Problems close to my heart

András Gyárfás*

August 11, 2020

Abstract

To celebrate the 40th anniversary of EJC, I share some problems close to my heart. They relate mainly to two areas I became acquainted with more than 50 years ago during my years as a student at the Eötvös University, Budapest. I was fascinated by Ramsey theory learned at the course of Vera T. Sós and the notion of perfect graphs and related structures learned from Erdős and Gallai. I was also impressed by the early developments of hypergraph theory led by Berge and his French school.

1 The world surrounding perfect graphs

Let $\chi(G)$ (resp. $\theta(G)$) denote the minimum number of independent sets (resp. cliques) needed to partition the vertices of G and let $\omega(G)$ (resp. $\alpha(G)$) denote the maximum size of a clique (resp. independent set) of G. A graph G is χ-*bounded* (resp. θ-*bounded*) by a function f if $\chi(H) \leq f(\omega(H))$ (resp. $\theta(H) \leq f(\alpha(H))$) for every induced subgraph H of G.

The above notions were inspired by the class of *perfect graphs*, the class of graphs χ-bounded by the identity function, $f(x) = x$. Lovász [27] proved that complements of perfect graphs are also perfect i.e. they are also θ-bounded by the identity function.

It was shown in [16] that the only graph family that is χ-bounded and also θ-bounded by the *same function* is the family of perfect graphs. However, as suggested in [16], graph families χ-bounded by a *small function* f might be θ-bounded as well by some other function, called a *complementary function*. The vague term "small" appears here because it is known that complementary functions do not exist if $f(x)$ is not close to the identity function. The best known example of a function without complementary function is $f(x) = x + \frac{x}{\log^j(x)}$, for arbitrary fixed j [22].

The smallest complementary function of f is denoted by f^*. In particular, I conjectured that the family of graphs χ-bounded by $f(x) = x + 1$ has complementary functions. This conjecture have been recently resolved by Scott and

*Alfréd Rényi Institute of Mathematics, Budapest, P.O. Box 127, Budapest, Hungary, H-1364. gyarfas.andras@renyi.mta.hu

Seymour [35] relying on their earlier result [34] that graphs without induced odd cycles of length at least five form a χ-bounded family (proving another conjecture from [16]). As the theory of χ-bounded classes has been developing rapidly, the reader should consult the survey of Scott and Seymour [33] which, among other works, summarizes results from their (presently) 13-part sequence of papers devoted to the subject. Nevertheless there are still mysteries in the world surrounding perfect graphs.

1.1 The closest relative of perfect graphs

In a sense the slightest extension of perfect graphs is the family of graphs χ-bounded by the function

$$f(x) = \begin{cases} 3 & \text{for } x = 2 \\ x & \text{for } x > 2. \end{cases}$$

As proved in [22], $f^*(2) = 3$ but for its next value only $4 \leq f^*(3) \leq 6$ is known, in general

$$\left\lfloor \frac{8}{5}x \right\rfloor \leq f^*(x) \leq \binom{x+1}{2}.$$

Problem 1.1. *([22]) Is $f^*(x)$ linear? Perhaps the lower bound gives the truth?*

1.2 Graphs χ-bounded by $x + 1$

A further extension (beyond the function in subsection 1.1) is provided by the family of graphs χ-bounded by the function $g(x) = x + 1$. Scott and Seymour [35] proved that $g^*(x)$ exists but even the value $g^*(2)$ is not immediate.

Proposition 1.2. $g^*(2) = 4$.

Proof. The smallest triangle-free 4-chromatic graph, the Grötzsch graph M_4 gives the lower bound for $g^*(2)$. Indeed, M_4 is obviously χ-bounded by the function $x+2$ and not χ-bounded by $x+1$. On the other hand, it is not difficult to check that M_4 is θ-bounded by $x + 1$. Then $g^*(2) \leq 4$ follows from the $k = 2$ case of the following deep result of Folkman [9] (conjectured by Erdős and Hajnal [8]).

