

Stability of extremal connected hypergraphs avoiding Berge-paths

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Abstract

A Berge-path of length k in a hypergraph \mathcal{H} is a sequence $v_1, e_1, v_2, e_2, \dots, v_k, e_k, v_{k+1}$ of distinct vertices and hyperedges with $v_{i+1} \in e_i, e_{i+1}$ for all $i \in [k]$. Füredi, Kostochka and Luo, and independently Győri, Salia and Zamora determined the maximum number of hyperedges in an n -vertex, connected, r -uniform hypergraph that does not contain a Berge-path of length k provided k is large enough compared to r . They also determined the unique extremal hypergraph \mathcal{H}_1 .

We prove a stability version of this result by presenting another construction \mathcal{H}_2 and showing that any n -vertex, connected, r -uniform hypergraph without a Berge-path of length k , that contains more than $|\mathcal{H}_2|$ hyperedges must be a subhypergraph of the extremal hypergraph \mathcal{H}_1 , provided k is large enough compared to r .

1 Introduction

In extremal graph theory, the Turán number $ex(n, G)$ of a graph G is the maximum number of edges that an n -vertex graph can have without containing G as a subgraph. If a class \mathcal{G} of graphs is forbidden, then the Turán number is denoted by $ex(n, \mathcal{G})$. The asymptotic behavior of the function $ex(n, G)$ is well-understood if G is not bipartite. However, much less is known if G is bipartite (see the survey [8]). One of the simplest classes of bipartite graphs is that of paths. Let P_k and C_k denote the path and the cycle with k edges and let $\mathcal{C}_{\geq k}$ denote the class of cycles of length at least k .

Erdős and Gallai [3] proved that for any $n \geq k \geq 1$, the Turán number satisfies $ex(n, P_k) \leq \frac{(k-1)n}{2}$. They obtained this result by first showing that for any $n \geq k \geq 3$, $ex(n, \mathcal{C}_{\geq k}) \leq \frac{(k-1)(n-1)}{2}$. The bounds are sharp for paths, if k divides n , and sharp for cycles, if $k-1$ divides $n-1$. These are shown by the example of n/k pairwise disjoint k -cliques for the path P_k , and adding an extra vertex joined by an edge to every other vertex for the class $\mathcal{C}_{\geq k+2}$ of cycles. Later, Faudree and Schelp [4] gave the exact value of $ex(n, P_k)$ for every n .

Observe that the extremal construction for the path is not connected. Kopylov [13] and independently Balister, Győri, Lehel, and Schelp [1] determined the maximum number of edges $ex^{conn}(n, P_k)$ that an n -vertex connected graph can have without containing a path of length k . The stability version of these results was proved by Füredi, Kostochka and Verstraëte [7]. To state their result, we need to define the following class of graphs.

Definition 1. For $n \geq k$ and $\frac{k}{2} > a \geq 1$ we define the graph $H_{n,k,a}$ as follows. The vertex set of $H_{n,k,a}$ is partitioned into three disjoint parts A, B and L such that $|A| = a$, $|B| = k - 2a$ and $|L| = n - k + a$. The edge set of $H_{n,k,a}$ consists of all the edges between L and A and also all the edges in $A \cup B$. Let us denote the number of edges in $H_{n,k,a}$ by $|H_{n,k,a}|$.

Theorem 2 (Füredi, Kostochka, Verstraëte [7], Theorem 1.6). *Let $t \geq 2$, $n \geq 3t - 1$ and $k \in \{2t, 2t + 1\}$. Suppose we have an n -vertex connected graph G with more edges than $|H_{n+1,k+1,t-1}| - n$. Then we have either*

- $k = 2t$, $k \neq 6$ and G is a subgraph of $H_{n,k,t-1}$, or
- $k = 2t + 1$ or $k = 6$, and $G \setminus A$ is a star forest for $A \subseteq V(G)$ of size at most $t - 1$.

The Turán numbers for hypergraphs $ex_r(n, \mathcal{H})$, $ex_r(n, \mathbb{H})$ can be defined analogously for r -uniform hypergraphs \mathcal{H} and classes \mathbb{H} of r -uniform hypergraphs. Note that there are several ways how one can define paths and cycles of higher uniformity. In this paper, we consider the definition due to Berge.

Definition 3. *A Berge-path of length t is an alternating sequence of $t + 1$ distinct vertices and t distinct hyperedges of the hypergraph, $v_1, e_1, v_2, e_2, v_3, \dots, e_t, v_{t+1}$ such that $v_i, v_{i+1} \in e_i$, for $i \in [t]$. The vertices v_1, v_2, \dots, v_{t+1} are called defining vertices and the hyperedges e_1, e_2, \dots, e_t are called defining hyperedges of the Berge-path. We denote the set of all Berge-paths of length t by \mathcal{BP}_t .*

Similarly, a Berge-cycle of length t is an alternating sequence of t distinct vertices and t distinct hyperedges of the hypergraph, $v_1, e_1, v_2, e_2, v_3, \dots, v_t, e_t$, such that $v_i, v_{i+1} \in e_i$, for $i \in [t]$, where indices are taken modulo t . The vertices v_1, v_2, \dots, v_t are called defining vertices and the hyperedges e_1, e_2, \dots, e_t are called defining hyperedges of the Berge-cycle.

As these are the only cycles and paths we consider in hypergraphs, we will often omit the word Berge.

The study of the Turán numbers $ex_r(n, \mathcal{BP}_k)$ was initiated by Győri, Katona and Lemons [9], who determined the quantity in almost every case. Later Davoodi, Győri, Methuku and Tompkins [2] settled the missing case $r = k + 1$. For results on the maximum number of hyperedges in r -uniform hypergraphs not containing Berge-cycles longer than k see [5, 10] and the references therein.

Analogously to graphs, a hypergraph is *connected*, if for any two of its vertices, there is a Berge-path containing both vertices. The connected Turán numbers for an r -uniform hypergraph \mathcal{H} and class of r -uniform hypergraphs \mathbb{H} can be defined analogously, they are denoted by the functions $ex_r^{conn}(n, \mathcal{H})$ and $ex_r^{conn}(n, \mathbb{H})$, respectively.

To describe the extremal result and to introduce our contributions, we need the following definition that can be considered as an analogue of Definition 1 for higher uniformity.

Definition 4. *For integers $n, a \geq 1$ and $b_1, \dots, b_t \geq 2$ with $n \geq 2a + \sum_{i=1}^t b_i$ let us denote by $\mathcal{H}_{n,a,b_1,b_2,\dots,b_t}$ the following r -uniform hypergraph.*

- *Let the vertex set of $\mathcal{H}_{n,a,b_1,b_2,\dots,b_t}$ be $A \cup L \cup \bigcup_{i=1}^t B_i$, where A, B_1, B_2, \dots, B_t and L are pairwise disjoint sets of sizes $|A| = a$, $|B_i| = b_i$ ($i = 1, 2, \dots, t$) and $|L| = n - a - \sum_{i=1}^t b_i$.*
- *Let the hyperedges of $\mathcal{H}_{n,a,b_1,b_2,\dots,b_t}$ be*

$$\binom{A}{r} \cup \bigcup_{i=1}^t \binom{A \cup B_i}{r} \cup \left\{ \{c\} \cup A' : c \in L, A' \in \binom{A}{r-1} \right\}.$$

Observe that the number of hyperedges in $\mathcal{H}_{n,a,b_1,b_2,\dots,b_t}$ is

$$\left(n - a - \sum_{i=1}^t b_i \right) \binom{a}{r-1} + \sum_{i=1}^t \binom{a + b_i}{r} - (t-1) \binom{a}{r}.$$

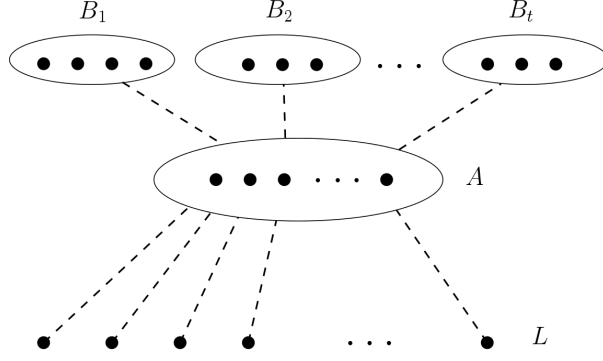


Figure 1: The hypergraph $\mathcal{H}_{n,a,b_1,b_2,\dots,b_t}$. Its hyperedges are all r -sets in the union of two sets connected by a dotted line.

Note that, if $a \leq a'$ and $b_i \leq b'_i$ for all $i = 1, 2, \dots, t$, then $\mathcal{H}_{n,a,b_1,b_2,\dots,b_t}$ is a subhypergraph of $\mathcal{H}_{n,a',b'_1,b'_2,\dots,b'_t}$. Finally, the length of the longest path in $\mathcal{H}_{n,a,b_1,b_2,\dots,b_t}$ is $2a - t + \sum_{i=1}^t b_i$ if $t \leq a + 1$, and $a - 1 + \sum_{i=1}^{a+1} b_i$ if $t > a + 1$ and the b_i 's are in non-increasing order.

With a slight abuse of notation, we define $\mathcal{H}_{n,a}^+$ to be a hypergraph obtained from $\mathcal{H}_{n,a}$ by adding an arbitrary hyperedge. Hyperedges containing at least $r - 1$ vertices from A are already in $\mathcal{H}_{n,a}$, therefore there are $r - 1$ pairwise different hypergraphs that we denote by $\mathcal{H}_{n,a}^+$ depending on the number of vertices from A in the extra hyperedge. Observe that the length of the longest path in $\mathcal{H}_{n,a}^+$ is one larger than in $\mathcal{H}_{n,a}$, in particular, if k is even, then $\mathcal{H}_{n,\lfloor \frac{k-1}{2} \rfloor}^+$ does not contain a Berge-path of length k .

The first attempt to determine the largest number of hyperedges in connected r -uniform hypergraphs without a Berge-path of length k can be found in [11], where the asymptotics of the extremal function was determined. The Turán number of Berge-paths in connected hypergraphs was determined by Füredi, Kostochka and Luo [6] for $k \geq 4r \geq 12$ and n large enough. Independently in a different range it was also given by Győri, Salia and Zamora [12], who also proved the uniqueness of the extremal structure. To state their result, let us introduce the following notation: for a hypergraph \mathcal{H} we denote by $|\mathcal{H}|$ the numbers of hyperedges in \mathcal{H} .

Theorem 5 (Győri, Salia, Zamora, [12]). *For all integers k, r with $k \geq 2r + 13 \geq 18$ there exists $n_{k,r}$ such that if $n > n_{k,r}$, then we have*

- $ex_r^{conn}(n, \mathcal{BP}_k) = |\mathcal{H}_{n,\lfloor \frac{k-1}{2} \rfloor}|$, if k is odd, and
- $ex_r^{conn}(n, \mathcal{BP}_k) = |\mathcal{H}_{n,\lfloor \frac{k-1}{2} \rfloor,2}|$, if k is even.

Depending on the parity of k , the unique extremal hypergraph is $\mathcal{H}_{n,\lfloor \frac{k-1}{2} \rfloor}$ or $\mathcal{H}_{n,\lfloor \frac{k-1}{2} \rfloor,2}$.

Our main result provides a stability version (and thus a strengthening) of Theorem 5 and also an extension of Theorem 2 for uniformity at least 3.

First we state it for hypergraphs with minimum degree at least 2, and then in full generality. In the proof, the hypergraphs $\mathcal{H}_{n,\frac{k-3}{2},3}$ and $\mathcal{H}_{n,\frac{k-3}{2},2,2}$ will play a crucial role in case k is odd, while if k is even, then the hypergraphs $\mathcal{H}_{n,\lfloor \frac{k-3}{2} \rfloor,4}$, $\mathcal{H}_{n,\lfloor \frac{k-3}{2} \rfloor,3,2}$ and $\mathcal{H}_{n,\lfloor \frac{k-3}{2} \rfloor,2,2,2}$ will be of importance, note that all of them are n -vertex, maximal, \mathcal{BP}_k -free hypergraphs. In both cases, the hypergraph listed first contains the largest number of hyperedges. This number gives the lower bound in the following theorem.

Theorem 6. For any $\varepsilon > 0$ there exist integers $q = q_\varepsilon$ and $n_{k,r}$ such that if $r \geq 3$, $k \geq (2 + \varepsilon)r + q$, $n \geq n_{k,r}$ and \mathcal{H} is a connected n -vertex, r -uniform hypergraph with minimum degree at least 2, without a Berge-path of length k , then we have the following.

- If k is odd and $|\mathcal{H}| > |\mathcal{H}_{n, \frac{k-3}{2}, 3}| = (n - \frac{k+3}{2}) \binom{\frac{k-3}{2}}{r-1} + \binom{\frac{k+3}{2}}{r}$, then \mathcal{H} is a subhypergraph of $\mathcal{H}_{n, \frac{k-1}{2}}$.
- If k is even and $|\mathcal{H}| > |\mathcal{H}_{n, \lfloor \frac{k-3}{2} \rfloor, 4}| = (n - \lfloor \frac{k+5}{2} \rfloor) \binom{\lfloor \frac{k-3}{2} \rfloor}{r-1} + \binom{\lfloor \frac{k+5}{2} \rfloor}{r}$, then \mathcal{H} is a subhypergraph of $\mathcal{H}_{n, \lfloor \frac{k-1}{2} \rfloor, 2}$ or $\mathcal{H}_{n, \lfloor \frac{k-1}{2} \rfloor}^+$.

