Concerning the Berge Turán function Gerbner and Palmer [5] showed that

$$\mathrm{ex}_r(n, \mathrm{B}F) \leq \mathrm{ex}(n, F)$$

for $r \geq |V(F)|$. So in this range $\mathrm{ex}_r(n, \mathrm{B}F) = O(n^2)$. For the complete graphs the two sides have the same order: $\mathrm{ex}_r(n, \mathrm{B}K_r) = \Theta(n^2)$ if $r \geq 3$. However this does not hold if r is large compared to $|V(F)|$. Grósz, Methuku, and Tompkins [6] proved that for any non-bipartite F and sufficiently large r, the order of $\mathrm{ex}_r(n, F)$ differs from that of $\mathrm{ex}(n, F)$: there exists some number $th(F)$ such that if $r \geq th(F)$ then $\mathrm{ex}_r(n, F) = o(n^2)$.

In contrast, the order of the induced Berge Turán function $\mathrm{ex}_r(n, \mathrm{B}_{\mathrm{ind}}F)$ is non-decreasing in r. Moreover, it is basically monotone. If $\bigcap E(F) = \emptyset$, i.e., F is not a star, then we will see later by

Lemma 3.1 that

$$\left(1-\frac{r-1}{n}\right)\mathrm{ex}_{r-1}(n,\mathrm{B}_{\mathrm{ind}}F)\le \mathrm{ex}_r(n,\mathrm{B}_{\mathrm{ind}}F). \tag{2}$$

2.2 Outerplanar graphs and more

We define the class of t-vertex graphs $\mathcal{G}_{\mathrm{tri}}^{(t)}$ by induction on t as follows. The class $\mathcal{G}_{\mathrm{tri}}^{(2)}$ has only a single member, K_2. For $t>2$ one obtains each member G of $\mathcal{G}_{\mathrm{tri}}^{(t)}$ by taking a $G^{(t-1)}\in\mathcal{G}_{\mathrm{tri}}^{(t-1)}$, taking an edge $xy\in G^{(t-1)}$, adding a new vertex $z\notin V(G^{(t-1)})$, and joining z to x and to y. Each $G\in\mathcal{G}_{\mathrm{tri}}^{(t)}$ has exactly t vertices and $2t-3$ edges. Finally, let $\mathcal{G}_{\mathrm{tri}}$ be the family of all non-empty subgraphs of the members of $\cup_{t\ge 2}\mathcal{G}_{\mathrm{tri}}^{(t)}$.

Note that $\mathcal{G}_{\mathrm{tri}}$ contains all outerplanar graphs, particuarly cycles, C_t, and forests. Each $G\in\mathcal{G}_{\mathrm{tri}}$ has chromatic number at most 3 and are obviously planar.

Theorem 2.2. *Let $r\ge 2$ be a positive integer. Fix a graph $F\in\mathcal{G}_{\mathrm{tri}}$. As $n\to\infty$ we have $\mathrm{ex}_r(n,\mathrm{B}_{\mathrm{ind}}F)=\Theta(\mathrm{ex}(n,F))$.*

This theorem reveals further gaps between $\mathrm{ex}_r(n,\mathrm{B}F)$ and $\mathrm{ex}_r(n,\mathrm{B}_{\mathrm{ind}}F)$. Győri and Lemons [9, 10] proved that for $r\ge 3$ an r-uniform hypergraph avoiding a Berge cycle C_{2t+1} has at most $O(\mathrm{ex}(n,C_{2t}))$ edges, which is known to be $O(n^{1+(1/t)})$. On the other hand, in the same range, we have $\mathrm{ex}_r(n,\mathrm{B}_{\mathrm{ind}}C_{2t+1})=\Theta(n^2)$.

Together, Theorems 2.1 and 2.2 show that $\mathrm{ex}(n,C_t)$ has the same order as $\max_{2\le s\le r}\{\mathrm{ex}(n,K_s,F)\}$. We obtain the following (known) corollary. For any $r\ge 2$ and $t\ge 3$

$$\mathrm{ex}(n,K_r,C_t)=O(\mathrm{ex}(n,C_t)).$$

We also state the case of trees.

