

Stability version of Dirac's theorem and its applications for generalized Turán problems

Xiutao Zhu^{1,2}, Ervin Győri¹, Zhen He^{1,3}, Zequn Lv^{1,3}, Nika Salia¹, and Chuanqi Xiao^{1,4}

¹Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences.

²Department of Mathematics, Nanjing University.

³Department of Mathematical Sciences, Tsinghua University.

⁴Central European University, Budapest.

Abstract

In 1952, Dirac proved that every 2-connected n -vertex graph with the minimum degree $k + 1$ contains a cycle of length at least $\min\{n, 2(k + 1)\}$. Here we obtain a stability version of this result by characterizing those graphs with minimum degree k and circumference at most $2k + 1$.

We present applications of the above-stated result by obtaining generalized Turán numbers. In particular, for all $\ell \geq 5$ we determine how many copies of a five-cycle as well as four-cycle are necessary to guarantee that the graph has circumference larger than ℓ . In addition, we give a new proof of Luo's Theorem for cliques using our stability result.

1 Introduction

Circumference of graphs

The problem of determining whether a graph contains a Hamiltonian cycle has been a fundamental question of graph theory. Deciding the Hamiltonicity for graphs is NP-complete. Therefore it is interesting to study sufficient conditions for Hamiltonicity. The natural generalization of this problem is to find sufficient conditions for a given circumference which is the length of a longest cycle. In 1952 Dirac obtained a bound on the circumference of 2-connected graphs in terms of the minimum degree. Let us denote the circumference of a graph G by $c(G)$.

Theorem 1. (Dirac [4]) *Let G be a 2-connected n -vertex graph with minimum degree at least $k + 1$, then*

$$c(G) \geq \min\{n, 2(k + 1)\}.$$

Later in 1977, Kopylov obtained a similar bound on the circumference of 2-connected graphs in terms of the average degree. Let us denote the number of edges of a graph G by $e(G)$.

Theorem 2. (Kopylov [17]) *Let G be a 2-connected n -vertex graph with $c(G) \leq \ell$ then*

$$e(G) \leq \max \left\{ \binom{\ell - 1}{2} + 2(n - \ell + 1), \binom{\lfloor \frac{\ell}{2} \rfloor}{2} + \lfloor \frac{\ell}{2} \rfloor \left(n - \lfloor \frac{\ell}{2} \rfloor \right) + \mathbb{1}_{2 | (\ell - 1)} \right\}.$$

Füredi, Kostochka and Verstraëte [9] obtained the stability result of Kopylov's Theorem. Before presenting the result we need to introduce a class of extremal graphs. Let K_k be the clique of k vertices and I_k be the independent set of k vertices. For a positive integer a , let aK_k be the graph consisting of a disjoint cliques of order k . For graphs G and H , we denote by $G \cup H$ the disjoint union of graphs G and H . We denote by $G + H$ the join of G and H , that is the graph obtained by connecting each pair of vertices between a vertex disjoint copies of G and H . For example $K_k + I_{n-k}$ has minimum degree k and circumference is $2k$ for $n \geq 2k$. For a set of vertices $A \subseteq V(G)$, let $G - A$ be the induced subgraph of G on the vertex set $V(G) \setminus A$, i.e. $G - A = G[V(G) \setminus A]$.

Introduction of some classes of extremal graphs. We denote the graph $K_k + I_{n-k}$ by $H(n, 2k)$ and let $H(n, 2k + 1)$ be a graph obtained from $H(n, 2k)$ by adding an additional edge incident to two vertices of the independent set I_{n-k} .

Here we define a class of graphs $\mathcal{H}_{1,n,k}$ for all integers k and n such that $n = b(k - 1) + 3$ for some positive integer b . Let $b = b_1 + b_2$ for some non-negative integers b_1 and b_2 . Then let G_0 be the graph $((b_1K_{k-1} + \{u_1\}) \cup (b_2K_{k-1} + \{u_2\})) + \{u\}$. Let G be the graph obtained from G_0 by adding the edge u_1u_2 , G_1 be the graph obtained from G by removing the edge uu_1 , G_2 be the graph obtained from G by removing the edge uu_2 and G_3 be the graph obtained from G by removing edges uu_1 and uu_2 . All such graphs G, G_1, G_2 and G_3 are from the class $\mathcal{H}_{1,n,k}$. Note that all graphs in $\mathcal{H}_{1,n,k}$ have circumference $2k + 1$.

For all integers k and n such that $n = b(k - 1) + 1$ for some positive integer b , let

$$\mathcal{H}_{2,n,k} = \{K_2 + bK_{k-1}, \overline{K}_2 + bK_{k-1}\}.$$

Note that, the graphs from $\mathcal{H}_{2,n,k}$ have circumference $2k$.

Theorem 3. (Füredi, Kostochka, Verstraëte [9]) *Let G be a 2-connected n -vertex graph such that $c(G) = \ell$ and $n \geq 3 \lfloor \ell/2 \rfloor$, then either*

$$e(G) < \binom{\lceil \ell/2 \rceil + 2}{2} + \left(\left\lfloor \frac{\ell}{2} \right\rfloor - 1 \right) \left(n - \left\lfloor \frac{\ell}{2} \right\rfloor - 2 \right),$$

or $G \subseteq H(n, \ell)$ or $G - A$ is a star forest for some $A \subseteq V(G)$ of size at most $\frac{\ell}{2}$.

Recently, Ma and Ning also obtained more general stability-type results of Kopylov's Theorem in [20]. In this work we prove the following stability version of Dirac's theorem.

Theorem 4. *Let G be a 2-connected graph of n vertices with $n \geq c(G) + 1$ and $\delta(G) = k$. Then either $c(G) \geq 2k + 2$, or*

- $c(G) = 2k + 1$ and $G \subseteq H(n, 2k + 1)$, or $G \in \mathcal{H}_{1,n,k}$, or $G \subseteq K_2 + (K_k \cup \frac{n-k-2}{k-1}K_{k-1})$, or $k = 4$ and $G \subseteq K_3 + \frac{n-3}{2}K_2$, or $k = 3$ and $G \subseteq K_2 + (S_{n-3-2t} \cup tK_2)$.
- $c(G) = 2k$ and $G \subseteq H(n, 2k)$ or $G \in \mathcal{H}_{2,n,k}$.

This theorem seems to have many applications. With this new tool, it is possible to re-prove some classical results in graph theory. Even more with this theorem we determined generalized Turán numbers of cycles.

Applications for Generalised Turán numbers.

A central topic of extremal combinatorics is to investigate sufficient conditions for the appearance of a given cycle. In particular, it is popular to maximize the number of cycles of length ℓ in

graphs of given order without a cycle of length k as a subgraph. For given integers $k > 3$ and m , Gishboliner and Shapira determined the order of magnitude of how many copies of k -cycle is enough to guarantee the appearance of a m -cycle. This problem was also settled independently in [10] for k and m even. Maximizing the number of triangles in k -cycle free graphs is still not settled, since this number is closely related to Turán number of even cycles see [13].

