# HITCHIN WKB-PROBLEM AND $P = W$ CONJECTURE IN LOWEST DEGREE FOR RANK 2 OVER THE 5-PUNCTURED SPHERE

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Abstract. We use abelianization of Higgs bundles away from the ramification divisor and fiducial solutions to analyze the large scale behaviour of Fenchel–Nielsen co-ordinates on the moduli space of rank 2 Higgs bundles on the Riemann sphere with 5 punctures. We solve the related Hitchin WKB problem and prove the lowest degree weighted pieces of the $P = W$ conjecture in this case.

## 1. Introduction and statement of the main result

In this paper we investigate the moduli space $\mathcal{M}_{\mathrm{Dol}}$ of Higgs bundles on $\mathbb{C}P^1$ with 5 logarithmic points in rank 2 and the corresponding character variety $\mathcal{M}_{\mathrm{B}}$, subject to specific choices of parameters. These spaces are complex varieties of dimension 4. The first aim of the paper is to give a complete answer (Propositions 10, 12, 14) for these spaces to the Hitchin WKB problem raised in [27]:

**Hitchin WKB problem** Consider a non-trivial $\mathbb{C}^{\times}$-orbit in the Hitchin base and a family of Higgs bundles lifting this orbit in the Dolbeault moduli space; determine then the asymptotic behaviour of the transport matrices of the associated family of representations in the character variety, as the point of $\mathbb{C}^{\times}$ converges to infinity.

For the classical theory of WKB approximation, see [48], [25, Section 2]. The second, closely related goal is to use these results in order to obtain for these spaces one extremal graded piece of the so-called $P = W$ conjecture:

**Theorem 1.** *Let $\mathcal{M}_{\mathrm{Dol}}$ and $\mathcal{M}_{\mathrm{B}}$ denote the Dolbeault moduli space and character variety of $\mathbb{C}P^1$ with 5 logarithmic points in rank 2. Then, for every $0 \leq k \leq 4$ the regular singular Riemann–Hilbert correspondence and the non-abelian Hodge correspondence induce an isomorphism*

$$\mathrm{Gr}_P^{-k-2} H^k(\mathcal{M}_{\mathrm{Dol}}, \mathbb{Q}) \cong \mathrm{Gr}_{2k}^W H^k(\mathcal{M}_{\mathrm{B}}, \mathbb{Q}).$$

The weights appearing in the theorem represent the lowest (respectively, highest) possibly non-trivial weights of $P$ (respectively, $W$). For our weight conventions, see Sections 2.3 and 2.7. Notice that since $\mathcal{M}_{\mathrm{B}}$ is a smooth 4-dimensional affine variety, by virtue of the Andreotti–Frankel theorem [2] the only degrees where it may have non-trivial cohomology are $0 \leq k \leq 4$.

Even though this paper contains a detailed study of just one special case of the $P = W$ conjecture in lowest weight, many of our technical results are valid for an arbitrary number $n \geq 5$ of parabolic points. We expect that it will be possible to treat more general cases along the same lines. Specifically, our results suggest

**Conjecture 1.** *The tropical geometry of the composition of the Riemann–Hilbert and non-abelian Hodge correspondences in the large scale limit is governed by the central charge function of the corresponding Donaldson–Thomas theory.*

By tropical geometry, we mean taking the maximum of the logarithms of the absolute values of local co-ordinates of a variety, see [36, Section 3.5]. A motivation for our conjecture is that in the so-called large scale limit, this procedure degenerates the usual additive structure of $\mathbb{R}$ to the tropical one

$$\lim_{R\to\infty} \frac{1}{R} \ln(e^{Rx} + e^{Ry}) = \max(x, y).$$

Now, according to Propositions 12, 17, some natural co-ordinates on the Betti side behave precisely as the expression on the left-hand side of this equation, with $x, y, \ldots$ real parts of integrals of the standard Liouville 1-form over some loops on the spectral curve. Such integrals are in turn called central charge in Donaldson–Thomas theory, see [4, Section 7.1], [5, Section 10.4], [29, Section 1.2]. We suspect that a generalization of the methods of the present paper, when combined with cluster co-ordinates on character varieties defined in [15] and [35], will turn out to be useful for the study of this question in further cases. For related work, see also [1, Theorem 1.5].

Let us now give some motivational background for this study. Until very recently, the $P = W$ conjecture was a major open problem in non-abelian Hodge theory, formulated by M. de Cataldo, T. Hausel and L. Migliorini [7] as a correspondence between the (decreasing) perverse Leray filtration $P$ induced by the Hitchin map on the cohomology of a Dolbeault moduli space and the (increasing) weight filtration $W$ of Deligne's mixed Hodge structure on the cohomology of the associated character variety (Betti space). First, in [7] the identity was proved in rank 2 over compact curves. Then, M. de Cataldo, D. Maulik and J. Shen [8] established it for curves of genus 2. Later, C. Felisetti and M. Mauri [14] proved it for character varieties admitting a symplectic resolution, i.e. in genus 1 and arbitrary rank, and in genus 2 and rank 2. The author has established the conjecture for complex 2-dimensional moduli spaces of rank 2 Higgs bundles with irregular singularities over $\mathbb{C}P^1$ corresponding to the Painlevé cases [46]. J. Shen and Z. Zhang [42] proved it for five infinite families of moduli spaces of parabolic Higgs bundles over $\mathbb{C}P^1$. Recently, two independent complete proofs using quite different methods have been announced [32], [22]. Both proofs start by converting the statement to one about Chern classes of the universal family using the results of Markman [31] and Shende [43]. Maulik and Shen then use vanishing cycle techniques, global Springer theory and a support theorem for a certain parabolic Hitchin system to proving it. On the other hand, Hausel, Mellit, Minets and Schiffmann deduce the claim from the observation that a polynomial ring over the cohomology ring of the Dolbeault moduli space carries the action of the algebra of polynomial Hamiltonian vector fields of the plane; their approach works in parabolic cases with generic stability parameters too. Our strategy differs from both of these. It is of more direct and geometrical nature, relying at the same time on recent progress on the asymptotic decoupling of the Hitchin system.

The $P = W$ conjecture has also been generalized in various interesting contexts. To name a few generalizations, A. Harder showed a similar statement for elliptic Lefschetz fibrations using methods coming from toric surfaces [20, Theorem 4.5].

Z. Zhang [49] found a related phenomenon for the weight filtration of certain 2-dimensional cluster varieties and the perverse filtration of elliptic fibrations with constrained singular fibers. A motivation for the $P = W$ conjecture was the so-called curious hard Lefschetz conjecture of T. Hausel, E. Letellier and F. Rodriguez-Villegas [21], that has been confirmed by A. Mellit [35]. A stacky version of the $P = W$ conjecture has been proposed (and proved in genera 0 and 1) by B. Davison [10].

Among the various generalizations and analogues of the $P = W$ conjecture of particular interest to us is an intriguing geometric counterpart formulated by L. Katzarkov, A. Noll, P. Pandit and C. Simpson [27, Conjecture 1.1] and C. Simpson [45, Conjecture 11.1]; this version is now called Geometric $P = W$ conjecture. Roughly speaking, the Geometric $P = W$ conjecture asserts the existence of a certain homotopy commutative diagram involving the Riemann–Hilbert map, non-abelian Hodge correspondence, the Hitchin map and the natural map from the character variety to the topological realization of its dual boundary complex. An immediate consequence of validity of this conjecture is that the homotopy type of the topological space of the dual boundary complex of the character variety is that of a sphere of given dimension, therefore finding this homotopy type is a first consistency check of the conjecture. The Geometric $P = W$ conjecture has also attracted considerable attention in recent times. A. Komyo [28] used an explicit geometric description to prove that the homotopy type of the dual boundary complex of the character variety for $\mathbb{C}P^1$ with 5 logarithmic points and group $\mathrm{GL}(2, \mathbb{C})$ (that is, the Betti space we will deal with in this paper) is that of the 3-sphere. C. Simpson [45] generalized Komyo's result to the case of arbitrarily many logarithmic points on $\mathbb{C}P^1$, in rank 2, by proving that the homotopy type of the dual boundary complex is that of $S^{2n-7}$; for this purpose, he introduced Fenchel–Nielsen type co-ordinates that will be widely used in this paper. T. Mochizuki [37] solved the closely related Hitchin WKB problem for non-critical paths. M. Mauri, E. Mazzon and M. Stevenson [33, Theorem 6.0.1] used Berkovich space techniques to show that the dual boundary complex of a log-Calabi–Yau compactification of the $\mathrm{GL}(n, \mathbb{C})$ character variety of a 2-torus is homeomorphic to $S^{2n-1}$. They also showed that Geometric $P = W$ conjecture implies the cohomological $P = W$ conjecture in top cohomological degree and lowest weight. L. Katzarkov, A. Harder and V. Przyjalkowski have formulated a version of the cohomological $P = W$ conjecture for log-Calabi–Yau manifolds and their mirror symmetric pairs, and in [26, Section 4] discussed a geometric version thereof. The author established the Geometric $P = W$ conjecture in the Painlevé cases in [46] via asymptotic abelianization of solutions of Hitchin's equations. In joint work with A. Némethi [40], the author gave a second proof for the same cases using different techniques, namely plumbing calculus. As far as the author is aware, up to date these latter articles are the only ones in which the full assertion of the Geometric $P = W$ conjecture has been confirmed, rather than just its implication on the homotopy type of the dual boundary complex. It is remarkable that the geometrical understanding of the moduli spaces developed in [40, Section 6] is quite reminiscent to the description of the weight filtration in terms of dual torus fibrations appearing in [26, Section 4] (up to the difference that the latter paper deals with the case of a smooth elliptic anti-canonical divisor rather than a singular one).

Previously, F. Loray and M. Saito [30] studied the algebraic geometric structure of the moduli space that we consider, endowed with its de Rham complex structure.

R. Donagi and T. Pantev [12] investigated Hecke transforms on this space and proved the Geometric Langlands correspondence for it. The paths that we need to consider in Propositions 10, 12, 14 for applications to the character variety are homologically non-trivial loops that do not satisfy the non-critical condition, therefore our results do not directly follow from previous study of T. Mochizuki [37] (though we make use of results of that paper).

We will achieve our goals by refining the approach pioneered in our previous paper [46]. Namely, using asymptotic abelianization we reduce the study to the classical abelian Hodge theory and Riemann–Hilbert correspondence treated in detail for instance in [19]. Specifically, we will make use of technical results of T. Mochizuki [37] describing the large-scale behaviour of solutions of Hitchin's equations away from the ramification divisor, and their extensions by R. Mazzeo, J. Swoboda, H. Weiss and F. Witt [34] and L. Fredrickson, R. Mazzeo, J. Swoboda and H. Weiss [16] in a neighbourhood of simple points of the ramification divisor and parabolic points respectively. As opposed to the non-parabolic case where the solutions (called fiducial solutions) of [34] give convenient local models, to deal with the parabolic case one needs the solutions given in [16] that generalize the original fiducial solutions of [34]. In this paper, we combine this understanding of the asymptotic behaviour of solutions of the self-duality equations with C. Simpson's Fenchel–Nielsen type co-ordinates of the character variety [45].

The studies in [16] and [34] were inspired by physical considerations pertinent to the WKB-analysis of Hitchin's equations given by D. Gaiotto, G. Moore and A. Neitzke [17], where the authors stated a conjecture about the large scale Riemannian structure of the Hodge moduli spaces. In a certain sense, our work therefore points out a connection between two seemingly unrelated circles of ideas: the $P = W$ conjecture on the algebraic topology of the Hodge moduli spaces on the one hand, and the Gaiotto–Moore–Neitzke conjecture on their Riemannian geometry on the other hand. This fits nicely into the broader picture of topology and Riemannian geometry having influence on one another, the bridge between them being built by geometric analysis.

One feature of the case we study is that the quadratic differentials at play may have at worst a double zero (see Proposition 3), giving rise to a transverse singular point of the spectral curve. The metric on the moduli space in a neighbourhood of the rays along which such singular fibers appear is believed to be approximately given by the Ooguri–Vafa metric [18], [39, Sections 6,7], [47].

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## 2. Basic notions and preparatory results

2.1. **Moduli spaces of tame harmonic bundles.** Consider $X = \mathbb{C}P^1$ with co-ordinates $z$ and $w = z^{-1}$, endowed with the standard Riemannian metric. We denote by $\mathcal{O}$ and $K$ the sheaves of holomorphic functions and holomorphic 1-forms respectively on $\mathbb{C}P^1$. We fix some values

$$t_1 < t_2 = -1, \quad t_3 = 0, \quad t_4 = 1 < t_0 \tag{2.1}$$

These choices will not be used until Section 6.1, so the results of all preceding sections are valid for any quintuple of distinct points in $\mathbb{C}P^1$. We consider the simple effective divisor

$$D = t_0 + t_1 + t_2 + t_3 + t_4$$

and set

$$L = K(D).$$

By an abuse of notation, we will also denote by $D$ the support set of $D$. Finally, we fix a point $x_0 \in \mathbb{C}P^1 \setminus D$. Much of the following discussion has a straightforward generalization to simple effective divisors of higher length too.

For $0 \leq j \leq 4$ we fix

$$\alpha_j^- = \frac{1}{4}, \quad \alpha_j^+ = \frac{3}{4} \tag{2.2}$$

that will serve as parabolic weights in the Dolbeault complex structure. These choices maximize the distance from the set of integer translates of $\alpha_j^-$ to those of $\alpha_j^+$, hence they lie at the center of the Weyl alcove describing the possible parabolic weights (in this case, an interval of length 1). Our choices will turn out to be important in the proof of Propositions (17) and (19); namely, they imply an unexpected cancellation. Notice that

$$\sum_{j=0}^{4} (\alpha_j^- + \alpha_j^+) = 5.$$

We will write

$$\alpha = (\alpha_j^-, \alpha_j^+)_{j=0}^4.$$

The basic object of our study will be a certain Hodge moduli space $\mathcal{M}_{\text{Hod}}$ of *tame harmonic bundles* [44] of rank 2 and parabolic degree 0 on $\mathbb{C}P^1$ with parabolic structure at $D$. We will describe this moduli space from two perspectives called the Dolbeault and the de Rham moduli spaces. Consider a smooth vector bundle $V$ of rank 2 and degree $-5$ over $\mathbb{C}P^1$. Then, the equations defining harmonic bundles are *Hitchin's equations* [24]

$$\bar{\partial}_{\mathcal{E}} \theta = 0 \tag{2.3}$$

$$F_h + [\theta, \theta^\dagger] = 0 \tag{2.4}$$

for a $(0,1)$-connection $\bar{\partial}_{\mathcal{E}}$ on $V$, a Hermitian metric $h$ on $V$ and a section $\theta$ of $End(V) \otimes \Omega^{1,0}_{\mathbb{C}P^1}$ over $\mathbb{C}P^1 \setminus D$, where $F_h$ is the curvature of the Chern connection $\nabla_h^+$ associated to $(\bar{\partial}_{\mathcal{E}}, h)$ and $\theta^\dagger$ is the section of $End(V) \otimes \Omega^{0,1}_{\mathbb{C}P^1}$ obtained by taking the adjoint of the endomorphism part of $\theta$ with respect to $h$ and the complex conjugate of its form-part. The reason of the terminology "harmonic bundle" is the fact that with respect to the de Rham complex structure, the equations imply that the map $h$ is equivariant harmonic from the universal cover of the Riemann surface to the Hermitian symmetric space $\mathrm{GL}(2, \mathbb{C}) / \mathrm{U}(2)$. The behaviour of $\theta$ and $h$ is assumed to satisfy the so-called *tameness* condition at each $t_j$, namely $h$ should admit a lift along any ray to $t_j$ which grows at most polynomially in Euclidean distance.

Hitchin's equations are presented above from the Dolbeault point of view. Let us first describe the boundary behaviour of the data from this perspective. Let us denote by $\mathcal{E}$ the holomorphic vector bundle $(V, \bar{\partial}_{\mathcal{E}})$ on $\mathbb{C}P^1 \setminus D$. It turns out that there exists an extension of the holomorphic bundle $\mathcal{E}$ over $D$ such that the Higgs field has at most logarithmic poles at $D$. A *parabolic structure* on $\mathcal{E}$ at $D$ is by definition a filtration

$$0 \subset \ell_j \subset \mathcal{E}|_{t_j} \tag{2.5}$$

of the fiber of $\mathcal{E}$ at every $t_j \in D$ that is stabilized by $\mathrm{res}_{t_j}\theta$. We assume that the Higgs field $\theta$ is *strongly parabolic*, meaning that the action of $\mathrm{res}_{t_j}\theta$ both on $\ell_j$ and on $\mathcal{E}|_{t_j}/\ell_j$ is trivial. Then, in the Dolbeault complex structure $\mathcal{M}_{\mathrm{Hod}}$ parameterizes $\alpha$-stable parabolic Higgs bundles with Higgs field having at most logarithmic poles at $D$ such that the eigenvalues of the residue of the associated Higgs field at $t_j$ vanish and the parabolic weights of the underlying holomorphic vector bundle in the Dolbeault picture at $t_j$ are equal to $\alpha_j^{\pm}$. The latter assumption on parabolic weights encodes a certain growth behaviour of the evaluation of the metric $h$ on elements of a local holomorphic trivialization. The moduli space of such logarithmic parabolic Higgs bundles is known to be a $\mathbb{C}$-analytic manifold

$$\mathcal{M}_{\mathrm{Dol}}(\mathbf{0}, \alpha)$$

called *Dolbeault moduli space*, whose underlying smooth manifold is $\mathcal{M}_{\mathrm{Hod}}$.

Let us now turn to the de Rham point of view. It is known that if $(\bar{\partial}_{\mathcal{E}}, h, \theta)$ is a tame harmonic bundle then the connection

$$\nabla = \nabla_h^+ + \theta + \theta^\dagger$$

is integrable, and the underlying holomorphic vector bundle admits an extension over $D$ with respect to which $\nabla^{1,0}$ has regular singularities. The associated de Rham moduli space parameterizes $\beta$-stable parabolic integrable connections on $V$ with regular singularities near the punctures $t_j$, with eigenvalues of its residue given by

$$\mu_j^{\pm} = \alpha_j^{\pm} \tag{2.6}$$

and parabolic weights given by

$$\beta_j^{\pm} = \alpha_j^{\pm}. \tag{2.7}$$

Again, a parabolic structure on the underlying holomorphic vector bundle at $D$ is defined as a flag of its fiber over $t_j \in D$ that is stabilized by $\mathrm{res}_{t_j}\nabla^{1,0}$ and such that its action on the first graded piece of the filtration be $\mu_j^-$. The *de Rham moduli space* of such parabolic integrable connections with regular singularities will be denoted by

$$\mathcal{M}_{\mathrm{dR}}(\alpha, \alpha);$$

it is a $\mathbb{C}$-analytic manifold with underlying smooth manifold $\mathcal{M}_{\mathrm{Hod}}$.

It follows from the above discussion that there exists a canonical diffeomorphism

$$\psi\colon \mathcal{M}_{\mathrm{Dol}}(\mathbf{0}, \alpha) \to \mathcal{M}_{\mathrm{dR}}(\alpha, \alpha) \tag{2.8}$$

called *non-abelian Hodge correspondence*.

2.2. **Character variety, Riemann–Hilbert correspondence, dual boundary complex.** We will need a third point of view of harmonic bundles, called Betti side. The *Betti moduli space* (or character variety) $\mathcal{M}_{\mathrm{B}}(\mathbf{c}, \mathbf{0})$ parameterizes filtered local systems on $\mathbb{C}P^1 \setminus D$ with prescribed conjugacy class of its monodromy around every $t_j$ and growth order of parallel sections on rays emanating from the punctures, up to simultaneous conjugation by elements of $\mathrm{PGL}(2, \mathbb{C})$. We will now describe the value and role of parameters $\mathbf{c}$. Namely, the monodromy transformation of an integrable connection $\nabla$ in $\mathcal{M}_{\mathrm{dR}}(\mu, \beta)$ along a positively oriented simple loop in $\mathbb{C}P^1$ separating $t_j$ from the other parabolic points has eigenvalues

$$c_j^{\pm} = \exp(-2\pi\sqrt{-1}\mu_j^{\pm}) = \exp(-2\pi\sqrt{-1}\alpha_j^{\pm}) = \pm\sqrt{-1} \tag{2.9}$$

and all weights of the associated filtration equal to 0. We notice that for any $\epsilon_j \in \{\pm 1\}$ for $0 \leq j \leq 4$ we have

$$c_0^{\epsilon_0} \cdots c_4^{\epsilon_4} \in \{\pm\sqrt{-1}\},$$

in particular

$$c_0^{\epsilon_0} \cdots c_4^{\epsilon_4} \notin \{\pm 1\}.$$

Said differently, the vector $\mathbf{c}$ satisfies the condition that Simpson calls Kostov-genericity (Condition [45, 4.3]).

It is known that the map

$$\mathrm{RH}\colon \mathcal{M}_{\mathrm{dR}}(\alpha, \alpha) \to \mathcal{M}_{\mathrm{B}}(\mathbf{c}, \mathbf{0}) \tag{2.10}$$

mapping any integrable connection to its (filtered) local system of vector spaces is a $\mathbb{C}$-analytic isomorphism, called *Riemann–Hilbert map.*

It is known that $\mathcal{M}_{\mathrm{B}}(\mathbf{c}, \mathbf{0})$ is an affine algebraic variety, which is smooth for generic choices of the parameters. We will denote by $\overline{\mathcal{M}}_{\mathrm{B}}(\mathbf{c}, \mathbf{0})$ a smooth compactification by a simple normal crossing divisor $D_{\mathrm{B}}$. Such a compactification exists by Nagata's compactification theorem [38] combined with Hironaka's theorem on the existence of resolutions of singularities in characteristic 0 [23].

