

ELUSIVE PROPERTIES OF INFINITE GRAPHS

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ABSTRACT. A graph property is said to be *elusive* (or *evasive*) if every algorithm testing this property by asking questions of the form "*is there an edge between vertices x and y* " requires, in the worst case, to ask about all pairs of vertices.

The unsettled Aanderaa–Karp–Rosenberg conjecture is that every non-trivial monotone graph property is elusive for finite vertex sets.

We show that the situation is completely different for infinite vertex sets: the monotone graph properties "*every vertex has degree at least n* " and "*every connected component has size at least m* ", where $n \geq 1$ and $m \geq 2$ are natural numbers, are not elusive for infinite vertex sets, but the monotone graph property "*the graph contains a cycle*" is elusive for arbitrary vertex set.

On the other hand, we also prove that every algorithm testing some natural monotone graph properties, e.g. "*every vertex has degree at least n* " or "*connected*" on the vertex set ω should check "lots of edges", more precisely, all the edges of an infinite complete subgraph.

1. INTRODUCTION

Given a graph property R and a vertex set V , let us consider the following game between two players, Alice, the seeker, and Bob, the hider. First the hider takes a graph G with vertex set V . Then, in each move of the game, the seeker asks the hider whether a certain edge e is in G or not. The game terminates when the seeker can decide whether G has property R .

Rosenberg [6] conjectured that for every non-trivial graph property R and for each finite vertex set V the seeker should ask $\Omega(|V|^2)$ many edges in the worst case. In [4] this conjecture was disproved by showing that the "*being a scorpion graph*" property can be decided by the seeker in $O(|V|)$ steps.

However, a weaker conjecture of Aanderaa and Rosenberg was proved by Rivest and Vuillemin in [5]: if R is a non-trivial *monotone* graph property, then for each finite vertex set V the seeker should ask $\Omega(|V|^2)$ many edges in the worst case; in this context, a property is monotone if it remains true when edges are added.

A graph property R is said to be *elusive on vertex set V* if the hider has a strategy such that the seeker needs query all the possible edges to decide whether G has property R or not.

A stronger, unsettled version of the Aanderaa–Rosenberg conjecture, called the evasiveness conjecture or the Aanderaa–Karp–Rosenberg conjecture, states that every non-trivial monotone graph property is elusive on each finite vertex set V .

Many partial results were proved in connection with this conjecture, see e.g. [1], [4]. Kahn, Saks and Sturtevant [3] proved it in the case when $|V|$ is a prime power, and Yao [7] proved an analogous conjecture for bipartite graphs.

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In this paper we show that the situation is quite different when the vertex set is infinite.

First, in section 3 we prove that the following natural monotone graph properties are not elusive for infinite vertex sets:

- (D_n) "the degree of each vertex is at least n ",
- (C_m) "the connected components have size at least m ",

for each natural number $n \geq 1$ and $m \geq 2$ (see Theorem 3.8).

On the other hand, we also prove that the monotone graph property "the graph contains a cycle" is elusive for arbitrary vertex sets (see Theorem 3.1).

At the moment we do not have any reasonable conjecture to characterize (large classes of) monotone graph properties which are elusive for infinite vertex sets.

In section 4 we formulate a theorem for infinite vertex sets which has the flavor of the result of Rivest and Vuillemin from [5]: although the properties D_n and C_m above are not elusive, but Alice, the seeker should check "lots of edges" before she can decide if a given graph has that property (see Theorem 4.12).

We also show that the "being a scorpion graph" property can be decided by the seeker asking "few" edges (see Theorem 4.7).

2. BASIC NOTIONS AND DEFINITIONS

First, we give a game-theoretic reformulation of the problem which also covers e.g. the bipartite case.

Based on the terminology of Bollobás [1], given a graph $H = \langle V, E^* \rangle$ we say that a triple $G = \langle V, E, N \rangle$ is an H -pregraphs iff $E \cup N \subset E^*$ and $E \cap N = \emptyset$. We will say that V is the set of *vertices*, E is the set of *edges*, and N is the set of *nonedges*, $P = E \cup N$ is the set of *determined pairs* of G , and $U = E^* \setminus P$ is the set of *undetermined pairs* of G . We will write $G = \langle V_G, E_G, N_G \rangle$, $P_G = P$, $U_G = U$, $\min G = \langle V_G, E_G \rangle$ and $\max G = \langle V_G, E_G \cup U_G \rangle$.

We say that a graph $G = \langle V, E \rangle$ is an H -graph iff $E \subset E^*$. If $G = \langle V, E \rangle$ is a graph, the set of *nonedges* of G is $N_G = [V]^2 \setminus E$.

If H is the complete graph on V , then we will write "pregraph on V " or simply "pregraph" instead of " H -pregraph".

If G_1 is a pregraph on V , and G_2 is either a graph or a pregraph on V , we say that G_2 *extends* G_1 , $G_1 \leq G_2$, provided $E_{G_2} \supseteq E_{G_1}$ and $N_{G_2} \supseteq N_{G_1}$.

Given a graph property R we say that an H -pregraph $G = \langle V, E, N \rangle$ has property R iff the graph $\min G = \langle V, E \rangle$ has property R .

Since we want to consider the elusiveness of properties of bipartite graphs as well, we should restrict the set of possible edges. That is the reason that we need an addition parameter E^* in the next definition: the set of allowed edges.

Definition 2.1. Let R be a graph property, V be the set of vertices, and $E^* \subset [V]^2$ be the set of *allowed edges*. Write $H = \langle V, E^* \rangle$.

Define the game $\mathbb{E}_{H,R}$ between two players, Alice and Bob, as follows:

- (1) During the game Alice and Bob construct a (transfinite) sequence $\langle G_\alpha : \alpha \leq \beta \rangle$ of H -pregraphs with $G_0 = \langle V, \emptyset, \emptyset \rangle$.
- (2) The game terminates when either every H -graph extending G_β has property R , or no H -graph extending G_β has property R .
- (3) If α is a limit ordinal, then let $E_{G_\alpha} = \bigcup_{\zeta < \alpha} E_{G_\zeta}$ and $N_{G_\alpha} = \bigcup_{\zeta < \alpha} N_{G_\zeta}$.
- (4) If $\alpha = \gamma + 1$ and the game has not terminated, then G_γ has undetermined pairs, and
 - (i) Alice picks an undetermined pair $e_\gamma \in U_{G_\gamma}$,
 - (ii) Bob decides if e_γ is an edge, or a nonedge in G_α , i.e. Bob selects an H -pregraph G_α such that $G_\alpha > G_\gamma$ and $P_{G_\alpha} = P_{G_\gamma} \cup \{e_\gamma\}$.

(5) Bob *wins* iff $P_\beta = E^*$, i.e. $\{e_\alpha : \alpha < \beta\} = E^*$ and so $U_\beta = \emptyset$.

If the property R is monotone, then (2) can be written in the following form:

(2') The game terminates in turn β if either $\min G_\beta = \langle V, E_\beta \rangle$ has property R or $\max G_\beta = \langle V, E_\beta \cup U_\beta \rangle$ does not have property R .

To simplify the notation, we will write E_α for E_{G_α} , N_α for N_{G_α} , P_α for P_{G_α} , and U_α for U_{G_α} .

If we do not have any restriction on edges, i.e. if $H = K_V = \langle V, [V]^2 \rangle$, then we write $\mathbb{E}_{V,R}$ for $\mathbb{E}_{K_V,R}$.

Definition 2.2. A graph property R is *elusive* in a graph H iff Alice does not have a winning strategy in the game $\mathbb{E}_{H,R}$. A graph property R is *elusive on a set V* iff Alice does not have a winning strategy in the game $\mathbb{E}_{V,R}$.

A graph property R is *strongly elusive* in a graph H iff Bob has a winning strategy in the game $\mathbb{E}_{H,R}$. A graph property R is *strongly elusive on a set V* iff Bob has a winning strategy in the game $\mathbb{E}_{V,R}$.

Definition 2.3. For natural numbers $n \geq 1$ and $m \geq 2$, let D_n and C_m denote the following graph properties:

- (D_n) "the degree of each vertex is at least n ",
- (C_m) "the connected components have size at least m ",

Our notation is standard, see e.g. [2]. Given sets X and Y , let $[X, Y] = \{\{a, b\} : a \in X, b \in Y, a \neq b\}$. If $G = \langle V, E \rangle$ is a graph and v is a vertex, $d_G(v)$ denotes the degree of the vertex v , $C_G(v)$ denotes the connected component of v , and $c_G(v) = |C_G(v)|$. If V is clear from the context, we will write $d_E(v)$ for $d_G(v)$, $C_E(v)$ for $C_G(v)$ and $c_E(v)$ for $c_G(v)$.