Theorem 1.3. *Assume that G is a graph such that every induced subgraph H of G satisfies $\alpha(H) \geq \frac{|V(H)|-k}{2}$. Then $\chi(G) \leq k + 2$.*

Returning to the upper bound of $g^*(2)$, assume that G is θ-bounded by $x+1$ and an induced subgraphs H of G satisfies $\omega(H) = 2$. Then

$$\frac{|V(H)|}{2} \leq \theta(H) \leq \alpha(H) + 1$$

and, according to Theorem 1.3 with $k = 2$, this condition implies $\chi(H) \leq 4$. □

The linear lower bound $\lfloor \frac{8}{5}x \rfloor$ of $f^*(x)$ is valid for $g^*(x)$ as well but a slightly better one comes from the following graph G. Consider two vertex disjoint six-cycles with vertex sets $\{A_i : i \in [6]\}, \{B_i : i \in [6]\}$. Moreover, for $i = 1, 3, 5$ let B_i be adjacent with A_i, A_{i+1} and for $i = 2, 4, 6$ let B_i be adjacent with $A_{i+2}, A_{i+3}, A_{i+4}, A_{i+5}$. Defining G_k as k disjoint copies of G we have $\omega(G_k) = 3, \chi(G_k) = 4, \alpha(G_k) = 3k, \theta(G_k) = 5k$ and can be easily seen that G_k is χ-bounded by $g = x+1$. However, since $\frac{8}{5} < \frac{5}{3}$, $\lfloor \frac{8}{5}x \rfloor$ is not a θ-bounding function for G_k.

Problem 1.4. *Is $g^*(x)$ linear?*

1.3 Almost perfect graphs

Another possibility to extend the world of perfect graphs a little bit is relaxing Lovász's characterization [28] of perfectness: a graph G is perfect if and only if its induced subgraphs H satisfy the condition

$$\alpha(H)\omega(H) \geq |V(H)|. \tag{1}$$

Define *almost perfect* graphs as those graphs G whose induced subgraphs H satisfy

$$\alpha(H)\omega(H) + 1 \geq |V(H)|. \tag{2}$$

In Problem 6.8 [16] I asked whether almost perfect graphs are χ-bounded (equivalently θ-bounded). It follows from the result of Scott and Seymour [35] that they are. Perhaps even the following question has an affirmative answer.

Question 1.5. *Are almost perfect graphs χ-bounded by the function $g(x) = x + 1$?*

The answer to Question 1.5 is yes when $\omega(G) = 2$ (or $\alpha(G) = 2$). Indeed, then (2) translates to $\alpha(H) \geq \frac{|V(H)|-1}{2}$ and by Theorem 1.3 $\chi(H) \leq 3$.

I close this section with a related (more than 20 years old) question distilled from a problem I heard from Erdős in a summer afternoon.

Question 1.6. *([18]) Consider the family of graphs in which the vertex set of every path induces a 3-colorable subgraph. Is this family χ-bounded?*

As far as I know the best bound in Question 1.6 is logarithmic in $|V(G)|$ [31].

2 Variations on the Ramsey theme

The problem of finding the 2-color Ramsey number of the path occurred to me in 1966. At the Combinatorics course of Vera T. Sós I learned that graphs with many edges have a long path and 2-colored complete graphs have a large monochromatic clique. This suggested to find how long monochromatic paths exist in 2-colored complete graphs. I shared the question with Laci Gerencsér and it was amazing how our conversations converged to a proof [13]. Forty years went by, then Gábor Sárközy, Szemerédi's ex-student fortified my research group and eventually we could solve the 3-color case as well [23]. While the 2-color case has a "down to earth" proof by induction and valid for all n, the 3-color case works only for "rocket high" n because the Regularity lemma is used in the proof. Both methods, sometimes their combinations, give important tools in many Ramsey type problems.

2.1 Vertex coverings by monochromatic paths

A footnote in my first paper [13] states that the vertex set of any 2-colored complete graph can be partitioned by the vertices of two monochromatic paths. This generated a lot of different extensions and variations surveyed in [21]. When I mentioned this in 1995 to Paul Erdős he said he did not believe it. In fact, he thought I meant that the two paths must have the same color. Within two weeks we arrived to a partial answer to his new problem.

Theorem 2.1. *([7]) The vertex set of a* 2*-colored* K_n *can be always covered by the vertices of at most* $2\sqrt{n}$ *monochromatic paths of the same color.*

Problem 2.2. *([7]) Is it possible to cover the vertex set of any* 2*-colored* K_n *with at most* $\sqrt{n}$ *monochromatic paths of the same color? This would be best possible.*

In Theorem 2.1 and Problem 2.2 the covering paths are not required to be even edge-disjoint. Returning to the original problem, I have the following.