**Definition 1.** *The* dual complex *of $D_{\mathrm{B}}$ is the simplicial complex $\mathbb{D}D_{\mathrm{B}}$ whose vertices are in bijection with irreducible components of $D_{\mathrm{B}}$, and whose $k$-faces are formed by $(k+1)$-tuples of vertices such that the intersection of the corresponding components is non-empty. We will denote the $k$-skeleton of $\mathbb{D}D_{\mathrm{B}}$ by $\mathbb{D}_k D_{\mathrm{B}}$, and the topological realization of $\mathbb{D}D_{\mathrm{B}}$ by $|\mathbb{D}D_{\mathrm{B}}|$.*

We will require $D_{\mathrm{B}}$ to be a very simple normal crossing divisor, meaning that any such non-empty intersection of components is connected. The above procedure may be applied to any quasi-projective smooth variety $X$, and an important result due to Danilov [6] states that the homotopy type of the simplicial complex is independent of the chosen compactification. We will apply it to $\mathcal{M}_{\mathrm{B}}(\mathbf{c}, \mathbf{0})$, and we will call the resulting simplicial complex its *dual boundary complex*, denoted by $\mathbb{D}\partial\mathcal{M}_{\mathrm{B}}(\mathbf{c}, \mathbf{0})$. A. Komyo [28] showed that for character varieties of rank 2 representations with $k = 5$ parabolic points the homotopy type of the dual boundary complex is that of the sphere $S^3$. C. Simpson [45] generalized this result to character varieties of the complement of $k \geq 5$ parabolic points, by showing that for $X = \mathcal{M}_{\mathrm{B}}(\mathbf{c}, \mathbf{0})$ the dual boundary complex is homotopy equivalent to the sphere $S^{2k-7}$.

2.3. **Topological description of the weights in mixed Hodge structure.** Another closely related consequence of the fact that $\mathcal{M}_{\mathrm{B}}(\mathbf{c}, \mathbf{0})$ is a smooth affine algebraic variety is that its cohomology spaces carry a mixed Hodge structure (MHS), defined by P. Deligne [11]. Let us recall the topological characterization of the weights in MHS, following [13, Section 6.5]. In what follows, we will often drop $\mathbf{c}, \mathbf{0}$ from the notation and write $\mathcal{M}_{\mathrm{B}}$ for the character variety.

In this section we adopt the point of view of [13] and consider homology groups rather than cohomology; application of the standard duality operation is implicitly meant whenever we compare a homology group with a cohomology group. This involves switching the signs of the degrees of the weight filtration. Let $\overline{\mathcal{M}}_{\mathrm{B}}$ be a smooth compactification of $\mathcal{M}_{\mathrm{B}}$ by a simple normal crossing divisor $D_{\mathrm{B}}$. We spell out the general construction of the mixed Hodge structure of $X \setminus Y$ given in [13] for $X = \overline{\mathcal{M}}_{\mathrm{B}}$ and $Y = D_{\mathrm{B}}$.

The filtration is the abutment of the spectral sequence associated to a double complex $A_{**}$ endowed with a filtration $W$. For any $p \geq 1$ we denote by $\tilde{D}^p$ the disjoint union of the $p$-fold intersections of the irreducible components of $D_{\mathrm{B}}$, and set $\tilde{D}^0 = \overline{\mathcal{M}}_{\mathrm{B}}$. We denote by $C_t^{\pitchfork}(\tilde{D}^s)$ the free abelian group generated by dimensionally transverse $t$-cycles in $\tilde{D}^s$, i.e. cycles for the 0-perversity function. We let

$$A_{s,t} = C_t^{\pitchfork}(\tilde{D}^{-s}),$$

where $s \leq 0, t \geq 0$. The filtration $W$ is defined by

$$W_s = \bigoplus_{p \leq s} A_{p,t}.$$

There exists a well-defined intersection morphism

$$\cap \colon C_t^{\pitchfork}(\tilde{D}^s) \to C_{t-2}^{\pitchfork}(\tilde{D}^{s+1})$$

compatible with $W$, turning $A_{**}$ into a filtered double complex. It is shown in [13, Theorem 1.5] that the associated spectral sequence $E_{st}^r$ degenerates at page $r = 2$ and abuts to the filtration

$$E_{st}^{\infty} \otimes \mathbb{Q} = \mathrm{Gr}_{-t}^W H_{s+t}(\mathcal{M}_{\mathrm{B}}, \mathbb{Q}).$$

The filtration $W$ on the right-hand side is then equal to Deligne's weight filtration.

The topological representatives of $\mathrm{Gr}_{-2k}^W H_k$ corresponding to the choices $t = 2k$ and $s = -k$ are generated by classes of the following form (for the similar cases $k = 1$ and $k = 2$ over surfaces see [13, Example 6.9]). Take a generic point $Q$ in the $k$-fold intersections of the divisors

$$Q \in \tilde{D}^k \setminus \tilde{D}^{k+1}.$$

Let the corresponding divisor components be denoted without loss of generality $Y_1, \dots, Y_k$. The preimage $\Pi^{-1}(Q)$ of $Q$ in the normal bundle of $Y_1 \cap \dots \cap Y_k$ in $X \setminus Y = \mathcal{M}_{\mathrm{B}}$ deformation retracts onto a $k$-dimensional real torus in the boundary of a tubular neighbourhood of $Y_1 \cap \dots \cap Y_k$. If one considers all $k$-tuple intersections of divisor components, then the classes of these tori generate $\mathrm{Gr}_{-2k}^W H_k$, and the dual cohomology classes generate $\mathrm{Gr}_{2k}^W H^k$.

2.4. **Hitchin maps and bases.** Let us set

$$\begin{aligned}\mathcal{B} &= H^0(\mathbb{C}P^1, K^{\otimes 2} \otimes \mathcal{O}(D)) \\ &= \{q : \quad q(t_j) = 0 \text{ for all } 0 \leq j \leq 4\} \\ &\subset H^0(\mathbb{C}P^1, L^{\otimes 2}) \cong \mathbb{C}^7.\end{aligned}$$

$\mathcal{B}$ is a linear subspace of dimension 2 over $\mathbb{C}$.

**Proposition 2.** (1) *For every strongly parabolic $\alpha$-stable Higgs bundle $(\mathcal{E}, \theta)$ with logarithmic singularities at $D$ we have*

$$tr(\theta) \equiv 0.$$

(2) *An $\alpha$-stable Higgs bundle $(\mathcal{E}, \theta)$ with logarithmic singularities at $D$ is strongly parabolic if and only if*
$$\det(\theta) \in \mathcal{B}.$$
(3) *The space $\mathcal{B}$ may be identified with quadratic differentials of the form*
$$Q(z) = \frac{(az - b)dz^{\otimes 2}}{\prod_{j=0}^{4}(z - t_j)}. \tag{2.11}$$
*where $a, b \in \mathbb{C}$ are scalars that do not simultaneously vanish.*

*Proof.* We have
$$\mathrm{tr}(\theta) \in H^0(\mathbb{C}P^1, L) \cong \mathbb{C}^4$$
$$\det(\theta) \in H^0(\mathbb{C}P^1, L^{\otimes 2}) \cong \mathbb{C}^7.$$
The requirement on the eigenvalues of the residues of $\theta$ together imposes 5 linear relations on $\mathrm{tr}(\theta)$; however, one of these conditions expresses that the sum of the eigenvalues is 0, and is therefore redundant. So, $\mathrm{tr}(\theta)$ is uniquely determined as $0 \in H^0(\mathbb{C}P^1, L)$, proving the first assertion.

The generic element of $\mathcal{B}$ can thus be denoted as
$$q \in H^0(\mathbb{C}P^1, L^{\otimes 2}).$$
We fix the isomorphism $\mathcal{O}(3) \cong L$ given on the affine open subset $w \neq 0$ by
$$s(z, w) = \sum_{i=0}^{3} s_i z^{3-i} w^i \mapsto S(z) = s(z, 1) \frac{\mathrm{d}z}{\prod_{j=0}^{4}(z - t_j)}. \tag{2.12}$$
Under this isomorphism, the value $s(t_j, 1)$ for $0 \leq j \leq 4$ is equal to the some non-zero multiple (only depending on the divisor $D$ and $j$) of the residue $\mathrm{res}_{t_j}(S)$. The isomorphism (2.12) induces the isomorphism $\mathcal{O}(6) \cong L^{\otimes 2}$ given by
$$q(z, w) = \sum_{i=0}^{6} q_i z^{6-i} w^i \mapsto Q(z) = q(z, 1) \frac{\mathrm{d}z^{\otimes 2}}{\prod_{j=0}^{4}(z - t_j)^2}. \tag{2.13}$$
The requirements on the eigenvalues of the residue of $\theta$ therefore impose 5 independent linear relations on $\det(\theta)$, namely that $q(t_j) = 0$ for all $0 \leq j \leq 4$. The second assertion follows.

The section $q$ is a homogeneous polynomial of degree 6, vanishing at the points of $D$ by part (2), hence is of the form
$$q(z, w) = (az - bw) \prod_{j=0}^{4} (z - t_j w)$$
for some $(a, b) \in \mathbb{C}^2 \setminus \{(0, 0)\}$. Using the isomorphism (2.13), the corresponding meromorphic quadratic differential reads as in (2.11). □

From now on, we will often let $\mathcal{M}_{\mathrm{Dol}}$ stand for $\mathcal{M}_{\mathrm{Dol}}(\mathbf{0}, \alpha)$. It follows from the Proposition that we have a well-defined map
$$H \colon \mathcal{M}_{\mathrm{Dol}} \to \mathcal{B} \tag{2.14}$$
$$(\mathcal{E}, \theta) \mapsto -\det(\theta)$$
called the *Hitchin map.* (The negative sign will simplify our later formulas.) The target space $\mathcal{B}$ of $h$ is called the *Hitchin base.*

2.5. **Spectral curve, Jacobian variety.** We consider the total space $\mathrm{Tot}(L)$ of $L$ with the natural projection

$$p_L \colon \mathrm{Tot}(L) \to \mathbb{C}P^1.$$

We denote by $\zeta$ the canonical section of $p_L^* L$. For any

$$q \in H^0(\mathbb{C}P^1, L^{\otimes 2})$$

we denote for simplicity $p_L^* q$ by $q$.

We endow $\mathcal{B}$ with a scalar product (we will be more precise in (2.23)), pick $R > 0$ and let $S^3_R$ denote the sphere of radius $R$ in $\mathcal{B} \cong \mathbb{C}^2$. For $q \in S^3_1$ we write $\zeta_\pm(Rq, z)$ for the roots of

$$\zeta^2 - Rq = 0,$$

specifically

$$\zeta_\pm(Rq, z) = \pm\sqrt{Rq(z, 1)}. \tag{2.15}$$

We denote by

$$X_{Rq} = \{([z : w], \pm\sqrt{Rq(z, w)})\} \subset \mathrm{Tot}(L) \tag{2.16}$$

the Riemann surface of the bivalued function $\zeta_\pm(Rq, z)$. For a generic choice of $q$ this curve is smooth and of genus

$$g(X_{Rq}) = 2.$$

For generic $q \in S^3_1$, the fiber $H^{-1}(q)$ is smooth, and known to be isomorphic to an abelian variety of dimension 2 over $\mathbb{C}$, namely (a torsor over) the Jacobian $\mathrm{Jac}(X_q)$ of $X_q$:

$$H^{-1}(q) \cong \mathrm{Jac}(X_q) = H^{0,1}(X_q)/\Lambda_q \tag{2.17}$$

for the period lattice $\Lambda_q \subset H^{0,1}(X_q) \cong \mathbb{C}^2$ of $X_{Rq}$. Recall that

$$\Lambda_q = \mathrm{Im}\left(p^{0,1} \circ \iota\right)$$

where the map

$$\iota \colon H^1(X_q, 2\pi\sqrt{-1}\mathbb{Z}) \to H^1(X_q, \mathbb{C})$$

is induced by the coefficient inclusion $2\pi\sqrt{-1}\mathbb{Z} \to \mathbb{C}$ and the map

$$p^{0,1} \colon H^1(X_q, \mathbb{C}) \to H^{0,1}(X_q) \tag{2.18}$$

is projection of harmonic forms to their antiholomorphic part. Then, for given $\mu_1, \mu_2 \in H^{0,1}(X_q)$ the relation

$$\mu_1 - \mu_2 \in \Lambda_q$$

is equivalent to the following condition: for every 1-cycle $A$ on $X_q$ with coefficients in $\mathbb{Z}$ we have

$$\int_A (\mu_1 - \mu_2) \in 2\pi\sqrt{-1}\mathbb{Z}.$$

The abelian version of the Hodge correspondence $\psi$ of (2.8) on $X_q$ states that any class in $H^{0,1}(X_q)$ may be represented by an anti-holomorphic form, i.e. $\mu \in \Omega^{0,1}(X_q)$ satisfying $\partial\mu = 0$, and that then the U(1)-connection on the trivial line bundle defined by the connection form

$$B = \mu - \bar{\mu} \in \Omega^1(X_q)$$

is flat, see [19, Proposition 4.1.5]. With this notation, fixing any basis $A_1, A_2, B_1, B_2 \in H_1(X_q, \mathbb{Z})$, the abelian version of $\mathrm{RH} \circ \psi$ (where RH is the Riemann–Hilbert correspondence (2.10)) is then the diffeomorphism between the Jacobian and the 4-torus given by

$$\begin{aligned} \mathrm{Jac}(X_q) &\to T^4 = (S^1)^4 \\ \mu &\mapsto \left( e^{\oint_{A_1} B}, e^{\oint_{A_2} B}, e^{\oint_{B_1} B}, e^{\oint_{B_2} B} \right). \end{aligned}$$

2.6. **Ramification of spectral curve.** Clearly, $X_{Rq}$ is ramified over $D$. Let $\tilde{D}$ denote the corresponding branch divisor, so $\tilde{D}$ consists of the preimages of the points of $D$ on $X_{Rq}$, all counted with multiplicity 1. We again let

$$Z_{\pm}(Rq, z) \tag{2.19}$$

stand for the bivalued meromorphic differentials corresponding to (2.15) over the chart $z$ under the isomorphism (2.12). In concrete terms, we have

$$Z_{\pm}(Rq, z) = \pm\sqrt{Rq(z,1)} \frac{\mathrm{d}z}{\prod_{j=0}^{4}(z - t_j)} \tag{2.20}$$

We set

$$\Delta_q = \{z \in \mathbb{C} : \quad q(z) = 0\}.$$

Regardless of the value of $R > 0$, $\Delta_q$ is the ramification divisor of the projection map

$$p_{Rq} : X_{Rq} \to \mathbb{C}P^1 \tag{2.21}$$

induced by $p_L$. $\Delta_q$ contains the points of $D$ by Proposition 2, and is of cardinality 6 because $\deg(L^{\otimes 2}) = 6$. It follows that it is of the form

$$\Delta_q = \{t_0, t_1, t_2, t_3, t_4, t(q)\} \tag{2.22}$$

for some $t(q) \in \mathbb{C}P^1$. In case $t(q) = t_{j_0}$ for some $0 \leq j_0 \leq 4$, we assign multiplicity 2 to $t_{j_0}$ in $\Delta_q$. On the other hand, for any fixed $t \in \mathbb{C}P^1 \setminus D$ we denote by $\Delta_t$ the set of $q \in S^3_1$ such that $t \in \Delta_q$. We denote by $\tilde{\Delta}_q$ the corresponding ramification points on $X_q$, and similarly by $\tilde{t}(q), \tilde{D}$ the lifts of $t(q)$ and of the divisor $D$, respectively.

**Proposition 3.** *For any fixed $t \in \mathbb{C}P^1 \setminus D$, the set $\Delta_t$ is diffeomorphic to $S^1$, and the map*

$$\begin{aligned} t \colon S^3_1 &\to \mathbb{C}P^1 \\ q &\mapsto t(q) \end{aligned}$$

*defined by* (2.22) *is the Hopf fibration.*

*Proof.* The coefficients $(a, b)$ appearing in (2.11) describe natural co-ordinates of the space $\mathcal{B} \cong \mathbb{C}^2$. For any fixed $[z_0 : w_0] \in \mathbb{C}P^1$ the condition $az_0 - bw_0 = 0$ is linear in $(a, b)$, hence $\Delta_t$ is the link of a line passing through 0 in $\mathcal{B}$. This shows the first assertion.

The map appearing in the second assertion is

$$(a, b) \mapsto [b : a].$$

This is just the canonical map from $\mathbb{C}^2 \setminus \{(0,0)\}$ to $\mathbb{C}P^1$. The result follows. $\square$

Using the factorization (2.11) we will assume that the norm of $\mathcal{B}$ is

$$|q| = \sqrt{|a|^2 + |b|^2}. \tag{2.23}$$

We use standard Hopf co-ordinates

$$a = \cos(\theta) e^{\sqrt{-1}(\varphi - \phi)}, \quad b = \sin(\theta) e^{\sqrt{-1}(\varphi + \phi)} \tag{2.24}$$

with $\theta \in [0, \frac{\pi}{2}]$ and $\varphi \in [0, 2\pi]$ and $\phi \in [0, \pi]$. Then, on the chart $\mathrm{Spec}(\mathbb{C}[z])$ the map $t$ reads as

$$t \colon (a, b) \mapsto \frac{b}{a} = \tan(\theta) e^{2\sqrt{-1}\phi}.$$

The parameter of the Hopf circles is $\varphi$.

2.7. **Perverse Leray filtration.** Consider a general quasi-projective variety $Y$ and denote by $D^b(Y, \mathbb{Q})$ the derived category of bounded complexes of $\mathbb{Q}$-vector spaces $K$ on $Y$ with constructible cohomology sheaves of finite rank. Beilinson, Bernstein and Deligne [3] defined truncation functors

$$^{\mathfrak{p}}\tau_{\leq i} : D^b(Y, \mathbb{Q}) \to {}^{\mathfrak{p}}D^{\leq i}(Y, \mathbb{Q})$$

encoding the support condition for the middle perversity function, giving rise to a system of truncations

$$0 \to \cdots \to {}^{\mathfrak{p}}\tau_{\leq -p} K \to {}^{\mathfrak{p}}\tau_{\leq -p+1} K \to \cdots \to K.$$

This gives rise to the *perverse filtration*

$$P^p H(Y, K) = \mathrm{Im}(H(Y, {}^{\mathfrak{p}}\tau_{\leq -p} K) \to H(Y, K)).$$

We will apply the above results to the following setup. Consider the right derived direct image functor

$$\mathbf{R}H_* \colon D^b(\mathcal{M}_{\mathrm{Dol}}, \mathbb{Q}) \to D^b(\mathcal{B}, \mathbb{Q})$$

and denote by $R^l H_*$ the $l$'th right derived direct image sheaf. Let $\mathbf{H}$ denote hypercohomology of a complex of sheaves and $H$ stand for cohomology of a single sheaf. (We hope that the two different usages of the symbol $H$ for the Hitchin map and for cohomology groups will not lead to confusion.) Let $\underline{\mathbb{Q}}_{\mathcal{M}_{\mathrm{Dol}}}$ denote the constant sheaf with fibers $\mathbb{Q}$ on $\mathcal{M}_{\mathrm{Dol}}$. With these notations, we will be interested in the perverse filtration on $K = \mathbf{R}H_* \underline{\mathbb{Q}}_{\mathcal{M}_{\mathrm{Dol}}}$ over $Y = \mathcal{B}$. We then have

$$\mathbf{H}^n(\mathcal{B}, \mathbf{R}H_* \underline{\mathbb{Q}}_{\mathcal{M}_{\mathrm{Dol}}}) \cong H^n(\mathcal{M}_{\mathrm{Dol}}, \mathbb{Q}).$$

We will make use of a geometric characterization of the perverse filtration provided by M. de Cataldo and L. Migliorini in [9, Theorem 4.1.1] in terms of the flag filtration $F$. Namely, let

$$Y_{-2} \subset Y_{-1} \subset Y = \mathcal{B} \tag{2.25}$$

be a generic full affine flag in $\mathcal{B}$, namely $Y_{-1}$ a generic line and $Y_{-2}$ a generic point within $Y_{-1}$. We then have the equality

$$\begin{aligned} P^p \mathbf{H}^n(Y, \mathbf{R}H_* \underline{\mathbb{Q}}_{\mathcal{M}_{\mathrm{Dol}}}) &= F^{p+n} \mathbf{H}^n(Y, \mathbf{R}H_* \underline{\mathbb{Q}}_{\mathcal{M}_{\mathrm{Dol}}}) \\ &= \mathrm{Ker}(\mathbf{H}^n(Y, \mathbf{R}H_* \underline{\mathbb{Q}}_{\mathcal{M}}) \to \mathbf{H}^n(Y_{p+n-1}, \mathbf{R}H_* \underline{\mathbb{Q}}_{\mathcal{M}}|_{Y_{p+n-1}})), \end{aligned}$$

where $F^\bullet$ stands to denote the flag filtration. It follows immediately from $Y_{-3} = \varnothing$ that $P^{1-n}\mathbf{H}^n = 0$ and $P^{-n-2}\mathbf{H}^n = \mathbf{H}^n$, so the only possibly non-trivial graded pieces live in degrees $-n-2, -n-1, -n$. Notice that for $p = -1-n$ we get

$$\mathbf{H}^*(Y_{-2}, \mathbf{R}H_* \underline{\mathbb{Q}}_{\mathcal{M}}|_{Y_{-2}}) \cong H^*(H^{-1}(Y_{-2}), \mathbb{Q}) \cong \Lambda^* H^1(H^{-1}(Y_{-2}), \mathbb{Q}),$$

the exterior algebra over $H^1(T^4, \mathbb{Q}) \cong \mathbb{Q}^4$. Moreover, by the isomorphism theorem

$$\mathrm{Gr}_P^{-n-2} H^n(\mathcal{M}_{\mathrm{Dol}}, \mathbb{Q}) \cong \mathrm{Im}(H^n(\mathcal{M}_{\mathrm{Dol}}, \mathbb{Q}) \to H^n(H^{-1}(Y_{-2}), \mathbb{Q})).$$

## 3. Large scale behaviour of solutions of Hitchin's equations

3.1. **Asymptotic abelianization, limiting configuration.** We fix a generic element $q \in S_1^3$ and consider $(\mathcal{E}, \theta) \in \mathcal{M}_{\mathrm{Dol}}(\mathbf{0}, \alpha)$ such that

$$H(\mathcal{E}, \theta) = q. \tag{3.1}$$

As we have explained in Section 2.5, choosing such a Higgs bundle $(\mathcal{E}, \theta)$ amounts to fixing a point in an abelian variety of complex dimension 2. Then, for any $t \in \mathbb{C}^\times$ we have $(\mathcal{E}, t\theta) \in \mathcal{M}_{\mathrm{Dol}}(\mathbf{0}, \alpha)$, i.e. $\mathbb{C}^\times$ acts on $\mathcal{M}_{\mathrm{Dol}}(\mathbf{0}, \alpha)$. Obviously,

$$H(\mathcal{E}, t\theta) = t^2 q \in S^3_{|t|^2}.$$

For any fixed value of $t$, there exists a unique solution $h_t$ of the real Hitchin's equation (2.4) associated to the pair $(\mathcal{E}, t\theta)$. We will summarize some results of [16] (partly based on [34] and [37]) regarding the asymptotic behaviour of the tame harmonic bundle associated to $(\mathcal{E}, \sqrt{R}\theta)$ (the parameter $t > 0$ of [16] thus being replaced by $\sqrt{R}$ with $R > 0$). The analysis in [16] relies on the assumption that $\theta$ is generically regular semisimple. This holds for generic $q \in S_1^3$. Indeed, if $\theta$ is not generically regular semisimple then the curve (2.16) is a section $s$ of $p_L$ with multiplicity 2, which is clearly not the case generically.