Let $\mathbb{H}'_{n', a, b_1, b_2, \dots, b_t}$ be the class of hypergraphs that can be obtained from $\mathcal{H}_{n, a, b_1, b_2, \dots, b_t}$ for some $n \leq n'$ by adding hyperedges of the form $A'_j \cup D_j$, where the D_j 's partition $[n'] \setminus [n]$, all D_j 's are of size at least 2 and $A'_j \subseteq A$ for all j . Let us define $\mathbb{H}^+_{n', \lfloor \frac{k-1}{2} \rfloor}$ analogously.

Theorem 7. For any $\varepsilon > 0$ there exist integers $q = q_\varepsilon$ and $n_{k,r}$ such that if $r \geq 3$, $k \geq (2 + \varepsilon)r + q$, $n \geq n_{k,r}$ and \mathcal{H} is a connected n -vertex, r -uniform hypergraph without a Berge-path of length k , then we have the following.

- If k is odd and $|\mathcal{H}| > |\mathcal{H}_{n, \frac{k-3}{2}, 3}|$, then \mathcal{H} is a subhypergraph of some $\mathcal{H}' \in \mathbb{H}'_{n, \frac{k-1}{2}}$.
- If k is even and $|\mathcal{H}| > |\mathcal{H}_{n, \lfloor \frac{k-3}{2} \rfloor, 4}|$, then \mathcal{H} is a subhypergraph of some $\mathcal{H}' \in \mathbb{H}'_{n, \lfloor \frac{k-1}{2} \rfloor, 2}$ or $\mathbb{H}^+_{n, \lfloor \frac{k-1}{2} \rfloor}$.

Notation

We use standard notation. The vertex set of an r -uniform hypergraph \mathcal{H} is denoted by $V(\mathcal{H})$, and we denote the set of its hyperedges by $\mathcal{E}(\mathcal{H})$. Sets of vertices that do not necessarily form a hyperedge are denoted by capital letters, while lower case letters u, v, x, y, z are used to denote vertices. Hyperedges of \mathcal{H} are usually denoted by lower case h . Hyperedges of some particular cycles are often denoted by e and those of paths by f . For a hypergraph \mathcal{H} and a set S of its vertices, the set of hyperedges of \mathcal{H} that contain at least one element of S is denoted by $E(S)$ (\mathcal{H} will always be clear from context). The (open) neighborhood of a vertex v in \mathcal{H} (i.e., the set of vertices u different from v for which there exists a hyperedge h of \mathcal{H} with $\{u, v\} \subset h$) is denoted by $N_{\mathcal{H}}(v)$ or simply $N(v)$ if \mathcal{H} is clear from context. For two hypergraphs \mathcal{H}_1 and \mathcal{H}_2 with $V(\mathcal{H}_2) \subseteq V(\mathcal{H}_1)$, we denote by $\mathcal{H}_1 \setminus \mathcal{H}_2$ the hypergraph with vertex set $V(\mathcal{H}_1)$ and hyperedge set $\mathcal{E}(\mathcal{H}_1) \setminus \mathcal{E}(\mathcal{H}_2)$. For two hypergraphs \mathcal{H}_1 and \mathcal{H}_2 we denote the fact that \mathcal{H}_1 is subhypergraph of \mathcal{H}_2 by $\mathcal{H}_1 \subseteq \mathcal{H}_2$.

2 Proofs

We start the proof of Theorem 6 with a technical lemma that will be crucial later.

Lemma 8. Let \mathcal{H} be a connected r -uniform hypergraph with minimum degree at least 2 and with longest Berge-path and Berge-cycle of length $\ell - 1$. Let C be a Berge-cycle of length $\ell - 1$ in \mathcal{H} , with defining vertices $V = \{v_1, v_2, \dots, v_{\ell-1}\}$ and defining edges $\mathcal{E}(C) = \{e_1, e_2, \dots, e_{\ell-1}\}$ with $v_i, v_{i+1} \in e_i$ (modulo $\ell - 1$). Then, we have

- (i) every hyperedge $h \in \mathcal{H} \setminus C$ contains at most one vertex from $V(\mathcal{H}) \setminus V$.

(ii) If u, v are not necessarily distinct vertices from $V(\mathcal{H}) \setminus V$, then there cannot exist distinct hyperedges $h_1, h_2 \in \mathcal{H} \setminus \mathcal{C}$ and an index i with $v, v_i \in h_1$ and $u, v_{i+1} \in h_2$.

(iii) If there exists a vertex $v \in V(\mathcal{H}) \setminus V$ and there exist different hyperedges $h_1, h_2 \in \mathcal{H} \setminus \mathcal{C}$ with $v, v_{i-1} \in h_1$ and $v, v_{i+1} \in h_2$, then there exists a cycle of length $\ell - 1$ not containing v_i as a defining vertex.

Proof. We prove (i) by contradiction. Suppose $h \in \mathcal{H} \setminus \mathcal{C}$ contains two vertices from $V(\mathcal{H}) \setminus V$. We distinguish two cases.

Case 1. Hyperedge h contains a vertex $u \notin V$ and a different vertex $v \in e_i \setminus V$ for some $i \leq \ell$. Then $v_{i+1}, e_{i+1}, v_{i+2}, \dots, v_\ell, e_\ell, v_1, e_1, \dots, v_i, e_i, v, h, u$ is a path of length ℓ , a contradiction.

Case 2. Hyperedge h contains two vertices u and v from $V(\mathcal{H}) \setminus V(C)$. We consider the hypergraph \mathcal{H}' obtained from \mathcal{H} by removing a hyperedge h .

Case 2.1. There is a Berge-path in \mathcal{H}' from $\{v, u\}$ to the cycle C , in particular to a defining vertex of C . Then let P be a shortest such path, let us assume P is from v to v_i . Without loss of generality we may suppose that P does not contain e_i as a defining hyperedge, (it is possible P contains e_{i-1} as a defining hyperedge). Then $u, h, v, P, v_i, e_i, v_{i+1}, \dots, e_{i-2}, v_{i-1}$ is a Berge-path of length at least ℓ , contradicting the assumption that the longest path in \mathcal{H} is of length $\ell - 1$.

Case 2.2. Suppose there is no Berge-path from the vertex v to the cycle C in \mathcal{H}' . However by connectivity of \mathcal{H} , there is a shortest path P from v to a defining vertex of C , say v_i and it does not use any defining hyperedge of C but possibly e_{i-1} . Also, h is not a hyperedge of P . There exists a hyperedge $h' \neq h$ containing v , as the minimum degree is at least 2 in \mathcal{H} . Note that h' is not a hyperedge of the path P , even more all vertices of h' different from v are not defining vertices of P or C . Fix a vertex $u' \in h' \setminus \{v\}$. Then $u', h', P, e_i, v_{i+1}, \dots, e_{i-2}, v_{i-1}$ is a Berge-path of length at least ℓ , a contradiction.

To prove (ii), assume first that $u = v$. Then one could enlarge C by removing e_i and adding h_1, v, h_2 to obtain a longer cycle, a contradiction. Assume now $u \neq v$. Then removing e_i and adding h_1, v and h_2, u , one would obtain a path of length ℓ , a contradiction.

Finally to show (iii), we can replace e_{i-1}, v_i, e_i in C by h_1, v, h_2 to obtain the desired cycle. \square

We say that an r -uniform hypergraph \mathcal{H} has the *set degree condition*, if for any set X of vertices with $|X| \leq k/2$, we have $|E(X)| \geq |X| \binom{\lfloor \frac{k-3}{2} \rfloor}{r-1}$, i.e., the number of those hyperedges that are incident to some vertex in X is at least $|X| \binom{\lfloor \frac{k-3}{2} \rfloor}{r-1}$. We first prove Theorem 6 for such hypergraphs.

Proof of Theorem 6 for hypergraphs having the set degree condition. Let \mathcal{H} be an n -vertex \mathcal{BP}_k -free hypergraph with the set degree condition. Also, assume $|\mathcal{H}|$ is as claimed in the statement of the theorem. However, for most part of the proof we will only use the set degree condition.

Claim 9. Let P be a longest Berge-path in \mathcal{H} with defining vertices $U = \{u_1, \dots, u_\ell\}$ and defining hyperedges $\mathcal{F} = \{f_1, f_2, \dots, f_{\ell-1}\}$ in this given order. Suppose P minimizes $x_1 + x_\ell$ among longest Berge-paths of \mathcal{H} , where x_i for $i \in [\ell]$, denotes the number of hyperedges in \mathcal{F} incident to u_i . Then the sizes of $N_{\mathcal{H} \setminus \mathcal{F}}(u_1)$ and $N_{\mathcal{H} \setminus \mathcal{F}}(u_\ell)$ are at least $\lfloor \frac{k-3}{2} \rfloor$.

Proof of Claim 9. Observe that the statement is trivially true for $r \geq 4$ and for arbitrary longest path, as by the set degree condition, there exist at least $\binom{\lfloor \frac{k-3}{2} \rfloor}{r-1} - k + 1$ hyperedges in $\mathcal{H} \setminus \mathcal{F}$

incident to u_1 . This is strictly greater than $\left\lfloor \frac{k-5}{r-1} \right\rfloor$ if $r \geq 4$ and $k \geq (2 + \varepsilon)r + q$, for large enough q , hence $|N_{\mathcal{H} \setminus \mathcal{F}}(u_1)| > \frac{k-5}{2}$, finishing the proof for $r \geq 4$.

Thus we can assume that $r = 3$. Let P be a longest Berge-path in \mathcal{H} , minimizing $x_1 + x_\ell$. First we claim that if $u_1 \in f_i$ then $x_i \geq x_1$. Note that the Berge-path

$$u_i, f_{i-1}, u_{i-1}, f_{i-2}, u_{i-2}, \dots, u_2, f_1, u_1, f_i, u_{i+1}, f_{i+1}, \dots, u_\ell, f_{\ell-1}, u_\ell$$

is also a longest Berge-path, with the same set of defining vertices and defining hyperedges and endpoint x_ℓ , hence by the minimality of the sum $x_1 + x_\ell$, the number of hyperedges from \mathcal{F} incident to u_i is at least x_1 .

This means that if we consider all possible Berge-paths obtained from P by the way described above (including itself), then the number of pairs (u, f) , where $u \in U$, $f \in \mathcal{F}$ and $u \in f$, is at least x_1^2 . On the other hand, this number is upper bounded by $3|\mathcal{F}| = 3(\ell - 1)$, hence we have $x_1^2 \leq 3(\ell - 1) \leq 3(k - 1)$, therefore $x_1 \leq \sqrt{3(k - 1)}$. The same holds for the other end vertex u_ℓ and so for x_ℓ by symmetry.

Since the degree of u_1 is at least $\left\lfloor \frac{k-3}{2} \right\rfloor$, out of which at most $\sqrt{3(k - 1)}$ of the hyperedges are defining hyperedges, the degree of u_1 in $\mathcal{H} \setminus \mathcal{F}$ is at least

$$\left(\left\lfloor \frac{k-3}{2} \right\rfloor \right) - \sqrt{3(k - 1)} > \left(\left\lfloor \frac{k-3}{2} \right\rfloor - 1 \right),$$

if $k \geq 21$. Thus $|N_{\mathcal{H} \setminus \mathcal{F}}(u_1)| \geq \left\lfloor \frac{k-3}{2} \right\rfloor$ and in the same way we have $|N_{\mathcal{H} \setminus \mathcal{F}}(u_\ell)| \geq \left\lfloor \frac{k-3}{2} \right\rfloor$. \square

Claim 10. *Let $\ell - 1$ be the length of the longest Berge-path in \mathcal{H} . Then $\ell \geq k - 3$ and \mathcal{H} contains a Berge-cycle of length $\ell - 1$.*

Proof of Claim 10. Let $u_1, f_1, u_2, f_2, \dots, u_{\ell-1}, f_{\ell-1}, u_\ell$ be a longest Berge-path given by Claim 9 with defining hyperedges $\mathcal{F} = \{f_1, f_2, \dots, f_{\ell-1}\}$ and defining vertices $U = \{u_1, u_2, \dots, u_\ell\}$.

Before the proof let us introduce some notations: for $\mathcal{E} \subseteq \mathcal{E}(\mathcal{H})$ and integer j with $1 \leq j \leq \ell$, let $S_{j,\mathcal{E}}$ denote the set of indices of vertices in $U \cap N_{\mathcal{H} \setminus \mathcal{E}}(u_j)$, and we simply denote $S_{j,\mathcal{F}}$ by S_j . In particular S_j denotes the set of indices i such that there is a hyperedge of \mathcal{H} that contains both u_i and u_j and is not a defining hyperedge of the path. For any set S of integers let $S^- := \{a : a > 0, a + 1 \in S\}$, $S^{--} = (S^-)^-$. The operations $^+$ and $^{++}$ are defined analogously.

To start the proof, observe first that \mathcal{H} cannot contain a Berge-cycle C of length ℓ . Indeed, the hyperedges of such a cycle contain at most $\ell(r - 1)$ vertices. Therefore there is a vertex $v \in V(\mathcal{H}) \setminus V(C)$, then as \mathcal{H} is connected, there exists a path from v to C and we obtain a path of length at least ℓ , contradicting our assumption on the length of the longest path.