3. ELUSIVE AND NON-ELUSIVE PROPERTIES

First we show that some natural monotone graph properties are elusive.

Theorem 3.1. *The monotone graph property R*

"the graph contains a cycle "

is strongly elusive on any vertex set V with $|V| \geq 3$.

Instead of Theorem 3.1 we prove the following stronger result.

Theorem 3.2. *Let $H = \langle V, E^* \rangle$ be a graph and let R denote the monotone graph property*

"the graph contains a cycle "

Then following statements are equivalent:

- (1) *every connected component of H is 2-edge connected,*
- (2) *R is elusive in H , i.e. Alice does not have a winning strategy in the game $\mathbb{E}_{H,R}$*
- (3) *R is strongly elusive in H , i.e. Bob has a winning strategy in the game $\mathbb{E}_{H,R}$.*

Proof of Theorem 3.2. (2) implies (1).

We show that $\neg(1)$ implies $\neg(2)$. Assume that K is a connected component of H , but K is not 2-edges connected, i.e. there is an edge $e \in E^*$ such that $K \setminus \{e\}$ is not connected.

Then Alice has the following winning strategy in the game $\mathbb{E}_{H,R}$: Alice enumerates all the edges of H as $\{e_\alpha : \alpha \leq \beta\}$ with $e_\beta = e$, and she asks e_α in the α th step. Then $\min G_\beta = \langle V, E_\beta \rangle$ contains a cycle iff $\max G_\beta = \langle V, E_\beta \cup \{e\} \rangle$ contains a cycle because there is no cycle in H which contains e . So the game terminates after at most β steps and $U_\beta = \{e\} \neq \emptyset$. Thus, Alice wins.

(3) implies (2). Trivial.

(1) implies (3).

We can assume that H is connected because Bob can imagine that he plays a separate game in each connected component of H , and if he wins in each component, then he also wins in $\mathbb{E}_{H,R}$.

So from now on we assume that H is connected. Let $V = V(H)$. We show that the following greedy algorithm gives a winning strategy for Bob in $\mathbb{E}_{H,R}$.

Assume that in the α th turn of the game we have a pregraph $G_\alpha = \langle V_\alpha, E_\alpha, N_\alpha \rangle$ and Alice picked the pair $e_\alpha \in U_\alpha = E^* \setminus (E_\alpha \cup N_\alpha)$.

If $\langle V, E_\alpha \cup \{e_\alpha\} \rangle$ is cycle-free, then Bob declares that e_α is an edge, i.e. $E_{\alpha+1} = E_\alpha \cup \{e_\alpha\}$. Otherwise, e_α will be a nonedge.

Assume that the game terminates after β turns.

Lemma 3.3. *For each $\alpha \leq \beta$, the graph $\langle V, E_\alpha \rangle$ is cycle-free.*

Proof. Trivial by transfinite induction. \square

Lemma 3.4. *For each $\alpha \leq \beta$, if $\langle V, E_\alpha \rangle$ is connected and $U_\alpha \neq \emptyset$, then $\max G_\alpha = \langle V, E_\alpha \cup U_\alpha \rangle$ contains a cycle.*

Proof. Indeed, if $\langle V, E_\alpha \rangle$ is connected and $e \in [V]^2 \setminus E_\alpha$ then $\langle V, E_\alpha \cup \{e\} \rangle$ contains a cycle. \square

Lemma 3.5. *For each $\alpha \leq \beta$, if $\langle V, E_\alpha \rangle$ is not connected then $\max G_\alpha = \langle V, E_\alpha \cup U_\alpha \rangle$ contains a cycle.*

Proof. Let $\langle A, B \rangle$ be a partition of V such that $[A, B] \cap E_\alpha = \emptyset$. Then there is no nonedge between A and B .

Pick $a \in A$ and $b \in B$. Since H is 2-edge-connected, there are two edge disjoint paths between a and b , $e_0 \dots e_n$ and $f_0 \dots f_m$ in H .

If e_i is an edge inside a connected component of $\langle V, E_\alpha \rangle$, then replace e_i with a path in E_α between the endpoints of e_i .

Since there is no non-edge between distinct connected components of $\langle V, E_\alpha \rangle$, we obtain the path $\bar{e}' = e'_0 \dots e'_k$ between a and b in $\max G_\alpha = \langle V, E_\alpha \cup U_\alpha \rangle$.

Similarly, we obtain the path $\bar{f}' = f'_0 \dots f'_\ell$ between a and b in $\max G_\alpha$ from $f_0 \dots f_m$.

Then \bar{e}' contains at least one edge e' between A and B . Since e' is not in \bar{f}' , there is a path $g_0 \dots g_s$ between the endpoints of e' containing edges from $\bar{e} \cup \bar{f} \setminus \{e'\}$.

Thus, $e'g_0 \dots g_s$ is a cycle in $\max G_\alpha = \langle V, E_\alpha \cup U_\alpha \rangle$. \square

By lemma 3.3 $\min G_\beta = \langle V, E_\beta \rangle$ does not contain a cycle. Since the game has terminated, $\max G_\beta = \langle V, E_\beta \cup U_\beta \rangle$ does not contain a cycle either. Thus, by lemma 3.5 the graph G_β is connected and so by Lemma 3.4 $U_{G_\beta} = \emptyset$.

So Bob wins. \square

The monotone graph property "every vertex has infinite degree" is clearly not elusive on infinite vertex sets. However, next we show that even certain natural, not "tailor made", monotone graphs properties are also not elusive. First we should recall some definition.

Definition 3.6. For a cardinal κ and a natural number $n \geq 2$, the *Turán graph* $T_{\kappa,n}$ is defined as follows: $V(T_{\kappa,n}) = \kappa \times n$ and $E(T_{\kappa,n}) = \{\{\langle \alpha, i \rangle, \langle \beta, j \rangle\} : \alpha, \beta \in \kappa, i \neq j < n\}$.

Let us remark that $T_{\kappa,2}$ is just the complete balanced bipartite graph of cardinality κ .

Definition 3.7. The *Cantor graph* C is defined as follows: its vertex set is the set of all finite 0-1 sequences, and $\{s, t\}$ is an edge iff $s \subsetneq t$ or $t \subsetneq s$.

Theorem 3.8. *If*

- (i) P denotes either the graph property D_n for some $n \geq 1$ or the graph property C_m for some $m \geq 2$, and
- (ii) H is either the complete graph K_κ or the Turan graph $T_{\kappa,k}$ for some infinite cardinal κ and $k \geq 2$, or the Cantor graph,

then Alice has a winning strategy in the game $\mathbb{E}_{H,P}$, i.e.

the property P is not elusive in H .

The previous theorem actually consists of 6 statements, but as we will see, one can find a common generalization of all of them (see Theorem 3.12 below).

First we find some property shared by the complete graphs, the Turan graphs and the Cantor graph (see Definition 3.9 below), then we isolate some properties of C_m and D_n which makes possible to prove the failure of elusiveness.

Definition 3.9. Let $H = \langle V, E^* \rangle$ be a graph. A vertex set $L \subset V$ is a *covering* set iff for each $v \in V \setminus L$ there is $a \in L$ with $\{v, a\} \in E^*$.

We say that H is *braided* iff it contains “lots of” finite covering sets: for each $W \in [V]^{<|V|}$ there is a finite covering set $L \in [V \setminus W]^{<\omega}$.