While Erdős was measuring how far are the triangle-free graphs from bipartite graphs, he naturally asked a question ‘*What is the maximum number of pentagons in a triangle-free graph*’ [5]. This question was settled half a century later by Grzesik [11] and independently by Hatami, Hladký, Král, Norine, Razborov [16], using flag algebras. In 1991, Györi, Pach, Simonovits [14], defined the generalized Turán number and obtained some results. In particular, they maximized copies of a bipartite graph with an almost one-factor in triangle-free graphs. While investigating pentagon-free 3-uniform hypergraphs Bollobás-Györi [3] initiated the study of the converse of the problem of Erdős. They asked the following question ‘*What is the maximum number of triangles in a pentagon-free graph*’. This problem is still open, for the improvements on the upper-bound see [3, 7, 8].

Grzesik and Kielak in [12] determined that every graph on n vertices without odd cycles of length less than k contains at most $(n/k)^k$ cycles of length k for all $k \geq 7$. This result is an extension of the previously mentioned problem of Erdős [5]. Erdős and Gallai determined the maximum number of edges in a graph not containing long paths and cycles as well in [6]. Luo [19] extended this result by determining the maximum number of cliques in a graph with a given circumference. The generalized Turán version of this problem for paths was studied in [15].

Notations. The cycle of length ℓ is denoted by C_ℓ . $\mathcal{C}_{\geq \ell}$ denotes the family of all cycles of length at least ℓ . For an integer n , a graph H and a family of graphs \mathcal{F} , Alon and Shikhelman denoted generalized Turán number by $\text{ex}(n, H, \mathcal{F})$ in [1,2]. Where $\text{ex}(n, H, \mathcal{F})$ denotes the maximum number of copies of H as a subgraph in an n -vertex graph not containing F as a subgraph for all $F \in \mathcal{F}$. When family \mathcal{F} consists of a single graph F , i.e. $\mathcal{F} = \{F\}$ we write $\text{ex}(n, H, F)$ instead of $\text{ex}(n, H, \{F\})$.

For graphs G and H let $H(G)$ be the number of copies of H in G . For example the number of cycles of length k in G is denoted by $C_k(G)$. For a vertex v in a graph G , let $C_t(v)$ be the number of cycles of length t containing the vertex v in G . For $v \in V(G)$, we denote the neighborhood of v by $N(v)$. For a vertex v , the closed neighbourhood of it $N(v) \cup \{v\}$ is denoted by $N[v]$.

Generalized Turán-type results. In this paper, by applying Theorem 4, we determine the maximum number of four-cycles and pentagons in graphs with bounded circumference. Even more we prove that the extremal graph is unique for large enough n .

Theorem 5. *For all integers $\ell \geq 6$ and $n \geq 100\ell^{3/2}$ we have*

$$\text{ex}(n, C_5, \mathcal{C}_{\geq \ell+1}) = C_5(H(n, \ell)),$$

and $H(n, \ell)$ is the unique extremal graph.

For $\ell = 5$ and $n \geq 200$, we have

$$\text{ex}(n, C_5, \mathcal{C}_{\geq \ell+1}) = \left\lfloor \frac{(n-3)^2}{2} \right\rfloor,$$

the extremal graph is a member of the family $\mathcal{H}_{1,n,k}$ with parameters $\left\lfloor \frac{n-3}{2} \right\rfloor, \left\lceil \frac{n-3}{2} \right\rceil$.

Theorem 6. For all integers n and ℓ such that $\ell \geq 4$ and $n \geq 10\ell^{3/2}$, we have

$$\text{ex}(n, C_4, \mathcal{C}_{\geq \ell+1}) = C_4(H(n, \ell)),$$

and $H(n, \ell)$ is the unique extremal graph.

In addition, we also give a new proof of Luo's following theorem by using Theorem 4.

Theorem 7. (Luo [19]) For all integers n and $\ell \geq 3$ we have

$$\text{ex}(n, K_s, \mathcal{C}_{\geq \ell+1}) \leq \frac{n-1}{\ell-1} \binom{\ell}{s}.$$

The equality holds if and only if $\ell-1 \mid n-1$.

We expect Theorem 5 holds not only for cycles of length four and five but for cycles of any length more than 3.

Conjecture 1. For all integers n , k and ℓ such that $k \geq 4$, $\ell > k$ and n large enough, we have

$$\text{ex}(n, C_k, \mathcal{C}_{\geq \ell+1}) = C_k(H(n, \ell)).$$

We also prove the following theorem which verifies Conjecture 1 asymptotically for large enough k and n .

Theorem 8. The following holds for every integer $k \geq 3$.

$$\lim_{\ell \rightarrow \infty} \left(\lim_{n \rightarrow \infty} \frac{\text{ex}(n, C_{2k}, \mathcal{C}_{\geq \ell+1})}{\left\lfloor \frac{\ell}{2} \right\rfloor^k n^k} \right) = \frac{1}{2k},$$

$$\lim_{\ell \rightarrow \infty} \left(\lim_{n \rightarrow \infty} \frac{\text{ex}(n, C_{2k+1}, \mathcal{C}_{\geq \ell+1})}{\left\lfloor \frac{\ell}{2} \right\rfloor^{k+1} n^k} \right) = \frac{1}{2}.$$

2 Preliminaries

Erdős and Gallai used the following robust lemma to find the extremal number of graphs with bounded circumference. We use the lemma to prove Theorem 4.

Lemma 9. (Erdős-Gallai [6]) Let G be a 2-connected graph and x, y be two given vertices. If every vertex other than x, y has a degree at least k in G , then there is an (x, y) -path of length at least k .

Even more, Li and Ning applied this lemma to prove the existence of (H, C, t) -fans under some conditions. For our proof of Theorem 4 we need the existence of (H, C, t) -brooms under the same conditions. Let us introduce the notion of (H, C, t) -brooms.

Definition. Let G be a graph, C be a cycle of G , and H be a component of $G - C$. A subgraph B of G is called an (H, C, t) -broom, if it consists of t paths P_1, P_2, \dots, P_t each starting at the same vertex of H and finishing at distinct vertices of C for some $t \geq 2$, such that

- (1) All vertices of P_1 except the last are in $V(H)$.
- (2) The paths P_i have length one for all $2 \leq i \leq t$.

The same proof of Theorem 2.1 in the paper of Li and Ning [18] proves the following theorem. Naturally, to refrain from repetition, we will not include their proof in this work.

Lemma 10. *(Li-Ning [18]) Let G be a 2-connected graph, C a cycle of G , and H a connected component of $G - C$. If each vertex $v \in V(H)$ has $d_G(v) \geq k$, then there is an (H, C, t) -broom with at least k edges.*

3 Proof of the Stability of Dirac's theorem

Here we present the proof of Theorem 4. Let G be an n -vertex 2-connected graph with minimum degree $k \geq 2$ and circumference at most $2k + 1$. By Theorem 1, G contains a cycle of length at least $\min\{n, 2k\}$. Since $n \geq c(G) + 1$, hence we have $c(G) \in \{2k + 1, 2k\}$. Let C be a longest cycle of G and H_1, H_2, \dots, H_s be connected components of $G - C$ for some $s \geq 1$, where $G - C$ is the induced subgraph of G on the vertex set $V(G) \setminus V(C)$. Since $\delta(G) \geq k$, each component H_j contains an (H_j, C, t) -broom with at least k edges by Lemma 10. In the following part of the proof we characterize the structure of each H_i .