If $\ell \in S_1$ or equivalently $1 \in S_\ell$, then a hyperedge showing this, together with \mathcal{F} forms a Berge-cycle of length ℓ in \mathcal{H} . So we can assume $S_1, S_\ell \subseteq \{2, \dots, \ell - 1\}$ and so $S_1^- \subseteq \{1, 2, \dots, \ell - 1\}$.

If $S_1^- \cap S_\ell \neq \emptyset$ (or symmetrically $S_1 \cap S_\ell^+ \neq \emptyset$), then \mathcal{H} contains a Berge-cycle of length ℓ . Indeed, if $i \in S_1^- \cap S_\ell$, then there are hyperedges e and e' in $\mathcal{H} \setminus \mathcal{F}$ with $u_1, u_i \in e$ and $u_\ell, u_{i-1} \in e'$. Then

$$u_{i-1}, f_{i-2}, u_{i-2}, \dots, f_2, u_2, f_1, u_1, e, u_i, f_{i+1}, u_{i+2}, \dots, f_{\ell-1}, u_\ell, e'$$

is a Berge-cycle of length ℓ . (Note that e and e' are distinct hyperedges as $\ell \notin S_1$.) Note that by Claim 9, we have $|S_\ell|, |S_1^-| \geq \left\lfloor \frac{k-3}{2} \right\rfloor$. So to avoid $S_1^- \cap S_\ell \neq \emptyset$, we have $\ell \geq k - 3$.

The exact same argument shows that if $S_1^{--} \cap S_\ell \neq \emptyset$ or symmetrically $S_1 \cap S_\ell^{++} \neq \emptyset$, then \mathcal{H} contains a Berge-cycle of length $\ell - 1$ and we are done in this case.

For two indices $x < y \in S_\ell$, let us introduce the relation $x \sim y$ if $S_1 \cap (x, y] = \emptyset$. Clearly, \sim is an equivalence relation. Assume S_ℓ has m_1 equivalence classes. Also, we say that a maximal

subset of consecutive integers in S_ℓ is an interval of S_ℓ . As $S_\ell^+ \cap S_1 = \emptyset$ by the above, elements of the same interval belong to the same equivalence class. Let m_2 be the number of intervals in S_ℓ . If \mathcal{H} does not contain cycles of length ℓ and $\ell - 1$, then for the maximal element z of each equivalence class, we have that $z + 1, z + 2 \notin S_1$ and so by the definition of equivalence classes $z + 1, z + 2 \notin S_\ell$. Moreover, if an element z belongs both to S_1 and S_ℓ , then z is the smallest element of an equivalence class. Also if z is the largest element of an interval that is not the rightmost interval in an equivalence class, then $z + 1 \notin S_1 \cup S_\ell$. These observations show that $2\lfloor \frac{k-3}{2} \rfloor + m_1 - 2 + (m_2 - m_1) \leq \ell - 2$ holds. As $\ell \leq k$, we must have $m_2 \leq 4$.

Similarly as in the proof of Claim 9 we can see that for any $j \in S_1^-$, the vertex u_j is the endpoint of a longest path \mathcal{F}_j with other end vertex u_ℓ and with defining vertex set U . Observe that the neighborhood S_ℓ of u_ℓ with respect to the non-defining hyperedges of \mathcal{F} and \mathcal{F}_j is the same, as the single hyperedge $h \in \mathcal{F}_j \setminus \mathcal{F}$ contains u_1 and therefore cannot contain u_ℓ without creating a cycle of length ℓ . Therefore $S_{j, \mathcal{F}_j} \subseteq U$ and similarly as above if $[(S_\ell^- \cup S_\ell^{--}) \cap (S_\ell^+ \cup S_\ell^{++})] \cap S_{j, \mathcal{F}_j} \neq \emptyset$, then \mathcal{H} contains a Berge-cycle of length ℓ or $\ell - 1$.

Let $S^* := (S_\ell^- \cup S_\ell^{--}) \cap (S_\ell^+ \cup S_\ell^{++})$ then $|S^*| \geq |S_\ell| - 2m_2 \geq \lfloor \frac{k-3}{2} \rfloor - 8$. Let $U_{S_1^-} := \{u_i : i \in S_1^-\}$ and consider $E(U_{S_1^-})$. Observe that all but one of the defining hyperedges of \mathcal{F}_i are in \mathcal{F} , thus there are at most $|\mathcal{F}| + |U_{S_1^-}| \leq k - 1 + |S_1^-|$ hyperedges altogether in $E(U_{S_1^-})$ that are defining hyperedges of \mathcal{F} or an \mathcal{F}_i . By the previous paragraph, all other hyperedges in $E(U_{S_1^-})$ are completely in $U \setminus S^*$, thus we have

$$E(U_{S_1^-}) \subseteq \binom{U \setminus S^*}{r} \cup \mathcal{F} \cup \bigcup_{x \in S_1^-} \mathcal{F}_x.$$

By the set degree condition and the above, we must have

$$|S_1^-| \binom{\lfloor \frac{k-3}{2} \rfloor}{r-1} \leq |E(U_{S_1^-})| \leq \binom{k - \lfloor \frac{k-3}{2} \rfloor + 8}{r} + k - 1 + |S_1^-|. \quad (1)$$

Using $\lfloor \frac{k-3}{2} \rfloor \leq |S_1^-|$, $\binom{a}{r} = \frac{a}{r} \binom{a-1}{r-1}$ and $\binom{\frac{a+1}{r-1}}{\frac{a}{r-1}} = \frac{a+1}{a-r+2} \leq \frac{a}{a-r}$, and writing $k = \alpha r$ we have

$$\begin{aligned} \binom{k - \lfloor \frac{k-3}{2} \rfloor + 8}{r} &= \frac{k - \lfloor \frac{k-3}{2} \rfloor + 8}{r} \binom{k - \lfloor \frac{k-3}{2} \rfloor + 7}{r-1} \\ &\leq \left(\frac{k/2 + 9}{r} \right) \binom{\lfloor \frac{k+17}{2} \rfloor}{r-1} = \left(\frac{\alpha}{2} + 9/r \right) \binom{\lfloor \frac{k+17}{2} \rfloor}{r-1} \\ &\leq \left(\frac{\alpha}{2} + 9/r \right) \left(\frac{\alpha}{\alpha-2} \right)^{10} \binom{\lfloor \frac{k-3}{2} \rfloor}{r-1}. \end{aligned} \quad (2)$$

Therefore (1), (2) and $k - 1 + |S_1^-| \leq 2k = 2\alpha r$ implies $\alpha r/2 - 2 \leq (\frac{\alpha}{2} + 9/r)(\frac{\alpha}{\alpha-2})^{10} + 2\alpha$. This shows that for any $\varepsilon > 0$, there is an r_0 such that if $r > r_0$, then $\alpha < 2 + \varepsilon$ must hold, a contradiction. For the finitely many smaller values of r , the above inequality gives an upper bound β_r for $\alpha = k/r$, which might be larger than $2 + \varepsilon$. In that case we can choose $q_\varepsilon := \max_{r \leq r_0} \beta_r r$. Then we have $k > q_\varepsilon \geq \alpha r = k$, a contradiction. \square

Note that the cycle C given by Claim 10 is a longest Berge-cycle in \mathcal{H} and let its defining vertices and defining hyperedges be $V := \{u_1, u_2, \dots, u_{\ell-1}\}$ and $E(C) := \{e_1, e_2, \dots, e_{\ell-1}\}$, respectively, with $u_i, u_{i+1} \in e_i$. We have ℓ is either $k - 3$, $k - 2$, $k - 1$ or k by Claim 10. Let us call u_{i-1} and u_{i+1} the *neighbors of u_i on C* .

2.1 Preliminary technical claims

By Lemma 8 (i), for any vertex $w \in V(\mathcal{H}) \setminus V$ we have $N_{\mathcal{H} \setminus C}(w) \subseteq V$. For any vertex $w \in V(\mathcal{H}) \setminus V$, we partition $N_{\mathcal{H} \setminus C}(w)$ into two parts the following way: let M_w denote the set of vertices $v \in V$ such that there exists exactly one hyperedge in $\mathcal{H} \setminus C$ containing both w and v , and let D_w denote the set of those vertices $v \in V$ for which there exist at least 2 hyperedges in $\mathcal{H} \setminus C$ containing both v and w .

Claim 11. *For any w and w' with $w, w' \in V(\mathcal{H}) \setminus V$ and not necessarily distinct, we have*

(i) *If $u_j \in N_{\mathcal{H} \setminus C}(w)$, $u_{j+1} \in N_{\mathcal{H} \setminus C}(w')$, then $w = w'$, $u_j, u_{j+1} \in M_w$ and there exists a non-defining hyperedge h with $w, u_j, u_{j+1} \in h$.*

(ii) *If $u_j \in N_{\mathcal{H} \setminus C}(w)$, $u_{j+2} \in D_w$, then there exists a cycle C' of length $\ell - 1$ in \mathcal{H} such that the defining vertices of C' are those of C but u_{j+1} replaced by w .*

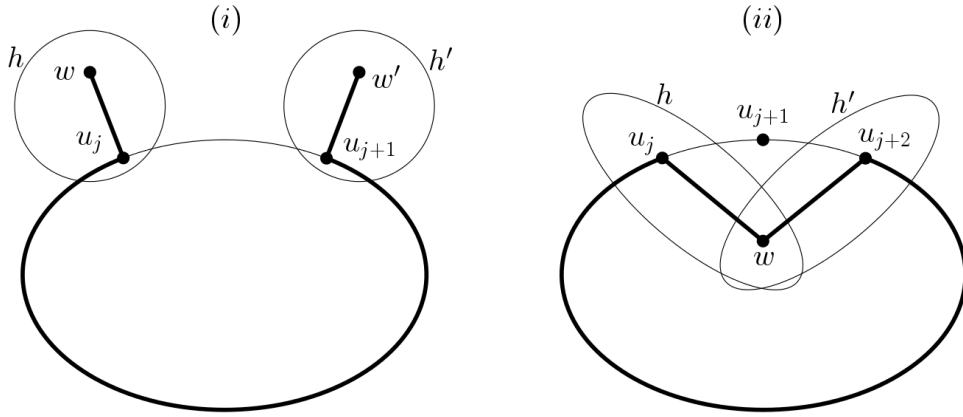


Figure 2: Sketch of the proof of Claim 11

Proof. Let $u_j \in N_{\mathcal{H} \setminus C}(w)$, $u_{j+1} \in N_{\mathcal{H} \setminus C}(w')$. If $w \neq w'$, then for the hyperedges $h, h' \in \mathcal{H} \setminus C$ with $u_j, w \in h$ and $u_{j+1}, w' \in h'$, we have $h \neq h'$, from Lemma 8 (i). But then

$$w', h', u_{j+1}, e_{j+1}, u_{j+2}, \dots, u_{\ell-1}, e_{\ell-1}, u_1, e_1, \dots, u_j, h, w$$

is a Berge-path of length ℓ , see Figure 2, a contradiction. So $w = w'$, and if there exist $h \neq h'$ with $u_j, w \in h$ and $u_{j+1}, w \in h'$, then the Berge-path presented above is in fact a Berge-cycle that is longer than C , a contradiction. This proves (i).

For the second part of the claim, observe that if $u_j \in N_{\mathcal{H} \setminus C}(w)$ and $u_{j+2} \in D_w$, then there exist two distinct hyperedges $h, h' \in \mathcal{H} \setminus C$ such that $u_j, w \in h$ and $u_{j+2}, w \in h'$, so in C we can replace e_j, u_{j+1}, e_{j+1} by h, w, h' to obtain desired cycle C' , see Figure 2. \square

Claim 12. *Suppose $u_{i-1}, u_{i+1}, u_j \in D_w$ are three distinct vertices for some $w \in V(\mathcal{H}) \setminus V$ and let $w^* \in V(\mathcal{H}) \setminus V$ be a vertex distinct from w . Then we have the following.*

- (i) *There is no hyperedge $h \in \mathcal{H} \setminus C$ with $u_i, u_{j-1} \in h$ nor with $u_i, u_{j+1} \in h$.*
- (ii) *If $u_{j+2} \in N_{\mathcal{H} \setminus C}(w)$, then e_{i-1}, e_i do not contain u_{j+1} .*
- (iii) *Hyperedges e_{i-1} and e_i are not incident with the vertices w, w^* .*
- (iv) *Suppose $u_{t+1} \in D_{w^*}$ or $u_{t-1} \in D_{w^*}$ for some $t \neq i$. Then there is no $h \in \mathcal{H} \setminus C$ incident to u_i and u_t .*
- (v) *The hyperedges e_{j-1}, e_j are not incident with u_i .*