Let $\mathcal{L}_{\mathcal{E}} \in \mathrm{Jac}(X_q)$ be the line bundle such that

$$\mathcal{E} = p_{q*}\mathcal{L}_{\mathcal{E}} \tag{3.2}$$

(see (2.21)) and for any $R > 0$ let us denote by

$$\rho\colon X_{Rq} \to X_{Rq}$$

the involution exchanging $Z_+(Rq, z)$ and $Z_-(Rq, z)$ (see (2.19)). As $\rho$ is the restriction to $X_{Rq}$ of an algebraic involution defined over all $\mathrm{Tot}(L)$, we will omit $Rq$ from its notation. Then, there exists a short exact sequence of sheaves on $X_{Rq}$

$$0 \to p^*_{Rq}\mathcal{E} \to \mathcal{L}_{\mathcal{E}} \oplus \rho^*\mathcal{L}_{\mathcal{E}} \to \mathcal{O}_{\Delta_q} \to 0. \tag{3.3}$$

Notice that for any $R > 0$ there is an isomorphism

$$X_{Rq} \cong X_q$$

commuting with $p_L$; we deduce that the restriction of the Hitchin map $H$ to the $\mathbb{R}^+$-orbit of $q$ is canonically isomorphic to a product

$$\mathbb{R}^+ \times H^{-1}(q).$$

Therefore, in the sequel we will often identify $H^{-1}(Rq)$ and $H^{-1}(q)$.

Let $\tilde{t}(q) \in X_{Rq}$ be the preimage of $t(q)$ under $p_{Rq}$.

**Proposition 4.** *Formulas* (2.20) *define univalued holomorphic differentials on* $X_{Rq}$*, vanishing to order* 2 *at* $\tilde{t}(q)$*.*

*Proof.* The fact that the forms are univalued is clear, as $X_{Rq}$ is by definition their Riemann surface.

For simplicity, let us work on the chart $z$ of $\mathbb{C}P^1$ and set $[z : 1] = z$, a similar analysis works over the chart $w$. Furthermore, in this proof we use the notation

$$\zeta_i = \zeta_i(Rq, z)$$

with $i \in \{\pm\}$. A holomorphic chart of $X_{Rq}$ near $t_j$ is given by $\zeta_i$, with local equation

$$\zeta_i^2 = R(z - t_j)h_j(z)$$

for some holomorphic function $h_j$ (depending on $q$) such that $h_j(t_j) \neq 0$. This shows that

$$2\zeta_i \mathrm{d}\zeta_i = R\mathrm{d}z h_j(z) + R(z - t_j)\mathrm{d}h_j.$$

We derive that the 1-form $\omega$ defined by

$$\begin{aligned} \omega = \frac{\mathrm{d}z}{\zeta_i} &= \frac{1}{h_j}\left(\frac{2\mathrm{d}\zeta_i}{R} - \frac{z - t_j}{\zeta_i}\mathrm{d}h_j\right) \\ &= \frac{1}{h_j}\left(\frac{\mathrm{d}\zeta_i}{R} - \zeta_i \frac{\mathrm{d}h_j}{h_j}\right) \end{aligned}$$

is holomorphic in $\zeta_i$. The formula shows that $\omega$ is holomorphic near $z = t(q)$ too. At any point away from the ramification divisor $\Delta_q$ the form $\omega$ is obviously regular.

Now, by (2.20) we have

$$\begin{aligned} Z_i &= \pm\sqrt{R}\sqrt{az - b}\frac{\mathrm{d}z}{\prod_{j=0}^4 \sqrt{z - t_j}} \\ &= \sqrt{R}(az - b)\frac{\mathrm{d}z}{\zeta_i} = \pm\sqrt{R}(az - b)\omega, \end{aligned}$$

where the root of the polynomial $az - b$ is $t(q)$. The first assertion immediately follows. For the second assertion, it is sufficient to notice that near $z = t(q)$ we have

$$az - b = \zeta_i^2 h(\zeta_i)$$

for some non-vanishing holomorphic function $h$. This finishes the proof. □

Fix some $q \in S^3_1$ and consider a Higgs bundle $(\mathcal{E}, \theta)$ satisfying (3.1), and recall the notation (2.20). Let $\mathcal{L}_{\mathcal{E}}$ be the line bundle satisfying (3.2). By abelian Hodge theory, there exists (up to multiplication by a constant) a unique Hermitian metric $h_{\det(\mathcal{E})}$ on $\det(\mathcal{E})$ over $\mathbb{C}P^1$ satisfying:

- the associated unitary connection $\nabla^+_{h_{\det(\mathcal{E})}}$ in $\det(\mathcal{E})$ is flat (i.e.,$h_{\det(\mathcal{E})}$ is Hermitian–Einstein),
- for some local holomorphic trivialization $\mathbf{e}_1 \wedge \mathbf{e}_2$ of $\det(\mathcal{E})$ at $t_j$ we have
$$\lim_{z \to t_j} |z - t_j|^{-1} |\mathbf{e}_1 \wedge \mathbf{e}_2|_{h_{\det(\mathcal{E})}} = 1$$
for every $0 \leq j \leq 4$.

Notice that the last condition is imposed by the choice of parabolic weights (2.2).

Moreover, there exists (up to a scalar) a unique abelian Hermitian metric $h_{\mathcal{L}_{\mathcal{E}}}$ on $\mathcal{L}_{\mathcal{E}}$ over $X_q$ with parabolic points at $\tilde{\Delta}_q$ such that

- the associated unitary connection $\nabla^+_{h_{\mathcal{L}_{\mathcal{E}}}}$ in $\mathcal{L}_{\mathcal{E}}$ is flat (i.e., $h_{\mathcal{L}_{\mathcal{E}}}$ is Hermitian–Einstein),
- we have $h_{\mathcal{L}_{\mathcal{E}}} \otimes \rho^* h_{\mathcal{L}_{\mathcal{E}}} = p_q^* h_{\det \mathcal{E}}$ over $\mathbb{C}P^1 \setminus \Delta_q$ (see (3.3)),
- for some trivialization $\mathbf{l}$ of $\mathcal{L}_{\mathcal{E}}$ at each point and some local chart $\zeta$ of $X_{Rq}$ centered at $\tilde{t}_j \in \tilde{D}$ we have
$$\lim_{\zeta \to 0} |\zeta|^{-1} |\mathbf{l}|^2_{h_{\mathcal{L}_{\mathcal{E}}}} = 1,$$

- for some trivialization $\mathbf{l}$ of $\mathcal{L}_{\mathcal{E}}$ at each point and some local chart $\zeta$ of $X_{Rq}$ centered at $\tilde{t}(q)$ we have

$$\lim_{\zeta\to 0} |\zeta||\mathbf{l}|^2_{h_{\mathcal{L}_{\mathcal{E}}}} = 1,$$

Let $h_{\mathcal{E},\infty}$ be the orthogonal push-forward of $h_{\mathcal{L}_{\mathcal{E}}}$ by $p_q$ over $\mathbb{C}P^1 \setminus \Delta_q$ multiplied by $\sqrt{h_{\det(\mathcal{E})}}$, so that in the direct sum decomposition (3.3) the summands $\mathcal{L}_{\mathcal{E}}, \rho^*\mathcal{L}_{\mathcal{E}}$ are orthogonal to each other and the restrictions of $h_{\mathcal{E},\infty}$ to these summands are respectively

$$\sqrt{h_{\det(\mathcal{E})}}h_{\mathcal{L}_{\mathcal{E}}}, \qquad \sqrt{h_{\det(\mathcal{E})}}\rho^* h_{\mathcal{L}_{\mathcal{E}}}.$$

Let $\nabla^+_{h_{\mathcal{E},\infty}}$ be the flat $\mathrm{U}(1)\times\mathrm{U}(1)$-connection in $\mathcal{E}$ associated to $h_{\mathcal{E},\infty}$ over $\mathbb{C}P^1\setminus\Delta_q$. Over any simply connected subset of $\mathbb{C}P^1 \setminus \Delta_q$, let $p_{q,*}$ stand for the inverse of $p_q^*$ on either branch of $X_q$. Let

$$B_{\det(\mathcal{E})} \in \Omega^1(\mathbb{C}P^1 \setminus D, \sqrt{-1}\mathbb{R}), \qquad \frac{1}{2}p_q^* B_{\det(\mathcal{E})} + B_{\mathcal{L}_{\mathcal{E}}} \in \Omega^1(X_q \setminus \tilde{\Delta}_q, \sqrt{-1}\mathbb{R})$$

stand for the connection forms of the flat abelian U(1)-connections $\nabla^+_{h_{\det(\mathcal{E})}}, \nabla^+_{h_{\mathcal{L}_{\mathcal{E}}}}$ with respect to some smooth unitary frames. The action of $\rho$ on the connection form of $\nabla^+_{h_{\mathcal{L}_{\mathcal{E}}}}$ with respect to frames corresponding to each other under $\rho$ is given by

$$\frac{1}{2}p_q^* B_{\det(\mathcal{E})} + B_{\mathcal{L}_{\mathcal{E}}} \mapsto \frac{1}{2}p_q^* B_{\det(\mathcal{E})} - B_{\mathcal{L}_{\mathcal{E}}}.$$

By the above properties, the connection form of $\nabla^+_{h_{\mathcal{E},\infty}}$ with respect to a smooth $\rho$-equivariant unitary trivialization

$$(\mathbf{f}_+, \mathbf{f}_-) \tag{3.4}$$

of $V$ compatibe with the decomposition (3.3) reads as

$$\nabla^+_{h_{\mathcal{E},\infty}} = \begin{pmatrix} \frac{1}{2}B_{\det(\mathcal{E})} + p_{q,*}B_{\mathcal{L}_{\mathcal{E}}} & 0 \\ 0 & \frac{1}{2}B_{\det(\mathcal{E})} - p_{q,*}B_{\mathcal{L}_{\mathcal{E}}} \end{pmatrix}. \tag{3.5}$$

Moreover, if one denotes by $\mu_{\det(\mathcal{E})}, \frac{1}{2}\mu_{\det(\mathcal{E})}+\mu_{\mathcal{L}_{\mathcal{E}}}$ the $(0,1)$-forms of the $\bar\partial$-operators of the corresponding line bundles with respect to smooth unitary frames, then we have

$$B_{\det(\mathcal{E})} = \mu_{\det(\mathcal{E})} - \bar\mu_{\det(\mathcal{E})}, \tag{3.6}$$

$$B_{\mathcal{L}_{\mathcal{E}}} = \mu_{\mathcal{L}_{\mathcal{E}}} - \bar\mu_{\mathcal{L}_{\mathcal{E}}}. \tag{3.7}$$

We then obviously have $p^{0,1}B_{\det(\mathcal{E})} = \mu_{\det(\mathcal{E})}$ and $p^{0,1}B_{\mathcal{L}_{\mathcal{E}}} = \mu_{\mathcal{L}_{\mathcal{E}}}$, see (2.18). We call $(\mathcal{E}, \theta, h_{\mathcal{E},\infty})$ the *limiting configuration* associated to $(\mathcal{E}, \theta)$. We introduce the model integrable connection

$$\nabla^{\text{limiting}}_{\sqrt{R}} = \nabla^+_{h_{\mathcal{E},\infty}} + \begin{pmatrix} 2\Re Z_+(Rq, z) & 0 \\ 0 & 2\Re Z_-(Rq, z) \end{pmatrix} \tag{3.8}$$

with respect to the trivialization (3.4). Notice that the Higgs field $\theta$ and the connection matrix of $\nabla^{\text{limiting}}_{\sqrt{R}}$ are simultaneously diagonal with respect to this frame. On the other hand, we denote by $h_{\sqrt{R}}$ the solution of (2.4) and $\nabla_{\sqrt{R}}$ the Hermitian–Einstein metric and integrable connection associated to $(\mathcal{E}, \sqrt{R}\theta)$.

**Theorem 2** (T. Mochizuki)**.** [37, Corollary 2.13] *Over any simply connected compact set $K \subset \mathbb{C} \setminus \Delta_q$ there exists a gauge transformation $g_{\sqrt{R}}$ such that*

$$g_{\sqrt{R}} \cdot \nabla_{\sqrt{R}} - \nabla_{\sqrt{R}}^{\text{limiting}} \to 0$$

*(measured with respect to $h_{\sqrt{R}}$) as $R \to \infty$, uniformly over $K$. More precisely, there exist $c_2, C_2 > 0$ (depending on $K, q$) such that for any $z \in K$ we have*

$$|g_{\sqrt{R}} \cdot \nabla_{\sqrt{R}}(z) - \nabla_{\sqrt{R}}^{\text{limiting}}(z)|_h < C_2 e^{-c_2\sqrt{R}}.$$

3.2. **Fiducial solution, approximate solutions.** We will equally need the asymptotic form of the solution of Hitchin's equations near the points of $\Delta_q$, where Theorem 2 does not apply. Such a description is provided by R. Mazzeo, J. Swoboda, H. Weiss, F. Witt in [34] over a smooth projective curve $X$ of arbitrary genus. This decription is extended by L. Fredrickson, R. Mazzeo, J. Swoboda, H. Weiss in [16] to the case of a smooth projective curve $X$ of arbitrary genus for solutions of Hitchin's equations with a finite number of logarithmic singularities and adapted parabolic structure. In accordance with our notations, we let $\sqrt{R}$ be the rescaling parameter of the Higgs field, equal to the parameter $t$ of [34] and [16]. We denote the standard holomorphic co-ordinate of $\mathbb{C}$ by $\tilde{z}$ and work in a fixed disc $B_{r_0}(0) = \{|\tilde{z}| \leq r_0\}$ for some $r_0 > 0$. We write $\tilde{z} = \tilde{r}e^{\sqrt{-1}\tilde{\varphi}}$ for polar co-ordinates of a point.

We first describe the solution in the case when a logarithmic singularity of a harmonic bundle is located at 0, with Dolbeault parabolic weights denoted by $\alpha^{\pm} \in [0, 1)$ as given in (2.2). Let

$$m_{\sqrt{R}} \colon \mathbb{R}_+ \to \mathbb{R}$$

be the unique solution of the Painlevé III type equation

$$\left( \frac{\mathrm{d}^2}{\mathrm{d}\tilde{r}^2} + \frac{1}{\tilde{r}} \frac{\mathrm{d}}{\mathrm{d}\tilde{r}} \right) m_{\sqrt{R}} = 8R\tilde{r}^{-1} \sinh(2m_{\sqrt{R}})$$

satisfying the boundary behaviours

$$m_{\sqrt{R}}(\tilde{r}) \approx \left( \frac{1}{2} + \alpha_j^- - \alpha_j^+ \right) \log(\tilde{r}) = 0$$
$$m_{\sqrt{R}}(\tilde{r}) \approx \frac{1}{\pi} K_0(8\sqrt{R\tilde{r}}) \approx \frac{1}{2\pi\sqrt{2}\sqrt[4]{R\tilde{r}}} e^{-8\sqrt{R\tilde{r}}}, \quad \tilde{r} \to \infty \tag{3.9}$$

where the sign $\approx$ stands for complete asymptotic expansion and $K_0$ is the modified Bessel function (or Bessel function of imaginary argument) of order 0. Furthermore, let us set

$$F_{\sqrt{R}}(\tilde{r}) = -\frac{1}{8} + \frac{1}{4}\tilde{r}\partial_{\tilde{r}} m_{\sqrt{R}}. \tag{3.10}$$

We now spell out a one-parameter family parameterized by $R > 0$ of so-called *fiducial solutions*

$$(\nabla^+_{h^{\text{fid}}_{\sqrt{R}}}, h^{\text{fid}}_{\sqrt{R}}, \theta^{\text{fid}}_{\sqrt{R}})$$

of Hitchin's equations (2.4) on $B_{r_0}(0)$, introduced in [16, Proposition 3.9]. Here, $h^{\text{fid}}_{\sqrt{R}}$ is a Hermitian metric on the rank 2 trivial holomorphic vector bundle over the disc, $\nabla^+_{h^{\text{fid}}_{\sqrt{R}}}$ is a unitary connection and $\theta^{\text{fid}}_{\sqrt{R}}$ is an endomorphism-valued $(1, 0)$-form. They are expressed with respect to a fixed unitary frame

$$(\mathbf{e}_1^{\text{fid}}, \mathbf{e}_2^{\text{fid}}) \tag{3.11}$$

called *fiducial frame*. So, with respect to the fiducial frame the Hermitian metric $h^{\mathrm{fid}}_{\sqrt{R}}$ of this solution is given by the identity matrix. We let $A^{\mathrm{fid}}_{\sqrt{R}}$ stand for the connection form of $\nabla^{+}_{h^{\mathrm{fid}}_{\sqrt{R}}}$, with respect to the fiducial frame. Using the functions $m_{\sqrt{R}}, F_{\sqrt{R}}$ introduced above and the values fixed in (2.2), the fiducial solutions are then given by the formulas

$$A^{\mathrm{fid}}_{\sqrt{R}} = \left( \frac{\alpha^+ + \alpha^-}{4} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + F_{\sqrt{R}}(\tilde{r}) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right) 2\sqrt{-1}\mathrm{d}\tilde{\varphi} \tag{3.12}$$

$$= \left( \frac{1}{4} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + F_{\sqrt{R}}(\tilde{r}) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right) 2\sqrt{-1}\mathrm{d}\tilde{\varphi}, \tag{3.13}$$

$$\theta^{\mathrm{fid}}_{\sqrt{R}} = \begin{pmatrix} 0 & \tilde{r}^{-1/2} e^{m_{\sqrt{R}}(\tilde{r})} \\ \tilde{z}^{-1}\tilde{r}^{1/2} e^{-m_{\sqrt{R}}(\tilde{r})} & 0 \end{pmatrix} \mathrm{d}\tilde{z}. \tag{3.14}$$

There is a similar family of solutions for the ramification point $t(q)$ of the spectral curve $X_q$ of the Higgs field. We fix a holomorphic chart $\tilde{z}$ centered at $t(q)$ with associated polar coordinates denoted by $\tilde{r}, \tilde{\varphi}$. Then, with respect to a unitary frame again denoted by

$$(\mathbf{e}_1^{\mathrm{fid}}, \mathbf{e}_2^{\mathrm{fid}}) \tag{3.15}$$

one can introduce

$$A^{\mathrm{fid}}_{\sqrt{R}} = \left( \frac{1}{8} + \frac{1}{4}\tilde{r}\partial_{\tilde{r}}\ell_{\sqrt{R}} \right) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} 2\sqrt{-1}\mathrm{d}\tilde{\varphi} \tag{3.16}$$

$$\theta^{\mathrm{fid}}_{\sqrt{R}} = \begin{pmatrix} 0 & \tilde{r}^{1/2} e^{\ell_{\sqrt{R}}(\tilde{r})} \\ \tilde{z}\tilde{r}^{-1/2} e^{-\ell_{\sqrt{R}}(\tilde{r})} & 0 \end{pmatrix} \mathrm{d}\tilde{z}, \tag{3.17}$$

where $\ell_{\sqrt{R}}$ is the solution of the equation

$$\left( \frac{\mathrm{d}^2}{\mathrm{d}\tilde{r}^2} + \frac{1}{\tilde{r}}\frac{\mathrm{d}}{\mathrm{d}\tilde{r}} \right) \ell_{\sqrt{R}} = 8R\tilde{r}\sinh(2\ell_{\sqrt{R}})$$

satisfying the boundary behaviours

$$\ell_{\sqrt{R}}(\tilde{r}) \approx -\frac{1}{2}\log(\tilde{r}), \quad \tilde{r} \to 0+$$

$$\ell_{\sqrt{R}}(\tilde{r}) \approx \frac{1}{\pi} K_0\left( \frac{8}{3}\sqrt{R\tilde{r}^3} \right) \approx \frac{\sqrt{3}}{2\pi\sqrt{2}\sqrt[4]{R\tilde{r}^3}} e^{-\frac{8}{3}\sqrt{R\tilde{r}^3}}, \quad \tilde{r} \to \infty.$$

The *limiting fiducial solution* is obtained by letting $R \to \infty$ in the above formulas, specifically

$$A^{\mathrm{fid}}_{\infty} = \frac{1}{8} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} 2\sqrt{-1}\mathrm{d}\tilde{\varphi}$$

$$\theta^{\mathrm{fid}}_{\infty} = \begin{pmatrix} 0 & \tilde{r}^{1/2} \\ \tilde{r}^{1/2} e^{\sqrt{-1}\tilde{\varphi}} & 0 \end{pmatrix} \mathrm{d}\tilde{z}.$$

In order to assemble the limiting configuration and the fiducial solutions into a family of approximate solutions $h^{\mathrm{app}}_{Rq}$, [16] perform a gluing construction. We describe this construction.