Let B be an edge-maximal (H_1, C, t) -broom consisting of following t paths vPu_1, vu_2, \dots , and vu_t . Recall that vertices $\{u_1, u_2, \dots, u_t\}$ are distinct vertices of the cycle C . Each cycle has a positive and a negative direction to visit their vertices, without loss of generality we assume that starting at the vertex u_1 going around C in the positive direction we visit terminal vertices of B in this given order $u_2, u_3 \dots u_t$. For a given vertex u of C , we denote its two neighbors on the cycle by u^+ and u^- , where u^-uu^+ is a sub-path of C in the positive direction. For two vertices x, y of C , $x\overrightarrow{C}y$ denotes the segment of C from x to y in the positive direction, $x\overleftarrow{C}y$ denotes the segment of C from x to y in the negative direction.

Recall the length of C is $2k$ or $2k + 1$ and $v(B) \geq k + 1$ by Lemma 10. On the other hand we have

$$v(C) = t + \sum_{i=1}^t v(u_i^+ \overrightarrow{C} u_{i+1}^-) \geq t + (t - 2) + 2(v(B) - t) = 2v(B) - 2,$$

where indices are taken modulo t . Hence we have $v(B) = k + 1$. Furthermore if $v(C) = 2k$, then the segments $u_1^+Cu_2^-$ and $u_t^+Cu_1^-$ contain exactly $v(vPu_1) - 1 = k - t + 1$ vertices while the rest of the segments contain exactly one vertex. If $v(c) = 2k + 1$, then one of the segments contains one more vertex.

Claim 1. *We have either $k = 4, c(G) = 9$ and $G \subseteq K_3 + \frac{n-3}{2}K_2$, or $t \in \{2, k\}$, $H_1 = K_{k-t+1}$ and each vertex of H_1 is incident with all vertices of $\{u_1, \dots, u_t\}$.*

Proof. At first we assume $t = k$. Therefore all paths of the broom B are single edges and all segments $u_i^+ \overrightarrow{C} u_{i+1}^-$ of cycle C contain exactly one vertex except if $c(G) = 2k$ and one segment containing two vertices if $c(G) = 2k + 1$. Without loss of generality, suppose $u_1^+ \overrightarrow{C} u_2^-$ contains two vertices. We have $V(H_1) = \{v\}$ since otherwise we could extend the cycle C given that G is $C \cup H_1$ is 2-connected. Hence we are done if $k = t$.

From here we assume $2 \leq t \leq k - 1$. The path vPu_1 is a path of $k - t + 2$ vertices, let vPu_1 be $v_0v_1 \dots v_{k-t}u_1$, where $v_0 = v$.

First we assume $V(H_1) \neq \{v, v_1, \dots, v_{k-t}\}$. Let H'_1 be a maximal connected component of $H_1 - \{v_0, v_1, \dots, v_{k-t}\}$. Since G is 2-connected there are at least two edges from H'_1 to the rest of

the graph. At first we suppose that there is a vertex y in $V(H'_1)$ with a neighbour u' on C . Since H'_1 is a subgraph of connected graph H_1 , there is an edge $v_i x$ between $V(H'_1)$ and $\{v_0, v_1, \dots, v_{k-t}\}$ for some i satisfying $0 \leq i \leq k-t$. Since x and y are vertices of $V(H'_1)$ there exists a path $xP'y$ from x to y in H'_1 . If u' is on the segment $u_j^+ \vec{C} u_{j+1}^-$ for some j satisfying $1 \leq j \leq t$, then

$$u_j P v_i x P' y u' \vec{C} u_j \text{ or } u_j \vec{C} u' y P' x v_i P v_0 u_2 C u_j$$

is a longer cycle. Since otherwise $v(u_j^+ \vec{C} (u')^-) \geq k-t-i+2$ and $v((u')^+ \vec{C} u_{j+1}^-) \geq i+2$ contradicting to $k-t+2 \geq v(u_j^+ \vec{C} u_{j+1}^-) = v(u_j^+ \vec{C} (u')^-) + 1 + v((u')^+ \vec{C} u_{j+1}^-) \geq k-t+5$. Moreover, if $u' = u_j$ for some j satisfying $3 \leq j \leq t$, then

$$u_{j-1} v_0 P v_i x P' y u_j \vec{C} u_{j-1}$$

is a longer cycle, a contradiction. Hence, $u' \in \{u_1, u_2\}$ and $N_G(V(H'_1)) \subseteq \{v_0, v_1, \dots, v_{k-t}, u_1, u_2\}$. Furthermore, if $t \geq 3$, $u' \neq u_2$ and $N_G(V(H'_1)) \subseteq \{v_0, v_1, \dots, v_{k-t}, u_1\}$. For otherwise,

$$u_3 v_0 P v_i x P' y u_2 \overleftarrow{C} u_3 \text{ or } u_1 P v_i x P' y u_2 \vec{C} u_1$$

is a longer cycle, a contradiction. (If it is not the first case, then $i = 0$ and $v(u_1^+ \vec{C} u_2^-) = k-t+1$)

Observe that no two consecutive vertices of the path $u_2 v P u_1$ are incident to a vertex of $V(H'_1)$. By the minimum degree condition, we have $|V(H'_1)| > 1$ since $k \geq 3$. Since G is 2-connected, there are at least two independent edges between $\{v_0, \dots, v_{k-t}, u_1, u_2\}$ and $V(H'_1)$. Note that $u_2, v_0, \dots, v_{k-t}, u_1$ is a path, for the technical reasons we denote $v_{-1} := u_2$ and $v_{k-t+1} := u_1$. From all such pairs of edges, we choose two independent edges $x_1 v_i$ and $x_2 v_j$ minimizing $j-i$ if H'_1 is 2-connected. Otherwise we still minimize $j-i$ such that that x_1 is in one of the 2-connected blocks of H'_1 containing exactly one cut vertex x' of H'_1 denote by B'_1 . The vertex x_2 is in any other 2-connected blocks of H'_1 . From minimality of $j-i$, vertices of $V(B'_1 \setminus \{x'\})$ are not incident with vertices from $\{v_{i+1}, \dots, v_{j-1}\}$. Every vertex of $B'_1 \setminus \{x'\}$ has degree at least k in G . On the other hand they are incident with vertices from $V(B'_1)$ and $\{v_{-1}, \dots, v_{k-t}, v_{k-t+1}\} \setminus \{v_{i+1}, \dots, v_{j-1}\}$. Hence we have the degree of vertices $B'_1 \setminus \{x'\}$ in B'_1 is at least

$$k - \left\lceil \frac{i+2}{2} \right\rceil - \left\lceil \frac{k-t-j+2}{2} \right\rceil \geq j-i-1.$$