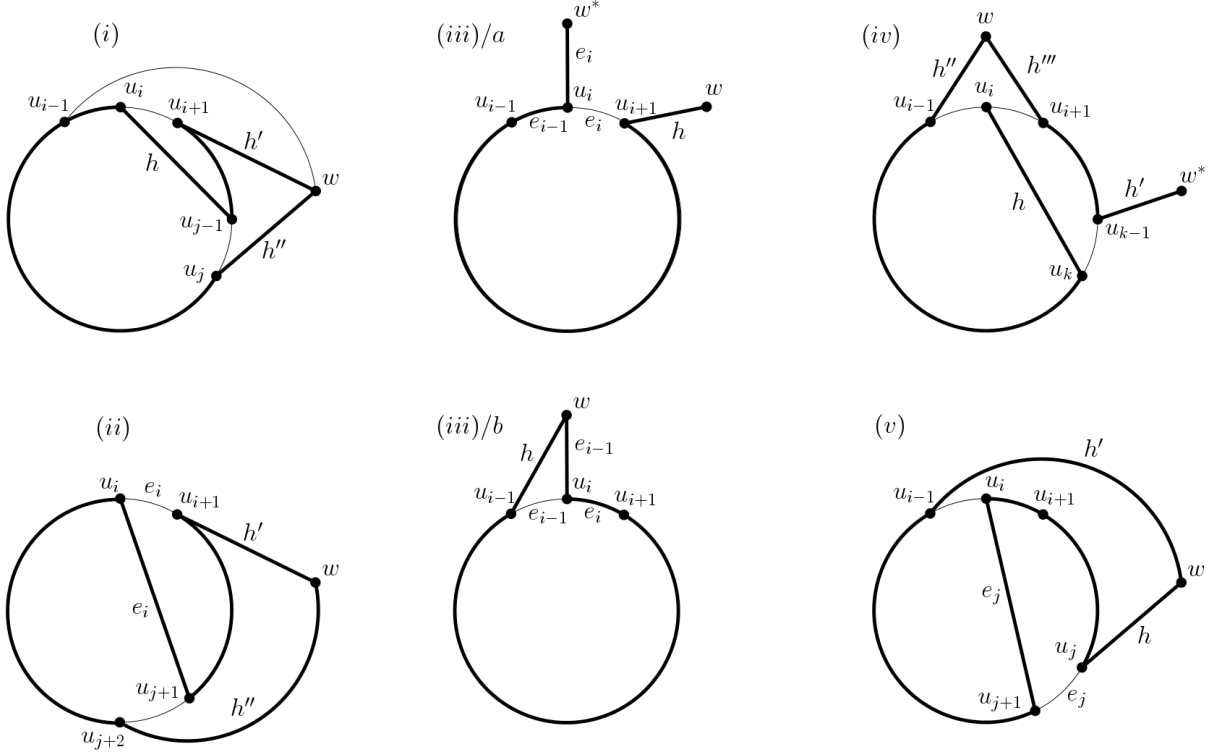


Figure 3: Sketch of the proof of Claim 12

Proof. We start with the proof of (i), see Figure 3 (i). Suppose by contradiction that $u_i, u_{j-1} \in h \in \mathcal{H} \setminus C$. Then by Claim 11 (i), we have $w \notin h$ (as otherwise $u_i, u_{i-1} \in M_w$, contradicting $u_{i-1} \in D_w$). Furthermore, as $u_{i+1}, u_j \in D_w$, there exist two distinct hyperedges $h', h'' \in \mathcal{H} \setminus C$ with $u_{i+1}, w \in h'$ and $u_j, w \in h''$. Using the fact that u_{j-1} and u_{i+1} are different vertices as there can not be neighboring vertices in D_w by Lemma 8 (ii), we have that

$$u_{i-1}, e_{i-1}, u_i, h, u_{j-1}, e_{j-2}, u_{j-2}, \dots, u_{i+1}, h', w, h'', u_j, e_j, u_{j+1}, \dots, e_{i-2}$$

is a Berge-cycle longer than C , a contradiction. Similarly we can extend the cycle C if $u_i, u_{j+1} \in h \in \mathcal{H} \setminus C$. This proves (i).

To show (ii) see Figure 3 (ii), it is enough to get a contradiction if e_i contains u_{j+1} , since the other case e_{i-1} contains u_{j+1} is symmetric. We have two non-defining distinct hyperedges, a hyperedge h'' incident to w and u_{j+2} and a hyperedge h' incident to w and u_{i+1} as $u_{i+1} \in D_w$. Then

$$u_i, e_i, u_{j+1}, e_j, u_j, e_{j-1}, \dots, e_{i+1}, u_{i+1}, h', w, h'', u_{j+2}, e_{j+2}, \dots, u_{i-2}, e_{i-2}, u_{i-1}, e_{i-1}$$

is a Berge-cycle longer than C , a contradiction.

To show statement (iii), suppose first $w^* \in e_i$. Then for a non-defining hyperedge h incident to w and u_{i+1} , we have that $w^*, e_i, u_i, e_{i-1}, u_{i-1}, \dots, u_{i+1}, h, w$ is a path of length ℓ - a contradiction. If $w^* \in e_{i-1}$, then similarly, for a non-defining hyperedge h incident to w and u_{i-1} , we have that $w^*, e_{i-1}, u_i, e_i, u_{i+1}, \dots, e_{i-2}, u_{i-1}, h, w$ is a path of length ℓ - a contradiction. If $w \in e_{i-1}$, then we have a contradiction since there exists a cycle longer than C , which is obtained from C by exchanging the edge e_{i-1} with h, w, e_{i-1} , where h is a non-defining hyperedge incident to w and u_{i-1} . Similarly we get a contradiction if $w \in e_i$.

To prove (iv) by a contradiction, suppose that we have a non-defining hyperedge h of C incident to u_i and u_t . Assume without loss of generality that $u_{t-1} \in D_w^*$ since the other case is symmetrical. Then there exists a non-defining hyperedge h' different from h , incident to u_{t-1} and w^* . Also there are two distinct non-defining hyperedges h'', h''' with $w, u_{i-1} \in h''$ and $w, u_{i+1} \in h'''$. At first note that hyperedge h is distinct from h'' and h''' by Claim 11 (i). From Lemma 8 (i) we have that hyperedges h'' and h''' distinct from h' . Finally we have a contradiction since the following is a Berge-path of length ℓ

$$w^*, h', u_{t-1}, e_{t-2}, \dots, u_{i+1}, h''', w, h'', u_{i-1}, e_{i-2}, \dots, u_{t+1}, e_t, u_t, h, u_i.$$

To prove (v) suppose by a contradiction that e_j contains u_i . There are distinct non-defining hyperedges h, h' with $w, u_j \in h$ and $w, u_{i-1} \in h'$. Then

$$u_{j+1}, e_j, u_i, e_i, u_{i+1}, e_{i+1}, \dots, e_{j-1}, u_j, h, w, h', u_{i-1}, e_{i-2}, u_{i-2}, \dots, e_{j+2}, u_{j+2}, e_{j+1}$$

is a Berge-cycle of length longer than C . This contradiction proves (v). The proof for the case $u_i \in e_{j-1}$ is analogous. \square

By Claim 11 and the set degree condition

$$\binom{\lfloor \frac{k-3}{2} \rfloor}{r-1} \leq |M_w| + \binom{\min \left\{ \left\lfloor \frac{\ell-1-|M_w|}{2} \right\rfloor, |D_w| \right\}}{r-1}$$

must hold for all $w \in V(\mathcal{H}) \setminus V(C)$. At first we observe that $|M_w| \leq 3$ as otherwise $\ell-1-|M_w| \leq k-5$. Therefore, we have $|D_w| \geq \lfloor \frac{k-3}{2} \rfloor$, if $k \geq 11$.

We say that a vertex $u_i \in V$ is *replaceable by w* , if $u_{i-1}, u_{i+1} \in D_w$, and we denote by R_w the set of vertices that are replaceable by w . A vertex is called *replaceable*, if it is replaceable by w for some $w \in V(\mathcal{H}) \setminus V$. For a replaceable vertex w' , we define $D_{w'}$ and $M_{w'}$ as for vertices in $V(\mathcal{H}) \setminus V$.

For a vertex $w \in V(\mathcal{H}) \setminus V$ let us call a maximal set I of consecutive defining vertices of C in $V \setminus D_w$ a *missing interval for w* (or just missing intervals, if w is clear from the context), if its size is at least two. Let I_1, I_2, \dots, I_s be the missing intervals of C for w and let us denote by $\overline{I}_1, \overline{I}_2, \dots, \overline{I}_s$ the same intervals without the terminal vertices (it is possible that $\overline{I}_j = \emptyset$). We have $\sum_{i=1}^s (|I_i| - 1) = \ell - 1 - 2|D_w|$. In particular, as $|D_w| \geq \lfloor \frac{k-3}{2} \rfloor$ by the set degree condition and Lemma 8 (i), we have $s \leq 3$, if k is even and $s \leq 2$, if k is odd. Let us consider a hyperedge $e_j \in C$ such that u_j or u_{j+1} is from a missing interval. The number of such hyperedges is $\sum_{i=1}^s (|I_i| + 1)$, which is at most 9, if k is even, and at most 6, if k is odd. Our next technical claim is about missing intervals.

Claim 13. *Suppose that $u_i, u_{i+1}, \dots, u_{i+t}$ form a missing interval for some $w \in V(\mathcal{H}) \setminus V$. Then*

- (i) e_{i-1} and e_{i+t} do not contain vertices $w^* \in V(\mathcal{H}) \setminus V$; and
- (ii) if $u_{i-1} \in D_{w'}$ (resp. $u_{i+t+1} \in D_{w'}$) for some $w' \neq w$, then e_{i-1} (resp. e_{i+t}) does not contain a vertex from R_w .

Proof. To prove (i) observe that there exists a Berge-path starting with the vertex w , a non-defining hyperedge h , the vertex u_{i-1} , going around C with defining vertices and hyperedges and finishing with a vertex u_i . Such h exists since u_{i-1} does not belong to the missing interval, so $u_{i-1} \in D_w$. Note that we did not use a hyperedge e_{i-1} which contains w^* . If $w = w^*$, then e_{i-1} closes a Berge-cycle longer than C , a contradiction, while if $w \neq w^*$, then finishing with e_{i-1}, w^* we obtain a Berge-path of length ℓ , a contradiction. This contradiction proves (i). Similar argument shows the statement for the hyperedge e_{i+t} .

We omit the proof of part (ii) since the same argument will provide the desired result after replacing a replaceable vertex with w . \square

Here we will show that $|D_{w^*}| \geq \lfloor \frac{k-3}{2} \rfloor$ holds even for vertices $w^* \in V(C) \setminus V$, therefore we have $|D_{w'}| \geq \lfloor \frac{k-3}{2} \rfloor$ for all $w' \in V(\mathcal{H}) \setminus V$.

By Claim 12 (iii) and Claim 13 (i), if $w^* \in V(C) \setminus V$ and $u_i \in D_w$, then $w^* \notin e_{i-1}, e_i$. Therefore the number of defining hyperedges that may contain w^* is at most 3. So Claim 11 and the set degree condition implies

$$\binom{\lfloor \frac{k-3}{2} \rfloor}{r-1} \leq 3 + |M_{w^*}| + \left(\min \left\{ \left\lfloor \frac{\ell-1-|M_{w^*}|}{2} \right\rfloor, |D_{w^*}| \right\} \right).$$

Just as for $w \in V(\mathcal{H}) \setminus V(C)$, in two steps we obtain $|D_w| \geq \lfloor \frac{k-3}{2} \rfloor$ for k large enough.

Before continuing with give possible embeddings of \mathcal{H} into some $\mathcal{H}_{n,a,b_1,\dots,b_s}$ let us state a last technical claim that will be used several times. Let us recall that a terminal vertex v is vertex of a missing interval that is adjacent to a vertex from D_w .

Claim 14. Suppose $D_w = D_{w'}$ for some $w' \in V(\mathcal{H}) \setminus V$ with $w' \neq w$.

(i) There does not exist $h \in \mathcal{H} \setminus C$ such that h contains terminal vertices of two distinct missing intervals of w .

(ii) If $\{u_i, u_{i+1}, u_{i+2}\}$ and $\{u_j, u_{j+1}\}$ form missing intervals of w and there exists $h \in \mathcal{H} \setminus C$ with $u_{i+1}, u_j \in h$ or $u_{i+1}, u_{j+1} \in h$, then there does not exist $h' \in \mathcal{H} \setminus C$, with $u_i, u_{i+2} \in h'$.

Proof. We prove (i) by contradiction. Suppose $\{u_i, u_{i+1}, \dots, u_{i+t}\}$ and $\{u_j, u_{j+1}, \dots, u_{j+z}\}$ are two distinct missing intervals of w .

• Suppose first $u_i, u_{j+z} \in h \in \mathcal{H} \setminus C$. We have $u_{i+t+1}, u_{i-1}, u_{j+z+1} \in D_w$, therefore there are three different hyperedges h_w, h'_w and $h_{w'}$, such that h_w is incident to w and u_{i+t+1}, h'_w is incident to w and u_{i-1} and $h_{w'}$ is incident to u_{j+z+1} and w' . Note that all those hyperedges are different from h by Claim 11 (i). Then we have a contradiction since the following Berge-path is of length ℓ , as it contains all the $\ell - 1$ defining vertices of C and w and w' :

$$u_{i+t}, \dots, u_i, h, u_{j+z}, e_{j+z-1}, \dots, u_{i+t+1}, h_w, w, h'_w, u_{i-1}, e_{i-2}, \dots, u_{j+z+1}, h_{w'}, w'.$$

• If $u_{i+t}, u_{j+z} \in h \in \mathcal{H} \setminus C$, then the Berge-path of length ℓ (using similar ideas as in the previous bullet) is

$$u_i, \dots, u_{i+t}, h, u_{j+z}, e_{j+z-1}, \dots, u_{i+t+1}, h_w, w, h'_w, u_{i-1}, e_{i-2}, \dots, u_{j+z+1}, h_{w'}, w',$$

and we are done with the proof of (i).