We start by describing normal forms of the solution of Hitchin's equations near the points of $\Delta_q$. By [16, Proposition 3.4] there exists a unique holomorphic coordinate $\tilde{z}_t$ defined in a neighbourhood of the ramification point $t = t(q)$ such

that
$$q(\tilde{z}_t) = -\tilde{z}_t (\mathrm{d}\tilde{z}_t)^2. \tag{3.18}$$
Furthermore, there exists a holomorphic gauge
$$(\mathbf{g}_1^{\mathrm{fid}}, \mathbf{g}_2^{\mathrm{fid}}) \tag{3.19}$$
of $\mathcal{E}$ near $t(q)$ with respect to which one has
$$\theta = \begin{pmatrix} 0 & 1 \\ \tilde{z}_t & 0 \end{pmatrix} \mathrm{d}\tilde{z}_t, \quad h_{\mathcal{E},\infty} = Q_t(\tilde{z}_t) \begin{pmatrix} |\tilde{z}_t|^{\frac{1}{2}} & 0 \\ 0 & |\tilde{z}_t|^{-\frac{1}{2}} \end{pmatrix}$$
where $Q_t$ is a locally-defined smooth function, completely determined by $h_{\det \mathcal{E}}$ and $q$. Similarly, for any $0 \leq j \leq 4$ [16, Proposition 3.5] shows that there exists some holomorphic co-ordinate $\tilde{z}_j$ of $\mathcal{E}$ near $t_j$ such that we have
$$q(\tilde{z}_j) = -\tilde{z}_j^{-1} (\mathrm{d}\tilde{z}_j)^2. \tag{3.20}$$
Furthermore, there exists a holomorphic gauge
$$(\mathbf{g}_1^{\mathrm{fid}}, \mathbf{g}_2^{\mathrm{fid}}) \tag{3.21}$$
of $\mathcal{E}$ near $t_j$ with respect to which one has
$$\theta = \begin{pmatrix} 0 & 1 \\ \tilde{z}_j^{-1} & 0 \end{pmatrix} \mathrm{d}\tilde{z}_j, \quad h_{\mathcal{E},\infty} = Q_j |\tilde{z}_j|^{\alpha_j^+ + \alpha_j^-} \begin{pmatrix} |\tilde{z}_j|^{-\frac{1}{2}} & 0 \\ 0 & |\tilde{z}_j|^{\frac{1}{2}} \end{pmatrix} = Q_j \begin{pmatrix} |\tilde{z}_j|^{\frac{1}{2}} & 0 \\ 0 & |\tilde{z}_j|^{\frac{3}{2}} \end{pmatrix}$$
where $Q_j$ is a locally-defined smooth function, completely determined by $h_{\det \mathcal{E}}$ and $q$. Let us fix $r_0 > 0$ and a cutoff function $\chi \colon [0, \infty) \to [0, 1]$ such that $\chi(\tilde{r}) = 1$ for all $\tilde{r} \leq r_0$ and $\chi(\tilde{r}) = 0$ for all $\tilde{r} \geq 2r_0$. We now take $(\mathcal{E}, \theta)$ to be exactly as in the above normal forms with respect to the frames (3.19) and (3.21), and define the smooth Hermitian metric $h_{\sqrt{R}}^{\mathrm{app}}$ to be equal

- to
$$Q_t(\tilde{z}_t) \begin{pmatrix} |\tilde{z}_t|^{\frac{1}{2}} e^{\ell_{\sqrt{R}}(|\tilde{z}_t|)\chi(|\tilde{z}_t|)} & 0 \\ 0 & |\tilde{z}_t|^{-\frac{1}{2}} e^{-\ell_{\sqrt{R}}(|\tilde{z}_t|)\chi(|\tilde{z}_t|)} \end{pmatrix}$$
on $|\tilde{z}_t| \leq 2r_0$ in a holomorphic co-ordinate and gauge (3.19);
- to
$$Q_j(\tilde{z}_j) \begin{pmatrix} |\tilde{z}_j|^{\frac{1}{2}} e^{m_{\sqrt{R}}(|\tilde{z}_j|)\chi(|\tilde{z}_j|)} & 0 \\ 0 & |\tilde{z}_j|^{\frac{3}{2}} e^{-m_{\sqrt{R}}(|\tilde{z}_j|)\chi(|\tilde{z}_j|)} \end{pmatrix}$$
on $|\tilde{z}_j| \leq 2r_0$ in a holomorphic co-ordinate and gauge (3.21);
- to $h_{\mathcal{E},\infty}$ on the complement of the above discs.

Fix a background Hermitian metric $h_0$ on $V$ and let us denote by $H_{\sqrt{R}}, H_{\sqrt{R}}^{\mathrm{app}}$ the $h_0$-Hermitian sections of $End(V)$ satisfying
$$h_{\sqrt{R}}(v, w) = h_0 \left( (H_{\sqrt{R}})^{\frac{1}{2}} v, (H_{\sqrt{R}})^{\frac{1}{2}} w \right) \tag{3.22}$$
and similarly
$$h_{\sqrt{R}}^{\mathrm{app}}(v, w) = h_0 \left( (H_{\sqrt{R}}^{\mathrm{app}})^{\frac{1}{2}} v, (H_{\sqrt{R}}^{\mathrm{app}})^{\frac{1}{2}} w \right).$$
Then, for a fixed Higgs bundle $(\mathcal{E}, \theta)$ one may look for solutions $(\mathcal{E}, \sqrt{R}\theta, h_{\sqrt{R}})$ of Hitchin's equations (i.e., the Hermite–Einstein equation for the pair $(\mathcal{E}, \sqrt{R}\theta)$) in the form
$$(H_{\sqrt{R}})^{\frac{1}{2}} = e^{\gamma_{\sqrt{R}}} (H_{\sqrt{R}}^{\mathrm{app}})^{\frac{1}{2}} \tag{3.23}$$
for some $\sqrt{-1}\mathfrak{su}(V, h_{\sqrt{R}}^{\mathrm{app}})$-valued section $\gamma_{\sqrt{R}}$.

**Theorem 3.** [34, Theorem 6.7],[16, Theorem 6.2] *Assume that all the zeroes of $q$ are simple. Then, there exists $C, \mu > 0$ and a unique section $\gamma_{\sqrt{R}}$ such that the Hermitian metric* (3.22) *with* (3.23) *satisfies the Hermite–Einstein equation, and*

$$\|\gamma_{\sqrt{R}}\|_{\mathcal{C}_b^{2,\alpha}} \leq Ce^{-(\mu/2)\sqrt{R}}$$

*for an appropriate Hölder norm $\mathcal{C}_b^{2,\alpha}$.*

The practical implication of this result for our purpose is that one may perturb the approximate solution by a term exponentially small in $\sqrt{R}$ so as to obtain the solution of Hitchin's equations. We will denote by $\nabla_{\sqrt{R}}$ the flat connection associated to the solution $(\mathcal{E}, \sqrt{R}\theta, h_{\sqrt{R}})$, i.e.

$$\nabla_{\sqrt{R}} = \bar{\partial}_{\mathcal{E}} + \partial^{h_{\sqrt{R}}} + \sqrt{R}\theta + \sqrt{R}\theta^{\dagger, h_{\sqrt{R}}} \tag{3.24}$$

where $\dagger, h_{\sqrt{R}}$ stands for adjoint with respect to $h_{\sqrt{R}}$. Then, $\nabla_{\sqrt{R}}$ is approximated up to exponentially decreasing error terms in $R$ by

$$\nabla^{\mathrm{app}}_{\sqrt{R}} = \bar{\partial}_{\mathcal{E}} + \partial^{h^{\mathrm{app}}_{\sqrt{R}}} + \sqrt{R}\theta + \sqrt{R}\theta^{\dagger, h^{\mathrm{app}}_{\sqrt{R}}}.$$

## 4. Simpson's Fenchel–Nielsen co-ordinates

Simpson has defined in [45, Section 10] co-ordinates of $\mathcal{M}_{\mathrm{B}}(\mathbf{c}, \gamma)$. In this section, we will recall the definition of these co-ordinates. The general element of the Betti moduli space is a local system $V$ on $\mathbb{C}P^1 \setminus D$, given by a representation $\chi$ of its fundamental group, with eigenvalues around the punctures $t_j$ equal to $c_j^{\pm} = \pm\sqrt{-1}$. For each $2 \leq i \leq 3$ there are two different co-ordinates: $l_i \in \mathbb{C}$ and $[p_i : q_i] \in \mathbb{C}P^1$. By analogy with classical Teichmüller theory, we will call co-ordinates of the first type $l_i$ the *complex length co-ordinates* and those of the second type $[p_i : q_i]$ the *complex twist co-ordinates*. Indeed, the traditional length co-ordinates in Teichmüller space belong to $\mathbb{R}$ and $\mathbb{C}$ is its complexification; similarly, the twist co-ordinates take values in $S^1$, which is the real part $\mathbb{R}P^1$ of $\mathbb{C}P^1$ for the canonical real structure.

**Remark 1.** *The construction of the co-ordinates depend on some choices, for instance radii of discs and marked points on pairs of pants. However, as we will see in Subsections 5.2, 5.3, 6.1, the asymptotic behaviour and homotopy type of the diffeomorphism* $\mathrm{RH} \circ \psi$ *do not depend on these choices.*

### 4.1. Complex length co-ordinates.

We fix disjoint open discs $D_j$ around the points $t_j$ for $0 \leq j \leq 4$; to fix our ideas we pick $D_j = B_{r_0}(t_j)^o = \{|z - t_j| < r_0\}$ for some $0 < r_0 \ll 1$ so that the different discs $D_j$ are disjoint. (Later, from Section 5.3 on, we will allow the radii of these discs to vary independently from one another.) We then set

$$\Sigma = \mathbb{C}P^1 \setminus (D_0 \cup \cdots \cup D_4). \tag{4.1}$$

Then $\Sigma$ is a smooth surface with boundary, inheriting an orientation from $\mathbb{C}P^1$. Let us denote by $\xi_j$ the boundary component $\partial D_j$, taken with the orientation induced from $\Sigma$. Specifically, we let

$$\xi_j(\varphi) = t_j + r_0 e^{\sqrt{-1}\varphi} \quad \text{for } \varphi \in [0, 2\pi]. \tag{4.2}$$

Thus, the base point of $\xi_j$ is $t_j + r_0$. Fix a simple loop $\rho_2$ in $\Sigma$ separating the boundary components $\xi_1, \xi_2$ from the remaining boundary components $\xi_3, \xi_4, \xi_0$, and a simple loop $\rho_3$ in $\Sigma$ separating the boundary components $\xi_4, \xi_0$ from the remaining boundary components $\xi_1, \xi_2, \xi_3$, so that $\rho_2$ and $\rho_3$ be disjoint from each other. These curves then decompose $\Sigma$ into the union

$$\Sigma = S_2 \cup S_3 \cup S_4 \tag{4.3}$$

of three pairs of pants:

- $S_2$ with boundary components $\xi_1, \xi_2, \rho_2$;
- $S_3$ with boundary components $\xi_3, \rho_2, \rho_3$;
- and $S_4$ with boundary components $\xi_4, \xi_0, \rho_3$.

This decomposition gives rise to a decomposition of $\mathbb{C}P^1$ into the three closed connected analytic subsets

$$X_2 = S_2 \cup D_1 \cup D_2 \tag{4.4}$$

$$X_3 = S_3 \cup D_3 \tag{4.5}$$

$$X_4 = S_4 \cup D_4 \cup D_0. \tag{4.6}$$

Furthermore, we fix

- base points $x_i \in \mathrm{int}(S_i)$ and $s_i \in \rho_i$;
- paths $\psi_i$ connecting $x_i$ to $x_{i+1}$ passing through $s_i$;
- paths $\eta_1, \eta_2$ connecting $x_2$ respectively to the base points $t_1 + r_0$ and $t_2 + r_0$ of $\xi_1, \xi_2$;
- a path $\eta_3$ connecting $x_3$ to the base point of $\xi_3$;
- paths $\eta_4, \eta_0$ connecting $x_4$ respectively to the base points $t_4 + r_0$ and $t_0 + r_0$ of $\xi_4, \xi_0$.

As the $D_j$ will actually depend on its radius $r_0$, we need to make a coherent choice for the paths $\eta_0, \dots, \eta_4$. We achieve this for instance for $\eta_1$ by first fixing a path starting at $x_2$ and ending at $t_1$, and then restricting this fixed path to the (uniquely determined) sub-interval of its domain such that the restriction connects $x_2$ to the base point of $\xi_1$. We apply a similar procedure to $\eta_j$ for all $0 \leq j \leq 4$. We set $\rho_1 = \xi_1$ and $\rho_4 = \xi_0$.

Following [45], for $2 \leq i \leq 3$ we set $l_i(V) = l_i(\chi)$ for the trace of $\chi$ evaluated on the class of the loop $\rho_i$:

$$l_i(V) = \mathrm{tr}\, \chi[\rho_i]$$

By definition, $l_i$ is the $i$'th *complex length co-ordinate.*

4.2. **Complex twist co-ordinates.** Twist co-ordinates are only defined over the part $\mathcal{M}_{\mathrm{B}}(\mathbf{c}, \gamma)'$ of the moduli space where we have $|l_i| \neq 2$ (equivalently, the eigenvalues of $\chi[\rho_i]$ are distinct) for both $2 \leq i \leq 3$, for the complex length co-ordinates $l_i$ introduced in Section 4.1, and a further stability condition holds (see [45, Definition 5.1]). It is proven in [45, Corollary 9.2] that the homotopy type of the dual boundary complex of $\mathcal{M}_{\mathrm{B}}(\mathbf{c}, \gamma)$ agrees with the one of $\mathcal{M}_{\mathrm{B}}(\mathbf{c}, \gamma)'$.

Let us introduce the scalar quantities

$$l_1 = c_1^+ + c_1^-$$

$$l_4 = c_4^+ + c_4^-$$

$$u_i = \frac{l_{i-1} - c_i^- l_i}{c_i^+ - c_i^-}$$

$$w_i = u_i(l_i - u_i) - 1$$

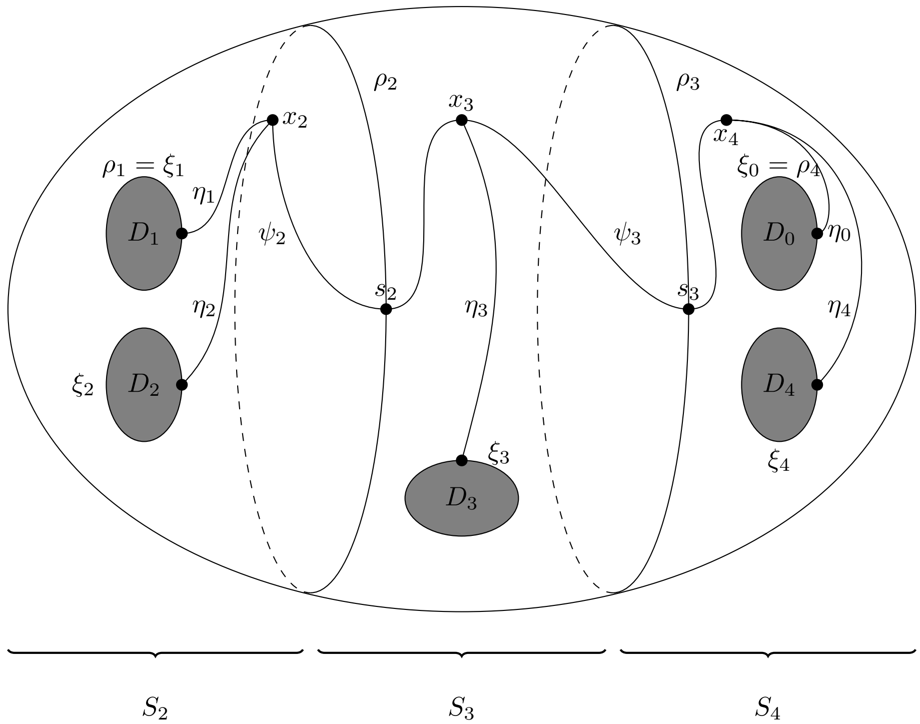



FIGURE 1. Decomposition of $S$ into three pairs of pants, indicating base points and paths. The shaded regions do not belong to $S$.

for $2 \leq i \leq 4$, where $l_i$ are the complex length co-ordinates associated to $V$ as in Section 4.1. Furthermore, introduce the matrices

$$
\begin{aligned}
A_i &= \begin{pmatrix} c_i^+ & 0 \\ 0 & c_i^- \end{pmatrix} \\
R_i &= \begin{pmatrix} u_i & 1 \\ w_i & (l_i - u_i) \end{pmatrix} \\
R'_{i-1} = A_i R_i &= \begin{pmatrix} c_i^+ u_i & c_i^+ \\ c_i^- w_i & c_i^- (l_i - u_i) \end{pmatrix} \\
T_i &= \begin{pmatrix} 0 & 1 \\ -1 & l_i \end{pmatrix} \\
U_i &= \begin{pmatrix} 1 & 0 \\ u_i & 1 \end{pmatrix}.
\end{aligned}
$$

These quantities are all determined by the fixed constants $c_i^\pm$ and the length co-ordinates $l_2, l_3$.

Let $V_i(l_{i-1}, l_i)$ denote the local system on $S_i$ whose monodromy matrices around $\rho_{i-1}, \rho_i$ and $\xi_i$, acting on its fiber over $x_i$, are respectively $R'_{i-1}, R_i, A_i$. [45, Corollary 10.3] implies that if $V|_{S_i}$ is stable then there exists a unique (up to a scalar) isomorphism

$$h_i \colon V|_{S_i} \to V_i(l_{i-1}, l_i). \tag{4.7}$$

By an abuse of notation, we denote by

$$\psi_i \colon V_{x_i} \to V_{x_{i+1}} \tag{4.8}$$

the parallel transport map of $V$ along the path $\psi_i$. Introduce

$$P_i = h_{i+1} \circ \psi_i \circ h_i^{-1} \colon V_i(l_{i-1}, l_i)_{x_i} \to V_{i+1}(l_i, l_{i+1})_{x_{i+1}} \tag{4.9}$$

and

$$Q_{i-1} = A_i^{-\frac{1}{2}} U_i P_{i-1} U_{i-1}^{-1}, \tag{4.10}$$

for any choice of the square root of $A_i$. It turns out that one has

$$Q_i = \begin{pmatrix} p_i & q_i \\ -q_i & p_i + l_i q_i \end{pmatrix} \tag{4.11}$$

for some $[p_i : q_i] \in \mathbb{C}P^1$ satisfying

$$p_i^2 + l_i p_i q_i + q_i^2 \neq 0.$$

By definition, $[p_i : q_i] \in \mathbb{C}P^1$ for $i \in \{2, 3\}$ is the $i$'th *complex twist co-ordinate.* Notice that a scalar factor on $Q_i$ has no impact on $[p_i : q_i]$. Let us introduce

$$\mathbf{Q} = \{(l, [p : q]) \in (\mathbb{C} \setminus \{\pm 2\}) \times \mathbb{C}P^1 \text{ satisfying } p^2 + lpq + q^2 \neq 0\}.$$

According to [45, Theorem 10.6], the map

$$\begin{aligned} \mathcal{M}_{\mathrm{B}}(\mathbf{c}, \gamma)' &\to \mathbf{Q}^2 \\ V &\mapsto ((l_2, [p_2 : q_2]), (l_3, [p_3 : q_3])) \end{aligned}$$

is a diffeomorphism.