Note that at least one of the vertices of $\{v_i, v_j\}$ is not from $\{u_1, u_2\}$, since H'_1 is subgraph of connected H_1 . By Lemma 9, there is a path $x_1 P_1 x'$ in the block B'_1 of length at least $j-i-1$. Therefore there is a path $x_1 P'' x_2$ of length at least $j-i+1$ in H'_1 , a contradiction to the maximality of the broom B . Since by exchanging $v_i P v_j$ with $v_i x_1 P'' x_2 v_j$, we would get a bigger broom. Therefore we have $V(H_1) = \{v_0, v_1, \dots, v_{k-t}\}$

Here we show $N(v_i) \subseteq V(H_1) \cup \{u_1, \dots, u_t\}$ for $0 \leq i \leq k-t$. The statement holds for v_0 , suppose some v_i is adjacent to a vertex u' which is on some segment $u_j^+ \vec{C} u_{j+1}^-$. Then one of the following cycles is longer than C

$$u' \vec{C} u_j P v_i u' \text{ or } u_{j+1} \vec{C} u' v_i P v_0 u_{j+1},$$

a contradiction. Hence we have $N(v_i) \subseteq V(H_1) \cup \{u_1, \dots, u_t\}$ for $0 \leq i \leq k-t$. From the minimum degree condition we have $k \leq d_G(v_i) \leq (v(H_1) - 1) + t = k$. Hence H_1 is a clique and each vertex of H_1 is incident with all vertices in $\{u_1, \dots, u_t\}$.

If $t = 2$, we have H_1 is a copy of K_{k-1} and each vertex is adjacent to both u_1, u_2 . Therefore we are done in this case.

If $t = 3$, then consider the following cycle

$$u_3 \xrightarrow{C} u_2 v_{k-t} P v_0 u_3.$$

Since the length of it is not greater than C and $v(u_2^+ \xrightarrow{C} u_3^-) \leq 2$, we have $k-t = 1$ and the segment $u_2^+ \xrightarrow{C} u_3^-$ contains exactly two vertices (This means $c(G) = 2k + 1$). From here it is straightforward to check that $G \subseteq K_3 + (\frac{n-3}{2} K_2)$.

If $k > t \geq 4$, one of segment $u_2^+ \xrightarrow{C} u_3^-$ or $u_3^+ \xrightarrow{C} u_4^-$ contains one vertex. Without loss of generality we may assume $v(u_2^+ \xrightarrow{C} u_3^-) = 1$. Therefore the cycle $u_3 \xrightarrow{C} u_2 v_{k-t} P v_0 u_3$ is a longer cycle than C , a contradiction. \square

From Claim 1 we have either $G \subseteq K_3 + (\frac{n-3}{2} K_2)$ and $k = 4$ or G contains a longest cycle C and each connected component of $G - C$ is either a vertex and adjacent to k vertices on C , or a clique of size $k-1$ and all vertices of the clique are adjacent to the same two vertices of C . If H_i is a $(k-1)$ -clique, we call the two neighbors of H_i lying on C the attached point.

First consider that each H_i is a clique of size $k-1$ and let w_i, w'_i denote the two attached points of H_i for all $1 \leq i \leq s$. If $c(G) = 2k$, one can easily check that $v(w_i^+ \xrightarrow{C} w_i'^-) = v(w_i'^+ \xrightarrow{C} w_i^-) = k-1$, $\{w_i, w'_i\} = \{w_1, w'_1\}$ and $w_i^+ \xrightarrow{C} w_i'^-, w_i'^+ \xrightarrow{C} w_i^-$ are both a copy of K_{k-1} , hence $G \in \mathcal{H}_{2,n,k}$. When $c(G) = 2k + 1$, by Claim 1, we say the segment $w_1 \xrightarrow{C} w'_1$ contains k vertices. If $s = 1$, then $G \subseteq K_2 + (K_k \cup 2K_{k-1})$. If $s \geq 2$, then since $c(G) = 2k + 1$, we have $\{w_1, w'_1\} \cap \{w_i, w'_i\} \neq \emptyset$ for any $2 \leq i \leq s$. Therefore either all H_i have the same two attached points $\{w_1, w'_1\}$ on C and we can see the segment $w_1 \xrightarrow{C} w'_1$ as a subgraph of K_k and we obtain $G \subseteq K_2 + (K_k \cup \frac{n-k-2}{k-1} K_{k-1})$. Or there are two of them such that their neighbours on C are w_1, w'_1 and w_1, w'_1 and $G \in \mathcal{H}_{1,n,k}$, this finishes the proof in this case.

Next consider the case there is a component of $G - C$ of size one. Let us denote this vertex by v . The vertex v has k neighbours on the cycle C and set $N(v) = \{u_1, \dots, u_k\}$. Even more the distance between any two consecutive neighbours of v is exactly two if $c(G) = 2k$ and with one has distance three if $c(G) = 2k + 1$ (if in such case, we assume $u_1 \xrightarrow{C} u_2$ is of distance 3). It is easy to see that for any other components of size 1, they have the same neighborhood with v since C is the longest. First assume there is no other component of size $k-1$. If $c(G) = 2k$, then $V(C) - N(v)$ is independent and hence $G \subseteq H(n, 2k)$. If $c(G) = 2k + 1$, then $V(C) - N(v)$ contains exactly one edge which lies on the segment of distance 3 between two consecutive neighbours of v . Hence $G \subseteq H(n, 2k + 1)$.

Hence we may assume that some component are $(k-1)$ -cliques with $k \geq 3$, saying H_i is one of such component with two attach points $\{u', u''\}$. If one of the attached points of H_i lies on $u_i^+ \xrightarrow{C} u_{i+1}^-$, we will find a longer cycle using H_i , a contradiction. Thus $\{u', u''\} \subseteq N(v)$ and we set $u' = u_a$, $u'' = u_b$ with $a, b \in [k]$. If $k \geq 4$, then by the distance of $u' \xrightarrow{C} u''$, we know $u_{b-1} \neq u_a$ and $u_{a+1} \neq u_b$. We have $u_a H_i u_b \xrightarrow{C} u_{a+1} v u_{b-1} \xrightarrow{C} u_a$ is a longer cycle, a contradiction. Then $k = 3$ and it

is easy to see that $c(G) = 7$ and $u_a = u_1$, $u_b = u_2$. That is $G - \{u_1, u_2\}$ is the disjoint union of a star and matching, $G = K_2 + (S_{n-3-2t} \cup tK_2)$. This finishes the proof of Theorem 4. \blacksquare

4 The applications for generalized Turán problems

In this chapter we present some applications of Theorem 4. In particular we determine the exact value of the generalized Turán number of pentagons or C_4 in graphs with bounded circumference and give a new proof of Theorem 7.

Proof of Theorem 5.

Throughout this subsection we denote $\lfloor \ell/2 \rfloor := k$ and $\lambda := \ell - 2k$.

Lemma 11. *Let F be a graph isomorphic to an n -vertex graph from the following set*

$$\left\{ H(n, \ell), K_2 + (K_k \cup bK_{k-1}), K_3 + \frac{n-3}{2}K_2, K_2 + (S_{n-3-2t} \cup tK_2) \right\} \cup \mathcal{H}_{1,n,k} \cup \mathcal{H}_{2,n,k}.$$

We have

- If $\ell \geq 6$ and $n \geq 3k$,

$$C_5(F) \leq C_5(H(n, \ell)).$$

The equality holds if and only if $F = H(n, \ell)$.