Proposition 3.10. *Given an infinite cardinal κ , the infinite complete graph K_κ , the Turan graphs $T_{\kappa,k}$ for $2 \leq k < \omega$, and the Cantor graph C are braided.*

Proof. The finite set of 0-1 sequences of length k is a covering set in C for $k \in \omega$. So C is braided. The other statements from this proposition are trivial. \square

Definition 3.11. Assume that V is an infinite set, $H = \langle V, E^* \rangle$ is a graph, $n \in \omega$, and $w : V \times \mathcal{P}(E^*) \rightarrow n + 1$ is a function. We say that w is

- (w1) *non-trivial* iff $w(a, \emptyset) = 0$ and $w(a, E^*) = n$ for each $a \in V$;
- (w2) *degree restricted* iff there is $K \in \omega$ such that $d_E(a) \geq K$ implies $w(a, E) = n$ for each $E \subset E^*$ and $a \in V$;
- (w3) *finitely determined* iff $w(a, E) = \max\{w(a, E') : E' \in [E]^{<\omega}\}$ for each $E \subset E^*$ and $a \in V$;
- (w4) *bounded* iff there is $M \in \omega$ and there is a function $W : V \times \mathcal{P}(E^*) \rightarrow [V]^{<M}$ such that
 - (i) $a \in W(a, E)$, and $a \in W(b, E)$ implies $W(a, E) = W(b, E)$ and $w(a, E) = w(b, E)$ for each $a, b \in V$ and $E \subset E^*$,
 - (ii) if $w(a, E) < w(a, E')$ for some $E \subset E' \subset E^*$ and $a \in V$, then $(E' \setminus E) \cap [W(a, E), V] \neq \emptyset$.

Theorem 3.12. *If $H = \langle V, E^* \rangle$ is an infinite braided graph, $n \in \omega$, and $w : V \times \mathcal{P}(E^*) \rightarrow n + 1$ is a non-trivial, degree restricted, finitely determined and bounded function, then the monotone graph property R_n*

$$“w(a, E) = n \text{ for each } a \in V”$$

is not elusive in H .

Proof of Theorem 3.8 from Theorem 3.12. If $P = D_m$ for some $m \geq 1$, define the function $w : V \times \mathcal{P}(E^*) \rightarrow m + 1$ as follows:

$$w(a, E) = \min(d_E(a), m).$$

Taking $K = m$, $M = 1$ and $W(a, E) = \{a\}$ we obtain that w is non-trivial, degree-restricted, finitely determined, and bounded.

By Proposition 3.10 the graph H is braided. Thus, property (P)

$$“w(a, E) = m \text{ for each } a \in V”$$

is not elusive in H by Theorem 3.12. But P holds iff D_m holds. Thus, we proved the theorem.

If $P = C_n$ for some $n \geq 2$, define the function $w : V \times \mathcal{P}(E^*) \rightarrow n$ as follows:

$$w(a, E) = \min(c_E(a) - 1, n - 1).$$

Taking $K = M = n - 1$ and

$$W(a, E) = \begin{cases} C_E(a) & \text{if } c_E(a) < n, \\ \{a\} & \text{if } c_E(a) \geq n, \end{cases}$$

we obtain that w is non-trivial, degree-restricted, finitely determined, and bounded.

By Proposition 3.10, the graph H is braided. Thus, the property (P)

$$“w(a, E) = n - 1 \text{ for each } a \in V”$$

is not elusive in H by Theorem 3.12. But P holds iff C_n holds. So we proved Theorem 3.8. \square

Before proving Theorem 3.12 we need some preparation.

Proposition 3.13. *Assume that V is an infinite set, $H = \langle V, E^* \rangle$ is a graph, $n \in \omega$, and $w : V \times \mathcal{P}(E^*) \rightarrow n + 1$ is non-trivial, finitely determined and bounded function. Then w is*

- (w5) monotone, i.e. $E_0 \subset E_1$ implies $w(a, E_0) \leq w(a, E_1)$ for $E_0 \subset E_1 \subset E^*$ and $a \in V$;
- (w6) continuous, i.e. if $\langle E_\alpha : \alpha < \mu \rangle \subset \mathcal{P}(E^*)$ is a \subset -increasing sequence, then for each $a \in V$, we have

$$w(a, \bigcup_{\alpha < \mu} E_\alpha) = \max_{\alpha < \mu} w(a, E_\alpha).$$

- (w7) stable i.e. if $E \subset E^*$ and $w(a, E) = w(b, E) = n$ for some $e = \{a, b\} \in E^*$, then $w(c, E \cup \{e\}) = w(c, E)$ for each $c \in V$.

Proof. If $E_0 \subset E_1$, then for each $a \in V$,

$$\begin{aligned} w(a, E_0) &= \max\{w(a, E) : E \in [E_0]^{<\omega}\} \leq \\ &\quad \max\{w(a, E) : E \in [E_1]^{<\omega}\} = w(a, E_1) \end{aligned}$$

by (w3). So (w5) holds.

If $\langle E_\alpha : \alpha < \mu \rangle \subset \mathcal{P}(E^*)$ is a \subset -increasing sequence, then for each $a \in V$,

$$\begin{aligned} w(a, \bigcup_{\alpha < \mu} E_\alpha) &= \max\{w(a, E') : E' \in [\bigcup_{\alpha < \mu} E_\alpha]^{<\omega}\} = \\ &\quad \max\{\max\{w(a, E') : E' \in [E_\alpha]^{<\omega}\} : \alpha < \mu\} = \max_{\alpha < \mu} w(a, E_\alpha), \end{aligned}$$

where the first and the third equality hold by (w3), and the second holds because the sequence $\langle E_\alpha : \alpha < \mu \rangle$ is increasing. Thus, (w6) holds.

To prove (w7) assume on the contrary that $E \subset E^*$, $w(a, E) = w(b, E) = n$ for some $e = \{a, b\} \in E^*$, and $w(c, E \cup \{e\}) > w(c, E)$. By (w4)(ii), $a \in W(c, E)$ or $b \in W(c, E)$. Thus, $w(c, E) = w(a, E)$ or $w(c, E) = w(b, E)$ by (w4)(i), i.e. $w(c, E) = n$. But $w(c, E \cup \{e\}) \leq n$. So $w(c, E \cup \{e\}) > w(c, E)$ is not possible, (w7) holds. \square

Proof of Theorem 3.12. Write $\kappa = |V|$. Fix $M, K \in \omega$ and $W : V \times \mathcal{P}(E^*) \rightarrow [V]^{\leq M}$ as in Definition 3.11.

We will give a winning strategy for Alice in the game \mathbb{E}_{H, R_n} . We divide the game into stages.

First Alice picks a finite covering set $L \subset V$.

Stage 1. This stage is divided into n substages.

Before the i^{th} substage, the players played m_i turns and determined a pregraph G_{m_i} , such that $m_i < \kappa$ and $w(\ell, E_{m_i}) \geq i$ for each $\ell \in L$.

Observe that the choice $m_0 = 0$ works because $w(\ell, \emptyset) = 0 \geq 0$ holds for each $\ell \in L$.

Substage i .

Alice enumerates the edges of H which contains at least one endpoint from the finite set $\bigcup\{W(\ell, E_{m_i}) : \ell \in L\}$ as $\{e'_\alpha : \alpha < \kappa\}$ and in the $m_i + \alpha$ th turn Alice plans to play the undetermined pair e'_α .

If for each $\alpha < \kappa$ we have $w(\ell, G_{m_i+\alpha}) \leq i$ for some $\ell \in L$, then $w(\ell, E_{m_i+\kappa}) \leq i$ for some $\ell \in L$ because w is monotone and continuous.

Since $[W(\ell, E_{m_i+\kappa}), V] \subset P_{m_i+\kappa}$, we have $[W(\ell, E_{m_i+\kappa}), V] \cap U_{m_i+\kappa} = \emptyset$, and so

$$w(\ell, E_{m_i+\kappa} \cup U_{m_i+\kappa}) = w(\ell, E_{m_i+\kappa}) \leq i < n$$

by (w4)(ii). Since $\max G_{m_i+\kappa} = \langle V, E_{m_i+\kappa} \cup U_{m_i+\kappa} \rangle$, it follows that $\max G_{m_i+\kappa}$ does not have property R_n . Thus, the game terminates in at most $m_i + \kappa$ steps, and so Alice wins because $G_{m_i+\kappa}$ has undetermined edges.

So we can assume that for some $\alpha < \kappa$ we have $w(\ell, E_{m_i+\alpha}) \geq i + 1$ for each $\ell \in L$, and after that step we declare that the i^{th} substage is terminated, and $m_{i+1} = m_i + \alpha$.

At the end of Stage 1, when we are after n many substages, write $\sigma = m_n$ and observe that we have a pregraph G_σ with $|P_{G_\sigma}| < \kappa$ such that $w(\ell, E_\sigma) \geq n$ for each $\ell \in L$.

For $\alpha < \sigma$ let e_α denote the pair Alice selected in the α th turn.