Corollary 2.3. *Let $r\ge 2$ and T be a forest with at least two edges. Then $\mathrm{ex}_r(n,\mathrm{B}_{\mathrm{ind}}T)=\Theta(\mathrm{ex}(n,T))=\Theta(n)$.*

Finally, we get better bounds for stars, $F=K_{1,t-1}$.

Theorem 2.4. *For any $r\ge 2$, $t\ge 3$, if $n=a(r+t-3)+b$ with $b\le r+t-4$ then*

$$a\binom{r+t-3}{r}+\binom{b}{r}\le \mathrm{ex}_r(n,\mathrm{B}_{\mathrm{ind}}K_{1,t-1})\le \frac{n}{r}\binom{r+t-3}{r-1}.$$

In particular, if n is divisible by $r+t-3$, the lower bound is $\frac{n}{r}\binom{r+t-4}{r-1}$.

3 Constructions and proofs

3.1 Simple constructions and a monotonicity of the induced Berge Turán function

If $E(F)$ has a single edge then for $n\ge |V(F)|+r-2$ we have $\mathrm{ex}(n,F)=\mathrm{ex}(n,K_r,F)=\mathrm{ex}_r(n,\mathrm{B}F)=\mathrm{ex}_r(n,\mathrm{B}_{\mathrm{ind}}F)=0$, so there is nothing to prove, all of our statements trivially hold.

In all other cases we have $\mathrm{ex}_r(n, \mathrm{B}_{\mathrm{ind}}F) = \Omega(n)$ as one can see from the following constructions. If F has two non-disjoint edges then a matching of r-sets gives $\mathrm{ex}_r(n, \mathrm{B}_{\mathrm{ind}}F) \geq \lfloor n/r \rfloor$. If F has two disjoint edges then the hypergraph consisting of $n-r+1$ sets sharing a common $(r-1)$-set yields $\mathrm{ex}_r(n, \mathrm{B}_{\mathrm{ind}}F) \geq n-r+1$.

If $x \in V(F)$ is an isolated vertex then $\mathrm{ex}_r(n, \mathrm{B}_{\mathrm{ind}}F) = \mathrm{ex}_r(n, \mathrm{B}_{\mathrm{ind}}(F \setminus \{x\}))$ for all $n > (r-2)|E(F)| + |V(F)|$. So we may delete isolated vertices and asymptotically get the same Turán number. From now on, we suppose that F has no isolated vertex and $|E(F)| \geq 2$.

Lemma 3.1. *Fix integers $r, t \geq 2$. If F is a graph on t vertices such that $F \neq K_{1,t-1}$ (and $e(F) \geq 2$ and F has no islated vertex), then $\mathrm{ex}_r(n, \mathrm{B}_{\mathrm{ind}}F) \geq \mathrm{ex}_{(r-1)}(n-1, \mathrm{B}_{\mathrm{ind}}F)$. In particular, $\mathrm{ex}_r(n, \mathrm{B}_{\mathrm{ind}}F) = \Omega(\mathrm{ex}(n, F))$.*

Proof. Let $\mathcal{H}$ be an $(r-1)$-uniform hypergraph on $n-1$ vertices with $\mathrm{ex}_r(n-1, \mathrm{B}_{\mathrm{ind}}F)$ edges and no induced Berge F. Construct an r-uniform hypergraph $\mathcal{H}'$ with $V(\mathcal{H}') = V(\mathcal{H}) \cup \{v\}$ such that the edges of $\mathcal{H}'$ are obtained by extending every edge of $\mathcal{H}$ to include the new vertex v. Suppose $\mathcal{H}'$ contains an induced Berge F. Since $\mathcal{H}$ was induced Berge F-free, v must be a base vertex. Because v is contained in every edge of $\mathcal{H}'$, there is a fixed vertex contained in every edge of F. I.e., $F = K_{1,t-1}$, a contradiction.