- If $\ell = 5$ and $n \geq 7$, then $F \in \mathcal{H}_{1,n,k}$ with parameters $\lfloor \frac{n-3}{2} \rfloor$ and $\lceil \frac{n-3}{2} \rceil$ contains most C_5 .

Proof. It is straightforward to determine the number of five cycles in $H(n, \ell)$.

$$\begin{aligned} C_5(H(n, \ell)) &= \binom{n-k}{2} \binom{k}{3} \cdot 3 \cdot 2 + (n-k) \binom{k}{4} \binom{4}{2} \cdot 2 + \binom{k}{5} \frac{5!}{10} \\ &\quad + \lambda \left\{ (n-k-2) \binom{k}{2} \cdot 2 + \binom{k}{3} \cdot 3 \cdot 2 \right\}. \end{aligned} \tag{1}$$

Suppose $F \in \mathcal{H}_{1,n,k}$, with parameters b_1 and b_2 . If $\ell \geq 6$ and $n \geq 3k$, then the number of pentagons in F is

$$\begin{aligned} C_5(F) &= \frac{n-3}{k-1} \binom{k+1}{5} \frac{5!}{10} + 2 \left(\binom{b_1}{2} + \binom{b_2}{2} \right) \binom{k-1}{2} (k-1) \\ &\quad + 2(b_1 + b_2) \binom{k-1}{2} + b_1 b_2 (k-1)^2 \\ &\leq \frac{n-3}{k-1} \binom{k+1}{5} \frac{5!}{10} + 2 \binom{\frac{n-3}{k-1}}{2} \binom{k-1}{2} (k-1) < C_5(H(n, \ell)). \end{aligned}$$

If $\ell = 5$, then $C_5(F) = b_1 b_2$. It is easy to see when $b_1 = \lfloor \frac{n-3}{2} \rfloor$ and $b_2 = \lceil \frac{n-3}{2} \rceil$, $C_5(F)$ attains maximum, which is greater than $C_5(H(n, 5)) = 2(n-4)$.

If $F \in \mathcal{H}_{2,n,k}$, then the number of pentagons in F is

$$C_5(F) = \frac{n-2}{k-1} \binom{k+1}{5} \frac{5!}{10} + 2 \binom{\frac{n-2}{k-1}}{2} \binom{k-1}{2} (k-1) < C_5(H(n, \ell)).$$

If $F = K_2 + (K_k \cup bK_{k-1})$ with parameters b_1 and b_2 , then the number of pentagons in F is

$$C_5(F) = \left(b \binom{k+1}{5} + \binom{k+2}{5} \right) \frac{5!}{10} + 2 \binom{b}{2} \binom{k-1}{2} (k-1) + 2 \left(kb \binom{k-1}{2} + b \binom{k}{2} \right) (k-1) < C_5(H(n, \ell)).$$

If $F = K_3 + \frac{n-3}{2} K_2$, then the number of pentagons in F is

$$C_5(F) = \frac{n-3}{2} \frac{5!}{10} + 2 \binom{\frac{n-3}{2}}{2} (2 * 3 * 2) + \left(\frac{n-3}{2} \right) (2 * 2 * 3 * 2) < C_5(H(n, \ell)).$$

If $F = K_2 + (S_{n-3-2t} \cup tK_2)$, then the number of pentagons in F is

$$2 \binom{t}{2} * 2 * 2 + \binom{s}{2} (2+4) + 4ts + 2(s+1)t < C_5(H(n, \ell))$$

□

Lemma 12. *Let G be a 2-connected $\mathcal{C}_{\geq \ell+1}$ -free graph with n vertices, such that $n \geq 3k$. For a vertex v of G with degree $d(v) \leq k-1$, we have*

$$C_5(v) \leq k(k-2)^2 n - \frac{1}{2} k^2 (k-2)^2.$$

Proof. We denote the set of vertices $V(G) - N[v]$ by $N_2(v)$. Let e_1 be the number of edges in $G[N(v)]$, e_2 be the number of edges between the sets of vertices $N(v)$ and $N_2(v)$ and e_3 be the number of edges in $G[N_2(v)]$ respectively. Since G is a 2-connected $\mathcal{C}_{\geq \ell+1}$ -free graph with n -vertices such that $n \geq 3k$, by Theorem 2 we have

$$e_1 + e_2 + e_3 \leq k(n-k) + \binom{k}{2} + \lambda. \quad (2)$$

Here we classify pentagons $vv_1v_2v_3v_4v$ incident with the vertex v in G . We say $vv_1v_2v_3v_4v$ is Type- i if $i = |\{v_2, v_3\} \cap N(v)|$.

In this paragraph, we estimate the maximum number of Type-2 pentagons. There are at most e_1 choices representing an edge v_2v_3 . After fixing such an edge, there are at most $\binom{|N(v)|-2}{2}$ choices for the pair of vertices v_1 and v_4 . Hence the number of Type-2 pentagons in G is at most

$$e_1 \binom{d(v)-2}{2} \cdot 2 \leq e_1 (k-3)(k-4).$$

Here we estimate the maximum number of Type-1 pentagons. Note that the opposite edge of v in the pentagon must be between $N(v)$ and $N_2(v)$. Hence there are at most e_2 choices for such

an edge. After fixing such an edge there are $\binom{|N(v)|-1}{2}$ choices for vertices v_1 and v_2 . Hence the number of Type-1 pentagons in G is at most

$$e_2 \binom{d(v)-1}{2} \cdot 2 \leq e_2(k-2)(k-3).$$

Here we estimate the maximum number of Type-0 pentagons. If each vertex of $N_2(v)$ has at most $k-2$ neighbors in $N(v)$, then the number of Type-0 pentagons in G is at most $e_3(k-2)(k-2)$. Therefore, by inequality (2), we have

$$\begin{aligned} C_5(v) &\leq (e_1 + e_2 + e_3)(k-2)^2 \leq \left(k(n-k) + \binom{k}{2} + \lambda \right) (k-2)^2 \\ &\leq k(k-2)^2 n - \frac{1}{2} k^2 (k-2)^2. \end{aligned}$$

If there is a vertex of $N_2(v)$ with $k-1$ neighbors in $N(v)$, then we have $d(v) = k-1$. We partition $N_2(v)$ into two sets A and B . Such that A contains all vertices in $N_2(v)$ with at least $k-2$ neighbors in $N(v)$. The set of remaining vertices $N_2(v) \setminus A$ is denoted by B . Let e'_3 denote the number of edges in $G[A]$ and $e''_3 := e_3 - e'_3$. In particular e''_3 denotes number of edges in $N_2(v)$ incident with at least one vertex from B . The number of Type-0 pentagons is at most

$$e'_3(k-1)(k-2) + e''_3(k-3)(k-2)$$

and

$$C_5(v) \leq e'_3(k-1)(k-2) + (e_1 + e_2 + e''_3)(k-2)(k-3). \quad (3)$$

If $|A| \leq k+1$, we have $e'_3 \leq \binom{k+1}{2}$. By inequality (2) and the above inequality we have

$$\begin{aligned} C_5(v) &\leq \binom{k+1}{2} (k-1)(k-2) + k(n-k)(k-2)(k-3) \\ &= k(k-2)(k-3)n - \frac{k^4}{2} + 4k^3 - \frac{13k^2}{2} + k \\ &\leq k(k-2)^2 n - \frac{1}{2} k^2 (k-2)^2. \end{aligned}$$

If $|A| \geq k+2$ then we distinguish two cases for estimating e'_3 depending on the value of λ . If $\ell = 2k$, then $G[A]$ is P_4 -free, $P_3 \cup P_2$ -free and $3P_2$ -free since G is $C_{\ell+1}$ -free.