Stage 2. Let $N = M \cdot (K + |L|) + 1$. Alice picks $N + 1$ pairwise disjoint finite covering sets $\mathcal{L} = \{L_0, \dots, L_N\}$ from $\kappa \setminus \bigcup\{e_\alpha : \alpha < \sigma\}$. Let

$$X = \bigcup \mathcal{L} \text{ and } A = \kappa \setminus X.$$

Next Alice asks all the undetermined pairs from $E^* \cap [A]^2$ in κ turns.

Since $\sigma + \kappa = \kappa$ and w is monotone, so

$$P_\kappa = E^* \cap [A]^2 \text{ and } w(v, E_\kappa) = n \text{ for each } v \in L \subset A.$$

Let

$$B = \{\beta \in A : w(\beta, E_\kappa) = n\} \text{ and } C = \{\gamma \in A : w(\gamma, E_\kappa) < n\}.$$

We have $L \subset B$, and so $B \neq \emptyset$.

Stage 3.

Alice should distinguish two cases.

Case 3.1. $|B| = \kappa$.

Since L_N is a covering set and $|B| = \kappa$, Alice can pick $x \in L_N$ such that $|[\{x\}, B] \cap E^*| = \kappa$. Let $Y = X \setminus \{x\}$.

Substage 3.1.1

First Alice asks all the undetermined pairs from $[Y]^2$ in finitely many steps, then the undetermined pairs $[C, Y]$ in at most κ turns. (C can be the empty set).

So for some $\rho \leq \kappa$

$$P_{\kappa+\rho} = E^* \cap ([A]^2 \cup [C \cup Y]^2) = E^* \cap ([\kappa \setminus \{x\}]^2 \setminus [B, Y]).$$

Claim 3.13.1. If $w(y, E_{\kappa+\rho}) = n$ for some $y \in Y$, then Alice can win.

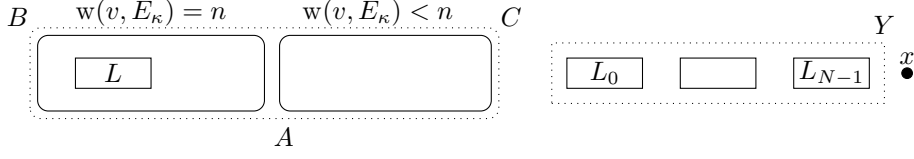


FIGURE 1.

Proof. Since L is a covering set, Alice can pick $\ell \in L$ with $\{y, \ell\} \in E^*$.

Alice enumerates the undetermined pairs of $G_{\kappa+\rho}$ with the exception of $\{y, \ell\}$ as $\{e'_\gamma : \gamma < \kappa\}$, and she asks the pair e'_γ in the turn $\kappa + \rho + \gamma$.

Writing $\delta = \kappa + \rho + \kappa$ we have

$$U_\delta = \{\{\ell, y\}\} \text{ and } w(y, E_\delta) = w(\ell, E_\delta) = n.$$

So $\min G_\delta = \langle V, E_\delta \rangle$ has property R_n iff $\max G_\delta = \langle V, E_\delta \cup \{\{\ell, y\}\} \rangle$ has property R_n by (w7). Thus, the game finishes after at most δ turns and $U_\delta \neq \emptyset$. So Alice wins. \square

So we can assume that

$$w(y, E_{\kappa+\rho}) < n \text{ for each } y \in Y.$$

Substage 3.1.2

Alice enumerates the pairs $[B, Y]$ in type κ as $\{e'_\zeta : \zeta < \kappa\}$, and in the turn $\kappa + \rho + \zeta$ she plans to play e'_ζ .

For each $i = 0, \dots, N-1$ pick $\ell_i \in L_i$ such that $|\{\{\ell_i\}, B\} \cap E^*| = \kappa$.

Claim 3.13.2. *If for some $\kappa + \rho \leq \eta < \kappa + \rho + \kappa$ we have that $w(\ell_i, E_\eta) = n$ for some $0 \leq i < N$, then Alice can win.*

Proof. After realizing that $w(\ell_i, E_\eta) = n$ after the turn η , Alice changes her plan how to play after that turn. Since $|\{\{\ell_i\}, B\} \cap E^*| = \kappa$, we can pick a pair $e = \{\ell_i, b\} \in [Y, B]$ such that e is undetermined in G_η . In the next κ turns Alice asks all the undetermined pairs of G_η with the exception of e . Then

$$U_{\eta+\kappa} = \{e\} \text{ and } w(\ell_i, G_{\eta+\kappa}) = n \text{ and } w(b, G_{\eta+\kappa}) = n.$$

So $\min G_{\eta+\kappa} = \langle V, E_{\eta+\kappa} \rangle$ has property R_n iff $\max G_{\eta+\kappa} = \langle V, E_{\eta+\kappa} \cup \{e\} \rangle$ has property R_n by (w7). Thus, the game finishes after at most $\eta + \kappa$ turns and $U_{\eta+\kappa} \neq \emptyset$. Thus, Alice wins. \square

So we can assume that Bob answered in such a way that after $\eta = \kappa + \rho + \kappa$ turns

$$w(\ell_i, E_\eta) < n \text{ for each } i < N,$$

and

$$P_{G_\eta} = E^* \cap [A]^2 \cup [Y]^2 \cup [A, Y] = E^* \cap [A \cup Y]^2.$$

Let

$$Z = \{\ell_i : i < N\}.$$

Case 3.2. $|B| < \kappa$.

Then $|C| = \kappa$. Pick an arbitrary $x \in L_N$. Let $Y = X \setminus \{x\}$.

Substage 3.2.1

First Alice enumerates all the pairs from $[Y]^2$ and the undetermined pairs in $[C, Y]$ as $\{e'_\gamma : \gamma < \kappa\}$, and she asks the pair e'_γ in the turn $\kappa + \gamma$. Consider the set

$$C' = \{y \in C : w(y, E_{\kappa+\kappa}) < n\}.$$

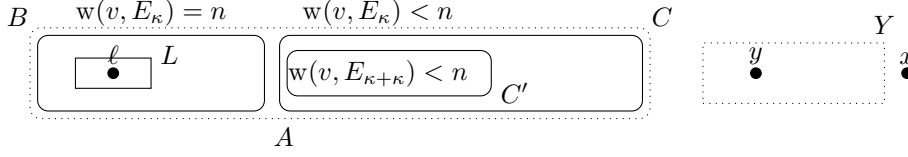


FIGURE 2.

Claim 3.13.3. *If $|C'| < \kappa$, then Alice can win.*

Proof. If $|C'| < \kappa$, then for each $c \in C \setminus C'$ we have $w(c, E_{\kappa+\kappa}) > w(c, E_{\kappa})$, so by (w4)(ii) we can find an edge $\{y_c, a_c\} \in E_{\kappa+\kappa} \setminus E_{\kappa}$ such that $a_c \in W(c, E_{\kappa})$.

Since $|C \setminus C'| = \kappa$, by (w4)(i) we can find $C'' \in [C \setminus C']^{\kappa}$ such that $\{W(c, E_{\kappa}) : c \in C''\}$ are pairwise disjoint and $Y \cap \bigcup \{W(c, E_{\kappa}) : c \in C''\} = \emptyset$.

Then $y_c \in Y$ for $c \in C''$. Since Y is finite, there is $y \in Y$ such that $|\{c \in C'' : y_c = y\}| = \kappa$ and so $d_{E_{\kappa+\kappa}}(y) = \kappa$. Thus, $w(y, E_{\kappa+\kappa}) = n$ by (w2).

Pick $\ell \in L$ such that $\{\ell, y\} \in E^*$.

Next Alice enumerates all undetermined pairs of $G_{\kappa+\kappa}$ with the exception of $\{\ell, y\}$ as $\{e_{\gamma} : \gamma < \kappa\}$, and she asks e_{γ} in the turn $\kappa + \kappa + \gamma$.

Let $\eta = \kappa + \kappa + \kappa$.

Then

$$U_{\eta} = \{\{\ell, y\}\} \text{ and } w(y, E_{\eta}) = n \text{ and } w(\ell, E_{\eta}) = n.$$

So $\min G_{\eta} = \langle V, E_{\eta} \rangle$ has property C_n iff $\max G_{\eta} = \langle V, E_{\eta} \cup \{e_{\gamma}\} \rangle$ has property R_n by (w7). Thus, the game terminates after at most η turns and $U_{\eta} \neq \emptyset$. Thus, Alice wins. \square

So we can assume that $|C'| = \kappa$.

Substage 3.2.2

Alice asks all the undetermined pairs from $[B, Y]$ in the next $\sigma \leq \kappa$ turns.