Inductively, we obtain $\mathrm{ex}_2(n-r+2, \mathrm{B}_{\mathrm{ind}}F) \leq \mathrm{ex}_r(n, \mathrm{B}_{\mathrm{ind}}F)$. But $\mathrm{ex}_2(n-r+2, \mathrm{B}_{\mathrm{ind}}F) = \mathrm{ex}(n-r+2, F) = \Theta(\mathrm{ex}(n, F))$. □

To show (2) let $\mathcal{H}$ be an induced Berge F-free $(r-1)$-uniform hypergraph on n vertices, $|\mathcal{H}| = \mathrm{ex}_{(r-1)}(n, \mathrm{B}_{\mathrm{ind}}F)$. For $x \in V := V(\mathcal{H})$ let $\mathcal{H}_x := \{e \in \mathcal{H} : e \subset V \setminus \{x\}\}$. Since each $\mathcal{H}_x$ is also induced Berge F-free we get

$$(n-r+1)\mathrm{ex}_{(r-1)}(n, \mathrm{B}_{\mathrm{ind}}F) = (n-r+1)|\mathcal{H}| = \sum_{x \in V} |\mathcal{H}_x| \leq n \times \mathrm{ex}_{(r-1)}(n-1, \mathrm{B}_{\mathrm{ind}}F).$$

By Lemma 3.1 the right hand side is at most $n \times \mathrm{ex}_r(n, \mathrm{B}_{\mathrm{ind}}F)$. Rearranging yields (2). □

3.2 The α-core of a hypergraph

Let $\mathcal{H}$ be an r-partite, r-uniform hypergraph with parts $V(\mathcal{H}) = V_1 \cup \ldots \cup V_r$. For some $1 \leq s \leq r$ and edge $e \in \mathcal{H}$, define $e[\overline{s}]$ to be the trace of e onto all parts other than V_s. That is, $e[\overline{s}] = e \setminus V_s$. Let $\mathcal{H}[\overline{s}] = \{e[\overline{s}] : e \in E(\mathcal{H})\}$.

Theorem 3.2. *For positive integers α, r, any r-uniform r-partite hypergraph $\mathcal{H}$ contains edge-disjoint subhypergraphs $\mathcal{A}$ and $\mathcal{B}$ such that*

(a) *For any $S \subseteq V(\mathcal{H})$, with $|S| = r-1$, either $\deg_{\mathcal{A}}(S) = 0$ or $\deg_{\mathcal{A}}(S) \geq \alpha$.*

(b) $|\mathcal{B}| \geq \frac{|\mathcal{H} \setminus \mathcal{A}|}{\alpha - 1}$ *and* $|\mathcal{B}| \leq \sum_{s=1}^{r} |\mathcal{B}[\overline{s}]|$.

Proof. We build $\mathcal{A}$ and $\mathcal{B}$ inductively. Initially set $\mathcal{H}_0 := \mathcal{H}$, $\mathcal{B}_0 := \{\emptyset\}$.

At step i, if there exists an $S \subseteq V(\mathcal{H}_{i-1})$ with $|S| = r-1$ and $1 \le \deg_{\mathcal{H}_{i-1}}(S) \le \alpha - 1$, then let E_S be the edges of $\mathcal{H}_{i-1}$ containing S. Set $\mathcal{H}_i = \mathcal{H}_{i-1} \setminus E_S$. Pick any edge, say $B_i \in E_S$, and set $\mathcal{B}_i = \mathcal{B}_{i-1} \cup \{B_i\}$.

The process ends after k steps when for every $S \subseteq V(\mathcal{H}_k)$ with $|S| = r-1$, either $\deg_{\mathcal{H}_k}(S) = 0$ or $\deg_{\mathcal{H}_k}(S) \ge \alpha$. Let $\mathcal{A} := \mathcal{H}_k$ and $\mathcal{B} := \mathcal{B}_k = \{B_1, \dots, B_k\}$. Then $\mathcal{A}$ satisfies (a).