This implies $e(G[A]) = e'_3 \leq n - k - 1$. If $\ell = 2k+1$, then $G[A]$ is P_5 -free, $P_4 \cup P_2$ -free, $P_3 \cup 2P_2$ -free, $2P_3$ -free and $4P_2$ -free. Which implies $e'_3 \leq n - k$. Hence by inequality (2) and the inequality (3) we have

$$\begin{aligned} C_5(v) &\leq (n-k)(k-1)(k-2) + \left(nk - \frac{k(k+1)}{2} - (n-k) + 1 \right) (k-2)(k-3) \\ &= k(k-2)(k-3)n + 2(k-2)(n-k) - \frac{1}{2}(k+1)k(k-2)(k-3) + (k-2)(k-3) \\ &\leq k(k-2)^2 n - \frac{1}{2} k^2 (k-2)^2. \end{aligned}$$

We are done. □

Here we finish the proof of Theorem 5 by means of progressive induction. Let G_n denote an extremal graph of $ex(n, C_5, \mathcal{C}_{\geq \ell+1})$. We may assume G_n is connected. First we prove the case $\ell \geq 6$. Let us define the following function.

$$\phi(n) = ex(n, C_5, \mathcal{C}_{\geq \ell+1}) - C_5(H(n, \ell)).$$

Note that $\phi(n) = C_5(G_n) - C_5(H(n, \ell))$ and it is a non-negative integer. In the following claim we find an upper-bound for $\phi(n)$.

Claim 2. *For all n such that $n \geq 100k$, either $G_n = H(n, \ell)$, or*

$$\phi(n) \leq \phi(n-1) - k(k-2)(n-4k).$$

Proof. By the definition of ϕ we have

$$\phi(n-1) - \phi(n) = (C_5(H(n, \ell)) - C_5(H(n-1, \ell))) - (C_5(G_n) - C_5(G_{n-1})).$$

Therefore from equality (1), we get

$$C_5(H(n, \ell)) - C_5(H(n-1, \ell)) = k(k-1)(k-2) \left(n - \frac{k+5}{2} \right) + \lambda k(k-1). \quad (4)$$

If G_n contains a cut vertex, let B_1 and B_2 be two end-blocks of G_n with $|V(B_2)| \geq |V(B_1)|$ and let b_1, b_2 be the cut vertices of B_1 and B_2 , respectively. At first we assume $V(B_2) \geq |V(B_1)| \geq 3k$ and $\delta(B_i) \geq k$ for each $i = 1, 2$. Since each B_i is 2-connected, combining Theorem 4 with Lemma 11, we have $B_i = H(|V(B_i)|, \ell)$. A contradiction to the maximality of the number of pentagons in G_n , since we have

$$C_5(H(v(B_1), \ell)) + C_5(H(v(B_2), \ell)) < C_5(H(v(B_1) - 1, \ell)) + C_5(H(v(B_2) + 1, \ell)),$$

by convexity. Note that we could exchange B_1 and B_2 with $H(v(B_1) - 1, \ell)$ and $H(v(B_2) + 1, \ell)$ since they are the end-blocks. Hence, either $v(B_1) \leq 3k$ or $\delta(B_i) \leq k-1$ for some B_i . If $v(B_1) \leq 3k$ then let v be a vertex other than b_1 in B_1 , then since $n \geq 100k$,

$$C_5(v) \leq 12 \binom{3k}{4} \leq k(k-2)^2 \left(n - \frac{k}{2} \right).$$

This implies

$$C_5(G_n) - C_5(G_{n-1}) \leq C_5(v) \leq k(k-2)^2 \left(n - \frac{k}{2} \right). \quad (5)$$

For the latter case $\delta(B_i) \leq k-1$ for some B_i without loss of generality, assume there is a vertex v in B_1 such that v has at most $k-1$ neighbors in B_1 . If $v \neq b_1$, then since B_1 is 2-connected and $v(B_1) \geq 3k$, inequality (5) holds by Lemma 12. If $v = b_1$, we remove all edges incident to b_1 in the subgraph B_1 . We destroyed at most $k(k-2)^2 n - \frac{1}{2} k^2 (k-2)^2$ copies of C_5 by Lemma 12. Even more the resulting graph is disconnected graph on n vertices. Therefore it contains at most $C_5(G_{n-1})$ pentagons, since we could identify a vertex from each connected component. Thus the inequality (5) holds in this case too.

Combining equality (4) and inequality (5), we get

$$\begin{aligned}\phi(n-1) - \phi(n) &\geq k(k-1)(k-2) \left(n - \frac{k+5}{2} \right) + \lambda k(k-1) - k(k-2)^2 \left(n - \frac{k}{2} \right) \\ &\geq k(k-2)(n-4k),\end{aligned}$$

therefore we are done if G_n is not 2-connected.

If G_n is 2-connected and it contains a vertex v of degree at most $k-1$, then by Lemma 12 we have $C_5(v) \leq k(k-2)^2 n - \frac{1}{2}k^2(k-2)^2$, hence $\phi(n) - \phi(n-1) \geq k(k-2)(n-4k)$ holds and we are done. If $\delta(G) \geq k$, then combining Theorem 4 and Lemma 11, we have $G_n = H(n, \ell)$. \square

The function $\phi(n)$ is decreasing non-negative function. We have a trivial bound

$$\phi(100k) \leq \binom{100k}{5} \frac{5!}{10} - C_5(H(100k, \ell)) \leq 10^9 k^5.$$

For each n such that $n > 100k$ we have either $C_5(H(n, \ell)) = ex(n, C_5, \mathcal{C}_{\geq \ell+1})$ and $\phi(n) = 0$ or $\phi(n) \neq 0$ and we have

$$\phi(n) \leq \phi(100k) - k(k-2) \sum_{i=100k}^n (i-4k) \leq 10^9 k^5 - \frac{n+92k}{2}(n-100k),$$

by Claim 2. Therefore for all $n \geq 10^5 k^{3/2}$ we have $\phi(n) = 0$. Hence we have $ex(n, C_5, \mathcal{C}_{\geq \ell+1}) = C_5(H(n, \ell))$.