Conjecture 2.3. *([17]) The vertex set of an* r*-colored* K_n *can be always partitioned into* r *monochromatic paths.*

Conjecture 2.3 was proved for $r = 3$ by Pokrovskiy [29]. Interestingly, Conjecture 2.3 is true for infinite graphs (Rado [30]) and k-uniform hypergraphs (Elekes, Soukup,Soukup, Szentmiklóssy [5]) even for *tight paths* where all consecutive k-sets of the vertices on the path form the edges of the path. For general r the best result known [25] gives a vertex partition with no more than $cr\log(r)$ monochromatic paths.

2.2 Balanced colorings

With Paul Erdős we called an edge coloring of K_n with r colors *balanced* if every subset of $\lceil n/r \rceil$ vertices contains at least one edge in each color ([6]). One can easily see that K_5 is the smallest complete graph with a balanced 2-coloring. This seems exceptional, for $r = 3, 4$ the smallest complete graphs with balanced r-colorings are K_{13}, K_{21}. In general we found balanced r-colorings of K_n when $n = r^2 + r + 1$ and $r + 1$ is a prime power. These colorings are derived from finite planes of order $r + 1$ and they have the property that in each color i there is a partition of the vertex set into $r + 1$ complete graphs of color i (r of them is a

K_r and one is a K_{r+1}. This property ensures that the coloring is balanced. We thought that $r^2 + r + 1$ is the smallest n for which balanced r-colorings exist. Since $\lceil \frac{r^2+r+1-i}{r} \rceil = r + 1$ for $i = 1, \ldots, r$, the conjecture can be formulated as follows.

Conjecture 2.4. *([6]) In every r-coloring of the edges of K_{r^2+1} there exist $r+1$ vertices with at least one missing color among them ($r \geq 3$, true for $r = 3, 4$).*

2.3 The chromatic Ramsey number of acyclic hypergraphs

It is tempting to extend results for K_n to any n-chromatic graph. Sometimes this is successful, I realized that my tree-packing conjecture [19] is equivalent for K_n and for n-chromatic graphs ([20], see Section 3). A famous case when the extension badly fails is the Alon-Saks-Seymour conjecture refuted by Huang and Sudakov [26]. It does not work for Ramsey numbers in general either, but if the target graph (or hypergraph) is acyclic, it has a chance. In fact, I do not know any acyclic graph (or hypergraph) G for which the 2-coloring of the edges of any graph (or hypergraph) with chromatic number at least $R(G)$ does not contain a monochromatic copy of G (where $R(G)$ is the 2-color Ramsey number of G). Nevertheless it is hard to believe that equality always holds...

This problem was initiated with Bialostocki in [2] and continued by Garrison [11], Alon at al. [1]. As proved in [24], *there exists a bound*, $f(G)$, such that any 2-coloring of the edges of any graph (or hypergraph) of chromatic number at least $f(G)$ contains a monochromatic copy of G. (The chromatic number of a hypergraph is the minimum k such that the vertices can be colored with k colors so that each edge has at least two colors.) A warm up case, when G is a two-edge path, the Ramsey number comes from [4] for any number colors and this extends to the chromatic version as well [11]. However, for 3-uniform hypergraphs, only the following is known (not an easy result, its proof uses the degree choosability version of Brooks theorem).

Theorem 2.5. *([24]) In every k-coloring of the edges of any $(k+1)$-chromatic 3-uniform hypergraph there are two edges of the same color, intersecting in one vertex ($k \geq 2$).*

Question 2.6. *Theorem 2.5 is best possible for $k = 2, 3$. What about larger values of k?*

The next step, to find how large chromatic number ensures that in every 2-coloring of the edges there are two edges of the same color intersecting in one vertex, seems interesting. Let the 1-*intersection graph* of a hypergraph H be the graph whose vertices correspond to edges of H and two vertices are adjacent if and only if the corresponding edges of H intersect in exactly one vertex.

Question 2.7. *Let H be an r-uniform hypergraph such that its 1-intersection graph is bipartite. Is H 2-colorable? (Open already for $r = 4$.)*

2.4 Coloring blocks of Steiner triple systems - the sail mystery

Classical results of Graham, Leeb, Rothchild [14] imply that in any 2-coloring of the blocks of large enough projective STS's (points and lines of $\mathrm{PG}(n, 2)$) there is a monochromatic Fano plane. In a recent paper [15] we looked at a similar problem with different quantifiers. Instead of looking for a monochromatic copy of a configuration C in 2-colorings of the blocks of a special families of STS's, we called C 2-Ramsey if there is a monochromatic C *in every 2-colored STS*(n) with sufficiently large admissible n. It turned out that among the unavoidable configurations of at most four blocks there is only one for which we could not decide whether C is 2-Ramsey. This is the *sail*: three blocks through a point p and a fourth block that intersects each of them in points different from p.