If $w(c, E_{\kappa+\kappa+\sigma}) > w(c, E_{\kappa+\kappa})$ for some $c \in C'$ then $W(c, E_{\kappa+\kappa}) \cap B = \emptyset$ by (w4)(i) and so $W(c, E_{\kappa+\kappa}) \cap Y \neq \emptyset$ by (w4)(ii) and so $c \in W(y, E_{\kappa+\kappa})$ for some $y \in Y$ by (w4)(i). So

$$|\{c \in C' : w(c, E_{\kappa+\kappa+\sigma}) > w(c, E_{\kappa+\kappa})\}| \leq |Y| \cdot M < \omega.$$

i.e. writing $\eta = \kappa + \kappa + \sigma$ we have that there is an (infinite) $C'' \subset C'$ with $|C' \setminus C''| \leq |Y| \cdot M$ such that

$$\forall v \in C'' \quad w(v, E_{\eta}) = w(v, E_{\kappa+\kappa}) < n.$$

Fix $Z \in [C'']^N$.

Stage 4.

Both in Case 3.1 and in Case 3.2 after η turns we have

$$P_{\eta} = E^* \cap [\kappa \setminus \{x\}]^2$$

and there is a set $Z \in [\kappa \setminus \{x\}]^N$ such that $w(z, E_{\eta}) < n$ for each $z \in Z$.

Since $|Z| > M(K + |L|)$, applying the fact that for each $\{z, z'\} \in [V]^2$ either $W(z, E_{\eta}) = W(z', E_{\eta})$ or $W(z, E_{\eta}) \cap W(z', E_{\eta}) = \emptyset$ by (w4)(i), we can pick elements $\{z_i : i \leq K + |L|\}$ from Z such that $\{W(z_i, E_{\eta}) : i \leq K + |L|\}$ are pairwise disjoint. Write $Z_i = W_{E_{\eta}}(z_i)$. We can assume that

$$(L \cup \{x\}) \cap \bigcup_{i < K} Z_i = \emptyset. \quad (x)$$

Pick $\ell \in L$ such that $\{x, \ell\} \in E^*$.

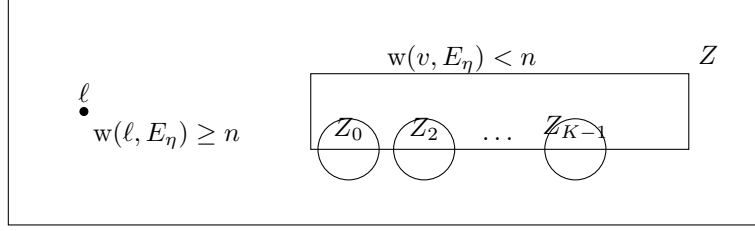


FIGURE 3.

Alice enumerates the undetermined pairs of G_η with the exception of $\{x, \ell\}$ as $\{e_\gamma : \gamma < \delta\}$, and she asks e_γ in the turn $\eta + \gamma$.

If $[Z_i, \{x\}] \cap E_{\eta+\delta} = \emptyset$, then $[Z_i, \{x\}] \cap (E_{\eta+\delta} \cup \{\{x, \ell\}\}) = \emptyset$ as well by (x), and so

$$w(z, E_{\eta+\delta} \cup \{\{x, \ell\}\}) = w(z, E_{\eta+\delta}) = w(z, E_\eta) < n$$

for each $z \in Z_i$ by (w4)(ii). Since

$U_{\eta+\delta} = \{\{x, \ell\}\}$ and $\max \langle V, E_{\eta+\delta} \rangle = \langle V, E_{\eta+\delta} \cup \{x, \ell\} \rangle$ does not have property R_n ,

so the game terminates at least $\eta + \delta$ steps and $U_{\eta+\delta} \neq \emptyset$, so Alice wins.

So we can assume that $[Z_i, \{x\}] \cap E_{\eta+\delta} \neq \emptyset$ for each $i < K$. So $d_{G_{\eta+\delta}}(x) \geq K$, and so $w(x, E_{\eta+\delta}) = n$ by (w2). Since $U_{\eta+\delta} = \{\{\ell, x\}\}$ and $w(\ell, E_{\eta+\delta}) = n$,

$\min G_{\eta+\delta} = \langle \kappa, E_{\eta+\delta} \rangle$ has property R_n iff

$$\max G_{\eta+\delta} = \langle \kappa, E_{\eta+\delta} \cup \{\{\ell, x\}\} \rangle \text{ has property } R_n$$

by (w7). Thus, the game terminates after at most $\eta + \delta$ turns and $U_{\eta+\delta} \neq \emptyset$. Thus, Alice wins. This completes the proof of Theorem 3.12. \square

Problems: (1) Are the following properties (strongly) elusive for infinite vertex sets:

- (i) “ G contains P_3 ”,
- (ii) “ G contains K_n ” for some $3 \leq n < \omega$,
- (iii) “ G is not bipartite”,
- (iv) “ G is connected.”

(2) Is it true that a property is elusive iff it is strongly elusive? Is it true under the Axiom of Determinacy?

(3) Find a reasonable conjecture to characterize (large classes of) monotone graph properties which are elusive for infinite vertex sets.

4. ON THE INFINITE VERSION OF THE AANDERAA-ROSENBERG CONJECTURE

We have seen in the previous section that Alice does not have to ask all the pairs in an infinite vertex set to decide if the hidden graph G has property D_n or C_m .

In contrast to these results, in this section we show that Alice, the seeker should check “lots of edges” before she can decide whether a graph has certain monotone graph properties.

Definition 4.1. Let R be a graph property, $H = \langle V, E^* \rangle$ be a graph, and $\mathcal{F} \subset \mathcal{P}(E^*)$ be upward closed. We will say that \mathcal{F} is the *family of large edge sets*.

Define the game $\mathbb{E}_{H,R,\mathcal{F}}$ between two players, Alice and Bob, as follows:

The gameplay is the same as in the game $\mathbb{E}_{H,R}$, but we modify the rule which determines when Bob wins: so we keep rules (1)-(4) from Definition 2.1, but we replace (5) with

(5') Bob wins iff $P_\beta \in \mathcal{F}$. (Informally, Bob wins if he can force Alice to ask a large set of edges.)

If we do not have any restriction on edges, i.e. if $H = K_V = \langle V, [V]^2 \rangle$, then we write $\mathbb{E}_{V,R,\mathcal{F}}$ for $\mathbb{E}_{K_V,R,\mathcal{F}}$.

Observe that $\mathbb{E}_{V,R} = \mathbb{E}_{V,R,\{[V]^2\}}$.

Definition 4.2. Given a vertex set V and a family $\mathcal{F} \subset \mathcal{P}([V]^2)$, a graph property R is said to be \mathcal{F} -hard on V iff Alice does not have a winning strategy in the game $\mathbb{E}_{V,R,\mathcal{F}}$.

As we recalled earlier, Rivest and Vuillemin [5] proved that if R is a non-trivial monotone graph property, then for each finite vertex set V the seeker should ask $\Omega(|V|^2)$ many edges in the worst case. We think that this result yields naturally the following problem for infinite sets:

Definition 4.3. Let

$$\mathcal{F}_{[\omega]^2} = \{F \subset [\omega]^2 : \exists W \in [\omega]^\omega [W]^2 \subset F\}.$$

Problem 4.4. Assume that P is a non-trivial, monotone graph property. Is it true that P is $\mathcal{F}_{[\omega]^2}$ -hard on ω ?

Definition 4.5. A *scorpion graph* on a vertex set V is a graph $G = \langle V, E \rangle$ such that it contains three special vertices, the *sting* denoted by s_G , the *tail* denoted by t_G , and the *body* denoted by b_G such that

- (a) the sting is connected only to the tail,
- (b) the tail is connected only to the sting and to the body, and
- (c) the body is connected to all vertices except the sting.

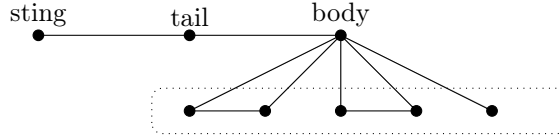


FIGURE 4. Scorpion graph

In [4] it was proved that that if V is a finite set, then the "being a scorpion graph on V " property can be decided by the seeker in $O(|V|)$ steps.