Next we prove the special case when $\ell = 5$ using progressive induction. Note that $k = 2$. Let the graph from $\mathcal{H}_{1,n,k}$ with parameters $\lfloor \frac{n-3}{2} \rfloor, \lceil \frac{n-3}{2} \rceil$ be denoted by F_n and $\phi(n) = ex(n, C_5, \mathcal{C}_{\geq \ell+1}) - C_5(F_n)$.

Claim 3. For all n such that $n \geq 29$, either $G_n = F_n$, or

$$\phi(n) \leq \phi(n-1) - \left\lfloor \frac{n-27}{2} \right\rfloor.$$

Proof. If the extremal graph G_n is 2-connected, then by Theorem 4 and Lemma 11 we have $G_n = F_n$ and we are done.

If G_n is not 2-connected then let B_1, B_2 be two distinct end-blocks of G_n such that $v(B_2) \geq v(B_1)$. If $v(B_1) \leq 5$, then by removing a vertex of degree at most four from B_1 we destroy at most 12 copies of C_5 . Hence we have $\phi(n-1) - \phi(n) \geq C_5(F_n) - C_5(F_{n-1}) - 12 = \left\lfloor \frac{n-3}{2} \right\rfloor - 12$.

If $v(B_1), v(B_2) \geq 6$, then note that $\delta(B_1), \delta(B_2) \geq 2$, we have $B_1 = F_{v(B_1)}, B_2 = F_{v(B_2)}$ by Theorem 4 and Lemma 11 or $B_1 = H(6, 5)$. By convexity of the number of pentagons in F_n and $H(n, 5)$, G_n is not the extremal graph, a contradiction. \square

By Claim 3, we start progressive induction from $n = 29$ and when $n \geq 200$, we get G_n is 2-connected and $G_n = F_n$. This completes the proof of Theorem 5. \blacksquare

Proof of Theorem 6

The proof of Theorem 6 is very similar to the proof of Theorem 5. At first we prove the following lemmas.

Lemma 13. *For all $n \geq \ell$, among all graphs in the set $\{H(n, \ell), K_2 + (K_k \cup bK_{k-1}), K_3 + \frac{n-3}{2}K_2\} \cup \mathcal{H}_{1,n,k} \cup \mathcal{H}_{2,n,k}$, $H(n, \ell)$ contains most copies of C_4 .*

We omit the proof since the proof is straightforward and similar to Lemma 11

Lemma 14. *Let G be a 2-connected $\mathcal{C}_{\geq \ell+1}$ -free graph on n vertices. If some vertex v has degree at most $k-1$, then*

$$C_4(v) \leq \binom{k-1}{2}n.$$

Proof. The number of ways to choose adjacent vertices of v in a C_4 is at most $\binom{k-1}{2}$ and the number of choices for the opposite vertex of v is at most n , hence we have $C_4(v) \leq \binom{k-1}{2}n$. \square

To finish the proof we also use progressive induction method. Let us define the following function

$$\phi(n) = \text{ex}(n, C_4, \mathcal{C}_{\geq \ell+1}) - C_4(H(n, \ell)).$$

Using the same technique as in Claim 2, we have either the extremal graph G_n is 2-connected with $\delta(G) \geq k$ hence $G_n = H(n, \ell)$, or

$$\begin{aligned} \phi(n-1) - \phi(n) &= C_4(H(n, \ell)) - C_4(H(n-1, \ell)) - C_4(v) \\ &\geq \binom{k}{2}(n-k-1) + 3\binom{k}{3} - \binom{k-1}{2}n \\ &\geq (k-1)n - \frac{3k(k-1)}{2} \geq (k-1)(n-2k). \end{aligned}$$

The function $\phi(n)$ is decreasing non-negative function. We have a trivial bound

$$\phi(4k) \leq 3\binom{4k}{4} - (k-1)n \binom{n-4k}{2}.$$

Therefore for all $n \geq 10k^{\frac{3}{2}}$ we have $\phi(n) = 0$. Hence we have $\text{ex}(n, C_4, \mathcal{C}_{\geq \ell+1}) = C_4(H(n, \ell))$, this completes the proof of Theorem 6. \blacksquare

A new proof of Luo's Theorem.

We prove Theorem 7 by induction on the number of vertices n . If $n \leq \ell$, then the theorem trivially holds. In what follows we prove the theorem for $n \geq \ell+1$ assuming it holds for all graphs with smaller number of vertices.

Note that we may assume that G is connected, otherwise, we are done by induction on each component. If G is 2-connected and $\delta(G) \geq \lfloor \frac{\ell}{2} \rfloor$, then by Theorem 4 we have

$$K_s(G) \leq K_s(H(n, \ell)) < \frac{n-1}{\ell-1} \binom{\ell}{s}.$$

If G is 2-connected and some vertex v has degree less than $\lfloor \frac{\ell}{2} \rfloor$, then

$$K_s(G) \leq K_s(G - v) + \binom{\lfloor \frac{\ell}{2} \rfloor - 1}{s - 1} < \frac{n - 1}{\ell - 1} \binom{\ell}{s},$$

by induction hypothesis.

If G is not 2-connected, let B_1 be the 2-connected end-block with the cut vertex v . Then by the induction hypothesis we have

$$\begin{aligned} K_s(G) &= K_s(B_1) + K_s(G - (V(B_1) \setminus \{v\})) \leq \frac{v(B_1) - 1}{\ell - 1} \binom{\ell}{s} + \frac{(n - v(B_1) + 1) - 1}{\ell - 1} \binom{\ell}{s} \\ &= \frac{n - 1}{\ell - 1} \binom{\ell}{s}. \end{aligned}$$

Equality holds if and only if $\ell - 1 \mid n - 1$ and each maximal 2-connected block is a copy of K_ℓ . \blacksquare

5 Counting general cycles

In this section we prove Theorem 8. At first note that $H(n, \ell)$ provides a lower-bound for the number of C_{2k} and C_{2k+1} as well.

At first we will show

$$\lim_{\ell \rightarrow \infty} \left(\lim_{n \rightarrow \infty} \frac{\text{ex}(n, C_{2k}, C_{\geq \ell+1})}{\lfloor \frac{\ell}{2} \rfloor^k n^k} \right) \leq \frac{1}{2k}.$$

Let G be a 2-connected graph with circumference at most ℓ . Then by Theorem 2 we have $e(G) \leq \lfloor \frac{\ell}{2} \rfloor n$. Let e_1, e_2, \dots, e_k be k independent edges such that there are no more than two cycles of length $2k$ containing edges e_1, e_2, \dots, e_k in this given order. Then the number of $2k$ -cycles on such k independent edges in G is at most

$$2 \frac{(\lfloor \frac{\ell}{2} \rfloor n)^k}{4k}.$$

Which is the desired upper bound in case the rest of the cycles are negligible. Indeed for independent edges e_1, e_2, \dots, e_k if there are more than two cycles of length $2k$ containing edges e_1, e_2, \dots, e_k in this given order then the induced graph on the vertex set $\cup_{i=1}^k \{v_i, u_i\}$ contains $2C_3 \cup (k - 3)P_2$ as a subgraph where $e_i = v_i u_i$. Hence the number of such cycles is at most

$$(2k)! \left(\frac{2}{3} \left\lfloor \frac{\ell}{2} \right\rfloor^2 n \right)^2 \left(\left\lfloor \frac{\ell}{2} \right\rfloor n \right)^{k-3}$$

where we use Theorem 7 to bound the number of cycles and Theorem 2 to bound the number of edges. This shows the desired upper-bound.