To see that $\mathcal{B}$ satisfies (b), at each step i when we choose $B_i \in E_S$, $|E_S| \le \alpha - 1$, so we obtain that $|\mathcal{B}|$ is at least a $1/(\alpha-1)$ portion of the deleted edges. Next, at each step, we associated with B_i a distinct set S_i of $r-1$ vertices. If B_i and B_j are associated with sets S_i and S_j respectively such that both sets are contained in $(V_1 \cup \ldots \cup V_r) \setminus V_s$, then in $\mathcal{B}[\overline{s}]$, $B_i[\overline{s}] = S_i$ and $B_j[\overline{s}] = S_j$ are distinct. Hence $\sum_{s=1}^{r} \mathcal{B}[\overline{s}] \ge |\{S_1, \dots, S_k\}| = |\mathcal{B}|$. □

Let any $\mathcal{A} \subseteq \mathcal{H}$ satisfying (a) be called an α-**core** of $\mathcal{H}$.

Lemma 3.3. *Let α, r be positive integers, and let F be a graph with $|V(F)| - 1 \le \alpha$. Let $\mathcal{H}$ be an r-uniform, r-partite hypergraph with an α-core $\mathcal{A}$. If the 2-shadow $\partial_2\mathcal{A}$ of $\mathcal{A}$ contains a copy of F then $\mathcal{A}$ (and therefore $\mathcal{H}$) contains an induced Berge F.*

Proof. We will find an induced Berge F on the same base vertex set $V(F)$. Let xy be an edge in the copy of F, and let e_{xy} be an edge of $\mathcal{A}$ containing $\{x, y\}$ with minimum $|e_{xy} \cap V(F)|$. Such an edge e_{xy} exists by the definition of the 2-shadow. If e_{xy} contains some vertex $z \in V(F) \setminus \{x, y\}$, then the $(r-1)$-set $e_{xy} \setminus \{z\}$ is contained in at least $\alpha - 1$ other edges in $\mathcal{A}$. Since there are $|V(F)| - 3 \le \alpha - 2$ vertices in $V(F) \setminus \{x, y, z\}$, we may find some $z' \notin V(F) - \{x, y, z\}$ such that $e_{xy} \setminus \{z\} \cup \{z'\} \in E(\mathcal{A})$, contradicting the choice of e_{xy}. Therefore $e_{xy} \cap V(F) = \{x, y\}$. We find such an edge of $\mathcal{A}$ for each edge of F. □

If $\alpha \ge e(F) + |V(F)|$, then with the same method one can find an induced Berge F in $\mathcal{A}$ such that each pair of hyperedges e_{xy} and e_{uv} intersect only at $\{x, y\} \cap \{u, v\}$. This is called an F-*expansion*. But this observation does not seem to help our purposes here.

Claim 3.4. *Suppose that $r \ge 3$ and $\mathcal{A}$ contains an induced Berge F, where $|V(F)| \le \alpha$ (and $E(F) \ne \emptyset$). Define a new graph $F^+ := F^+_{xy}$ by adding a new vertex $z \notin V(F)$, taking an edge $xy \in E(F)$, and joining z to x and to y. Then $\mathcal{A}$ also contains an induced Berge F^+.*

Proof. By Lemma 3.3, there exists a hyperedge $e_{xy} \in \mathcal{A}$ such that $e_{xy} \cap V(F) = \{x, y\}$. Then for any $z' \in e_{xy} \setminus \{x, y\}$ we have that xz' and $yz' \in \partial_2\mathcal{A}$, so F^+ is a subgraph of $\partial_2\mathcal{A}$. Then Lemma 3.3 completes the Claim. □

Lemma 3.5. *Suppose that $G \in \mathcal{G}_{\text{tri}}$ with $t = |V(T)| \ge 3$. Then $G \in \mathcal{G}^{(t)}_{\text{tri}}$.*