In (ii) we can assume that $u_{i+1}, u_{j+1} \in h$ holds since the case $u_{i+1}, u_j \in h$ is identical. The proof of this part is similar, at first we observe from part (i) that we have $h \neq h'$. Then the following Berge-path of length ℓ gives us a contradiction:

$$u_i, h', u_{i+2}, e_{i+1}, u_{i+1}, h, u_{j+1}, e_j, u_j, e_{j-1}, \dots, u_{i+3}, h_w, w, h'_w, u_{i-1}, e_{i-2}, \dots, u_{j+2}, h_{w'}, w'.$$

\square

2.2 Possible embeddings of \mathcal{H}

Now we are in the situation to be able to give possible embeddings of \mathcal{H} into some $\mathcal{H}_{n,a,b_1,\dots,b_s}$. In this subsection we gather all the information that we know about these embeddings so far and in the next subsection we analyze further the different cases to finish the proof.

Let us fix $w \in V(\mathcal{H}) \setminus V$ with D_w of maximum size and let \mathcal{H}^* denote the subhypergraph of \mathcal{H} that we obtain by removing those defining hyperedges e_i of C for which at least one of u_i or u_{i+1} is a vertex of a missing interval for w . By the above, $|\mathcal{H}| \leq |\mathcal{H}^*| + 9$.

If we are in a case when for all $w' \in V(\mathcal{H}) \setminus V$ we have $D_{w'} \subseteq D_w$, then let $A = D_w$, $B_i = I_i$ for $i = 1, 2, \dots, s$ and $L = V(\mathcal{H}) \setminus (D_w \cup \bigcup_{i=1}^s I_i)$. Let us summarize the findings of the technical claims and enumerate the types of different hyperedges in $\mathcal{H} \setminus \mathcal{H}_{n,a,b_1,b_2,\dots,b_s}$ in this scenario.

Summary 1.

If $h \in \mathcal{H} \setminus \mathcal{H}_{n,a,b_1,b_2,\dots,b_s}$ is not a defining hyperedge of C (i.e., $h \in \mathcal{H} \setminus C$), then

1. either there exists $v \in (V(\mathcal{H}) \setminus V) \cup R_w$ such that $h \setminus \{v\} \subseteq D_w \cup \bigcup_{i=1}^s \bar{I}_i$ and $h \cap \bigcup_{i=1}^s \bar{I}_i \neq \emptyset$; We refer to these hyperedges as type 1 hyperedges in what follows.
2. $h \subseteq V \setminus R_w$ and h contains vertices from at least two distinct missing intervals. We refer to these hyperedges as type 2 hyperedges in what follows.

If $e_i \in \mathcal{H} \setminus \mathcal{H}_{n,a,b_1,b_2,\dots,b_s}$ is a defining hyperedge of C , then

3. either $e_i \in \mathcal{H} \setminus \mathcal{H}^*$; or
4. u_i or u_{i+1} belongs to R_w , $e_i \setminus \{u_i, u_{i+1}\} \subseteq D_w \cup \bigcup_{i=1}^s I_i$ and $e_i \cap \bigcup_{i=1}^s I_i \neq \emptyset$.

Proof. Suppose first that h is not a defining hyperedge of C and h contains a vertex $v \in (V(\mathcal{H}) \setminus V) \cup R_w$. We claim that h cannot contain any $v' \in V(\mathcal{H}) \setminus V$ with $v' \neq v$. Indeed, if $v \notin V$, then it follows from Lemma 8 (i). If $v \in R_w$ and $v' = w$, then w can be inserted to obtain a longer cycle than C , while if $w \neq v'$, then using h , the defining vertices and hyperedges of C one can create a Berge-path of length ℓ from v' to w .

We also claim that h cannot contain a neighbor of a vertex in D_w on C . Indeed, if $v \notin V$, then it follows from Lemma 8 (ii) and (iii). If $v \in R_w$, then it follows from Claim 12 (i). Therefore, h cannot contain other vertices of R_w , nor terminal vertices of missing intervals. This gives possibility 1.

Otherwise if $h \in \mathcal{H} \setminus \mathcal{H}_{n,a,b_1,b_2,\dots,b_s}$ is not a defining hyperedge of C , then we must have $h \subseteq V \setminus R_w$. As all hyperedges in $\binom{A \cup I_j}{r}$ belong to $\mathcal{H}_{n,a,b_1,b_2,\dots,b_s}$, there must exist two distinct missing intervals meeting h . This gives possibility 2.

Let $e_i \in \mathcal{H} \setminus \mathcal{H}_{n,a,b_1,b_2,\dots,b_s}$ be a defining hyperedge of C . If at least one of u_i or u_{i+1} belongs to a missing interval, then $e_i \in \mathcal{H} \setminus \mathcal{H}^*$ by definition of \mathcal{H}^* . This gives possibility 3. Note that we have more information on some of these hyperedges by Claim 13.

Otherwise u_i or u_{i+1} belongs to R_w . By Claim 12 (ii), e_i does not contain any other vertex from R_w , and by Claim 12 (iii) e_i cannot contain any vertex from $V(\mathcal{H}) \setminus V$. This gives us possibility 4. Even more, if the unique element of $e_i \cap R_w$ is also replaceable by some $w' \neq w$, then e_i cannot contain w either. \square

If we are in a case when we have vertices $w, w' \in V(\mathcal{H}) \setminus V$ with $D_w \not\subseteq D_{w'}$ and $D_{w'} \not\subseteq D_w$, then as $\lfloor \frac{k-3}{2} \rfloor \leq |D_w|, |D_{w'}|$, we will have $\lfloor \frac{k-1}{2} \rfloor \leq |D_w \cup D_{w'}|$. Since the elements of $D_w \cup D_{w'}$ cannot be neighbors on C by Claim 11 (i) and $|C| \leq k-1$, we must have $|D_w \cup D_{w'}| = \lfloor \frac{k-1}{2} \rfloor$.

If $|C| = 2 \lfloor \frac{k-3}{2} \rfloor + 2$, then we will embed \mathcal{H} to $\mathcal{H}_{n, \lfloor \frac{k-1}{2} \rfloor}$, with $A = D_w \cup D_{w'}$ and all the other vertices are going to L .

If $|C| = 2 \lfloor \frac{k-3}{2} \rfloor + 3$, then we will embed \mathcal{H}^* to $\mathcal{H}_{n, \lfloor \frac{k-1}{2} \rfloor, 2}$ with $A = D_w \cup D_{w'}$, the unique missing interval goes to B_1 and all the remaining vertices are going to L .

Summary 2. *If for $w, w' \in V(\mathcal{H}) \setminus V$ we have $D_w \not\subseteq D_{w'}$ and $D_{w'} \not\subseteq D_w$, then*

1. *there is no hyperedge $h \in \mathcal{H} \setminus C$ with $h \in \mathcal{H} \setminus \mathcal{H}_{n, \lfloor \frac{k-1}{2} \rfloor, 2}$ or $h \in \mathcal{H} \setminus \mathcal{H}_{n, \lfloor \frac{k-1}{2} \rfloor, 2}$ depending on whether $|V| = 2 \lfloor \frac{k-3}{2} \rfloor + 2$ or $|V| = 2 \lfloor \frac{k-3}{2} \rfloor + 3$; and*
2. *if $u_{i-1}, u_{i+1} \in D_w \cup D_{w'}$, then $e_{i-1} \setminus \{u_i\}, e_i \setminus \{u_i\} \subseteq D_w \cup D_{w'} \cup I$, where I is the unique possible interval u_j, u_{j+1} of size two disjoint with $D_w \cup D_{w'}$. Furthermore, if u_i is replaceable by either w or w' , then $e_{i-1} \setminus \{u_i\}, e_i \setminus \{u_i\} \subseteq D_w \cup D_{w'}$.*

Proof. Note that every $u \in V \setminus (D_w \cup D_{w'})$ has a neighbor on C in $D_w \cup D_{w'}$. Therefore, if $v \in h \in \mathcal{H} \setminus C$ with $v \in V(\mathcal{H}) \setminus V$, then Claim 11 (i) yields $h \setminus \{v\} \subseteq D_w \cup D_{w'}$. So we only have to consider hyperedges $h \subset V$. If u_i is replaceable by either w or w' and $u_i \in h \in \mathcal{H} \setminus C$, then Claim 12 (i) and (iv) yield $h \setminus \{u_i\} \subseteq D_w \cup D_{w'}$. Finally, if u_j, u_{j+1} form the unique interval of $V \setminus (D_w \cup D_{w'})$, and u_i is neither replaceable by w nor by w' , then one of u_{i-1}, u_{i+1} belong to D_w , the other to $D_{w'}$. Suppose that $u_i, u_j \in h \in \mathcal{H} \setminus C$, the other case $u_i, u_{j+1} \in h \in \mathcal{H} \setminus C$ is symmetric. Then $u_{j-1} \in D_{w^*}$ and $u_{i-1} \in D_{w^{**}}$ for some $w^*, w^{**} \in \{w, w'\}$. Therefore

$$w^*, h', u_{j-1}, e_{j-2}, \dots, u_{i+1}, e_i, u_i, h, u_j, e_j, u_{j+1}, \dots, e_{i-2}, u_{i-1}, h'', w^{**}$$

is either a cycle (if $w^* = w^{**}$) or a path (if $w^* \neq w^{**}$) of length k . Such distinct hyperedges h', h'' exist from the definition of $D_{w^*}, D_{w^{**}}$ as well as they are different from the hyperedge h since $h \subset V$. This settles part 1.

For part 2, let us consider defining hyperedges e_{i-1}, e_i of C with $u_{i-1}, u_{i+1} \in D_w \cup D_{w'}$. Observe first that all but at most one of the u_i 's are replaceable either by w or by w' . If u_i is indeed replaceable by w or by w' , then Claim 12 (iii) yields $e_{i-1} \setminus \{u_i\}, e_i \setminus \{u_i\} \subseteq D_w \cup D_{w'}$. For the at most one exception u_i , we have that one of u_{i-1}, u_{i+1} is in D_w , the other one is in $D_{w'}$ and by Claim 12 (v) we are done. \square

2.3 Case-by-case analysis

We finish the proof with a case-by-case analysis according to the length of the longest Berge-cycle C and subcases will be defined according to the size of D_w . Let us remind the reader that the length of the cycle C , $\ell - 1$, might take the values $2 \lfloor \frac{k-3}{2} \rfloor$, $2 \lfloor \frac{k-3}{2} \rfloor + 1$, $2 \lfloor \frac{k-3}{2} \rfloor + 2$ or $2 \lfloor \frac{k-3}{2} \rfloor + 3$, and in the last case k is even. In each case we will use the summaries from the previous subsection.

Case I $\ell - 1 = 2 \lfloor \frac{k-3}{2} \rfloor$.

As $|D_w| \geq \lfloor \frac{k-3}{2} \rfloor$, then by Claim 11 (i), D_w must consist of every second vertex of V , so there are no missing intervals. Summary 1 implies $\mathcal{H} \subseteq \mathcal{H}_{n, \lfloor \frac{k-3}{2} \rfloor}$ thus

$$|\mathcal{H}| \leq |\mathcal{H}_{n, \lfloor \frac{k-3}{2} \rfloor}| < |\mathcal{H}_{n, \lfloor \frac{k-3}{2} \rfloor, 3}|,$$

which contradicts the assumption on $|\mathcal{H}|$.

Case II $\ell - 1 = 2 \lfloor \frac{k-3}{2} \rfloor + 1$.

$|D_w| \geq \lfloor \frac{k-3}{2} \rfloor$ and Claim 11 (i) imply that, after a possible relabelling we have $D_w = \{u_1, u_4, \dots, u_{2 \lfloor \frac{k-3}{2} \rfloor}\}$ and thus $\{u_2, u_3\}$ is the only missing interval for w , and all other vertices in $V \setminus D_w$ are in R_w . As all vertices in $V \setminus D_w$ are neighbors to some vertex in D_w , by Summary 1, all hyperedges in $\mathcal{H} \setminus C$ belong to $\mathcal{H}_{n, \lfloor \frac{k-3}{2} \rfloor, 2}$.

To consider the defining hyperedges of C , let us analyze those that contain an $u_i \in R_w$. Observe that by Claim 11 (i) a vertex in D_w cannot be a neighbor on C of a vertex in $D_{w'}$ for some $w' \in V(\mathcal{H}) \setminus V$, so $D_w = D_{w'}$ for any two $w, w' \in V(\mathcal{H}) \setminus V$. Hence we have that e_{i-1} and e_i cannot contain any of u_2 and u_3 by Claim 11 (i) applied to the cycle C' we obtain by Claim 11 (ii). Therefore by Summary 1 we have that $\mathcal{H}^* \subseteq \mathcal{H}_{n, \lfloor \frac{k-3}{2} \rfloor, 2}$, thus

$$|\mathcal{H}| \leq |\mathcal{H}_{n, \lfloor \frac{k-3}{2} \rfloor, 2}| + 3 < |\mathcal{H}_{n, \lfloor \frac{k-3}{2} \rfloor, 3}|,$$

which contradicts the assumption on $|\mathcal{H}|$.

Case III $\ell - 1 = 2 \lfloor \frac{k-3}{2} \rfloor + 2$.

The three subcases below cover this case.

Case III/A There exists $w \in V(\mathcal{H}) \setminus V$ with $|D_w| = \lfloor \frac{k-3}{2} \rfloor + 1$.

Then there is no missing interval for w , and so $V \setminus D_w \subseteq R_w$, so by Summary 1 we have $\mathcal{H} = \mathcal{H}^* \subseteq \mathcal{H}_{n, \lfloor \frac{k-1}{2} \rfloor}$.