4.3. **Homotopy type of compactifying divisor.** According to [45, Lemma 10.7] and [41, Lemma 6.2] we have homotopy equivalences

$$\mathbb{D}\partial\mathbf{Q} \sim S^1 \tag{4.12}$$

$$\mathbb{D}\partial\mathbf{Q}^2 \sim S^1 * S^1 \sim S^3, \tag{4.13}$$

where $X * Y$ stands for the join of the topological spaces $X, Y$. Combining these arguments, [45, Corollary 10.8] shows that

$$\mathbb{D}\partial\mathcal{M}_{\mathrm{B}}(\mathbf{c}, \gamma) \sim S^3.$$

Let us spell out explicitly the homotopy equivalence (4.12). We now consider two copies of

$$\mathbf{Q} \subset \mathbb{C}P^1 \times \mathbb{C}P^1,$$

that we will denote by $\mathbf{Q}_i$ for $i \in \{2, 3\}$. A compactification of $\mathbf{Q}_i$ is $\mathbb{C}P^1 \times \mathbb{C}P^1$, an open affine of the first component being parametrised by $l_i$, and the second component being parametrised by $[p_i : q_i]$. Let us denote by $F_{i,+}, F_{i,-}, F_{i,\infty}$ the fibers of the first projection over $2, -2$ and $\infty$ respectively. The irreducible decomposition of the compactifying divisor of $\mathbf{Q}_i$ in $\mathbb{C}P^1 \times \mathbb{C}P^1$ reads as

$$\partial\mathbf{Q}_i = \mathbb{C}P^1 \times \mathbb{C}P^1 \setminus \mathbf{Q}_i = C_i \cup F_{i,+} \cup F_{i,-} \cup F_{i,\infty}$$

where $C_i$ is the quadric defined by $p_i^2 + l_i p_i q_i + q_i^2 = 0$, see Figure 2. Clearly, $C_i$ is generically $2 : 1$ over $\mathbb{C}P^1_t$, with ramification points in the fibers $F_{i,+}, F_{i,-}$. Therefore, the compactifying divisor in $\mathbb{C}P^1 \times \mathbb{C}P^1$ is not normal crossing. To remedy this failure, we consider the blow up of $\mathbb{C}P^1 \times \mathbb{C}P^1$ in the intersection

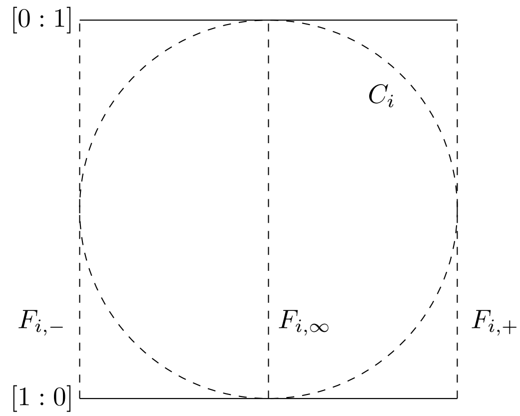



FIGURE 2. Compactifying divisor in $\mathbb{C}P^1 \times \mathbb{C}P^1$.

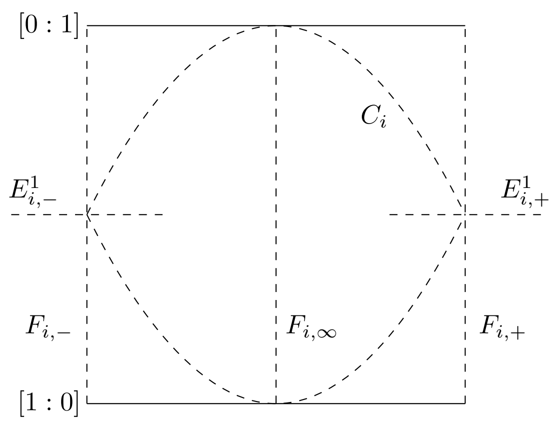



FIGURE 3. Compactifying divisor in first blow-up.

points $(2, [1 : -1])$ and $(-2, [1 : 1])$, see Figure 3. We continue to denote by $C_i, F_{i,+}, F_{i,-}, F_{i,\infty}$ the proper transforms in $X$ of the named divisors, and we denote by $E^1_{i,+}, E^1_{i,-}$ the exceptional divisors. The compactifying divisor in the blow-up is

$$C_i \cup F_{i,+} \cup F_{i,-} \cup F_{i,\infty} \cup E^1_{i,+} \cup E^1_{i,-}.$$

However, this is still not simple normal crossing, because of the triple intersection points of $C_i, E^1_{i,+}$ and $F_{i,+}$ on the one hand, and $C_i, E^1_{i,-}$ and $F_{i,-}$ on the other hand. Therefore, we need to blow up again in these intersection points, see Figure 4. The compactifying divisor in this surface is of normal crossing. Dropping the subscripts $i$ for simplicity, its dual complex is

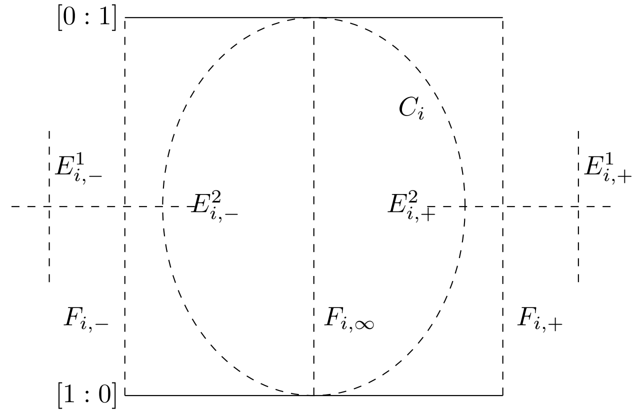
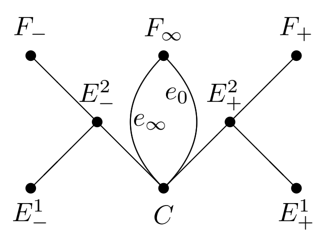



FIGURE 4. Compactifying divisor in second blow-up.




Obviously, this graph deformation retracts to the cycle defined by the vertices $F_{i,\infty}, C_i$ together with the edges $e_0, e_\infty$ connecting them. Notice that in $\mathbb{C}P^1 \times \mathbb{C}P^1$, this cycle reduces to the normal crossing components $F_{i,\infty}, C_i$.

Next, let us be more precise about the homotopy equivalence (4.13) following [41, Lemma 6.2]. The compactifying divisor of $\mathbf{Q}^2 = \mathbf{Q}_2 \times \mathbf{Q}_3$ in

$$(\mathbb{C}P^1)^4 = (\mathbb{C}P^1)^2 \times (\mathbb{C}P^1)^2$$

can be given as

$$(\partial \mathbf{Q}_2 \times (\mathbb{C}P^1)^2) \cup ((\mathbb{C}P^1)^2 \times \partial \mathbf{Q}_3).$$

As we have explained above, up to homotopy of $\mathbb{D}\partial\mathbf{Q}_i$ we only need to consider the divisor components $C_i$ and $F_{i,\infty}$ of $\partial\mathbf{Q}_i$ and the edges connecting them; in the rest of this section, we will thus replace $\mathbb{D}\partial\mathbf{Q}_i$ by this subcomplex without changing the notation. Our notation in the dual graph of $\partial\mathbf{Q}_i$ is that $e_{i,\infty}$ stands for the edge corresponding to the point

$$(l_i, [p_i : q_i]) = (\infty, [1 : 0]) \in (\mathbb{C}P^1)^2,$$

and $e_{i,0}$ for the edge corresponding to

$$(l_i, [p_i : q_i]) = (\infty, [0 : 1]) \in (\mathbb{C}P^1)^2.$$

Now, to each of the four points

$$\begin{aligned}&(\infty, [1:0]), (\infty, [1:0])\\&(\infty, [1:0]), (\infty, [0:1])\\&(\infty, [0:1]), (\infty, [1:0])\\&(\infty, [0:1]), (\infty, [0:1])\end{aligned} \tag{4.14}$$

of $(\mathbb{C}P^1)^4$, there corresponds in $\mathbb{D}\partial\mathbf{Q}^2$ a 3-dimensional simplex, namely the join of the edges in $\mathbb{D}\partial\mathbf{Q}_i$ corresponding to each component. Thus, the natural $\Delta$-complex structure of $\mathbb{D}\partial\mathbf{Q}^2 \sim S^3$ contains these four 3-simplices, which in order are

$$\begin{aligned}&e_{2,\infty} * e_{3,\infty}\\&e_{2,\infty} * e_{3,0}\\&e_{2,0} * e_{3,\infty}\\&e_{2,0} * e_{3,0}.\end{aligned}$$

## 5. Asymptotic behaviour of Fenchel–Nielsen co-ordinates

In this section we will determine the asymptotic behaviour of the co-ordinates reviewed in Section 4 as $R \to \infty$, for fixed $q \in S^3_1$. The constants we will find in this section may all depend on the divisor $D$. On the other hand, their dependence on $q$ is crucial, hence we will indicate when a constant depends on $q$.

5.1. **Monodromy of diagonalizing frames.** For our purpose, we first need to determine the monodromy transformation of a diagonalizing frame of the solution to Hitchin's equations along a loop around a logarithmic point or a ramification point.

Clearly, the gauge transformation $g_{\sqrt{R}}$ provided by Theorem 2 is unique up to a reducible transformation, i.e. one preserving the decomposition of (3.8) into abelian summands. Consider now any simple loop

$$\gamma\colon [0,1] \to \mathbb{C} \setminus \Delta_q.$$

**Definition 5.** *Let $k(\gamma, q) \in \mathbb{Z}_2$ be the number of points of $\Delta_q$ contained in one of the connected components of $\mathbb{C}P^1 \setminus \gamma$, counted with multiplicity and modulo 2.*

Notice that the number of points of $\Delta_q$ in the two connected components of $\mathbb{C}P^1 \setminus \gamma$ add up to 6, so $k(\gamma, q)$ is independent of the chosen component.

The loop $\gamma$ may be covered by a finite union of compact discs $K_1, \ldots, K_N$ as in Theorem 2, so we get for each $K_l$ a local holomorphic trivialization $(\mathbf{f}^l_1, \mathbf{f}^l_2)$ of $\mathcal{E}$ specified by the local gauges $g_{\sqrt{R}}$ provided by Theorem 2. We assume that for each $1 \leq l \leq N$ we have $K_l \cap K_{l+1} \cap \gamma([0,1]) \neq \varnothing$ (where $l = N+1$ is identified with $l = 1$), and pick any point $\gamma(\tau_l) \in K_l \cap K_{l+1}$. For $1 \leq l \leq N-1$, up to applying a constant gauge transformation over $K_{l+1}$ we may assume that

$$(\mathbf{f}^l_1(\gamma(\tau_l)), \mathbf{f}^l_2(\gamma(\tau_l))) = (\mathbf{f}^{l+1}_1(\gamma(\tau_l)), \mathbf{f}^{l+1}_2(\gamma(\tau_l))).$$

Let $M(\gamma, Rq)$ be the monodromy transformation of the local trivializations, defined by

$$(\mathbf{f}^N_1(\gamma(\tau_N)), \mathbf{f}^N_2(\gamma(\tau_N))) = (\mathbf{f}^1_1(\gamma(\tau_N)), \mathbf{f}^1_2(\gamma(\tau_N)))M(\gamma, Rq).$$

Let $T$ stand for the transposition matrix

$$T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

**Proposition 6.** *For any simple loop $\gamma$ we have*

$$M(\gamma, Rq) = \begin{pmatrix} \alpha(\gamma, Rq) & 0 \\ 0 & \delta(\gamma, Rq) \end{pmatrix} T^{k(\gamma,q)}$$

*for some $\alpha(\gamma, Rq), \delta(\gamma, Rq) \in \mathbb{C}^\times$.*

*Proof.* Recall from (2.22) that $\Delta_q = D \cup \{t(q)\}$. Assume first that $t(q) \notin D$. Then $X_{Rq}$ is smooth. Now, since all ramification points of $p|_L \colon X_{Rq} \to \mathbb{C}P^1$ are of index 2, the lift $\tilde{\gamma}$ of $\gamma$ to $X_{Rq}$ is a loop if and only if $k(\gamma, q) = 0$. Let $\tilde{\zeta}_\pm$ be a continuous lift of $\zeta_\pm$ to $\tilde{\gamma}$. Then, we have

$$\tilde{\zeta}_\pm(Rq, \tilde{\gamma}(1)) = (-1)^{k(\gamma,q)} \tilde{\zeta}_\pm(Rq, \tilde{\gamma}(0)).$$

Now, $\mathbf{f}_1^l$ and $\mathbf{f}_2^l$ belong to the $\tilde{\zeta}_+$- and $\tilde{\zeta}_-$-eigenspaces of $\theta$ respectively over $K_l$. In the case $k(\gamma, q) = 0$, eigenvectors $\mathbf{f}_1^0$ and $\mathbf{f}_1^N$ are both eigenvectors of $\theta$ for the same eigenvalue $\tilde{\zeta}_+$. It follows that they are related by some nonzero multiplicative scalar $\alpha$, which shows the result. Similarly, in case $k(\gamma, q) = 1$ the vectors $\mathbf{f}_1^0$ and $\mathbf{f}_2^N$ are eigenvectors of $\theta$ for the same eigenvalue, so they only differ by some nonzero scalar.

In case $t(q) \in D$, i.e. $t(q) = t_j$ for some $0 \leq j \leq 4$, the curve $X_{Rq}$ has an ordinary double point at $(t_j, 0)$, hence the form $\omega$ is unramified over $t_j$. If $\gamma$ is a loop enclosing $t_j$ and no other point of $D$ then $M(\gamma) = \mathrm{I}$ and $k(\gamma, q) \equiv 0 \pmod 2$ (because the points of $D$ are counted with multiplicity), so we conclude by the equality

$$T^2 = \mathrm{I}.$$

In the case of an arbitrary loop $\gamma$, one concludes by a combination of the above arguments. □

Now, assume that $t(q) \notin D$. Using the notations of (2.11), let us set

$$\tau_j = \tau_j(q) = \mathrm{res}_{z=t_j}(\partial_z^{\otimes 2} \angle Q(z)) = \frac{at_j - b}{\prod_{0 \leq k \leq 4, k \neq j}(t_j - t_k)} \in \mathbb{C} \tag{5.1}$$

(where $\angle$ stands for contraction of tensor fields) and introduce the local holomorphic co-ordinate

$$\tilde{z}_j = \tau_j(z - t_j). \tag{5.2}$$

This is indeed a local co-ordinate by the assumption that the root $t(q)$ of the linear functional $az - b$ does not belong to $D$, which means $\tau_j \neq 0$. Notice that as $\tau_j$ depends continuously on $q$, there exists some $M > 0$ only depending on $t_0, \ldots, t_4$ such that for all $q \in S_1^3(\mathbf{0})$ and all $0 \leq j \leq 4$ we have

$$|\tau_j(q)| \leq M. \tag{5.3}$$

Then, a simple computation shows that up to holomorphic terms in $z - t_j$ we have

$$\frac{\mathrm{d}\tilde{z}_j^{\otimes 2}}{\tilde{z}_j} \approx Q(z).$$

We write $\tilde{z}_j = \tilde{r}_j e^{\sqrt{-1}\tilde{\varphi}_j}$ for the polar co-ordinates of the local parameter. With respect to these polar co-ordinates, the circle $\xi_j$ of radius $r_0$ centered at $t_j$ then has the equation

$$\tilde{r}_j = |\tau_j| r_0. \tag{5.4}$$

More precisely, it follows from (5.1) that we have

$$\arg(\tilde{z}_j) = \arg(\tau_j) + \arg(z - t_j),$$

so the parameterization (4.2) of $\xi_j$ becomes

$$\xi_j(\tilde{\varphi}_j) = t_j + r_0 e^{\sqrt{-1}\tilde{\varphi}_j} \quad \text{for } \tilde{\varphi}_j \in [\arg(\tau_j), 2\pi + \arg(\tau_j)]. \tag{5.5}$$

Let $\gamma$ denote the positively oriented simple loop around $t_j$ defined by $\tilde{r}_j = r_j$ for some fixed $0 < r_j \ll 1$ chosen so that one of the connected components of $\mathbb{C}P^1 \setminus \gamma$ contains no other point of $\Delta_q$ than $t_j$. (In the case $r_j = |\tau_j| r_0$ we get $\gamma = \xi_j$, however we are not guaranteed that for a given $q$ this choice of $r_j$ satisfies the above requirement.) Let us define the unit norm trivialization

$$\mathbf{f}_1^{\mathrm{fid}}(\tilde{z}_j) = \frac{1}{\sqrt{e^{2m_{\sqrt{R}}(\tilde{r}_j)} + 1}} \begin{pmatrix} e^{m_{\sqrt{R}}(\tilde{r}_j)} \\ e^{-\sqrt{-1}\tilde{\varphi}_j/2} \end{pmatrix} \tag{5.6}$$

$$\mathbf{f}_2^{\mathrm{fid}}(\tilde{z}_j) = \frac{1}{\sqrt{e^{2m_{\sqrt{R}}(\tilde{r}_j)} + 1}} \begin{pmatrix} e^{m_{\sqrt{R}}(\tilde{r}_j)} \\ -e^{-\sqrt{-1}\tilde{\varphi}_j/2} \end{pmatrix} \tag{5.7}$$

in the disc $\tilde{r}_j \leq r_j$ with respect to the fiducial frame (3.11).

**Proposition 7.** *Let $\gamma$ be the positive simple loop defined by $\tilde{r}_j = r_j$.*

(1) *We have*

$$h_{\sqrt{R}}^{\mathrm{fid}}(\mathbf{f}_1^{\mathrm{fid}}, \mathbf{f}_2^{\mathrm{fid}}) \to 0 \quad (R \to \infty).$$

(2) *The frame $\mathbf{f}_1^{\mathrm{fid}}, \mathbf{f}_2^{\mathrm{fid}}$ diagonalizes the fiducial Higgs field (3.14) with eigenvalues*

$$\pm\sqrt{R}\tilde{r}_j^{-\frac{1}{2}} e^{-\sqrt{-1}\tilde{\varphi}_j/2} d\tilde{z}_j, \tag{5.8}$$

*where we take the determination of the angle $\tilde{\varphi}_j \in [0, 2\pi)$.*

(3) *The corresponding factors found in Proposition 6 fulfill*

$$\alpha(\gamma, Rq) = 1 = \delta(\gamma, Rq).$$

*Proof.* For part (1), as the vectors $\mathbf{f}_1^{\mathrm{fid}}, \mathbf{f}_2^{\mathrm{fid}}$ are written in a unitary frame, we simply compute

$$h_{\sqrt{R}}^{\mathrm{fid}}(\mathbf{f}_1^{\mathrm{fid}}, \mathbf{f}_2^{\mathrm{fid}}) = \frac{e^{2m_{\sqrt{R}}(\tilde{r}_j)} - 1}{e^{2m_{\sqrt{R}}(\tilde{r}_j)} + 1}.$$

Now, observe that by (3.9) we have

$$\begin{aligned} e^{m_{\sqrt{R}}(r_j)} &\approx \exp\left(\frac{1}{\pi} K_0(8\sqrt{Rr_j})\right) \\ &\approx \exp\left(\frac{1}{2\pi\sqrt{2}\sqrt[4]{Rr_j}} e^{-8\sqrt{Rr_j}}\right) \\ &\to 1 \end{aligned} \tag{5.9}$$

as $R \to \infty$, since

$$(Rr_j)^{-\frac{1}{4}} e^{-8\sqrt{Rr_j}} \to 0.$$

For part (2), we first need to determine the eigendirections of the fiducial Higgs field $\sqrt{R}\theta_{\sqrt{R}}^{\mathrm{fid}}$ with respect to the fiducial frame. We need to find the eigenvalues $\lambda_\pm$ of

$$\begin{aligned} &\sqrt{R} \begin{pmatrix} 0 & \tilde{r}_j^{-1/2} e^{m_{\sqrt{R}}(\tilde{r}_j)} \\ \tilde{z}_j^{-1} \tilde{r}_j^{1/2} e^{-m_{\sqrt{R}}(\tilde{r}_j)} & 0 \end{pmatrix} \mathrm{d}\tilde{z}_j \\ = &\sqrt{R} \begin{pmatrix} 0 & \tilde{r}_j^{-1/2} e^{m_{\sqrt{R}}(\tilde{r}_j)} \\ \tilde{r}_j^{-1/2} e^{-\sqrt{-1}\tilde{\varphi}_j - m_{\sqrt{R}}(\tilde{r}_j)} & 0 \end{pmatrix} \mathrm{d}\tilde{z}_j. \end{aligned}$$

A direct computation gives that $\lambda_\pm$ are given by (5.8), with corresponding eigenspaces spanned by unit vectors $\mathbf{f}_1^{\mathrm{fid}}, \mathbf{f}_2^{\mathrm{fid}}$ introduced in (5.6), (5.7).

For part (3), we fix $\tilde{r}_j = r_j$ and let $\tilde{\varphi}_j$ range over $[0, 2\pi]$, with a branch cut at $\tilde{\varphi}_j = 0$, and we write

$$\mathbf{f}_i^{\mathrm{fid}}(\tilde{\varphi}_j) = \mathbf{f}_i^{\mathrm{fid}}(\tilde{z}_j).$$

We find

$$\mathbf{f}_1^{\mathrm{fid}}(2\pi) = \frac{1}{\sqrt{e^{2m_{\sqrt{R}}(\tilde{r}_j)} + 1}} \begin{pmatrix} e^{m_{\sqrt{R}}(\tilde{r}_j)} \\ -1 \end{pmatrix} = \mathbf{f}_2^{\mathrm{fid}}(0)$$

$$\mathbf{f}_2^{\mathrm{fid}}(2\pi) = \frac{1}{\sqrt{e^{2m_{\sqrt{R}}(\tilde{r}_j)} + 1}} \begin{pmatrix} e^{m_{\sqrt{R}}(\tilde{r}_j)} \\ 1 \end{pmatrix} = \mathbf{f}_1^{\mathrm{fid}}(0).$$

□

Finally, let us study the neighbourhood of the ramification point $t = t(q) = \frac{b}{a}$. Here, using the notation of (2.11), we introduce the local holomorphic co-ordinate

$$\tilde{z}_t = \left( \frac{a}{\prod_{j=0}^4 (t - t_j)} \right)^{\frac{1}{3}} \left( z - \frac{b}{a} \right).$$

Then, up to at least quadratic terms in $\tilde{z}_t$ we have

$$\tilde{z}_t \mathrm{d}\tilde{z}_t^{\otimes 2} \approx Q(z).$$

We then write $\tilde{z}_t = \tilde{r}_t e^{\sqrt{-1}\tilde{\varphi}_t}$ for polar co-ordinates. Let $\gamma$ be the simple positive loop defined by $\tilde{r}_t = r_5$ for some $0 < r_5 \ll 1$ so that $\gamma$ separates $t(q)$ from the logarithmic points. Finally, introduce the unit length trivialization

$$\mathbf{f}_1^{\mathrm{fid}}(\tilde{z}_t) = \frac{1}{\sqrt{e^{2\ell_{\sqrt{R}}(\tilde{r}_t)} + 1}} \begin{pmatrix} e^{\ell_{\sqrt{R}}(\tilde{r}_t)} \\ e^{\sqrt{-1}\tilde{\varphi}_t/2} \end{pmatrix},$$

$$\mathbf{f}_2^{\mathrm{fid}}(\tilde{z}_t) = \frac{1}{\sqrt{e^{2\ell_{\sqrt{R}}(\tilde{r}_t)} + 1}} \begin{pmatrix} e^{\ell_{\sqrt{R}}(\tilde{r}_t)} \\ -e^{\sqrt{-1}\tilde{\varphi}_t/2} \end{pmatrix}$$

over the disc $\tilde{r}_t \leq r_5$ with respect to the fiducial frame (3.15).