Proof. This statement seems to be evident, but still needs a proof. By definition, there exists an $s \ge t$ such that $G \in \mathcal{G}^{(s)}_{\text{tri}}$. Let $s = s(G)$ be the smallest such s. We will show by induction on t that $s(G) = t$. The base case $t = 3$ is obvious. Suppose $t > 3$ and that G is a subgraph of $H \in \mathcal{G}^{(s)}_{\text{tri}}$, where the vertices of H are $\{v_1, \dots, v_s\}$ and each v_i (with $i \ge 3$) has exactly two H-neighbors in $\{v_1, \dots, v_{i-1}\}$. Moreover, these two neighbors (call them $v_{\alpha(i)}$ and $v_{\beta(i)}$) are joined by an edge in H. Let $I \subseteq [s]$, $I := \{i_1, \dots, i_t\}$, $1 \le i_1 < \cdots < i_t \le s$, $V_I := \{v_i : i \in I\}$, and suppose that G

is a spanning subgraph of $H[V_I]$. Since s is minimal, we have $i_t = s$ and $N_H(v_s) = \{v_{\alpha(s)}, v_{\beta(s)}\}$. $G' := H[V_I]\backslash\{v_s\}$ has $t-1$ vertices, and it belongs to $\mathcal{G}_{\text{tri}}$. By our induction hypothesis there exists a $H' \in \mathcal{G}_{\text{tri}}^{t-1}$ such that G' is a subgraph of H' on the same vertex set $V_I\backslash\{v_s\}$. If $\{v_{\alpha(s)}, v_{\beta(s)}\} \subseteq V(H')$ then by adjoining a new vertex z' to H' and connecting it to $v_{\alpha(s)}$ and $v_{\beta(s)}$ we obtain a t-vertex graph H'' from $\mathcal{G}_{\text{tri}}^{(t)}$ containing G. If $|N_H(v_s) \cap V(H')| \leq 1$ then it is even simpler to find such a graph H''. □

3.3 Proofs of the upper bounds for induced Berge F problems

We prove a version of Theorem 2.1 with more precise bounds. For positive integers a and b, $(a)_b = (a)(a-1)\cdots(a-b+1)$ denotes the falling factorial.

Theorem 3.6. *Let t, r, n be positive integers, and let F be any graph with $|V(F)| = t$. Let $\mathcal{H}$ be an n-vertex r-uniform hypergraph with no induced Berge F. If $\mathcal{H}$ is r-partite, then*

$$e(\mathcal{H}) \leq \sum_{i=2}^{r} (t-2)^{r-i}(r)_{r-i}\text{ex}(n, K_i, F).$$

Proof. We proceed by induction on r. The base case $r = 2$ is trivial since an induced Berge F is just a copy of F. Thus $\text{ex}_2(n, \text{B}_{\text{ind}}F) = \text{ex}(n, K_2, F) = \text{ex}(n, F)$. Now let $r \geq 3$. Let $\mathcal{A}$ and $\mathcal{B}$ be subhypergraphs of $\mathcal{H}$ obtained from Theorem 3.2 with $\alpha = t - 1$. So we have

$$|\mathcal{H}| = |\mathcal{A}| + |\mathcal{H} \setminus \mathcal{A}| \leq |\mathcal{A}| + (t-2)\sum_{s=1}^{r} |B[\overline{s}]| \leq |\mathcal{A}| + (t-2)(r)\text{ex}_{r-1}(n, \text{B}_{\text{ind}}F),$$

where the last inequality holds because each $B[\overline{s}]$ is $(r-1)$-uniform, $(r-1)$-partite and does not contain an induced Berge F.

By Lemma 3.3, $\partial_2\mathcal{A}$ contains no copy of F. Furthermore, since each edge in $\mathcal{A}$ creates a K_r in $\partial_2\mathcal{A}$, $|\mathcal{A}| \leq \text{ex}(n, K_r, F)$. Applying the induction hypothesis, we obtain

$$|\mathcal{H}| \leq \text{ex}(n, K_r, F) + (t-2)r\sum_{i=2}^{r-1} (t-2)^{r-1-i}(r-1)_{r-1-i}\text{ex}(n, K_i, F)$$

and we are done. □

Corollary 3.7. *Let t, r, n be positive integers, and let F be any graph with $V(F) = t$. Then*

$$\max_{2\leq s\leq r}\{\text{ex}(n-(r-s), K_s, F)\} \leq \text{ex}_r(n, \text{B}_{\text{ind}}F) \leq \frac{r^r}{r!}\sum_{i=2}^{r}(t-2)^{r-i}(r)_{r-i}\text{ex}(n, K_i, F).$$

In particular, $\text{ex}_r(n, \text{B}_{\text{ind}}F) = \Theta(\max_{s\leq r}\{\text{ex}(n, K_s, F)\})$.