Question 2.8. *Is the sail* 2*-Ramsey?*

It is worth noting that a positive answer cannot come from a density argument because there are sail-free partial STS(n)'s with much more than $\frac{n(n-1)}{12}$, the half of the total number of blocks. In fact, the maximum number of blocks in a sail-free partial STS(n) is asymptotic to $n^2/9$, [10].

3 The tree packing conjecture

During the problem section of the Hungarian combinatorial colloquium at Keszthely in 1976 Richard Guy kindly recommended a question from my talk to the audience: is it true that K_n has an edge-disjoint decomposition into $T_1, T_2, \dots, T_{n-1}$, where T_i is any tree with i edges. This became known as the "tree packing conjecture" [36].

We proved with Lehel [19] that the tree packing conjecture is true if all trees are stars or paths. However, the five page proof can be replaced by a "proof without words" discovered by Zaks and Liu [37]. I completely forgot another nice special case asked 44 years ago in [36], I think it worth restating.

Conjecture 3.1. *The tree packing conjecture is true if T_{n-1} is arbitrary and $T_1, T_2, \dots, T_{n-2}$ are paths.*

The tree packing conjecture is true if all but two trees are stars [19] and even if all but three are stars as proved by Roditty [32].

Gerbner, Keszegh and Palmer proposed [12] a natural extension of the tree packing conjecture: *is it true if K_n is replaced by any n-chromatic graph?* They proved [12] that it is true if all but three trees are stars. However, the case when all trees are stars and paths resisted, it seemed that neither proofs ([19], [37]) can be generalized to n-chromatic graphs. Then, to my great surprise, it turned out that an easy "black box" argument settles this generalization.

Theorem 3.2. *([20]) Assume that K_n has an edge disjoint decomposition into trees $T_1, \dots, T_{n-1}$. Then any n-chromatic graph contains edge disjoint copies of these trees.*

For many interesting developments related to the tree packing conjecture (packing small trees, packing large trees, packing bounded degree trees etc.) I recommend [3] and its references.

Acknowledgement. Careful reading and advices of the referees are appreciated.

References

[1] N. Alon, M. Amir, P. Haxell,D. Hefetz, Z. Jiang, G. Kronenberg, A. Naor, Ramsey-nice families of graphs, *Eur. J. of Combinatorics* **72** (2018) 29–44.

[2] A. Bialostocki, A. Gyárfás, Replacing the host K_n by n-chromatic graphs in Ramsey-type results, *arXiv:1506.04495* (2015).

[3] J. Böttcher, J. Hladky, D. Piguet, A. Taraz, An approximate version of the tree packing conjecture, *Israel Journal of Mathematics* **211** (2016) 391–446.

[4] S. A. Burr, J. A. Roberts, On Ramsey numbers for stars, Utilitas Math. **4** (1973) 217–220.

[5] M. Elekes, D. Soukup, L. Soukup, Z. Szentmiklóssy, Decomposition of edge-colored infinite complete graphs into monochromatic paths, Discrete Mathematics **340** (2017) 2053–2069.

[6] P. Erdős, A. Gyárfás, Split and balanced colorings of complete graphs, *Discrete Mathematics* **200.** (1999) 79–86.

[7] P. Erdős, A. Gyárfás, Vertex covering with monochromatic paths, *Mathematica Pannonica* **6** (1995), 7–10.

[8] P. Erdős, A. Hajnal, Problem 3, *Theory of Graphs. Proc. of the Colloquium at Tihany, Akadémiai Kiadó, P. Erdős and G. Katona editors.* (1968) 362.

[9] J. H. Folkman, An upper bound for the chromatic number of a graph, *Coll. Math. Soc. J. Bolyai 4, Combinatorial theory and its applications*, (1969) 437–457.

[10] Z. Füredi, A. Gyárfás, An extension of Mantel's theorem to k-graphs, *American Mathematical Monthly* **127** (2020) 263–268.

[11] P. Garrison, Good graph hunting, *arXiv:1508.01833* (2015).

[12] D. Gerbner, B. Keszegh, C. Palmer, Generalizations of the tree packing conjecture, *Dicussiones Math. Graph Theory* **32** (2012) 569–582.