Definition 4.6. If $E \subset [\omega]^2$, let

$$\Omega(E) = \{v \in \omega : \deg_E(v) = \omega\},$$

and for $n \in \omega$ let

$$\mathcal{J}_n = \{E \subset [\omega]^2 : |\Omega(E)| \geq n\}.$$

Theorem 4.7. The property (S)

"being a scorpion graph"

is not \mathcal{J}_5 -hard on the vertex set ω .

Before proving the theorem we need some preparation.

Definition 4.8. If $G = \langle \omega, E, N \rangle$ is a pregraph, let

$$B_G = \{v \in \omega : \deg_N(v) \leq 1\},$$

$$S_G = \{v \in \omega : \deg_E(v) \leq 1\},$$

$$T_G = \{v \in \omega : \deg_E(v) \leq 2\}.$$

The following two lemmas are straightforward from the definition.

Lemma 4.9. *If $G = \langle \omega, E, N \rangle$ is a pregraph, and $G \leq H$ is a scorpion graph, then $b_H \in B_G$, $s_H \in S_G$ and $t_H \in T_G$.*

Lemma 4.10. *If $G = \langle \omega, E, N \rangle$ is a pregraph, $s', t', b' \in \omega$, $[\{s', t', b'\}, \omega] \subset E \cup N$, then the following statements are equivalent:*

- (1) $G' = \langle \omega, E \rangle$ is a scorpion graph with $s_{G'} = s'$, $t_{G'} = t'$ and $b_{G'} = b'$,
- (2) some extension H of G is a scorpion graph with $s_H = s'$, $t_H = t'$ and $b_H = b'$,
- (3) every extension H of G is a scorpion graph with $s_H = s'$, $t_H = t'$ and $b_H = b'$.

Proof of theorem 4.7. We will use the following notation. If the players construct the sequence $\langle G_\alpha : \alpha \leq \beta \rangle$ of pregraphs, then we write $G_\alpha = \langle \omega, E_\alpha, N_\alpha \rangle$, $P_\alpha = P_{G_\alpha} = E_\alpha \cup N_\alpha$, $U_\alpha = U_{G_\alpha} = [\omega]^2 \setminus P_\alpha$, $S_\alpha = S_{G_\alpha}$, $B_\alpha = B_{G_\alpha}$ and $T_\alpha = T_{G_\alpha}$.

Let

$$\mathbb{S} = \{G : G \text{ is a scorpion graph on } \omega\}.$$

Before describing a winning strategy of Alice in the game $\mathbb{E}_{S, \omega, \mathcal{J}_5}$, we need to prove the following lemma.

Lemma 4.11. *Assume that Alice and Bob played so far α turn in the game $\mathbb{E}_{S, \omega, \mathcal{J}_5}$, and they constructed the pregraph G_α .*

If either

- (a) *there is $s' \in \omega$ such that $s_G = s'$ for each $G \in \mathbb{S}$ with $G_\alpha \leq G$,*

or

- (b) *there is $b' \in \omega$ such that $b_G = b'$ for each $G \in \mathbb{S}$ with $G_\alpha \leq G$,*

then Alice can play in such a way that the game terminates in $\beta \leq \alpha + \omega + \omega + \omega$ turns and $|\Omega(P_\beta) \setminus \Omega(P_\alpha)| \leq 3$.

Proof of Lemma 4.11. Assume first that (a) holds: $s' \in \omega$ such that $s_G = s'$ for each $G \in \mathbb{S}$ with $G_\alpha \leq G$.

The strategy of Alice will be divided into three stages.

Stage A.

Alice enumerates the undetermined pairs $U_\alpha \cap [\{s'\}, \omega]$ as $\{e'_\ell : \ell < M_s\}$ for some $M_s \leq \omega$, and Alice plays e'_ℓ in step $\alpha + \ell$ for $\ell < M_s$.

Write $\sigma = \alpha + M_s$. Clearly $\Omega(P_\sigma) \setminus \Omega(P_\alpha) \subset \{s'\}$. Let

$$T = \{t \in \omega : \{t, s'\} \in E_\sigma\}.$$

If $G \geq G_\sigma$ is a scorpion graph, then $s_G = s'$, and so $T = \{t_G\}$.

Thus, if $|T| \neq 1$, then there is no scorpion graph G which extends G_σ . So the game terminates after at most σ turns and $\Omega(P_\sigma) \setminus \Omega(P_\alpha) \subset \{s'\}$.

So we can assume that $T = \{t'\}$, and we know that

$$\text{if } G \geq G_\sigma \text{ is a scorpion graph, then } s_G = s' \text{ and } t_G = t'. \quad (\text{B})$$

Stage B.

Alice enumerates the undetermined pairs $U_\sigma \cap [\{t'\}, \omega]$ as $\{e''_\ell : \ell < M_t\}$ for some $M_t \leq \omega$, and Alice plays e''_ℓ in step $\sigma + \ell$ for $\ell < M_t$.

Write $\rho = \sigma + M_t$. Clearly $\Omega(P_\rho) \setminus \Omega(P_\alpha) \subset \{s', t'\}$.

Let

$$B = \{b \in \omega : \{b, t'\} \in E_\rho, b \neq s'\}.$$

If $G \geq G_\rho$ is a scorpion graph, then $s_G = s'$ and $t_G = t'$, and so $B = \{b_G\}$.

Thus, if $|B| \neq 1$, then there is no scorpion graph G which extends G_ρ . So the game terminates after at most ρ turns and $\Omega(P_\rho) \setminus \Omega(P_\alpha) \subset \{s', t'\}$.

So we can assume that $B = \{b'\}$, and we know that

$$\text{if } G \geq G_\rho \text{ is a scorpion graph, then } s_G = s', t_G = t' \text{ and } b_G = b'. \quad (\text{C})$$

Stage C.

Alice enumerates the undetermined pairs $U_\rho \cap [\{b'\}, \omega]$ as $\{e_\ell^* : \ell < M_b\}$ for some $M_b \leq \omega$, and Alice plays e_ℓ^* in step $\rho + \ell$ for $\ell < M_b$.

Write $\nu = \rho + M_b$. Clearly $\Omega(P_\nu) \setminus \Omega(P_\alpha) \subset \{s', t', b'\}$.

Since (C) holds and $[\{s', t', b'\}, \omega] \subset P_\nu$, by Lemma 4.10 either every extension of G_ν is a scorpion graph, or no extension of G_ν is a scorpion graph.

So the game terminates after at most ν turns, $\nu \leq \alpha + \omega + \omega + \omega$ and $\Omega(P_\nu) \setminus \Omega(P_\alpha) \subset \{s', t', b'\}$.

Assume now that (b) holds: $b' \in \omega$ such that $b_G = b'$ for each $G \in \mathbb{S}$ with $G_\alpha \leq G$.

The strategy of Alice will be divided into stages.

Stage A. Enumerate $U_\alpha \cap [\{b'\}, \omega]$ as $\{e_i' : i < M_b\}$ for some $M_b \leq \omega$ and in the next M_b steps Alice asks $\{e_i' : i < M_b\}$.

Write $\sigma = \alpha + M_b$. Clearly $\Omega(P_\sigma) \setminus \Omega(P_\alpha) \subset \{b'\}$. Let

$$S = \{s \in \omega : \{b', s'\} \in N_\sigma\}.$$

If $G \geq G_\sigma$ is a scorpion graph, then $b_G = b'$, and so $S = \{s_G\}$.

Thus, if $|S| \neq 1$, then there is no scorpion graph G which extends G_σ , and so the game terminates after at most σ turns and $\Omega(P_\sigma) \setminus \Omega(P_\alpha) \subset \{s'\}$.

So we can assume that $S = \{s'\}$, and we know that

$$\text{if } G \geq G_\sigma \text{ is a scorpion graph, then } b_G = b' \text{ and } s_G = s'. \quad (\text{B''})$$

Stage B.

Alice enumerates the undetermined pairs $U_\sigma \cap [\{s'\}, \omega]$ as $\{e_\ell'' : \ell < M_s\}$ for some $M_s \leq \omega$, and Alice plays e_ℓ'' in step $\sigma + \ell$ for $\ell < M_s$.

Write $\rho = \sigma + M_s$. Clearly $\Omega(P_\rho) \setminus \Omega(P_\alpha) \subset \{b', s'\}$.

Let

$$T = \{t \in \omega : \{t, s'\} \in E_\rho\}.$$

If $G \geq G_\rho$ is a scorpion graph, then $s_G = s'$ and so $T = \{t_G\}$.