**Proposition 8.** *Let $\gamma$ be a simple loop enclosing the ramification point $t(q)$ in counterclockwise direction such that the component of $\mathbb{C}P^1 \setminus \gamma$ containing $t(q)$ contains no logarithmic point $t_j$. Then the frame $\mathbf{f}_1^{\mathrm{fid}}, \mathbf{f}_2^{\mathrm{fid}}$ diagonalizes the fiducial Higgs field* (3.17)*, and the corresponding factors found in Proposition 6 fulfill*

$$\alpha(\gamma, Rq) = 1 = \delta(\gamma, Rq).$$

*Proof.* Similar to Proposition 7, up to the following modifications: the eigenvalues of the fiducial Higgs field are $\pm\sqrt{R\tilde{r}_t} e^{\sqrt{-1}\tilde{\varphi}_t/2}\mathrm{d}z$, with corresponding eigendirections $\mathbf{f}_1^{\mathrm{fid}}, \mathbf{f}_2^{\mathrm{fid}}$. □

5.2. **Asymptotics of complex length co-ordinates.** Here, we will study the behaviour of the complex length co-ordinates $(l_2, l_3)$ of $\mathcal{M}_{\mathrm{B}}(\mathbf{c}, \mathbf{0})$ introduced in Section 4.1 for the local systems obtained by applying the non-abelian Hodge and Riemann–Hilbert correspondences to a Higgs bundle in a Hitchin fiber close to infinity. More precisely, we set

$$l_i(\mathcal{E}, \sqrt{R}\theta) = \operatorname{tr} \mathrm{RH}(\nabla_{\sqrt{R}})[\rho_i]. \tag{5.10}$$

Notice that the connection $\nabla_{\sqrt{R}}$ depends on $(\mathcal{E}, \sqrt{R}\theta)$, hence it is justified to include the dependence of $l_i$ on $(\mathcal{E}, \sqrt{R}\theta)$ in the notation. However, to lighten notation we will sometimes simply write $l_i$. With these notations, we will determine the asymptotic behaviour as $R \to \infty$ of $l_2(\mathcal{E}, \sqrt{R}\theta), l_3(\mathcal{E}, \sqrt{R}\theta)$ for any $(\mathcal{E}, \theta) \in H^{-1}(q)$, where $q \in S^3_1$ is fixed. Throughout, by the phase of $z \in \mathbb{C}^\times$ we will mean its image under the natural projection

$$\mathbb{C}^\times \to \mathrm{U}(1).$$

According to Theorem 2 for $R \gg 0$ and at any point in the complement of $\Delta_q$ there exists a 1-parameter family of frames that asymptotically diagonalize $\theta$. The family is obtained by rescaling a given frame by diagonal elements of $\mathrm{SL}(2, \mathbb{C})$. (One may additionally apply the only non-trivial element $T$ of the Weyl group, with the effect of exchanging the two trivializations of the frame.) It follows that there exists (up to permutation and the action of the Cartan subgroup $S^1 \subset \mathrm{SL}(2, \mathbb{C})$) a unique such orthonormal frame. For any loop $\gamma$ in $\mathbb{C}P^1 \setminus \Delta_q$ let us write

$$\mathrm{RH}(\nabla_{\sqrt{R}})[\gamma] = \begin{pmatrix} a(\gamma, (\mathcal{E}, \sqrt{R}\theta)) & b(\gamma, (\mathcal{E}, \sqrt{R}\theta)) \\ c(\gamma, (\mathcal{E}, \sqrt{R}\theta)) & d(\gamma, (\mathcal{E}, \sqrt{R}\theta)) \end{pmatrix} \tag{5.11}$$

with respect to this (essentially) unique orthonormal base of the fiber $V|_{\gamma(0)}$ of the underlying smooth vector bundle $V$ at $\gamma(0)$. Notice that the effect of the action by the Cartan subgroup means that the off-diagonal entries are only defined up to a common phase factor. Our aim in this section is to study the asymptotic behaviour of the entries of $\mathrm{RH}(\nabla_{\sqrt{R}})[\rho_i]$ for $2 \leq i \leq 3$, and in particular their trace. Clearly, the set of eigenvalues (hence the trace) is invariant with respect to the action of the Weyl group. In order to achieve this, we will decompose the class of $\rho_i$ in $\pi_1(\mathbb{C}P^1 \setminus \Delta_q, \rho_i(0))$ into a concatenation of several loops (see Figure 1). The number of loops appearing in this decomposition will be 2 or 3, depending on the position of the ramification point $t(q)$ with respect to the decomposition of $S$ into pairs of pants (4.3). Around each of the loops appearing in the decomposition we will explicitly determine the monodromy, and the monodromy around $\rho_i$ is essentially the product of the monodromies of the constituent loops.

**Proposition 9.** *For any fixed $q \in S^3_1$ such that $t(q) \notin D_j$, the connection form of the flat connection associated to the fiducial solution* (3.12), (3.14) *restricted to the curve $\xi_j$ (given by $\tilde{r}_j = r_j$) with respect to the unit diagonalizing frame* (5.6), (5.7) *of the Higgs field reads as*

$$\begin{pmatrix} \frac{3}{4} + 2\sqrt{-1}\sqrt{Rr_j}\sin\left(\frac{\tilde{\varphi}_j}{2}\right) & -\frac{1}{2} r_j \partial_{\tilde{r}} m_{\sqrt{R}}(r_j) \\ -\frac{1}{2} r_j \partial_{\tilde{r}} m_{\sqrt{R}}(r_j) & \frac{3}{4} - 2\sqrt{-1}\sqrt{Rr_j}\sin\left(\frac{\tilde{\varphi}_j}{2}\right) \end{pmatrix} \sqrt{-1} d\tilde{\varphi}_j.$$

*Proof.* This follows from a straightforward computation. Set

$$F = \frac{1}{\sqrt{e^{2m_{\sqrt{R}}(r_j)} + 1}} \begin{pmatrix} e^{m_{\sqrt{R}}(r_j)} & e^{m_{\sqrt{R}}(r_j)} \\ e^{-\sqrt{-1}\tilde{\varphi}_j/2} & -e^{-\sqrt{-1}\tilde{\varphi}_j/2} \end{pmatrix}$$

for the matrix formed by the restrictions of the column vectors (5.6), (5.7) to $\xi_j$, with determinant

$$\det(F) = -\frac{2e^{m_{\sqrt{R}}(r_j) - \sqrt{-1}\tilde{\varphi}_j/2}}{e^{2m_{\sqrt{R}}(\tilde{r}_j)} + 1}$$

and inverse matrix given by

$$F^{-1} = \frac{\sqrt{e^{2m_{\sqrt{R}}(r_j)}+1}}{2e^{m_{\sqrt{R}}(r_j)-\sqrt{-1}\tilde{\varphi}_j/2}} \begin{pmatrix} e^{-\sqrt{-1}\tilde{\varphi}_j/2} & e^{m_{\sqrt{R}}(r_j)} \\ e^{-\sqrt{-1}\tilde{\varphi}_j/2} & -e^{m_{\sqrt{R}}(r_j)} \end{pmatrix}.$$

Since on $\xi_j$ we have $\mathrm{d}\tilde{r}_j = 0$, we need to compute the $\mathrm{d}\tilde{\varphi}_j$-part of

$$F^{-1}\cdot(\mathrm{d} + A^{\mathrm{fid}}_{\sqrt{R}} + \theta^{\mathrm{fid}}_{\sqrt{R}} + (\theta^{\mathrm{fid}}_{\sqrt{R}})^\dagger) = -F^{-1}\mathrm{d}F + \mathrm{Ad}_{F^{-1}}(A^{\mathrm{fid}}_{\sqrt{R}} + \theta^{\mathrm{fid}}_{\sqrt{R}} + (\theta^{\mathrm{fid}}_{\sqrt{R}})^\dagger).$$

Note first that $\mathrm{Ad}_{F^{-1}}$ acts trivially on the central part of (3.12). On the other hand, a computation shows that

$$\begin{aligned} & F^{-1}\begin{pmatrix} F_{\sqrt{R}}(r_j) & 0 \\ 0 & -F_{\sqrt{R}}(r_j) \end{pmatrix} F \\ &= \frac{1}{2e^{m_{\sqrt{R}}(r_j)-\sqrt{-1}\tilde{\varphi}_j/2}} \begin{pmatrix} 0 & -2e^{m_{\sqrt{R}}(r_j)-\sqrt{-1}\tilde{\varphi}_j/2}F_{\sqrt{R}}(r_j) \\ -2e^{m_{\sqrt{R}}(r_j)-\sqrt{-1}\tilde{\varphi}_j/2}F_{\sqrt{R}}(r_j) & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -F_{\sqrt{R}}(r_j) \\ -F_{\sqrt{R}}(r_j) & 0 \end{pmatrix}. \end{aligned}$$

By Proposition 7, $\mathrm{Ad}_{F^{-1}}(\theta^{\mathrm{fid}}_{\sqrt{R}} + (\theta^{\mathrm{fid}}_{\sqrt{R}})^\dagger)$ is diagonal with eigenvalues given by

$$\pm\sqrt{Rr_j}\sqrt{-1}(e^{\sqrt{-1}\tilde{\varphi}_j/2} - e^{-\sqrt{-1}\tilde{\varphi}_j/2})\mathrm{d}\tilde{\varphi}_j = \mp 2\sqrt{Rr_j}\sin\left(\frac{\tilde{\varphi}_j}{2}\right)\mathrm{d}\tilde{\varphi}_j.$$

Lastly, restricted to $\xi_j$ we find

$$\begin{aligned} F^{-1}\mathrm{d}F =& \frac{1}{2e^{m_{\sqrt{R}}(r_j)-\sqrt{-1}\tilde{\varphi}_j/2}} \begin{pmatrix} e^{-\sqrt{-1}\tilde{\varphi}_j/2} & e^{m_{\sqrt{R}}(r_j)} \\ e^{-\sqrt{-1}\tilde{\varphi}_j/2} & -e^{m_{\sqrt{R}}(r_j)} \end{pmatrix} \\ & \cdot \begin{pmatrix} 0 & 0 \\ -\frac{\sqrt{-1}}{2}e^{-\sqrt{-1}\tilde{\varphi}_j/2} & \frac{\sqrt{-1}}{2}e^{-\sqrt{-1}\tilde{\varphi}_j/2} \end{pmatrix}\mathrm{d}\tilde{\varphi}_j \\ =& \frac{\sqrt{-1}}{4}\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\mathrm{d}\tilde{\varphi}_j. \end{aligned}$$

We conclude combining the above computations and using the identity (see (3.10))

$$2F_{\sqrt{R}}(r_j) + \frac{1}{4} = \frac{1}{2}r_j\partial_{\tilde{r}}m_{\sqrt{R}}(r_j).$$

□

The behaviour of the entries of the matrix (5.11) for $R \gg 0$ and the choice $\gamma = \xi_j$ is given by the following.

**Proposition 10.** *Fix any $q \in S^3_1$ and consider the loop $\gamma = \xi_j$.*

(1) *The behaviour of the diagonal entries of (5.11) as $R \to \infty$ is given by the limits*

$$a(\xi_j, (\mathcal{E}, \sqrt{R}\theta)) \to 0$$
$$d(\xi_j, (\mathcal{E}, \sqrt{R}\theta)) \to 0$$

*as $R \to \infty$, at exponential rate in $\sqrt{R}$.*

(2) *The behaviour of the off-diagonal entries of* (5.11) *as* $R \to \infty$ *is given by the limits*

$$b(\xi_j, (\mathcal{E}, \sqrt{R}\theta))e^{8\Re\sqrt{\tau_j}\sqrt{Rr_0}} \to \sqrt{-1}$$
$$c(\xi_j, (\mathcal{E}, \sqrt{R}\theta))e^{-8\Re\sqrt{\tau_j}\sqrt{Rr_0}} \to \sqrt{-1}$$

*where* $\tau_j$ *is defined in* (5.1) *and* $r_0 > 0$ *is the radius of* $\xi_j$ *in the Euclidean metric.*

*Proof.* Let us set

$$r_j = |\tau_j| r_0 \in \mathbb{R}_+$$

(see (5.4)). Recall the reparameterization (5.5) of $\xi_j$ with respect to the polar coordinates of the local holomorphic chart (5.2). Integrating the connection form of the flat connection found in Proposition 9 from $\arg(\tau_j)$ to $2\pi + \arg(\tau_j)$ with respect to $\tilde{\varphi}_j$ we find the matrix

$$\begin{pmatrix} \sqrt{-1}\frac{3\pi}{2} - 8\cos\left(\frac{\arg(\tau_j)}{2}\right)\sqrt{Rr_j} & -\sqrt{-1}\pi r_j \partial_{\tilde{r}} m_{\sqrt{R}}(r_j) \\ -\sqrt{-1}\pi r_j \partial_{\tilde{r}} m_{\sqrt{R}}(r_j) & \sqrt{-1}\frac{3\pi}{2} + 8\cos\left(\frac{\arg(\tau_j)}{2}\right)\sqrt{Rr_j} \end{pmatrix}. \tag{5.12}$$

There are two cases to consider depending on whether $\Re\sqrt{\tau_j} = 0$ or $\Re\sqrt{\tau_j} \neq 0$.

We first treat the case $\Re\sqrt{\tau_j} \neq 0$. This condition is equivalent to the pair of conditions $|\tau_j| \neq 0$ (equivalently, $t(q) \notin D$) and

$$\cos\left(\frac{\arg(\tau_j)}{2}\right) \neq 0,$$

equivalently $\arg(\tau_j) \neq \pi + 2k\pi$ for $k \in \mathbb{Z}$. The matrix (5.12) is then of the form

$$\begin{pmatrix} A - C & B \\ B & A + C \end{pmatrix}$$

with

$$\begin{aligned} A = A_j &= \sqrt{-1}\frac{3\pi}{2} \\ B = B_j &= -\sqrt{-1}\pi r_j \partial_{\tilde{r}} m_{\sqrt{R}}(r_j), \\ C = C_j &= 8\cos\left(\frac{\arg(\tau_j)}{2}\right)\sqrt{Rr_j} = 8\Re\sqrt{\tau_j}\sqrt{Rr_0} \neq 0, \end{aligned}$$

where we have used (5.4) in the third line. A straightforward computation shows that setting $D = \sqrt{B^2 + C^2}$ the exponential of the negative of the above matrix is

$$\frac{-\sqrt{-1}}{2D}\begin{pmatrix} (-C + D)e^{-D} + (C + D)e^D & -2B\sinh D \\ -2B\sinh D & (C + D)e^{-D} + (-C + D)e^D \end{pmatrix}. \tag{5.13}$$

According to (3.9) as $R \to \infty$ for fixed $r_j$, we have

$$B\frac{e^{8\sqrt{Rr_j}}}{\sqrt{\pi}\sqrt[4]{Rr_j}} \to \sqrt{-1} \tag{5.14}$$

$$\frac{C}{\sqrt{Rr_j}} \to \upsilon_j \tag{5.15}$$

where we have set

$$\upsilon_j = 8\cos\left(\frac{\arg(\tau_j)}{2}\right) \in [-8, 8] \setminus \{0\}. \tag{5.16}$$

As a consequence we find

$$D \approx |C|.$$

Moreover, notice that up to higher order terms we have

$$\sqrt{B^2 + C^2} = C\sqrt{1 + \frac{B^2}{C^2}} \approx C\left(1 + \frac{B^2}{2C^2}\right),$$

implying

$$\frac{-C + \sqrt{B^2 + C^2}}{2\sqrt{B^2 + C^2}} e^{\pm\sqrt{B^2+C^2}} \approx \frac{B^2}{4C^2} e^{\pm C} \approx -\frac{\pi}{4\upsilon_j^2} \frac{e^{(-16\pm\upsilon_j)\sqrt{Rr_j}}}{\sqrt{Rr_j}}.$$

We infer that the leading order terms of the matrix (5.13) are equal to

$$\sqrt{-1}\begin{pmatrix} \frac{B^2}{4C^2}e^{-C} + e^C & -\frac{B}{C}e^{|C|} \\ -\frac{B}{C}e^{|C|} & e^{-C} + \frac{B^2}{4C^2}e^C \end{pmatrix}$$

$$\approx \begin{pmatrix} \frac{-\sqrt{-1}\pi}{4\upsilon_j^2\sqrt{Rr_j}} e^{(-16-\upsilon_j)\sqrt{Rr_j}} + \sqrt{-1}e^{\upsilon_j\sqrt{Rr_j}} & \frac{\sqrt{\pi}e^{-(8-|\upsilon_j|)\sqrt{Rr_j}}}{\upsilon_j\sqrt[4]{Rr_j}} \\ \frac{\sqrt{\pi}e^{-(8-|\upsilon_j|)\sqrt{Rr_j}}}{\upsilon_j\sqrt[4]{Rr_j}} & \sqrt{-1}e^{-\upsilon_j\sqrt{Rr_j}} + \frac{-\sqrt{-1}\pi}{4\upsilon_j^2\sqrt{Rr_j}} e^{(-16+\upsilon_j)\sqrt{Rr_j}} \end{pmatrix}$$

This matrix describes the action of $\mathrm{RH}(\nabla_{\sqrt{R}})[\xi_j]$ with respect to the bases

$$\mathbf{f}_1^{\mathrm{fid}}(0), \mathbf{f}_2^{\mathrm{fid}}(0) \quad \text{and} \quad \mathbf{f}_1^{\mathrm{fid}}(2\pi), \mathbf{f}_2^{\mathrm{fid}}(2\pi)$$

of the fiber $V|_{\xi_j(0)} = V|_{\xi_j(1)}$ (recall from the proof of Proposition 7 our convention that the argument of $\mathbf{f}_i^{\mathrm{fid}}$ is angular co-ordinate $\tilde{\varphi}_j$). In order to find the matrix of $\mathrm{RH}(\nabla_{\sqrt{R}})[\xi_j]$ with respect to the single basis $\mathbf{f}_1^{\mathrm{fid}}(0), \mathbf{f}_2^{\mathrm{fid}}(0)$, we need to multiply the above matrix from the left by the inverse of $M(\xi_j, Rq)$. By Propositions 6 and 7, $M(\xi_j, Rq) = T$ and the product is

$$\mathrm{RH}(\nabla_{\sqrt{R}})[\xi_j] = \tag{5.17}$$

$$\begin{pmatrix} \frac{\sqrt{\pi}e^{-(8-|\upsilon_j|)\sqrt{Rr_j}}}{\upsilon_j\sqrt[4]{Rr_j}} & \sqrt{-1}e^{-\upsilon_j\sqrt{Rr_j}} - \frac{\sqrt{-1}\pi}{4\upsilon_j^2\sqrt{Rr_j}} e^{(-16+\upsilon_j)\sqrt{Rr_j}} \\ -\frac{\sqrt{-1}\pi}{4\upsilon_j^2\sqrt{Rr_j}} e^{(-16-\upsilon_j)\sqrt{Rr_j}} + \sqrt{-1}e^{\upsilon_j\sqrt{Rr_j}} & \frac{\sqrt{\pi}e^{-(8-|\upsilon_j|)\sqrt{Rr_j}}}{\upsilon_j\sqrt[4]{Rr_j}} \end{pmatrix}.$$

By (5.16) we have $8 - |\upsilon_j| \geq 0$, whence we immediately get part (1) (in the case $8 - |\upsilon_j| = 0$, the assertion follows from the factor $\sqrt[4]{R}$ in the denominator). On the other hand, (5.16) also shows that $-16 - \upsilon_j \leq \upsilon_j$, with equality if and only if $\cos\left(\frac{\arg(\tau_j)}{2}\right) = -1$. In the case where $-16-\upsilon_j < \upsilon_j$, the first term of $c(\xi_j, (\mathcal{E}, \sqrt{R}\theta))$ is negligible compared to the second one, and we get (2) for $c(\xi_j, (\mathcal{E}, \sqrt{R}\theta))$. In case $\cos\left(\frac{\arg(\tau_j)}{2}\right) = -1$, the exponential factors in the two terms of $c(\xi_j, (\mathcal{E}, \sqrt{R}\theta))$ agree, however the polynomial term in $R$ converges to 0 for the first term, while is constant for the second term, again implying (2) for $c(\xi_j, (\mathcal{E}, \sqrt{R}\theta))$. A similar argument may be applied to get (2) for $b(\xi_j, (\mathcal{E}, \sqrt{R}\theta))$.