Case III/B There exists $w \in V(\mathcal{H}) \setminus V$, for which there are two missing intervals, $\{u_i, u_{i+1}\}$ and $\{u_j, u_{j+1}\}$.

Note that there is no type 1 hyperedge of $\mathcal{H} \setminus C$, as each vertex of the missing intervals is terminal. Observe that all the vertices in $V \setminus D_w$ have neighbors in D_w , therefore the fact that $|D_{w'}| \geq \lfloor \frac{k-3}{2} \rfloor$, together with $|D_w| = \lfloor \frac{k-3}{2} \rfloor$ and Claim 11 (i) imply $D_w = D_{w'}$ for all $w, w' \in V(\mathcal{H}) \setminus V$. This enables us to conclude that

- by Claim 14 (i), there is no hyperedge $h \in \mathcal{H} \setminus C$ of type 2; and
- by Claim 12 (v), if $u_l \in R_w$, then e_{l-1}, e_l do not contain vertices of missing intervals.

So by Summary 1 we have $\mathcal{H}^* \subseteq \mathcal{H}_{n, \lfloor \frac{k-3}{2} \rfloor, 2, 2}$ and thus

$$|\mathcal{H}| \leq |\mathcal{H}_{n, \lfloor \frac{k-3}{2} \rfloor, 2, 2}| + 6 < |\mathcal{H}_{n, \lfloor \frac{k-3}{2} \rfloor, 3}|,$$

contradicting the assumption on $|\mathcal{H}|$.

Case III/C For all $w \in V(\mathcal{H}) \setminus V$, there is only one missing interval containing three vertices $\{u_{i(w)}, u_{i(w)+1}, u_{i(w)+2}\}$.

If there exist two vertices $w, w' \in V(\mathcal{H}) \setminus V$ with $i(w) \neq i(w')$, then $D_w \cup D_{w'}$ must contain every second vertex of C . So by Summary 2, we have $\mathcal{H} \subseteq \mathcal{H}_{n, \lfloor \frac{k-1}{2} \rfloor}$ as claimed by the theorem.

So we can assume that the D_w s are the same and without loss of generality suppose that for every $w \in V(\mathcal{H}) \setminus V$, the missing interval is $\{u_1, u_2, u_3\}$. Moreover, as every replaceable vertex u_i is replaceable by any $w \in V(\mathcal{H}) \setminus V$, replaceable vertices and defining hyperedges e_{i-1}, e_i behave as vertices in $V(\mathcal{H}) \setminus V$ and hyperedges in $\mathcal{H} \setminus C$. By Summary 1 and the above, we have to deal with type 1 hyperedges of $\mathcal{H} \setminus C$ and $\mathcal{H} \setminus \mathcal{H}^* = \{e_2 \lfloor \frac{k-3}{2} \rfloor + 2, e_1, e_2, e_3\}$.

• At first suppose that there exists a type 1 hyperedge of $\mathcal{H} \setminus C$, i.e., $h \in \mathcal{H} \setminus C$ with $v, u_2 \in h$ for some $v \in (V(\mathcal{H}) \setminus V) \cup R_w$. Without loss of generality we may assume $v \in V(\mathcal{H}) \setminus V$. Then we claim that there is no hyperedge $h' \in \mathcal{H}$ with $u_1, u_3 \in h'$. Suppose by a contradiction that such h' exists, then observe that $h' \neq h$, as otherwise we would have $v, u_1, u_3 \in h'$ that is not possible by Summary 1. Also, either $h' \notin \{e_1, e_3\}$ or $h' \notin \{e_2 \lfloor \frac{k-3}{2} \rfloor + 2, e_2\}$, so we may assume $h' \notin \{e_1, e_3\}$ without loss of generality. Since $u_2 \lfloor \frac{k-3}{2} \rfloor + 2 \in D_v$, there is a hyperedge h'' different from the hyperedges h and h' , incident to the vertices v and $u_2 \lfloor \frac{k-3}{2} \rfloor + 2$. We have a contradiction since the following is a longer Berge-cycle than C , containing all defining vertices of C and v :

$$v, h, u_2, e_1, u_1, h', u_3, e_3, u_4, \dots, u_2 \lfloor \frac{k-3}{2} \rfloor + 2, h''.$$

As no hyperedge contains both u_1 and u_3 , we obtained $\mathcal{H} \subseteq \mathcal{H}_{n, \lfloor \frac{k-1}{2} \rfloor}$ in this case.

• Suppose next that there is no type 1 hyperedge of $\mathcal{H} \setminus C$, i.e., by Summary 1, we have $\mathcal{H}^* \subseteq \mathcal{H}_{n, \lfloor \frac{k-3}{2} \rfloor, 3}$. Observe that $e_2 \lfloor \frac{k-3}{2} \rfloor + 2$ and e_3 do not contain vertices from $(V(\mathcal{H}) \setminus V) \cup R_w$ by Claim 13 (i) and (ii). If the same holds for e_1, e_2 , then $\mathcal{H} \subseteq \mathcal{H}_{n, 2 \lfloor \frac{k-3}{2} \rfloor, 3}$ contradicting the assumption $|\mathcal{H}| > |\mathcal{H}_{n, 2 \lfloor \frac{k-3}{2} \rfloor, 3}|$. So we can assume that e_2 contains a vertex $v \in (V(\mathcal{H}) \setminus V) \cup R_w$. Then we claim that there is no hyperedge $h \in \mathcal{H} \setminus C$ with $u_1, u_3 \in h$. In here we get a contradiction as in the previous settings with a longer Berge-cycle, therefore we omit the proof. We obtained the following contradiction

$$|\mathcal{H}| \leq 2 + |\mathcal{H}_{n, \lfloor \frac{k-3}{2} \rfloor, 3}| - \binom{\lfloor \frac{k-3}{2} \rfloor}{r-2} < |\mathcal{H}_{n, \lfloor \frac{k-3}{2} \rfloor, 3}|.$$

Case IV $\ell - 1 = 2 \lfloor \frac{k-3}{2} \rfloor + 3$.

Note that in this case k is even and the length of C is $k - 1$. We again distinguish several subcases.

Case IV/A $|D_w| = \lfloor \frac{k-1}{2} \rfloor$.

Then as D_w does not contain neighboring vertices on C , after relabelling, we can suppose that we have $D_w = \{u_1, u_4, u_6, \dots, u_{k-2}\}$. So there is one missing interval $\{u_2, u_3\}$, therefore there does not exist a type 1 or type 2 hyperedge $h \in \mathcal{H} \setminus C$. If $u_i \in R_w$, then by Claim 12 (iii) e_{i-1} and e_i do not contain vertices from $V(\mathcal{H}) \setminus V$. We claim that e_{i-1} and e_i do not contain vertices from the missing interval $\{u_2, u_3\}$. Indeed, if there exists $w^* \neq w$ with $u_1 \in N_{\mathcal{H} \setminus C}(w^*)$ and $u_2 \in e_i$ or e_{i-1} , then the following is a Berge-path of length k :

$$u_i, e_i \text{ (or } e_{i-1}), u_2, e_2, u_3, \dots, u_{i-1}, h, w, h', u_{i+1}, e_{i+1}, \dots, u_{k-1}, e_{k-1}, u_1, h'', w^*.$$

Here h and h' exist and are distinct as u_i is in R_w and h'' exists by the choice of w^* .

Similarly, if there exists $w^{**} \neq w$ with $u_4 \in N_{\mathcal{H} \setminus C}(w^{**})$, then e_{i-1}, e_i cannot contain u_3 . As all D_{w^*} is of size at least $\lfloor \frac{k-3}{2} \rfloor$, the only cases when we are not yet done is when $u_1 \notin N_{\mathcal{H} \setminus C}(w^*)$ and $D_{w^*} = \{u_4, u_6, \dots, u_{2 \lfloor \frac{k-3}{2} \rfloor + 2}\}$ or $u_4 \notin N_{\mathcal{H} \setminus C}(w^*)$ and $D_{w^*} = \{u_6, u_8, \dots, u_{2 \lfloor \frac{k-3}{2} \rfloor + 2}, u_1\}$. By symmetry, we can assume the first. But then any replaceable u_i but $u_{2 \lfloor \frac{k-3}{2} \rfloor + 3}$ can be replaced with some $w^* \neq w$, and the above arguments applied to the new cycle C' show that any $u_i \in h \in \mathcal{H} \setminus C'$ (in particular, it applies to e_i and e_{i-1} !) cannot contain u_3 , and by Summary 1, we already know that e_{i-1}, e_i cannot contain $u_{2 \lfloor \frac{k-3}{2} \rfloor + 3}$. Therefore setting $A = D_w \setminus \{u_1\}$, $B_1 = \{u_{2 \lfloor \frac{k-3}{2} \rfloor + 3}, u_1, u_2, u_3\}$ we have that \mathcal{H} is a subfamily of $\mathcal{H}_{n, \lfloor \frac{k-3}{2} \rfloor, 4}$ apart from $e_2 \lfloor \frac{k-3}{2} \rfloor + 2, e_{2 \lfloor \frac{k-3}{2} \rfloor + 3}, e_1, e_2, e_3$ and the hyperedges containing both w and u_1 . On the other hand, there cannot exist $h \in \mathcal{H} \setminus C$ with $u_{2 \lfloor \frac{k-3}{2} \rfloor + 3}, u_2 \in h$ nor with $u_{2 \lfloor \frac{k-3}{2} \rfloor + 3}, u_3 \in h$ as in the former case

$$w, h', u_1, e_{2 \lfloor \frac{k-3}{2} \rfloor + 3}, u_{2 \lfloor \frac{k-3}{2} \rfloor + 3}, h, u_2, e_2, \dots, u_{2 \lfloor \frac{k-3}{2} \rfloor + 2}, h'', w^*,$$

while in the latter case

$$w, h', u_1, e_1, u_2, e_2, u_3, h, u_{2 \lfloor \frac{k-3}{2} \rfloor + 3}, e_{2 \lfloor \frac{k-3}{2} \rfloor + 2}, u_{2 \lfloor \frac{k-3}{2} \rfloor + 2}, \dots, e_4, u_4, h'', w^*$$

is a Berge-path of length k . So we have

$$|\mathcal{H}| \leq |\mathcal{H}_{n, \lfloor \frac{k-3}{2} \rfloor, 4}| + 5 + \binom{\lfloor \frac{k-3}{2} \rfloor}{r-2} - 2 \binom{\lfloor \frac{k-3}{2} \rfloor}{r-2} < |\mathcal{H}_{n, \lfloor \frac{k-3}{2} \rfloor, 4}|,$$

contradicting the assumption on $|\mathcal{H}|$. So we obtained that e_{i-1}, e_i cannot contain u_2, u_3 and thus so far by Summary 1 we have $\mathcal{H}^* \subseteq \mathcal{H}_{n, \lfloor \frac{k-1}{2} \rfloor, 2}$.

Now let us concentrate on the hyperedges in $\mathcal{H} \setminus \mathcal{H}^*$. So $\{u_2, u_3\}$ is the unique missing interval (all other vertices of $V \setminus D_w$ are in R_w), and thus $\mathcal{H} \setminus \mathcal{H}^*$ contains three hyperedges: e_1, e_2 and e_3 . Observe that by Claim 13 (i), e_1 and e_3 do not contain any $w' \in V(\mathcal{H}) \setminus V$. By Claim 12 (v), e_1 and e_3 do not contain any vertex in R_w .

- If e_2 does not contain any vertex in $R_w \cup (V(\mathcal{H}) \setminus V)$, then we are done, since $\mathcal{H} \subseteq \mathcal{H}_{n, \lfloor \frac{k-1}{2} \rfloor, 2}$.

- If e_2 does contain a vertex from $R_w \cup (V(\mathcal{H}) \setminus V)$, then there does not exist any other hyperedge h that contains both u_2 and u_3 . Indeed, if e_2 contained w , then w could be inserted in between u_2 and u_3 in the Berge-cycle C to form a longer cycle than C , a contradiction. If e_2 contains some $w' \neq w$ from $V(\mathcal{H}) \setminus V$, then we can reach a contradiction as before: we would find a Berge-path of length k starting with w', e_2, u_2, h, u_3 , then going through C and ending with u_1, h', w as $u_1 \in D_w$.

Finally, if e_2 contains a replaceable u_i , then at least one of u_1, u_4 belongs to $D_{w'}$ for some $w' \in V(\mathcal{H}) \setminus V$ with $w' \neq w$, since $D_{w'} \subseteq D_w$ from Claim 11 (i) and $|D_w \setminus D_{w'}| \leq 1$. By symmetry, we may assume that $u_1 \in D_{w'}$. Then we have a contradiction since the following Berge-path has length k . The Berge-path is $u_i, e_2, u_2, h, u_3, u_4, \dots$ that goes around the cycle C , replaces u_i by w and finishes with $u_1, h_{w'}, w'$, such $h_{w'}$ exists from the definition of $D_{w'}$. Therefore, if e_2 does contain a vertex from $R_w \cup (V(\mathcal{H}) \setminus V)$, then there does not exist any other hyperedge h that contains both u_2 and u_3 . Hence, $\mathcal{H} \subseteq \mathcal{H}_{n, \lfloor \frac{k-1}{2} \rfloor}^+$ with $A = D_w$, $L = V(\mathcal{H}) \setminus D_w$ and e_2 being the unique hyperedge of $\mathcal{H}_{n, \lfloor \frac{k-1}{2} \rfloor}^+$ that contains less than $r - 1$ vertices of A .