We use induction on the number of vertices to show

$$\lim_{\ell \rightarrow \infty} \left(\lim_{n \rightarrow \infty} \frac{\text{ex}(n, C_{2k+1}, \mathcal{C}_{\geq \ell+1})}{\left\lfloor \frac{\ell}{2} \right\rfloor^{k+1} n^k} \right) \leq \frac{1}{2}.$$

Observe that it is enough to show that there exists a vertex incident to at most

$$\frac{k}{2} \left\lfloor \frac{\ell}{2} \right\rfloor^{k+1} n^{k-1} + N_\ell k \ell^{k+1} n^{k-2}$$

cycles of length $2k+1$, for some constant N_ℓ . By Dirac's theorem we have a vertex of G with degree at most $\left\lfloor \frac{\ell}{2} \right\rfloor$. Let v be a vertex of minimum degree. Let us fix two vertices w_1 and w_2 adjacent to v .

Claim 4. *The number of paths of length $2k-1$ from w_1 to w_2 is at most*

$$k \left\lfloor \frac{\ell}{2} n \right\rfloor^{k-1} + N'_\ell \ell^{k-1} n^{k-2}$$

for some constant N'_ℓ dependent on ℓ .

Proof. The number of such $2k-1$ -paths with terminal vertices w_1 and w_2 with a subgraph isomorphic to $K_4 \cup (k-3)K_2$ or $2K_3 \cup (k-4)K_2$ is bounded by $N'_\ell \ell^{k-1} n^{k-2}$ by Theorem 7, for some constant N'_ℓ .

The number of $2k-1$ -paths with terminal vertices w_1 and w_2 using the fixed $k-1$ independent edges in the given order without having a subgraph $K_4 \cup (k-3)K_2$ or $2K_3 \cup (k-4)K_2$ is at most k . Hence we have the number of $2k-1$ -paths with terminal vertices w_1 and w_2 is at most

$$k \left\lfloor \frac{\ell}{2} \right\rfloor^{k-1} n^{k-1} + N'_\ell \ell^{k-1} n^{k-2}$$

□

The number of cycles of length $2k+1$ incident with this vertex is at most

$$\binom{\left\lfloor \frac{\ell}{2} \right\rfloor}{2} k \left(\left\lfloor \frac{\ell}{2} \right\rfloor^{k-1} n^{k-1} + N'_\ell \ell^{k-1} n^{k-2} \right).$$

This finishes the proof. ■

6 Acknowledgements

We would like to thank Yixiao Zhang for useful remarks on the manuscript. The research of Gyóri and Salia was supported by the National Research, Development and Innovation Office NKFIH, grants K132696 and SNN-135643.

References

- [1] Noga Alon and Clara Shikhelman. Many T copies in H-free graphs. *Journal of Combinatorial Theory, Series B*, 121:146–172, 2016.
- [2] Noga Alon and Clara Shikhelman. Additive approximation of generalized Turán questions. *arXiv preprint arXiv:1811.08750*, 2018.
- [3] Béla Bollobás and Ervin Győri. Pentagons vs. triangles. *Discrete Mathematics*, 308(19):4332–4336, 2008.
- [4] Gabriel Andrew Dirac. Some theorems on abstract graphs. *Proceedings of the London Mathematical Society*, 3(1):69–81, 1952.
- [5] Paul Erdős. On some problems in graph theory, combinatorial analysis and combinatorial number theory. *Graph Theory and Combinatorics (Cambridge, 1983)*, Academic Press, London, pages 1–17, 1984.
- [6] Paul Erdős and Tibor Gallai. On maximal paths and circuits of graphs. *Acta Mathematica Academiae Scientiarum Hungarica*, 10(3-4):337–356, 1959.
- [7] Beka Ergemlidze, Ervin Győri, Abhishek Methuku, and Nika Salia. A note on the maximum number of triangles in a C_5 -free graph. *Journal of Graph Theory*, 90(3):227–230, 2019.
- [8] Beka Ergemlidze and Abhishek Methuku. Triangles in C_5 -free graphs and hypergraphs of girth six. *arXiv preprint arXiv:1811.11873*, 2018.
- [9] Zoltán Füredi, Alexandr Kostochka, and Jacques Verstraëte. Stability in the Erdős–Gallai theorems on cycles and paths. *Journal of Combinatorial Theory, Series B*, 121:197–228, 2016.
- [10] Dániel Gerbner, Ervin Győri, Abhishek Methuku, and Máté Vizer. Generalized Turán problems for even cycles. *arXiv preprint arXiv:1712.07079*, 2017.
- [11] Andrzej Grzesik. On the maximum number of five-cycles in a triangle-free graph. *Journal of Combinatorial Theory, Series B*, 102(5):1061–1066, 2012.
- [12] Andrzej Grzesik and Bartłomiej Kielak. On the maximum number of odd cycles in graphs without smaller odd cycles. *Journal of Graph Theory*, 99(2):240–246, 2022.
- [13] Ervin Győri and Hao Li. The maximum number of triangles in C_{2k+1} -free graphs. *Combinatorics, Probability and Computing*, 21(1-2):187–191, 2012.
- [14] Ervin Győri, János Pach, and Miklós Simonovits. On the maximal number of certain subgraphs in K_r -free graphs. *Graphs and Combinatorics*, 7(1):31–37, 1991.
- [15] Ervin Győri, Nika Salia, Casey Tompkins, and Oscar Zamora. The maximum number of P_ℓ copies in P_k -free graphs. *Discrete Mathematics & Theoretical Computer Science*, 21(1), 2018.
- [16] Hamed Hatami, Jan Hladký, Daniel Král, Serguei Norine, and Alexander Razborov. On the number of pentagons in triangle-free graphs. *Journal of Combinatorial Theory, Series A*, 120(3):722–732, 2013.
- [17] GN Kopylov. On maximal paths and cycles in a graph. In *Doklady Akademii Nauk*, volume 234, pages 19–21. Russian Academy of Sciences, 1977.
- [18] Binlong Li and Bo Ning. A strengthening of erdős-gallai theorem and proof of woodall’s conjecture. *Journal of Combinatorial Theory, Series B*, 146:76–95, 2021.
- [19] Ruth Luo. The maximum number of cliques in graphs without long cycles. *Journal of Com-*

binatorial Theory, Series B, 128:219–226, 2018.

- [20] Jie Ma and Bo Ning. Stability results on the circumference of a graph. *Combinatorica*, 40(1):105–147, 2020.

E-mail addresses:

X. Zhu: zhuxt@smail.nju.edu.cn

E. Györi: gyori.ervin@renyi.hu

Z. He: hz18@mails.tsinghua.edu.cn

J. Lv: lvzq19@mails.tsinghua.edu.cn

N. Salia: nikasalia@yahoo.com

C. Xiao: chuanqixm@gmail.com