We now turn to the case $\Re\sqrt{\tau_j} = 0$. In this case, (5.12) simplifies to

$$\sqrt{-1}\begin{pmatrix} \frac{3\pi}{2} & -\pi r_j \partial_{\tilde{r}} m_{\sqrt{R}}(r_j) \\ -\pi r_j \partial_{\tilde{r}} m_{\sqrt{R}}(r_j) & \frac{3\pi}{2} \end{pmatrix}.$$

The diagonal entries of this matrix are constant, and its off-diagonal ones converge to 0 as $R \to \infty$. By continuity, the matrix exponential of the negative of this matrix converges to

$$\sqrt{-1}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

In order to obtain the matrix of $\mathrm{RH}(\nabla_{\sqrt{R}})[\xi_j]$ with respect to the single basis $\mathbf{f}_1(0), \mathbf{f}_2(0)$, we again need to multiply by the transposition matrix $T$, and this gives the desired formulas. □

Next, we will consider the loop based at $x_2$

$$\zeta_2 = \eta_1 * \xi_1 * \eta_1^{-1} * \eta_2 * \xi_2 * \eta_2^{-1}$$

enclosing the punctures $t_1, t_2$ once in counterclockwise direction. Clearly, the classes

$$[\rho_2] \in \pi_1(\mathbb{C}P^1 \setminus D, s_2) \quad \text{and} \quad [\zeta_2] \in \pi_1(\mathbb{C}P^1 \setminus D, x_2)$$

are conjugate to each other by $\psi_2$, so that $\mathrm{RH}(\nabla_{\sqrt{R}})[\zeta_2]$ is conjugate in $\mathrm{SL}(2,\mathbb{C})$ to $\mathrm{RH}(\nabla_{\sqrt{R}})[\rho_2]$ by the parallel transport map of $\nabla_{\sqrt{R}}$ along $\psi_2$. In particular, this implies that the trace of $\mathrm{RH}(\nabla_{\sqrt{R}})[\zeta_2]$ agrees with the trace $l_2(\mathcal{E}, \sqrt{R}\theta)$ of $\mathrm{RH}(\nabla_{\sqrt{R}})[\rho_2]$. Our next aim is to compute the asymptotic behaviour of the coefficients of $\mathrm{RH}(\nabla_{\sqrt{R}})[\zeta_2]$ as $R \to \infty$. In order to state the result, we need some preparation. We will consider the part lying between $t_j$ and $t_j + r_j$ of the ray emanating out from $t_j$ with direction parallel to the positive real line, and denote this path by $\sigma_j$ (see Figure 5). We then set for $j \in \{1,2\}$

$$\pi_j = \pi_j(q) = \int_{x_2}^{t_j} Z_+(q,z), \tag{5.18}$$

the contour of the line integral being $\eta_j * \sigma_j^{-1}$. Notice that this is a convergent improper integral; indeed, by (2.20) and (2.11) the integrand grows as $|z - t_j|^{-\frac{1}{2}}$ near $t_j$. We will indicate the dependence of $\pi_j$ on $q$ whenever we vary it. Notice that by its definition (2.20), $Z_+$ (hence $\pi_j$) is only defined up to a sign. We take $Z_+$ to be the square root that is the continuous extension to $\eta_j$ of the square root corresponding to the negative sign in (5.8). We will return to our choice of sign in (5.25).

**Proposition 11.** *For some choice of the diagonalizing frame at $x_2$, as $R \to \infty$, for $1 \le j \le 2$ we have*

$$\begin{aligned} a(\eta_j * \xi_j * \eta_j^{-1}, (\mathcal{E}, \sqrt{R}\theta)) &\to 0 \\ b(\eta_j * \xi_j * \eta_j^{-1}, (\mathcal{E}, \sqrt{R}\theta)) e^{4\sqrt{R}\Re\pi_j} &\to \sqrt{-1} e^{-2\int_{\eta_j} B_{\mathcal{L}_{\mathcal{E}}}} \\ c(\eta_j * \xi_j * \eta_j^{-1}, (\mathcal{E}, \sqrt{R}\theta)) e^{-4\sqrt{R}\Re\pi_j} &\to \sqrt{-1} e^{2\int_{\eta_j} B_{\mathcal{L}_{\mathcal{E}}}} \\ d(\eta_j * \xi_j * \eta_j^{-1}, (\mathcal{E}, \sqrt{R}\theta)) &\to 0. \end{aligned}$$

*Proof.* By Theorem 2, parallel transport map of $\nabla_{\sqrt{R}}$ along $\eta_j$ with respect to a diagonalizing frame of the Higgs field is approximated by the matrix

$$P_j(\mathcal{E}, \sqrt{R}\theta) = \begin{pmatrix} e^{\int_{\eta_j} \frac{1}{2} B_{\det(\mathcal{E})} + B_{\mathcal{L}_{\mathcal{E}}} + \sqrt{R}\Re Z_+(q,z)} & 0 \\ 0 & e^{\int_{\eta_j} \frac{1}{2} B_{\det(\mathcal{E})} - B_{\mathcal{L}_{\mathcal{E}}} + \sqrt{R}\Re Z_-(q,z)} \end{pmatrix}. \tag{5.19}$$

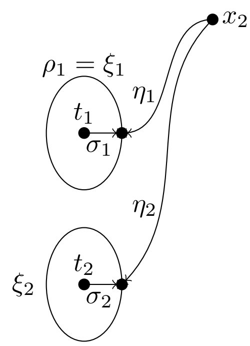



FIGURE 5. Paths $\sigma_1, \sigma_2$.

Parallel transport $P_j(\mathcal{E}, \sqrt{R}\theta)$ carries a unit length diagonalizing frame

$$(\mathbf{f}_+(x_2), \mathbf{f}_-(x_2)) \tag{5.20}$$

of $\theta_{\sqrt{R}}$ at $\eta_j(0)$ to a diagonalizing frame

$$(\mathbf{f}_+(\eta_j(1)), \mathbf{f}_-(\eta_j(1))) \tag{5.21}$$

at $\eta_j(1)$. This latter, however, is not of unit length; instead, the lengths of its vectors are given for $i \in \{\pm\}$ by

$$|\mathbf{f}_i(\eta_j(1))| = e^{\sqrt{R}\int_{\eta_j} \Re Z_i(q,z)}. \tag{5.22}$$

On the other hand, the frame (5.6), (5.7) with $\tilde{r}_j = r_0, \tilde{\varphi}_j = 0$ is an orthonormal diagonalizing frame of the fiducial Higgs field at the same point $\eta_j(1)$.

**Lemma 1.** *For suitable choices of the phases of the vectors* (5.20)*, the matrix expressing the basis elements of* (5.21) *with respect to* (5.6), (5.7) *is given by*

$$Q_j(\mathcal{E}, \sqrt{R}\theta) \approx \begin{pmatrix} e^{\int_{\eta_j} \sqrt{R}\Re Z_+(q,z)} & 0 \\ 0 & e^{\int_{\eta_j} \sqrt{R}\Re Z_-(q,z)} \end{pmatrix}$$

*Proof.* With respect to the frames (3.19) and (3.21), $\theta$ is in normal form, equal to the fiducial Higgs field. It follows that the frame (5.6), (5.7) diagonalizes both $\theta$ and the fiducial Higgs field. The same holds for (5.21). Any two diagonalizing bases of a given semi-simple (but not simple) endomorphism of a 2-dimensional vector space over $\mathbb{C}$ are related to one another by a diagonal automorphism with some diagonal elements in $\mathbb{C}^\times$. The norms of the diagonal elements have been determined in (5.22). We get

$$Q_j(\mathcal{E}, \sqrt{R}\theta) \approx \begin{pmatrix} a_j e^{\int_{\eta_j} \sqrt{R}\Re Z_+(q,z)} & 0 \\ 0 & d_j e^{\int_{\eta_j} \sqrt{R}\Re Z_-(q,z)} \end{pmatrix}$$

for some $a_j, d_j \in \mathrm{U}(1)$.

Now, the union of sufficiently narrow tubular neighbourhoods of $\eta_j$ being simply connected, Theorem 2 may be applied to it. It follows that we may choose the vectors (5.20) so as to simultaneously get rid of all the phase factors $a_j, d_j$. □

It follows that the monodromy matrix of $\nabla_{\sqrt{R}}$ along the loop $\eta_j * \xi_j * \eta_j^{-1}$ with respect to the frame (5.21) is equal to

$$\mathrm{Ad}_{P_j(\mathcal{E},\sqrt{R}\theta)^{-1}} \circ \mathrm{Ad}_{Q_j(\mathcal{E},\sqrt{R}\theta)^{-1}}(\mathrm{RH}(\nabla_{\sqrt{R}})[\xi_j]).$$

In view of Proposition 10 we find

$$a(\eta_j * \xi_j * \eta_j^{-1}, (\mathcal{E}, \sqrt{R}\theta))e^{(8-\upsilon_j)\sqrt{Rr_j}}\upsilon_j \sqrt[4]{Rr_j} \to \sqrt{\pi}$$

$$b(\eta_j * \xi_j * \eta_j^{-1}, (\mathcal{E}, \sqrt{R}\theta))e^{\sqrt{Rr_j}\upsilon_j + 4\int_{\eta_j}\sqrt{R}\Re Z_+(q,z)} \to \sqrt{-1}e^{-2\int_{\eta_j} B_{\mathcal{L}_\mathcal{E}}} \tag{5.23}$$

$$c(\eta_j * \xi_j * \eta_j^{-1}, (\mathcal{E}, \sqrt{R}\theta))e^{-\sqrt{Rr_j}\upsilon_j - 4\int_{\eta_j}\sqrt{R}\Re Z_+(q,z)} \to \sqrt{-1}e^{2\int_{\eta_j} B_{\mathcal{L}_\mathcal{E}}} \tag{5.24}$$

$$d(\eta_j * \xi_j * \eta_j^{-1}, (\mathcal{E}, \sqrt{R}\theta))e^{(8-\upsilon_j)\sqrt{Rr_j}}\upsilon_j \sqrt[4]{Rr_j} \to \sqrt{\pi}$$

with

$$\upsilon_j = 8\cos\left(\frac{\arg(\tau_j)}{2}\right),$$
$$r_j = |\tau_j| r_0.$$

Given that $B_{\mathcal{L}_\mathcal{E}}$ is a $\sqrt{-1}\mathbb{R}$-valued 1-form, the above limiting values are of length 1 and $\sqrt{\pi}$ respectively; the phases appearing in (5.23) and (5.24) will play a fundamental role in Section 6.

We now make the observation that by (2.20) and (2.11) for $0 < r_0 \ll 1$ we have

$$\begin{aligned}\int_{\sigma_j} Z_+(q,z) &= \pm \int_{\sigma_j} \sqrt{\frac{(az-b)}{\prod_{k=0}^4 (z-t_k)}}\mathrm{d}z \\ &\approx \pm \sqrt{\frac{at_j - b}{\prod_{0\le k\le 4, k\neq j}(t_j - t_k)}} \int_{t_j}^{t_j+r_0} \frac{\mathrm{d}z}{\sqrt{z-t_j}} \\ &= \pm 2\sqrt{\tau_j r_0}\end{aligned}$$

where we have used (5.1) in the last line. Notice that we have a freedom of sign in choosing both $Z_+(q,z)$ and $\sqrt{\tau_j}$. We require that the sign of $Z_+$ is chosen so that the precise form of the above equality be

$$\int_{\sigma_j} Z_+(q,z) = -2\sqrt{\tau_j r_0}. \tag{5.25}$$

We infer that the integral appearing in the exponent of the off-diagonal terms (5.23)–(5.24) is of the form

$$\begin{aligned}\sqrt{Rr_j}\upsilon_j + 4\int_{\eta_j} \sqrt{R}\Re Z_+(q,z) &= 8\sqrt{Rr_0}\Re\sqrt{\tau_j} + 4\sqrt{R}\int_{\eta_j} \Re Z_+(q,z) \\ &= 4\sqrt{R}\left(-\int_{\sigma_j} \Re Z_+(q,z) + \int_{\eta_j} \Re Z_+(q,z)\right) \\ &= 4\sqrt{R}\int_{x_2}^{t_j} \Re Z_+(q,z) \\ &= 4\sqrt{R}\Re\pi_j, \end{aligned} \tag{5.26}$$

which allows us to recast the limits (5.23)–(5.24) in the desired form. The assertion about the diagonal terms follows as in Proposition 10 (see the paragraph following formula (5.17)). $\square$

**Proposition 12.** *Fix $q \in S^3_1$ and consider the loop $\gamma = \rho_2$. In case $\Re(\pi_1 - \pi_2) \neq 0$ we have the limit*

$$l_2(\mathcal{E}, \sqrt{R}\theta)^{-1} 2\cosh\left(2\int_{\eta_2-\eta_1} B_{\mathcal{L}_\mathcal{E}} + 4\sqrt{R}\Re(\pi_2 - \pi_1)\right) \to -1$$

*as $R \to \infty$. In case $\Re(\pi_1 - \pi_2) = 0$ the limit of $l_2(\mathcal{E}, \sqrt{R}\theta)$ as $R \to \infty$ exists and is finite.*

*Proof.* As mentioned above, it is sufficient to find the asymptotic behaviour of the diagonal entries of $\mathrm{RH}(\nabla_{\sqrt{R}})[\zeta_2]$. Now, we have

$$\begin{aligned}&\mathrm{RH}(\nabla_{\sqrt{R}})[\zeta_2] \\ &= \mathrm{RH}(\nabla_{\sqrt{R}})[\eta_1 * \xi_1 * \eta_1^{-1}]\,\mathrm{RH}(\nabla_{\sqrt{R}})[\eta_2 * \xi_2 * \eta_2^{-1}].\end{aligned} \tag{5.27}$$

By definition, $l_2(\mathcal{E}, \sqrt{R}\theta)$ is the trace of this matrix, hence we need to compute the diagonal entries of the product (5.27).

Its entry $a(\zeta_2, (\mathcal{E}, \sqrt{R}\theta))$ of index $(1,1)$ is the sum

$$a(\eta_1*\xi_1*\eta_1^{-1}, (\mathcal{E}, \sqrt{R}\theta))a(\eta_2*\xi_2*\eta_2^{-1}, (\mathcal{E}, \sqrt{R}\theta)) + b(\eta_1*\xi_1*\eta_1^{-1}, (\mathcal{E}, \sqrt{R}\theta))c(\eta_2*\xi_2*\eta_2^{-1}, (\mathcal{E}, \sqrt{R}\theta)).$$

According to (5.17) the leading order term of the asymptotic expansion of its first term as $R \to \infty$ is given by

$$\pi \frac{e^{-(8-\upsilon_1)\sqrt{Rr_1}-(8-\upsilon_2)\sqrt{Rr_2}}}{\upsilon_1\upsilon_2\sqrt[4]{r_1 r_2 R^2}}. \tag{5.28}$$

The leading order term of the asymptotic expansion of the second term of $a(\zeta_2, (\mathcal{E}, \sqrt{R}\theta))$ is

$$-\exp\left(2\int_{\eta_2-\eta_1} B_{\mathcal{L}_\mathcal{E}} + 4\sqrt{R}\Re(\pi_2 - \pi_1)\right). \tag{5.29}$$

We again emphasize that this formula gives the polar decomposition of the corresponding term, as $\int_{\eta_2-\eta_1} B_{\mathcal{L}_\mathcal{E}}$ is purely imaginary.

The terms of the $(2,2)$-entry $d(\zeta_2, (\mathcal{E}, \sqrt{R}\theta))$ of (5.27) are similar to those of $a(\zeta_2, (\mathcal{E}, \sqrt{R}\theta))$, up to exchanging the subscripts $j = 1$ and $j = 2$ of $\upsilon_j, r_j, \eta_j$. Namely, the term coming from the product of diagonal entries of the factors has leading order term in its asymptotic expansion given by (5.28), and the leading order term of its other term is

$$-\exp\left(2\int_{\eta_1-\eta_2} B_{\mathcal{L}_\mathcal{E}} + 4\sqrt{R}\Re(\pi_1-\pi_2)\right). \tag{5.30}$$

Notice that the product of (5.29) and (5.30) equals 1.

Now, we observe that by (5.16) the coefficient of $\sqrt{R}$ in the exponent of (5.28) is never positive (as it has already been pointed out in the proof of Proposition 10). On the other hand, at least one of the coefficients of $\sqrt{R}$ in the exponent of (5.29) and in the exponent of (5.30) is non-negative. In the extreme case where the coefficients of $\sqrt{R}$ in the exponent of all terms (5.28), (5.29) and (5.30) vanish, then the $\sqrt{R}$ in the denominator of (5.28) guarantees that it is negligible compared to the sum of the other two terms. To sum up, this shows that in the trace, the leading-order term may not be (5.28), rather it is equal to (5.29) or (5.30) according as $\Re(\pi_1-\pi_2)<0$ or $\Re(\pi_1-\pi_2)>0$, and to the sum of these terms if $\Re(\pi_1-\pi_2)=0$. In any case, the term (5.28) converges to 0 as $R\to\infty$, and if $\Re(\pi_1-\pi_2)>0$ (respectively, $\Re(\pi_1-\pi_2)<0$) then the same limit holds for the term (5.29) (respectively, (5.30)). Finally, we conclude using that for a ray $\mathbb{C}$ of the form $te^{\sqrt{-1}\phi_0}$ with fixed $-\frac{\pi}{2}<\phi_0<\frac{\pi}{2}$ and variable $t>0$ we have

$$\lim_{t\to\infty} 2\cosh(te^{\sqrt{-1}\phi_0})e^{-te^{\sqrt{-1}\phi_0}} = 1.$$

□

Recall Hopf co-ordinates (2.24) on $S^3_1$, $\varphi$ being the co-ordinate along the Hopf fibers.

**Proposition 13.** *Fix $q\in S^3_1$ and assume $\pi_1(q)\neq\pi_2(q)$. Then there exists a unique $\varphi_2\in[0,2\pi)$ such that for every $\mathcal{L}_\mathcal{E}\in \mathrm{Jac}(X_q)$ the co-ordinate $l_2(\mathcal{E}, e^{\sqrt{-1}\varphi_2}\sqrt{R}\theta)$ remains bounded as $R\to\infty$.*

*Proof.* According to Proposition 12, $l_2(\mathcal{E}, e^{\sqrt{-1}\varphi_2}\sqrt{R}\theta)$ is bounded as $R\to\infty$ if and only if the equation

$$\Re(\pi_1(e^{\sqrt{-1}\varphi}q)-\pi_2(e^{\sqrt{-1}\varphi}q))=0$$

holds for the variable $\varphi\in[0,2\pi)$. This quantity is the horizontal projection of

$$\int_{t_2}^{t_1} Z_+(e^{\sqrt{-1}\varphi}q,z),$$

where the contour of integration is $\sigma_1*\eta_1^{-1}*\eta_2*\sigma_2^{-1}$. Now, taking into account the definition (2.20), we have

$$Z_+(e^{\sqrt{-1}\varphi}q,z)=e^{\sqrt{-1}\varphi/2}Z_+(q,z).$$

Clearly, there exists a unique value $\varphi_2\in[0,2\pi)$ satisfying the property that the non-zero complex number $\pi_1(q)-\pi_2(q)$ multiplied by the unit length complex number $e^{\sqrt{-1}\varphi_2/2}$ has horizontal projection equal to 0. □

The results of this subsection have been stated for the case of $l_2(\mathcal{E},\sqrt{R}\theta)$. We now proceed to stating the results analogous to Propositions 12 and 13 for the case of $l_3(\mathcal{E}, \sqrt{R}\theta)$, whose proofs are straightforward modifications of the ones proven so far. For $j \in \{0, 4\}$ introduce

$$\pi_j = \pi_j(q) = \int_{x_4}^{t_j} Z_+(q, z)$$

along some path in $S_3$.

**Proposition 14.** *Fix $q \in S_1^3$.*

(1) *In case $\Re(\pi_4(q) - \pi_0(q)) \neq 0$ we have the limit*

$$l_3(\mathcal{E}, \sqrt{R}\theta)^{-1} 2 \cosh \left( 2 \int_{\eta_4 - \eta_0} B_{\mathcal{L}_\mathcal{E}} + 4\sqrt{R}\Re(\pi_4 - \pi_0) \right) \to -1$$

*as $R \to \infty$.*

(2) *In case $\Re(\pi_4(q) - \pi_0(q)) = 0$ the limit of $l_3(\mathcal{E}, \sqrt{R}\theta)$ as $R \to \infty$ exists and is finite.*

(3) *If $\pi_4(q) \neq \pi_0(q)$, then there exists a unique $\varphi_3 \in [0, 2\pi)$ such that for every $\mathcal{L}_\mathcal{E} \in \mathrm{Jac}(X_q)$ the co-ordinate $l_3(\mathcal{E}, e^{\sqrt{-1}\varphi_3}\sqrt{R}\theta)$ remains bounded as $R \to \infty$.*

5.3. **Asymptotic behaviour of complex twist co-ordinates.** For $i \in \{2, 3\}$ we let

$$[p_i(\mathcal{E}, \sqrt{R}\theta) : q_i(\mathcal{E}, \sqrt{R}\theta)]$$

stand for the complex twist co-ordinates $[p_i : q_i]$ introduced in Section 4.2 of the local system $\mathrm{RH} \circ \psi(\mathcal{E}, \sqrt{R}\theta)$. In this section we will determine the asymptotic behaviour of these co-ordinates as $R \to \infty$. For this purpose, we first determine the asymptotic behaviour of the quantities

$$u_i, w_i, A_i, R_i, R'_{i-1}, T_i, U_i, h_i, \psi_i, P_i, Q_i$$

introduced in Section 4.2. For ease of notation, we will often omit to indicate the dependence of the co-ordinates on $(\mathcal{E}, \sqrt{R}\theta)$.