Case IV/B For all $w' \in V(\mathcal{H}) \setminus V$, we have $|D'_w| = |D_w| = \lfloor \frac{k-3}{2} \rfloor$.

As the length of C is $k - 1$, k is even and vertices of D_w are not neighbors on C , we have at most three missing intervals. If there are three missing intervals, then each of them contains two vertices. If there are two missing intervals, then they contain two and three vertices and if there is only one missing interval, then it contains 4 vertices. According to this structure we are going to consider the following three subcases.

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Case IV/B/1 There exists $w \in V(\mathcal{H}) \setminus V$ with $V \setminus D_w$ containing 3 intervals of length 2.

Observe that as all the missing intervals are of size 2, we do not have type 1 hyperedges $h \in \mathcal{H} \setminus C$. As all vertices in $V \setminus D_w$ have neighbors in D_w , we obtain that for any $w' \in V(\mathcal{H}) \setminus V$ we have $D_w = D_{w'}$. So Claim 14 (i) implies that there does not exist any type 2 hyperedges $h \in \mathcal{H} \setminus C$. Finally, Claim 12 (v) implies that defining hyperedges of C , apart from those in $\mathcal{H} \setminus \mathcal{H}^*$, are in $\mathcal{H}_{n, \lfloor \frac{k-3}{2} \rfloor, 2, 2, 2}$. So we obtained a contradiction as

$$|\mathcal{H}| \leq 9 + |\mathcal{H}_{n, \lfloor \frac{k-3}{2} \rfloor, 2, 2, 2}| < |\mathcal{H}_{n, \lfloor \frac{k-3}{2} \rfloor, 4}|.$$

Case IV/B/2 For all $w \in V(\mathcal{H}) \setminus V$, the number of missing intervals is at most 2 and there exist $w, w' \in V(\mathcal{H}) \setminus V$ with $D_w \neq D_{w'}$.

By relabeling, we can assume that $\{u_2, u_3\}$ forms the unique missing interval for both w and w' , i.e., the unique interval of length more than 1 in $V \setminus (D_w \cup D_{w'})$. According to Summary 2, if every $u_i \notin D_w \cup D_{w'} \cup \{u_2, u_3\}$ is replaceable, then we have $\mathcal{H} \setminus \{e_1, e_2, e_3\} \subseteq \mathcal{H}_{n, \lfloor \frac{k-1}{2} \rfloor, 2}$, while if there is $u_i \in V \setminus (D_w \cup D_{w'})$ ($i \neq 2, 3$) that is not in $R_w \cup R_{w'}$, then we know $e_{i-1} \setminus \{u_i\}, e_i \setminus \{u_i\} \subseteq D_w \cup D_{w'} \cup \{u_2, u_3\}$.

- At first we suppose that there exists a $u \in D_w \cup D_{w'}$ such that $|\{w^* \in (V(\mathcal{H}) \setminus V) \cup R_w \cup R_{w'} : u \in N_{\mathcal{H} \setminus C}(w^*)\}| = 1$. In that case the unique w^* must be either w or w' , say w .

Consider the hypergraph $\mathcal{H} \setminus \mathcal{H}_{n, \lfloor \frac{k-3}{2} \rfloor, 3}$ with $\mathcal{H}_{n, \lfloor \frac{k-3}{2} \rfloor, 3}$ having $A = D_w \cup D_{w'} \setminus \{u\} = D_{w'}$ and $B_1 = \{u, u_2, u_3\}$. Then, by Summary 2, the hyperedges left are incident with the vertex u , thus the number of hyperedges is at most $\binom{\lfloor \frac{k-3}{2} \rfloor}{r-2} + 5$. Here the first term is an upper bound for those hyperedges that are incident with both u and w , while the second term is 5 for $\{e_{i-1}, e_i, e_1, e_2, e_3\}$. So we have a contradiction as

$$|\mathcal{H}| \leq \binom{\lfloor \frac{k-3}{2} \rfloor}{r-2} + 5 + |\mathcal{H}_{n, \lfloor \frac{k-3}{2} \rfloor, 3}| < |\mathcal{H}_{n, \lfloor \frac{k-3}{2} \rfloor, 4}|.$$

• Suppose now that for all $u \in D_w \cup D_{w'}$, $|\{w^* \in (V(\mathcal{H}) \setminus V) \cup R_w \cup R_{w'} : u \in N_{\mathcal{H} \setminus C}(w^*)\}| \geq 2$. At first we show that $u_2, u_3 \notin e_{i-1}, e_i$ if $u_i \in V \setminus (D_w \cup D_{w'})$ ($i \neq 2, 3$). This holds by Summary 2, if u_i is replaceable by either w or w' . Therefore without loss of generality we may assume $u_{i+1} \in D_w \setminus D_{w'}$ and $u_{i-1} \in D_{w'} \setminus D_w$. Note that $D_w = (D_w \cup D_{w'}) \setminus \{u_{i-1}\}$ and $D_{w'} = (D_w \cup D_{w'}) \setminus \{u_{i+1}\}$. Because of symmetry, it is enough to show a contradiction only if $u_2 \in e_i$, the three remaining cases are similar to this one. The following is a Berge-path of length k

$$u_i, e_i, u_2, e_2, u_3, e_3, \dots, u_{i-1}, h, w', h', u_1, e_{k-1}, u_{k-1}, e_{k-2}, \dots, e_{i+1}, u_{i+1}, h'', w,$$

a contradiction. The hyperedges h, h', h'' can be chosen distinct as $u_1, u_{i-1} \in D_{w'}$ and $u_{i+1} \in D_w$ and by Lemma 8 (i), $h^* \in \mathcal{H} \setminus C$ cannot contain distinct vertices from outside V .

By Claim 13 (i) and (ii), e_1 and e_3 are not incident with vertices in $V(\mathcal{H}) \setminus V$ or in $R_w \cup R_{w'}$. Even more, they are not incident with u_i either, since otherwise if $u_i \in e_1$, the following path is of length k , a contradiction:

$$u_i, e_1, u_2, e_2, u_3, e_3, \dots, u_{i-1}, h, w', h', u_1, e_{k-1}, u_{k-1}, \dots, e_{i+1}, u_{i+1}, h'', w.$$

An analogous argument shows $u_i \notin e_3$.

Finally, if e_2 contains any vertex from $V(\mathcal{H}) \setminus (D_w \cup D_{w'})$, then similarly to previous cases a hyperedge $e_2 \neq h \in \mathcal{H}$ containing both u_2, u_3 would lead to a Berge-path of length k . So if no such hyperedge h exists, then $\mathcal{H} \subseteq \mathcal{H}_{n, \lfloor \frac{k-1}{2} \rfloor, 2}$. Otherwise, we have $\mathcal{H} \subseteq \mathcal{H}_{n, \lfloor \frac{k-1}{2} \rfloor}^+$. Both possibilities are as claimed by the theorem.

Case IV/B/3 For all $w' \in V(\mathcal{H}) \setminus V \cup R_w$, the number of missing intervals is at most 2 and $D_w = D_{w'}$.

As $D_w = D_{w'}$ for all $w, w' \in V(\mathcal{H}) \setminus V$, it follows that we do not have to distinguish between vertices in $V(\mathcal{H}) \setminus V$ and vertices in R_w . Also, anything that we prove for hyperedges $h \in \mathcal{H} \setminus C$ is valid for all e_i, e_{i-1} if $u_i \in R_w$, by Claim 11 (ii).

Case IV/B/3/1 Let us consider first the case when for every $v \in V(\mathcal{H}) \setminus V \cup R_w$, the missing intervals for v are $\{u_2, u_3, u_4\}$ and $\{u_i, u_{i+1}\}$ for some $6 \leq i \leq k-2$, after possible relabeling. By Summary 1 and Claim 14 (i), we need to consider the 7 hyperedges in $\mathcal{H} \setminus \mathcal{H}^*$, the hyperedges in $\mathcal{H} \setminus C$ containing u_3, u_i or u_3, u_{i+1} and the hyperedges in $\mathcal{H} \setminus C$ containing u_3 and some $v \in V(\mathcal{H}) \setminus V \cup R_w$.

• If there are no hyperedges in $\mathcal{H} \setminus C$ containing u_3, u_i or u_3, u_{i+1} or u_3 and some $v \in V(\mathcal{H}) \setminus V \cup R_w$, then $\mathcal{H}^* \subseteq \mathcal{H}_{n, \lfloor \frac{k-3}{2} \rfloor, 3, 2}$, with embedding $A = D_w$, $B_1 = \{u_2, u_3, u_4\}$, $B_2 = \{u_i, u_{i+1}\}$ and

$$|\mathcal{H}| \leq |\mathcal{H}^*| + 7 \leq |\mathcal{H}_{n, \lfloor \frac{k-3}{2} \rfloor, 3, 2}| + 7 < |\mathcal{H}_{n, \lfloor \frac{k-3}{2} \rfloor, 4}|,$$

contradicting the assumption on $|\mathcal{H}|$.

• If there are no hyperedges in $\mathcal{H} \setminus C$ containing u_3 and some $v \in V(\mathcal{H}) \setminus V \cup R_w$, but there exist a hyperedge $h \in \mathcal{H} \setminus C$ containing u_3, u_i or u_3, u_{i+1} , then by Claim 14 (ii), there is no hyperedge containing both u_2 and u_4 . In particular, with embedding $A = D_w, B_1 = \{u_2, u_3, u_4\}, B_2 = \{u_i, u_{i+1}\}$ we have $|\mathcal{H}_{n, \lfloor \frac{k-3}{2} \rfloor, 3, 2} \setminus \mathcal{H}| \geq \binom{\lfloor \frac{k-3}{2} \rfloor}{r-2}$. Also, by Summary 1, the hypergraph $\mathcal{H} \setminus \mathcal{H}_{n, \lfloor \frac{k-3}{2} \rfloor, 3, 2}$ may contain the 7 hyperedges of $\mathcal{H} \setminus \mathcal{H}^*$ and at most $2\binom{\lfloor \frac{k-3}{2} \rfloor}{r-2} + \binom{\lfloor \frac{k-3}{2} \rfloor}{r-3}$ hyperedges containing u_i or/and u_{i+1} and u_3 . So we have

$$|\mathcal{H}| \leq |\mathcal{H}_{n, \lfloor \frac{k-3}{2} \rfloor, 3, 2}| + 7 + 2\binom{\lfloor \frac{k-3}{2} \rfloor}{r-2} + \binom{\lfloor \frac{k-3}{2} \rfloor}{r-3} - \binom{\lfloor \frac{k-3}{2} \rfloor}{r-2} < |\mathcal{H}_{n, \lfloor \frac{k-3}{2} \rfloor, 4}|,$$

which contradicts the assumption on $|\mathcal{H}|$.

• Suppose that there is a hyperedge $h \in \mathcal{H} \setminus C$ containing u_3 and some $v \in V(\mathcal{H}) \setminus V \cup R_w$. There is no $h' \in \mathcal{H} \setminus C$ incident with u_2 and u_4 . Indeed, otherwise

$$v, u_3, e_2, u_2, h', u_4, e_4, \dots, u_1, h_w, w$$

is a Berge-path of length k , a contradiction.

By the above, Summary 1 and Claim 14 (i), we have that $\mathcal{H}^* \subseteq \mathcal{H}_{n, \lfloor \frac{k-1}{2} \rfloor, 2}$ with embedding $A = D_w \cup \{u_3\}, B_1 = \{u_i, u_{i+1}\}$. Even more, since $D_v = D_w \ni u_1, u_5$, by Lemma 8 (iii) there exist cycles C_2, C_4 with v replacing u_2 and u_4 , respectively. Observe that the set D_w^* does not change when we apply these changes from C to C_2 and C to C_4 . In C_2 , e_1, e_2 are not defining hyperedges, while in C_4 , e_3, e_4 are not defining hyperedges. Therefore, applying Lemma 8 (ii), we obtain that e_1, e_2 do not contain u_4, u_i, u_{i+1} and e_3, e_4 do not contain u_2, u_i, u_{i+1} . Hence hyperedges e_1, e_2, e_3, e_4 are also from $\mathcal{H}_{n, \lfloor \frac{k-1}{2} \rfloor, 2}$ by Summary 1. By Claim 13 (i) and Claim 14 (i), we have that the hyperedges e_{i-1} and e_{i+1} are also from $\mathcal{H}_{n, \lfloor \frac{k-1}{2} \rfloor, 2}$. Finally, if e_i does not contain any vertex from $(V(\mathcal{H}) \setminus V) \cup R_w \cup \{u_3\}$, then we have $\mathcal{H} \subseteq \mathcal{H}_{n, \lfloor \frac{k-1}{2} \rfloor, 2}$. Otherwise, as in Case IV/A, one can see that there does not exist $h \neq e_i$ with $u_i, u_{i+1} \in h$ and thus $\mathcal{H} = \mathcal{H}_{n, \lfloor \frac{k-1}{2} \rfloor}^+$ with $A = D_w \cup \{u_3\}$ and e_i being the unique hyperedge with less than $r-1$ elements in A .

Case IV/B/3/2 For all $v \in V(\mathcal{H}) \setminus V \cup R_w$, the only missing interval consists of $\{u_2, u_3, u_4, u_5\}$, after possible relabelling.

By Summary 1, we need to handle hyperedges e_1, e_2, e_3, e_4, e_5 and those $h \in \mathcal{H} \setminus C$ that contain a $v \in V(\mathcal{H}) \setminus V \cup R_w$ and u_3 and/or u_4 .