Thus, if $|T| \neq 1$, then there is no scorpion graph G which extends G_ρ , and so the game terminates after at most ρ turns and $\Omega(P_\rho) \setminus \Omega(P_\alpha) \subset \{b', s'\}$.

So we can assume that $T = \{t'\}$, and we know that

$$\text{If } G \geq G_\rho \text{ is a scorpion graph, then } s_G = s', t_G = t' \text{ and } b_G = b'. \quad (\text{C''})$$

Stage C.

Alice enumerates the undetermined pairs $U_\rho \cap [\{t'\}, \omega]$ as $\{e_\ell^* : \ell < M_t\}$ for some $M_t \leq \omega$, and Alice plays e_ℓ^* in step $\rho + \ell$ for $\ell < M_t$.

Write $\nu = \rho + M_t$. Clearly $\Omega(P_\nu) \setminus \Omega(P_\alpha) \subset \{s', t', b'\}$.

Since (C'') holds and $[\{s', t', b'\}, \omega] \subset P_\nu$, by Lemma 4.10 either every extension of G_ν is a scorpion graph, or no extension of G_ν is a scorpion graph.

So the game terminates after at most ν turns, $\nu \leq \alpha + \omega + \omega + \omega$ and $\Omega(P_\nu) \setminus \Omega(P_\alpha) \subset \{s', t', b'\}$.

So we completed the proof of Lemma 4.11. \square

After this preparation we can describe the winning strategy of Alice. We divide the game into stages.

Stage 1.

Let $F \subset [\omega]^2$ be fixed such that $d_F(v) = 4$ for each $v \in \omega$. Write $F = \{f_\ell : \ell < \omega\}$. For $\ell < \omega$ let Alice ask the pair f_ℓ in the ℓ^{th} turn.

Stage 1 terminates after ω turns. Then $\deg_{P_\omega}(n) = 4$ for each $n \in \omega$. So

$$B_\omega \cap T_\omega = \emptyset$$

and clearly $S_\omega \subset T_\omega$.

Stage 2. This stage is divided into substages. Before Substage i the players played $\eta_i = \omega + n_i$ turns for some $n_i < \omega$. Let $n_0 = 0$.

Substage i.

If $B_{\eta_i} = \emptyset$ or $S_{\eta_i} = \emptyset$, then there is no scorpion graph G which extends G_{η_i} , so the game has terminated. Since $\langle \omega, P_{\eta_i} \rangle$ is locally finite, i.e. $P_{\eta_i} \notin \mathcal{J}_1$, Alice wins.

So we can assume that $B_{\eta_i} \neq \emptyset$ and $S_{\eta_i} \neq \emptyset$. Let

$$k_i = \min(B_{\eta_i} \cup S_{\eta_i}),$$

and let

$$C_i = \begin{cases} B_{\eta_i} & \text{if } k_i \in S_{\eta_i}, \\ S_{\eta_i} & \text{if } k_i \in B_{\eta_i}. \end{cases}$$

Fix an enumeration $\{e_j^i : j < M_i\}$ of $U_{\eta_i} \cap [\{k_i\}, C_i]$ for some $M_i \leq \omega$, and let Alice play the pair e_j^i in the turn $\eta_i + j$.

If $k_i \in B_{\eta_i+j} \setminus B_{\eta_i+j+1}$, or $k_i \in S_{\eta_i+j} \setminus S_{\eta_i+j+1}$, then Substage i terminates, and we put $\eta_{i+1} = \eta_i + j + 1$.

If the Substage i has not terminated for any $j < M_i$, then we declare that the Substage i was the last substage and Stage 2 terminates.

Assume that Stage 2 has not terminated after Substage i for any $i < \omega$. Then we have

$$\min(B_{\eta_i} \cup S_{\eta_i}) \notin B_{\eta_{i+1}} \cup S_{\eta_{i+1}},$$

for each $i < \omega$. So taking $\eta = \sup\{\eta_i : i < \omega\}$ we have

$$B_\eta \cup S_\eta \subset \bigcap_{i < \omega} (B_{\eta_{i+1}} \cup S_{\eta_{i+1}}) = \emptyset.$$

Thus, there is no scorpion graph G which extends G_η , so the game terminates after at most η steps. Since $\langle \omega, P_\eta \rangle$ is locally finite, i.e. $P_\eta \notin \mathcal{J}_1$, Alice wins.

Stage 3.

Assume that Stage 2 terminated after Substage i , and we are after turn η . Let us observe that $\eta = \eta_i + \omega$ if $M_i = \omega$, and $\eta = \eta_i + M_i + 1$ if $M_i < \omega$. Moreover,

$$\Omega(P_\eta) \subset \{k_i\}.$$

We should distinguish two cases.

Case 1. $k_i \in B_{\eta_i}$.

Then $k_i \in B_\eta$ as well, and $[\{k_i\}, S_\eta] \subset P_\eta$.

Let

$$S = \{s \in S_\eta : \{s, k_i\} \in N_\eta\}.$$

Assume G is a scorpion graph which extends G_η . Then $s_G \in S_\eta$, and so $\{s_G, k_i\} \notin E_G$, and so $\{s_G, k_i\} \in N_\eta$. Since $k_i \in B_\eta$ and so $\deg_{N_\eta}(k_i) \leq 1$, it follows that $|S| \leq 1$. So $S = \{s_G\}$.

Thus, if $|S| \neq 1$, then there is no scorpion graph G which extends G_η , and so the game terminates after at most η turns. Since $\Omega(P_\eta) \subset \{k_i\}$ and so $P_\eta \notin \mathcal{J}_2$, Alice wins.

So we can assume that $S = \{s'\}$, and we know that

$$\text{if } G \geq G_\eta \text{ is a scorpion graph, then } s_G = s'. \quad (\text{A})$$

Thus, we can apply Lemma 4.11(a): Alice can play in such a way that the game terminates in $\beta \leq \eta + \omega + \omega + \omega$ turns and $|\Omega(P_\beta) \setminus \Omega(P_\eta)| \leq 3$. Since $\Omega(P_\eta) \subset \{k_i\}$, we have $|\Omega(P_\beta)| \leq 4$ and so $P_\beta \notin \mathcal{J}_5$, Alice wins.

So we completed the investigation of Case 1.

Case 2. $k_i \in S_{\eta_i}$.

Then $k_i \in S_\eta$ as well, and $[\{k_i\}, B_\eta] \subset P_\eta$. Consider the set

$$B = \{b \in B_\eta : \{k_i, b\} \in E_\eta\}.$$

Since $k_i \in S_\eta$, we have $\deg_{E_\eta}(k_i) \leq 1$, and so $|B| \leq 1$.

Subcase 2.1 $B = \emptyset$.

First observe that

$$\text{if } G \geq G_\eta \text{ is a scorpion graph, then } s_G = k_i. \quad (\text{A}')$$

Indeed, in this case $b_G \in B_\eta$ and so $B = \emptyset$ implies $\{k_i, b_G\} \in N_\eta$, which yields $k_i = s_G$.

Thus, we can apply Lemma 4.11(a): Alice can play in such a way that the game terminates in $\beta \leq \eta + \omega + \omega + \omega$ turns and $|\Omega(P_\beta) \setminus \Omega(P_\eta)| \leq 3$. Since $\Omega(P_\eta) \subset \{k_i\}$, $|\Omega(G_\beta)| \leq 4$ and so $G_\beta \notin \mathcal{J}_5$, Alice wins.

So we completed the investigation of Subcase 2.1.

Subcase 2.2 $B = \{b'\}$ for some $b' \in B_\eta$.

First observe that

$$\text{if } G \geq G_\eta \text{ is a scorpion graph, then } b_G = b'. \quad (\text{A}'')$$

Indeed, since $t_G \notin B_\eta$ because $B_\eta \cap T_\eta \subset B_\omega \cap T_\omega = \emptyset$, $B \neq \emptyset$ implies $k_i \neq s_G$, and so $b_G \in B$. Since $|B| = 1$, we have $b' = b_G$.

Thus, we can apply Lemma 4.11(b): Alice can play in such a way that the game terminates in $\beta \leq \eta + \omega + \omega + \omega$ turns and $|\Omega(P_\beta) \setminus \Omega(P_\eta)| \leq 3$. Since $\Omega(P_\eta) \subset \{k_i\}$, we have $|\Omega(G_\beta)| \leq 4$ and so $P_\beta \notin \mathcal{J}_5$, Alice wins.

So we completed the investigation of Subcase 2.2.