Proof. The lower bound follows from Lemma 3.1 and (1). For the upper bound, we use the fact that any r-uniform hypergraph $\mathcal{H}$ has an r-partite subhypergraph with at least $\frac{r!}{r^r}e(\mathcal{H})$ edges. Apply Theorem 3.6 to any such subhypergraph. □

Proof of Theorem 2.2. The lower bound comes from Lemma 3.1. For the upper bound, we proceed by induction on r. First we show that if $\mathcal{H}$ is r-partite with no induced Berge $F \in \mathcal{G}_{\text{tri}}$ then

$$|\mathcal{H}| \leq (t-2)^{r-2}\frac{r!}{2}\text{ex}(n, F). \tag{3}$$

The base case $r = 2$ is trivial, so let $r \geq 3$. Let $\mathcal{A}$ and $\mathcal{B}$ be subhypergraphs of $\mathcal{H}$ obtained from Theorem 3.2 with $\alpha = t - 1$. Again we have

$$|\mathcal{H}| \leq |\mathcal{A}| + (t-2)\sum_{s=1}^{r}|B[\overline{s}]| \leq |\mathcal{A}| + (t-2)(r)\text{ex}_{r-1}(n, \text{B}_{\text{ind}}F). \tag{4}$$

Observe that $\mathcal{A}$ is empty. Indeed, if $\mathcal{A}$ contains at least one edge, then the 2-shadow $\partial_2\mathcal{A}$ contains a K_r. So Claim 3.4 and Lemma 3.5 imply that $\partial_2\mathcal{A}$ contains a copy of F. Then we apply Lemma 3.3 to find an induced Berge F, a contradiction. Hence $|\mathcal{A}| = 0$. Applying induction hypothesis, (4) yields (3).

Finally, if $\mathcal{H}$ is not r-partite, then we apply the previous proof to an r-partite subgraph $\mathcal{H}'$ of $\mathcal{H}$ with at least $\frac{r!}{r^r}|\mathcal{H}|$ edges to obtain $|\mathcal{H}| \leq \frac{1}{2}r^r(t-2)^{r-2}\text{ex}(n, F)$. □

Proof of Theorem 2.4. For the lower bound, let each component of $\mathcal{H}$ be a clique such that there are as many cliques of size $r+t-3$ as possible. If $n = a(r+t-3)+b$ where $0 \leq b < r+t-3$, then $|\mathcal{H}| = a\binom{r+t-3}{r} + \binom{b}{r}$. Suppose $\mathcal{H}$ contains an induced Berge $K_{1,t-1}$. Then its base vertices, say $\{v_1, \ldots, v_t\}$ must be contained in a single component of $\mathcal{H}$. But each edge in a component contains at least 3 base vertices, a contradiction.

For the upper bound, let $\mathcal{H}$ be an n-vertex, r-uniform hypergraph with no induced Berge $K_{1,t-1}$. We say that a set system $\{f_1, \ldots, f_s\}$ is *strongly representable* if for every $f_i \in \mathcal{F}$, there exists a $v_i \in f_i$ such that $v_i \notin f_j$ for all $j \neq i$. Füredi and Tuza [4] proved that if a set system $\mathcal{F}$ with $|f| \leq r$ for all $f \in \mathcal{F}$ does not contain a strongly representable subfamily of size s then $|\mathcal{F}| \leq \binom{r+s-1}{r}$. For any vertex $v \in V(\mathcal{H})$, let $E_v := \{e \setminus \{v\} : v \in e \in \mathcal{H}\}$. The $(r-1)$-uniform set system E_v cannot contain a strongly representable subfamily of size $t-1$, otherwise the corresponding edges in $\mathcal{H}$ and their representative vertices would yield an induced Berge $K_{1,t-1}$ in $\mathcal{H}$ with vertex v as the center vertex. Therefore $\deg(v) \leq \binom{(r-1)+(t-2)}{r-1}$ so $|\mathcal{H}| \leq \frac{n}{r}\binom{r+t-3}{r-1}$. □

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