• If there are no such hyperedges and $e_1, e_2, e_3, e_4, e_5 \subseteq D_w \cup \{u_2, u_3, u_4, u_5\}$, then $\mathcal{H} \subseteq \mathcal{H}_{n, \lfloor \frac{k-3}{2} \rfloor, 4}$ contradicting the assumption on $|\mathcal{H}|$.

• Suppose next there is no $h \in \mathcal{H} \setminus C$ with a vertex from $V(\mathcal{H}) \setminus V \cup R_w$ containing u_3 or u_4 , but some e_i ($i = 1, 2, 3, 4, 5$) does contain a vertex from outside V . By Claim 13 (i), it is neither e_1 nor e_5 . If e_i contains a vertex v from outside V , then there cannot exist $h \in \mathcal{H} \setminus C$ with $u_2, u_{i+1} \in h$, as then

$$v, e_i, u_i, e_{i-1}, \dots, u_2, h, u_{i+1}, e_{i+1}, u_{i+2}, e_{i+2}, \dots, u_{k-1}, e_{k-1}, u_1, h'$$

is a Berge-cycle of length k . For the existence of h' we used $D_v = D_w \ni u_1$. Therefore we have

$$|\mathcal{H}| \leq 3 + |\mathcal{H}_{n, \lfloor \frac{k-3}{2} \rfloor, 4}| - \binom{\lfloor \frac{k-3}{2} \rfloor}{r-2} < |\mathcal{H}_{n, \lfloor \frac{k-3}{2} \rfloor, 4}|,$$

contradicting the assumption on $|\mathcal{H}|$.

• If there exists a hyperedge $h \in \mathcal{H} \setminus C$ incident with some vertex $v \in V(\mathcal{H}) \setminus V \cup R_w$ and u_3 , then there is no $h' \neq h$, $h' \in \mathcal{H} \setminus C$ incident with some vertex from $V(\mathcal{H}) \setminus V \cup R_w$ and u_4 , by Claim 11 (i). Even more, there is no $h'' \in \mathcal{H} \setminus C$ with $u_2, u_4 \in h''$. The argument is the same as if e_3 contained v from the previous bullet. Similarly one can get that there is no hyperedge $h'' \in \mathcal{H} \setminus C$ with $u_2, u_5 \in h''$.

Observe that there should exist at least two distinct $v_1, v_2 \in V(\mathcal{H}) \setminus V \cup R_w$ for which hyperedges h_{v_1}, h_{v_2} with $v_1, u_3 \in h_{v_1}$ and $v_2, u_3 \in h_{v_2}$ exist. Indeed, otherwise using that there is no non-defining edge incident to u_2, u_4 , we have

$$|\mathcal{H}| \leq 5 + 1 + |\mathcal{H}_{n, \lfloor \frac{k-3}{2} \rfloor, 4}| - \binom{\lfloor \frac{k-3}{2} \rfloor}{r-2} < |\mathcal{H}_{n, \lfloor \frac{k-3}{2} \rfloor, 4}|.$$

We will show that either $\mathcal{H} \subseteq \mathcal{H}_{n, \lfloor \frac{k-1}{2} \rfloor, 2}$ or $\mathcal{H} \subseteq \mathcal{H}_{n, \lfloor \frac{k-1}{2} \rfloor}^+$ with $A = D_w \cup \{u_3\}$ and $B_1 = \{u_4, u_5\}$. Let w^* denote an arbitrary vertex $w^* \in V(\mathcal{H}) \setminus V$ with $w^* \neq v_1, v_2$. We will use that $u_1, u_6 \in D_w = D_{w^*} = D_{v_1} = D_{v_2}$ thus there exists a hyperedge that is not a defining hyperedge of C and is different from h_{v_1} and h_{v_2} , containing either u_1 or u_6 together with v_1 or v_2 or w^* .

We need to prove that $u_4, u_5 \notin e_1, e_2$ and $u_2 \notin e_3, e_5$. In each of the cases we present a Berge-path of length k below, which is a contradiction.

If $u_4 \in e_1$, then the path is

$$v_1, h_{v_1}, u_3, e_2, u_2, e_1, u_4, e_4, u_5, \dots, u_{k-1}, e_{k-1}, u_1, h, w^*.$$

If $u_4 \in e_2$, then the path is

$$u_2, e_2, u_4, e_4, u_5, e_5, \dots, u_{k-1}, e_{k-1}, u_1, h, v_1, h_{v_1}, u_3, h_{v_2}, v_2.$$

If $u_5 \in e_1$ or e_2 , then the path is

$$u_2, e_1 \text{ or } e_2, u_5, e_4, u_4, e_3, u_3, h_{v_1}, v_1, h, u_6, e_6, \dots, u_{k-1}, e_{k-1}, u_1, h', w^*.$$

If $u_2 \in e_3$, then the path is

$$v_1, h_{v_1}, u_3, e_2, u_2, e_3, u_4, e_4, u_5, \dots, u_{k-1}, e_{k-1}, u_1, h, w^*.$$

If $u_2 \in e_5$, then the path is

$$u_2, e_5, u_5, e_4, u_4, e_3, u_3, h_{v_1}, v_1, h, u_6, e_6, \dots, u_{k-1}, e_{k-1}, u_1, h', w^*.$$

From here, one can conclude to $\mathcal{H} \subseteq \mathcal{H}_{n, \lfloor \frac{k-1}{2} \rfloor, 2}$ or $\mathcal{H} \subseteq \mathcal{H}_{n, \lfloor \frac{k-1}{2} \rfloor}^+$ as in Case IV/A, depending on whether $e_4 \subseteq D_w \cup \{u_3, u_4, u_5\}$ or not.

The above case-by-case analysis concludes the proof of Theorem 6 under the set degree condition, i.e., for any set X of vertices with $|X| \leq k/2$ the number of hyperedges incident with some vertex in X , $|E(X)|$, is at least $|X| \binom{\lfloor \frac{k-3}{2} \rfloor}{r-1}$. \square

Let $n'_{k,r}$ denote the threshold such that the statement of Theorem 6 holds for hypergraphs with the set degree condition if $n \geq n'_{k,r}$. We are now ready to prove the general statements.

Proof of Theorem 6 and Theorem 7. Let \mathcal{H} be a connected n -vertex r -uniform hypergraph without a Berge-path of length k , and suppose that if k is odd, then

$$|\mathcal{H}| > |\mathcal{H}_{n, \lfloor \frac{k-3}{2} \rfloor, 3}| = \left(n - \frac{k+3}{2}\right) \binom{\lfloor \frac{k-3}{2} \rfloor}{r-1} + \binom{\lfloor \frac{k+3}{2} \rfloor}{r},$$

while if k is even, then

$$|\mathcal{H}| > |\mathcal{H}_{n, \lfloor \frac{k-3}{2} \rfloor, 4}| = \left(n - \left\lfloor \frac{k+5}{2} \right\rfloor\right) \binom{\lfloor \frac{k-3}{2} \rfloor}{r-1} + \binom{\lfloor \frac{k+5}{2} \rfloor}{r}.$$

We obtain a subhypergraph \mathcal{H}' of \mathcal{H} using a standard greedy process: as long as there exists a set S of vertices with $|S| \leq k/2$ such that $|E(S)| < |S| \binom{\lfloor \frac{k-3}{2} \rfloor}{r-1}$, we remove S from \mathcal{H} and all hyperedges in $E(S)$. Let \mathcal{H}' denote the subhypergraph at the end of this process.

Claim 15. *There exists a threshold $n''_{k,r}$, such that if $|V(\mathcal{H})| \geq n''_{k,r}$, then \mathcal{H}' is connected and contains at least $n'_{k,r}$ vertices.*

Proof. To see that \mathcal{H}' is connected, observe that every component of \mathcal{H}' possesses the set degree condition. Therefore Claim 10 yields that every component contains a cycle of length at least $k-4$. Therefore, as \mathcal{H} is connected, \mathcal{H} contains a Berge-path with at least $2k-8$ vertices from two different components of \mathcal{H}' , a contradiction as $k \geq 9$.

Suppose to the contrary that \mathcal{H}' has less than $n'_{k,r}$ vertices. Observe that, by definition of the process, $|\mathcal{E}(\mathcal{H}')| - |V(\mathcal{H}')| \binom{\lfloor \frac{k-3}{2} \rfloor}{r-1}$ strictly increases at every removal of some set X of at most k vertices. Therefore if $n > n'_{k,r} + k \binom{n'_{k,r}}{r} = n''_{k,r}$ and $|V(\mathcal{H}')| < n'_{k,r}$, then at the end we would have more hyperedges than those in the complete r -uniform hypergraph on $|V(\mathcal{H}')|$ vertices, a contradiction. \square

By Claim 15 and the statement for hypergraphs with the set degree property, we know that \mathcal{H}' has $n_1 \geq n'_{k,r}$ vertices, and $\mathcal{H}' \subseteq \mathcal{H}_{n_1, \lfloor \frac{k-1}{2} \rfloor}$ if k is odd, and $\mathcal{H}' \subseteq \mathcal{H}_{n_1, \lfloor \frac{k-1}{2} \rfloor, 2}$ or $\mathcal{H}_{n_1, \lfloor \frac{k-1}{2} \rfloor}^+$ if k is even. Then for any hyperedge $h \in \mathcal{E}(\mathcal{H}) \setminus \mathcal{E}(\mathcal{H}')$ that contain at least one vertex from $V(\mathcal{H}) \setminus V(\mathcal{H}')$ with degree at least two, we can apply Lemma 8 (i) to obtain that all such h must meet the A of \mathcal{H}' in $r-1$ vertices. This shows that if the minimum degree of \mathcal{H} is at least 2, then $\mathcal{H} \subseteq \mathcal{H}_{n_2, \lfloor \frac{k-1}{2} \rfloor}$ if k is odd, and $\mathcal{H} \subseteq \mathcal{H}_{n_2, \lfloor \frac{k-1}{2} \rfloor, 2}$ or $\mathcal{H} \subseteq \mathcal{H}_{n_2, \lfloor \frac{k-1}{2} \rfloor}^+$ if k is even, where $n_2 \leq n$ is the number of vertices that are contained in a hyperedge of \mathcal{H} that is either in \mathcal{H}' or has a vertex in $V(\mathcal{H}) \setminus V(\mathcal{H}')$ with degree at least 2. This finishes the proof of Theorem 6.

Finally, consider the hyperedges that contain the remaining $n - n_2$ vertices. As all these vertices are of degree 1, they are partitioned by these edges. For such a hyperedge h let D_h denote the subset of such vertices. Observe that for such a hyperedge h , we have that $h \setminus D_h \subseteq A$. Indeed if $v \in h \setminus (D_h \cup A)$, then there exists a cycle C of length $k-1$ in \mathcal{H}' not containing v . Thus there is a path of length at least k starting at an arbitrary $d \in D_h$, continuing with h, v , and having $k-1$ more vertices as it goes around C with defining hyperedges and vertices. This contradicts Claim 10 and finishes the proof of Theorem 7. \square

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References

- [1] P.N. Balister, E. Győri, J. Lehel, R.H. Schelp. Connected graphs without long paths. *Discrete Mathematics* **308**(19), (2008) 4487–4494.
- [2] A. Davoodi, E. Győri, A. Methuku, C. Tompkins. An Erdős–Gallai type theorem for uniform hypergraphs. *European Journal of Combinatorics* **69**, (2018) 159–162.
- [3] P. Erdős, T. Gallai. On maximal paths and circuits of graphs. *Acta Math. Acad. Sci. Hungar.* **10**, (1959) 337–356.
- [4] R.J. Faudree, R.H. Schelp. Path Ramsey numbers in multicolorings. *Journal of Combinatorial Theory, Series B*, **19**(2), (1975) 150–160.
- [5] Z. Füredi, A. Kostochka, R. Luo. Avoiding long Berge-cycles. *Journal of Combinatorial Theory, Series B*, **137**, (2019) 55–64.
- [6] Z. Füredi, A. Kostochka, R. Luo. On 2-connected hypergraphs with no long cycles. *Electronic Journal of Combinatorics*, **26**, (2019) P.4.31.
- [7] Z. Füredi, A. Kostochka, J. Verstraëte. Stability Erdős–Gallai Theorems on cycles and paths. *Journal of Combinatorial Theory, Series B*, **121**, (2016) 197–228.
- [8] Z. Füredi, M. Simonovits. The history of degenerate (bipartite) extremal graph problems. In *Erdős Centennial* (pp. 169–264). Springer, Berlin, Heidelberg. (2013)
- [9] E. Győri, G. Y. Katona, N. Lemons. Hypergraph extensions of the Erdős–Gallai Theorem. *European Journal of Combinatorics* **58**, (2016) 238–246.
- [10] E. Győri, N. Lemons, N. Salia, O. Zamora. The Structure of Hypergraphs without long Berge-cycles. *Journal of Combinatorial Theory, Series B*, to appear.
- [11] E. Győri, A. Methuku, N. Salia, C. Tompkins, M. Vizer. On the maximum size of connected hypergraphs without a path of given length. *Discrete Mathematics* **341**(9), (2018) 2602–2605.
- [12] E. Győri, N. Salia, O. Zamora. Connected Hypergraphs without long paths, *arXiv:1910.01322*.
- [13] G.N. Kopylov. On maximal paths and cycles in a graph. *Soviet Math.* (1977) 593–596.