Since there are no more cases, we proved Theorem 4.7. \square

Theorem 4.12. (1) For each natural number $n \geq 1$ the monotone graph property D_n is $\mathcal{F}_{[\omega]^2}$ -hard on the vertex set ω .

(2) For each natural number $m \geq 2$ the monotone graph property C_m is $\mathcal{F}_{[\omega]^2}$ -hard on the vertex set ω .

Proof. (1) If $F \subset [\omega]^2$ and $j \in \omega$, write $d_F(j) = |\{i \in \omega : \{i, j\} \in F\}|$ and $d_F^<(j) = |\{i < j : \{i, j\} \in F\}|$.

Fix $n \in \omega$. Let Bob play using the following strategy: in the α th step, if Alice asks the undetermined pair $e_\alpha = \{i, j\}$ with $i < j < \omega$, then Bob says "yes" iff either

(a) $i \leq n$ and $d_{E_\alpha}(i) < n$, or $j \leq n$ and $d_{E_\alpha}(j) < n$,

or

(b) $d_{E_\alpha}(j) + d_{U_\alpha}^<(j) = n$.

Assume that the game terminates after β steps.

Lemma 4.13. For each $\alpha \leq \beta$, we have $d_{\max G_\alpha}(v) \geq n$ for each $v \in \omega$.

Proof. Fix $j \geq n$. Then $d_{E_0}(j) + d_{U_0}^<(j) = 0 + j \geq n$. By transfinite induction on α we can see that we have $d_{E_\alpha}(j) + d_{U_\alpha}^<(j) \geq n$ for each $\alpha \leq \beta$.

Moreover, for each $i < n$ we have $d_{E_\alpha}(i) \geq \min(n, d_{P_\alpha}(i))$, so if $d_{E_\alpha}(i) < n$, then $d_{U_\alpha}(i)$ is infinite.

Thus, $d_{\max G_\alpha}(v) \geq n$ for each $v \in \omega$ and $\alpha \leq \beta$. \square

Lemma 4.14. If $d_{E_\alpha}(v) \geq 1$ for each $v \in \omega$ for some $\alpha \leq \beta$, then $P_{G_\alpha} \in \mathcal{F}_{[\omega]^2}$.

Proof of the Lemma. E_α contains just finitely many edges which were obtained by applying rule (a). Let $J \in \omega$ such that $[J]^2$ contains all of these edges. Let $\{i, j\} \in E_\alpha \setminus [J]^2$. Then there is $\gamma < \alpha$ such that $e_\gamma = \{i, j\}$ and rule (b) was applied, i.e. $d_{E_\gamma}(j) + d_{U_\gamma}^<(j) = n$, and so $d_{U_\alpha}^<(j) \leq n$.

Let $A = \{j : J < j \wedge \{i, j\} \in E_\alpha \text{ for some } i < j\}$.

Consider the following graph $K = \langle A, F \rangle$: $\{i, j\} \in F$ iff $i < j$ and $\{i, j\} \in U_\alpha$. Since $d_F^\leq(j) \leq n$, the chromatic number of K is at most $n + 1$, and so there is $B \in [A]^\omega$ such that $[B]^2 \cap F = \emptyset$, i.e. $[B]^2 \subset P_\alpha$. So $P_\alpha \in \mathcal{F}_{[\omega]^2}$. \square

By Lemma 4.13 $\max G_\beta$ has property D_n . Since the game has terminated, $\min G_\beta = \langle \omega, E_\beta \rangle$ also has property D_n . Since $n \geq 1$, we can apply Lemma 4.14 for $\alpha = \beta$ to obtain $P_\beta \in \mathcal{F}_{[\omega]^2}$, which proves (1).

(2) Let Bob use the strategy he applied again property D_{m-1} .

Assume that the game terminates after β moves.

Then for each $\alpha \leq \beta$ we have $d_{\max G_\alpha}(v) \geq m - 1$ for each $v \in \omega$ by Lemma 4.13. Thus, $c_{\max G_\beta}(v) \geq m$ for each $v \in \omega$ and so $\max G_\beta$ has property C_m .

Since the game has terminated, $\min G_\beta = \langle V, E_\beta \rangle$ has property C_m as well. Then $d_{G_\beta}(v) \geq 1$ for each $v \in \omega$ because $m \geq 2$. Thus, by Lemma 4.14, $P_\beta \in \mathcal{F}_{[\omega]^2}$. Thus, Bob wins. \square

Theorem 4.15. *The monotone graph property "connected" is $\mathcal{F}_{[\omega]^2}$ -hard on ω .*

Proof. Let Bob play according to the following strategy.

If $e_\alpha = \{i, j\}$, then Bob says yes iff both $A = C_{G_\alpha}(i)$, the connected component of i in G_α , and $B = C_{G_\alpha}(j)$, the connected component of j in G_α , are finite, $A \neq B$ and $[A, B] \subset P_\alpha \cup \{e_\alpha\}$.

Assume that the game terminates after β steps.

Assume first that G_β contains an infinite connected component, and let

$$\gamma = \min\{\gamma' \leq \beta : G_{\gamma'} \text{ contains an infinite connected component}\},$$

and consider an infinite component A of G_γ .

Fix $\{i, j\} \in [A]^2$, and let α be the minimal ordinal such that $C_{G_\alpha}(i) = C_{G_\alpha}(j)$. Then $\alpha = \delta + 1 < \gamma$ and $[C_{G_\delta}(i), C_{G_\delta}(j)] \subset P_\alpha$ because Bob declared an edge between $C_{G_\delta}(i)$ and $C_{G_\delta}(j)$ in the δ^{th} turn. Thus, $\{i, j\} \in P_\alpha$. Since $\{i, j\}$ was arbitrary, hence $[A]^2 \subset P_\gamma \subset P_\beta$. Thus, $P_\beta \in \mathcal{F}_{[\omega]^2}$.

So we can assume that every connected component of G_β is finite. So G_β is not connected. Since the game has terminated, $\max G_\beta$ is not connected as well. So ω has a partition $\omega = X_0 \cup X_1$ such that $(E_\beta \cup U_\beta) \cap [X_0, X_1] = \emptyset$. Pick $v_i \in X_i$ and let $K_i = C_{G_\beta}(v_i)$ for $i < 2$. Since $K_i \subset X_i$, $[K_0, K_1] \subset N_\beta$.

But $[K_0, K_1]$ is finite, so let α be the maximal ordinal such that $e_\alpha \in [K_0, K_1]$. Let $\ell_i = e_\alpha \cap K_i$ for $i < 2$. Then

$$e_\alpha \in [C_{G_\alpha}(\ell_0), C_{G_\alpha}(\ell_1)] \subset [K_0, K_1] \subset N_\alpha \cup \{e_\alpha\},$$

and so Bob declared that e_α is an edge. Contradiction, which proves that it is not possible that every connected component of G_β is finite.

Thus, we proved $P_\beta \in \mathcal{F}_{[\omega]^2}$. \square

Theorem 4.16. *For each natural number $n \geq 1$ the monotone graph property*

$$"G \text{ contains } K_{1,n}"$$

is \mathcal{F}_n -hard on the vertex set ω , where

$$\mathcal{F}_n = \{E \subset [\omega]^2 : \exists B \in [\omega]^n \text{ such that } [\omega]^2 \setminus E \subset [B]^2\}.$$

Proof. Let Bob play according to the following greedy strategy: in the α th step if $e_\alpha = \{i, j\}$ then Bob says yes iff $d_{G_\alpha}(i) < n - 1$ and $d_{G_\alpha}(j) < n - 1$.

Assume that the game terminates after β steps.

Clearly G_β does not contain $K_{1,n}$. Since the game has terminated, $\max G_\beta$ can not contain $K_{1,n}$ as well.

Let $A = \{v \in \omega : d_{E_\beta}(v) = n - 1\}$ and $B = \{v \in \omega : d_{E_\beta}(v) \leq n - 2\}$. Then $N_\beta \cap [B]^2 = \emptyset$. Thus, $|B| \leq n$ or $d_{E_\beta \cup U_\beta}(b) \geq |B| - 1 \geq n$ for each $b \in B$.

Moreover, $U_\beta \subset [B]^2$, or $\{i, j\} \in U_\beta \cap [A, \omega]$ with $i \in A$ implies $d_{E_\beta \cup U_\beta}(i) \geq n$. Thus, $P_\beta \in \mathcal{F}_n$.

Thus, Bob wins. \square

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