[13] L. Gerencsér, A. Gyárfás, On Ramsey-type problems, *Ann. Univ. Eötvös, Sect. Math.* **10** (1967) 167–170.

[14] R. L. Graham, K. Leeb, B. L. Rothschild, Ramsey's theorem for a class of categories, *Adv. Math.* **8** (1972) 417–433.

[15] E. Granath, A. Gyárfás, J. Hardee, T. Watson, W. Xiaoze, Ramsey theory on Steiner systems, *Journal of Combinatorial Designs* **26** (2018) 5–11.

[16] A. Gyárfás, Problems from the world surrounding perfect graphs, *Zastowania Matematyki Applicationes Mathematicae* (1987) 413–441.

[17] A. Gyárfás, Covering complete graphs by monochromatic paths, *In: Irregularities of Partitions, Springer Verlag, Algorithms and Combinatorics,* **8** (1989) 89–91.

[18] A. Gyárfás, Fruit salad, Electronic J. of Combinatorics **4** (1997) R8.

[19] A. Gyárfás, J. Lehel, Packing trees of different order into K_n, *Coll. Math. Soc. J. Bolyai* **18** (1976) 463–469.

[20] A. Gyárfás, Packing trees in n-chromatic graphs, *Discussiones Mathematicae, Graph Theory* **34** (2014) 199–201.

[21] A. Gyárfás, Vertex covers by monochromatic pieces - a survey of results and problems, *Discrete Mathematics* **339** (2016) 1970–1977.

[22] A. Gyárfás, Z. Li, R. Machado, A. Sebő, S. Thomassé and N. Trotignon, Complements of nearly perfect graphs, *Journal of Combinatorics* **4** (2013) 299–310.

[23] A. Gyárfás, M. Ruszinkó, G. N. Sárközy, E. Szemerédi, Three-color Ramsey numbers for paths, *Combinatorica* **27** (2007) 35–69.

[24] A. Gyárfás, A. W. N. Riasanovsky, M. U. Sherman-Bennett, Chromatic Ramsey number of acyclic hypergraphs, *Discrete Mathematics* **340** (2017) 373–378.

[25] A. Gyárfás, M. Ruszinkó, G. N. Sárközy, E. Szemerédi, An improved bound for the monochromatic cycle partition number *Journal of Combinatorial Theory B* **96** (2006) 855–873.

[26] H. Huang, B. Sudakov, A counterexample to the Alon-Saks-Seymour conjecture and related problems, *Combinatorica* **32** (2012), 205–219.

[27] L. Lovász, Normal hypergraphs and the perfect graph conjecture, *Discrete Mathematics* **2** (1972) 253–267.

[28] L. Lovász, A characterization of perfect graphs, *Journal of Combinatorial Theory* **13** (1972) 95–98.

[29] A. Pokrovskiy, Partitioning edge-coloured complete graphs into monochromatic cycles and paths, *Journal of Combinatorial Theory B* **106** (2014) 70–97.

[30] R. Rado, Monochromatic paths in graphs, *Annals of Discrete Mathematics* **3** (1987) 89–91.

[31] B. Randerath, I. Schiermeyer, Chromatic number of graphs each path of which is 3-colorable, *Results in Mathematics, Birkhäuser Verlag, Basel* **41** (2002) 150–155.

[32] Y. Roditty, Packing and covering of the complete graph. III. On the tree packing conjecture. *Sci. Ser. A. Math. Sci. (N.S.)* **1** (1988), 81-85.

[33] A. Scott, P. Seymour, A survey of χ-boundedness, *ArXiv:1812.07500v2.*

[34] A. Scott, P. Seymour, Induced subgraphs of graphs with large chromatic number I. Odd holes, *Journal of Combinatorial Theory B* **121**, (2016) 68–84.

[35] A. Scott, P. Seymour, Induced subgraphs of graphs with large chromatic number VII. Gyárfás's complementation conjecture. *Journal of Combinatorial Theory B* **142** (2020) 43–55.

[36] A. Gyárfás, J. Lehel, Packing trees of different order into K_n, *Coll. Math. Soc. J. Bolyai* **18** (1976) 463–469.

[37] S. Zaks, C. Liu, Decompositions of graphs into trees, *Proceedings of the 8-th Southeastern Conference on Combinatorics, Graph Theory and Computing, Utilitas Math. Congr. Numerantium XIX* (1977) 643–654.