In what follows, for $2 \times 2$ matrices $A, B$ with non-vanishing entries depending on a parameter $R \in \mathbb{R}$ we write $A \approx B$ whenever the limit of each entry of $A$ divided by the corresponding entry of $B$ converges to 1 as $R \to \infty$. Similarly, for two scalar quantities $a, b$ depending on $R \in \mathbb{R}$ we write $a \approx b$ to express that $\frac{a}{b} \to 1$ as $R \to \infty$. We say $a$ is negligible compared to $b$ if $\frac{a}{b} \to 0$ as $R \to \infty$.

Recall from Section 2.2 that $c_j^\pm = \pm\sqrt{-1}$. We start by recording some asymptotic behaviours as $R \to \infty$ ensuing from Sections 4.2 and 5.2 in the case $\Re(\pi_1 - \pi_2) \neq 0$

$$l_1 = c_1^+ + c_1^- = 0$$

$$l_2 \approx -2\cosh\left(4\sqrt{R}\Re(\pi_2 - \pi_1) + 2\int_{\eta_2-\eta_1} B_{\mathcal{L}_\mathcal{E}}\right) \tag{5.31}$$

$$u_2 = \frac{l_1 - c_2^- l_2}{c_2^+ - c_2^-} \approx -\cosh\left(4\sqrt{R}\Re(\pi_2 - \pi_1) + 2\int_{\eta_2-\eta_1} B_{\mathcal{L}_\mathcal{E}}\right) \tag{5.32}$$

$$U_2 \approx \begin{pmatrix} 1 & 0 \\ -\cosh\left(4\sqrt{R}\Re(\pi_2 - \pi_1) + 2\int_{\eta_2-\eta_1} B_{\mathcal{L}_\mathcal{E}}\right) & 1 \end{pmatrix} \tag{5.33}$$

$$A_2 = \begin{pmatrix} c_2^+ & 0 \\ 0 & c_2^- \end{pmatrix} = \sqrt{-1}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{5.34}$$

$$R_2 \approx \begin{pmatrix} -\cosh\left(4\sqrt{R}\Re(\pi_2 - \pi_1) + 2\int_{\eta_2-\eta_1} B_{\mathcal{L}_\mathcal{E}}\right) & 1 \\ * & -\cosh\left(4\sqrt{R}\Re(\pi_2 - \pi_1) + 2\int_{\eta_2-\eta_1} B_{\mathcal{L}_\mathcal{E}}\right) \end{pmatrix} \tag{5.35}$$

$$R_1' \approx \begin{pmatrix} -\sqrt{-1}\cosh\left(4\sqrt{R}\Re(\pi_2 - \pi_1) + 2\int_{\eta_2-\eta_1} B_{\mathcal{L}_\mathcal{E}}\right) & \sqrt{-1} \\ * & \sqrt{-1}\cosh\left(4\sqrt{R}\Re(\pi_2 - \pi_1) + 2\int_{\eta_2-\eta_1} B_{\mathcal{L}_\mathcal{E}}\right) \end{pmatrix} \tag{5.36}$$

where the entries marked by $*$ may be determined by the (known) determinant of the matrices, but we refrain from spelling them out as they will be irrelevant for our purposes. We set

$$\pi_3 = \pi_3(q) = \int_{x_3}^{t_3} Z_+(q, z).$$

**Proposition 15.** *For $2 \leq i \leq 4$ we have the asymptotic behaviours*

$$h_i^{-1} \approx \begin{pmatrix} v_i & w_i e^{-4\sqrt{R}\Re\pi_i - 2\int_{\eta_i} B_{\mathcal{L}_\mathcal{E}}} \\ -v_i e^{4\sqrt{R}\Re\pi_i + 2\int_{\eta_i} B_{\mathcal{L}_\mathcal{E}}} & w_i \end{pmatrix}$$

*and*

$$h_i \approx \frac{1}{2v_i w_i}\begin{pmatrix} w_i & -w_i e^{-4\sqrt{R}\Re\pi_i - 2\int_{\eta_i} B_{\mathcal{L}_\mathcal{E}}} \\ v_i e^{4\sqrt{R}\Re\pi_i + 2\int_{\eta_i} B_{\mathcal{L}_\mathcal{E}}} & v_i \end{pmatrix}$$

*for some $v_i, w_i \in \mathbb{C}^\times$.*

*Proof.* Let us recall from Subsection 4.2 that $h_i$ is the constant matrix that identifies $V|_{S_i}$ with the model local system $V_i(l_{i-1}, l_i)$ admitting the same monodromies (up to conjugacy) around the punctures. These local systems are both given in terms of their fiber $\mathbb{C}^2$ over $x_i$ endowed with an action of $\pi_1(S_i, x_i)$, where we identify $V|_{x_i}$ with $\mathbb{C}^2$ using an orthonormal diagonalizing frame $(\mathbf{f}_+(x_i), \mathbf{f}_-(x_i))$ of $\theta$.

By definition, the monodromy matrix of $V_i(l_{i-1}, l_i)$ around the loop $\eta_i * \xi_i * \eta_i^{-1}$ centered at $x_i$ is the diagonal matrix $A_i$. On the other hand, by Proposition 11 the monodromy matrix of $V|_{S_i}$ around the same loop with respect to $(\mathbf{f}_+(x_i), \mathbf{f}_-(x_i))$

is given by

$$\begin{pmatrix} \frac{\sqrt{\pi}e^{-(8-\upsilon_i)\sqrt{Rr_i}}}{\upsilon_i\sqrt[4]{Rr_i}} & -\sqrt{-1}e^{-4\sqrt{R}\Re\pi_i-2\int_{\eta_i}B_{\mathcal{L}\mathcal{E}}} \\ -\sqrt{-1}e^{4\sqrt{R}\Re\pi_i+2\int_{\eta_i}B_{\mathcal{L}\mathcal{E}}} & \frac{\sqrt{\pi}e^{-(8-\upsilon_i)\sqrt{Rr_i}}}{\upsilon_i\sqrt[4]{Rr_i}} \end{pmatrix}. \tag{5.37}$$

Now, in order to determine $h_i^{-1}$ we need to find the eigenvectors of the above matrix. As we have shown in Proposition 11, the diagonal entries converge to 0 and its determinant obviously converges to 1. Therefore, the eigenvalues are $\pm\sqrt{-1}$, and a direct computation then shows that $h_i^{-1}$ is of the desired form. We conclude by taking matrix inverse. □

**Remark 2.** *Remember from* (5.18) *that $\pi_i$ is only defined up to a sign because the same holds for $Z_+$. In case we change the sign of $Z_+$, the vectors of the frame $(\mathbf{f}_+(x_i), \mathbf{f}_-(x_i))$ get interchanged with each other, because the first of these spans the $Z_+$-eigenspace of $\theta$. Writing the monodromy matrix of $V|_{S_i}$ around the loop $\eta_i * \xi_i * \eta_i^{-1}$ with respect to the frame $(\mathbf{f}_-(x_i), \mathbf{f}_+(x_i))$ can be obtained by conjugating* (5.37) *by the transposition matrix $T$. This gives the exact same matrix* (5.37) *(up to the change of sign of $\pi_i$), so the value of its diagonalizing endomorphism $h_i$ is independent of the choice of sign of $\pi_i$.*

**Proposition 16.** (1) *If $\Re(\pi_2-\pi_1)>0$ then we have*

$$h_2^{-1} \approx \upsilon_2 \begin{pmatrix} 1 & -2e^{-4\sqrt{R}\Re(\pi_2-\pi_1)-2\int_{\eta_2-\eta_1}B_{\mathcal{L}\mathcal{E}}} \\ -e^{4\sqrt{R}\Re\pi_2+2\int_{\eta_2}B_{\mathcal{L}\mathcal{E}}} & -2e^{4\sqrt{R}\Re\pi_1+2\int_{\eta_1}B_{\mathcal{L}\mathcal{E}}} \end{pmatrix}$$

*for some $\upsilon_2 \in \mathbb{C}^\times$.*

(2) *If $\Re(\pi_2-\pi_1)<0$ then we have*

$$h_2^{-1} \approx \upsilon_2 \begin{pmatrix} 1 & 2e^{4\sqrt{R}\Re(\pi_2-\pi_1)+2\int_{\eta_2-\eta_1}B_{\mathcal{L}\mathcal{E}}} \\ -e^{4\sqrt{R}\Re\pi_2+2\int_{\eta_2}B_{\mathcal{L}\mathcal{E}}} & 2e^{4\sqrt{R}\Re(2\pi_2-\pi_1)+2\int_{2\eta_2-\eta_1}B_{\mathcal{L}\mathcal{E}}} \end{pmatrix}$$

*for some $\upsilon_2 \in \mathbb{C}^\times$.*

*Proof.* By Proposition 15, we just need to find the values of $\upsilon_2, w_2$; for this purpose, we will use the monodromy around the loop $\eta_1 * \xi_1 * \eta_1^{-1}$. Indeed, it is required that the $(1,2)$-entry of

$$h_2\,\mathrm{RH}(\nabla_{\sqrt{R}})[\eta_1 * \xi_1 * \eta_1^{-1}]h_2^{-1}$$

be equal to $\sqrt{-1}$. After elementary algebra, this entry is asymptotic to

$$-\sqrt{-1}\frac{w_2}{2\upsilon_2}\left(e^{-4\sqrt{R}\Re\pi_1-2\int_{\eta_1}B_{\mathcal{L}\mathcal{E}}} - e^{4\sqrt{R}\Re(\pi_1-2\pi_2)+2\int_{\eta_1-2\eta_2}B_{\mathcal{L}\mathcal{E}}}\right).$$

Now, if $\Re(\pi_2-\pi_1)>0$ then the second term in this expression is negligible compared to the first one, therefore the condition for this entry to be equal to $\sqrt{-1}$ reads as

$$w_2 \approx -2\upsilon_2 e^{4\sqrt{R}\Re\pi_1+2\int_{\eta_1}B_{\mathcal{L}\mathcal{E}}}\left(1+e^{-8\sqrt{R}\Re(\pi_2-\pi_1)-4\int_{\eta_2-\eta_1}B_{\mathcal{L}\mathcal{E}}}+\cdots\right),$$

the first term being dominant. Plugging this value into Proposition 15, we find the desired result.

The second case can be proven similarly. □

**Proposition 17.** *Fix $q \in S_1^3$.*

(1) *Assume $\int_{t_1}^{t_2}\Re Z_+ = \Re(\pi_2-\pi_1) > 0$.*

(a) *If*

$$2\int_{t_1}^{t_2} \Re Z_+ < \int_{t_2}^{t_3} \Re Z_+$$

*then we have*

$$\frac{p_2}{q_2} \approx \frac{1}{2} e^{-\int_{t_1}^{t_2}(4\sqrt{R}\Re Z_+ + 2B_{\mathcal{L}_\mathcal{E}})} \to 0$$

(b) *If*

$$\int_{t_1}^{t_2} \Re Z_+ < \int_{t_2}^{t_3} \Re Z_+ < 2\int_{t_1}^{t_2} \Re Z_+$$

*then we have*

$$\frac{p_2}{q_2} \approx -e^{-\int_{t_2}^{t_3}(4\sqrt{R}\Re Z_+ + 2B_{\mathcal{L}_\mathcal{E}}) + \int_{t_1}^{t_2}(4\sqrt{R}\Re Z_+ + 2B_{\mathcal{L}_\mathcal{E}})} \to 0$$

(c) *If*

$$0 < \int_{t_2}^{t_3} \Re Z_+ < \int_{t_1}^{t_2} \Re Z_+$$

*then we have*

$$\frac{p_2}{q_2} \approx -e^{-\int_{t_2}^{t_3}(4\sqrt{R}\Re Z_+ + 2B_{\mathcal{L}_\mathcal{E}}) + \int_{t_1}^{t_2}(4\sqrt{R}\Re Z_+ + 2B_{\mathcal{L}_\mathcal{E}})} \to \infty$$

(d) *If*

$$\int_{t_2}^{t_3} \Re Z_+ < 0$$

*then we have*

$$\frac{p_2}{q_2} \approx \frac{1}{2} e^{\int_{t_1}^{t_2}(4\sqrt{R}\Re Z_+ + 2B_{\mathcal{L}_\mathcal{E}})} \to \infty$$

*as* $R \to \infty$.

(2) *Assume* $\int_{t_1}^{t_2} \Re Z_+ = \Re(\pi_2 - \pi_1) < 0$.

(a) *If*

$$\int_{t_2}^{t_3} \Re Z_+ < 2\int_{t_1}^{t_2} \Re Z_+$$

*then we have*

$$\frac{p_2}{q_2} \approx \frac{1}{2} e^{\int_{t_1}^{t_2}(4\sqrt{R}\Re Z_+ + 2B_{\mathcal{L}_\mathcal{E}})} \to 0$$

(b) *If*

$$2\int_{t_1}^{t_2} \Re Z_+ < \int_{t_2}^{t_3} \Re Z_+ < \int_{t_1}^{t_2} \Re Z_+$$

*then we have*

$$\frac{p_2}{q_2} \approx -e^{\int_{t_2}^{t_3}(4\sqrt{R}\Re Z_+ + 2B_{\mathcal{L}_\mathcal{E}}) - \int_{t_1}^{t_2}(4\sqrt{R}\Re Z_+ + 2B_{\mathcal{L}_\mathcal{E}})} \to 0$$

(c) *If*

$$\int_{t_1}^{t_2} \Re Z_+ < \int_{t_2}^{t_3} \Re Z_+ < 0$$

*then we have*

$$\frac{p_2}{q_2} \approx -e^{\int_{t_2}^{t_3}(4\sqrt{R}\Re Z_+ + 2B_{\mathcal{L}_\mathcal{E}}) - \int_{t_1}^{t_2}(4\sqrt{R}\Re Z_+ + 2B_{\mathcal{L}_\mathcal{E}})} \to \infty$$

(d) *If*
$$0 < \int_{t_2}^{t_3} \Re Z_+$$
*then we have*
$$\frac{p_2}{q_2} \approx \frac{1}{2} e^{-\int_{t_1}^{t_2} (4\sqrt{R}\Re Z_+ + 2B_{\mathcal{L}_\mathcal{E}})} \to \infty$$
*as* $R \to \infty$.

*In the above formulas, the contour of* $\int_{t_1}^{t_2}$ *is*
$$\sigma_1 * \eta_1^{-1} * \eta_2 * \sigma_2^{-1},$$
*and the one of* $\int_{t_2}^{t_3}$ *is*
$$\sigma_2 * \eta_2^{-1} * \psi_2 * \eta_3 * \sigma_3^{-1}.$$

**Remark 3.** *Just as in Remark 2, cases* (1) *and* (2) *are symmetric under change of sign of* $Z_+$.

*Proof.* Our task is to compute
$$P_2 = h_3 \circ \psi_2 \circ h_2^{-1}, \tag{5.38}$$
where $\psi_2$ stands for parallel transport map along the path $\psi_2$. Since constant factors $v_2, v_3$ multiplying $Q_i$ (or $P_i$) have no influence on the definition (4.11) of the complex twist co-ordinates, from now on we will ignore constant factors; said differently, the formulas of the rest of this section are valid in $\mathrm{PGL}(2,\mathbb{C})$. In particular, the exact value of the constants $v_3, w_3$ appearing in Proposition 15 will not be relevant, the essential information is that their ratio is well-defined. Just as in the proof of Proposition 11, with respect to suitable diagonalizing bases we have
$$\psi_2 \approx \begin{pmatrix} e^{\int_{\psi_2}\left(\frac{1}{2}B_{\det(\mathcal{E})} + B_{\mathcal{L}_\mathcal{E}} + \sqrt{R}\Re Z_+(q,z)\right)} & 0 \\ 0 & e^{\int_{\psi_2}\left(\frac{1}{2}B_{\det(\mathcal{E})} - B_{\mathcal{L}_\mathcal{E}} - \sqrt{R}\Re Z_+(q,z)\right)} \end{pmatrix}.$$
We take the frame at $\psi_2(0) = x_2$ to consist of unit vectors, and then the above matrix expresses the action of parallel transport with respect to a basis at $\psi_2(1) = x_3$ whose first and second vectors are respectively of length
$$\exp\left(\pm\sqrt{R}\int_{x_2}^{x_3} \Re Z_+(q,z)\right).$$
It follows that the action of parallel transport along $\psi_2$, written in unit-length diagonalizing bases both at $x_2, x_3$, is described by the matrix
$$\begin{pmatrix} e^{\int_{\psi_2}\left(\frac{1}{2}B_{\det(\mathcal{E})} + B_{\mathcal{L}_\mathcal{E}} + 2\sqrt{R}\Re Z_+(q,z)\right)} & 0 \\ 0 & e^{\int_{\psi_2}\left(\frac{1}{2}B_{\det(\mathcal{E})} - B_{\mathcal{L}_\mathcal{E}} - 2\sqrt{R}\Re Z_+(q,z)\right)} \end{pmatrix}.$$
For ease of notation, from now on we will drop the argument of $Z_+$ and the term $\frac{1}{2}B_{\det(\mathcal{E})}$ in the argument of the exponential (which has no effect in $\mathrm{PGL}(2,\mathbb{C})$).

Using Proposition 15, the product $h_3\psi_2$ reads as
$$h_3\psi_2 \approx \frac{1}{2v_3w_3}\begin{pmatrix} w_3 e^{2\sqrt{R}\int_{\psi_2}\Re Z_+ + \int_{\psi_2} B_{\mathcal{L}_\mathcal{E}}} & -w_3 e^{-4\sqrt{R}\Re\pi_3 - 2\sqrt{R}\int_{\psi_2}\Re Z_+ - \int_{\psi_2+2\eta_3} B_{\mathcal{L}_\mathcal{E}}} \\ v_3 e^{4\sqrt{R}\Re\pi_3 + 2\sqrt{R}\int_{\psi_2}\Re Z_+ + \int_{\psi_2+2\eta_3} B_{\mathcal{L}_\mathcal{E}}} & v_3 e^{-2\sqrt{R}\int_{\psi_2}\Re Z_+ - \int_{\psi_2} B_{\mathcal{L}_\mathcal{E}}} \end{pmatrix}$$

We first treat the case (2), i.e. we assume $\Re(\pi_2 - \pi_1) < 0$. According to Proposition 16, the $(1,1)$-entry of (5.38) is then of the form

$$\frac{v_2}{2v_3} e^{\int_{\psi_2}\left(B_{\mathcal{L}_{\mathcal{E}}} + 2\sqrt{R}\Re Z_+\right)} + \frac{v_2}{2v_3} e^{-\int_{\psi_2}\left(B_{\mathcal{L}_{\mathcal{E}}} + 2\sqrt{R}\Re Z_+\right) + 4\sqrt{R}\Re(\pi_2 - \pi_3) + 2\int_{\eta_2 - \eta_3} B_{\mathcal{L}_{\mathcal{E}}}}, \tag{5.39}$$

and its $(1,2)$-entry is

$$q_2 = \frac{v_2}{v_3} e^{\int_{\psi_2}\left(B_{\mathcal{L}_{\mathcal{E}}} + 2\sqrt{R}\Re Z_+\right) + 4\sqrt{R}\Re(\pi_2 - \pi_1) + 2\int_{\eta_2 - \eta_1} B_{\mathcal{L}_{\mathcal{E}}}} \tag{5.40}$$

$$- \frac{v_2}{v_3} e^{-\int_{\psi_2}\left(B_{\mathcal{L}_{\mathcal{E}}} + 2\sqrt{R}\Re Z_+\right) + 4\sqrt{R}\Re(2\pi_2 - \pi_1 - \pi_3) + 2\int_{2\eta_2 - \eta_1 - \eta_3} B_{\mathcal{L}_{\mathcal{E}}}} \tag{5.41}$$

We now turn to computing the ratio of the entries of the first row of the matrix

$$Q_2 = A_3^{-\frac{1}{2}} U_3 P_2 U_2^{-1}. \tag{5.42}$$

Since $A_3^{-\frac{1}{2}} U_3$ is lower triangular, left multiplication by this matrix does not affect the quotient of the entries in the first row, so we may ignore this factor. On the other hand, since $\Re(\pi_1 - \pi_2) \neq 0$, we have by (5.33)

$$U_2^{-1} = \begin{pmatrix} 1 & 0 \\ -u_2 & 1 \end{pmatrix} \approx \begin{pmatrix} 1 & 0 \\ \cosh\left(4\sqrt{R}\Re(\pi_2 - \pi_1) + 2\int_{\eta_2 - \eta_1} B_{\mathcal{L}_{\mathcal{E}}}\right) & 1 \end{pmatrix}.$$

The $(1,1)$-entry of (5.42) reads as

$$p_2 \approx \frac{v_2}{2v_3} e^{\int_{\psi_2}\left(B_{\mathcal{L}_{\mathcal{E}}} + 2\sqrt{R}\Re Z_+\right)} \tag{5.43}$$

$$+ \frac{v_2}{2v_3} e^{-\int_{\psi_2}\left(B_{\mathcal{L}_{\mathcal{E}}} + 2\sqrt{R}\Re Z_+\right) + 4\sqrt{R}\Re(\pi_2 - \pi_3) + 2\int_{\eta_2 - \eta_3} B_{\mathcal{L}_{\mathcal{E}}}} \tag{5.44}$$

$$+ \frac{v_2}{v_3} \cosh\left(4\sqrt{R}\Re(\pi_2 - \pi_1) + 2\int_{\eta_2 - \eta_1} B_{\mathcal{L}_{\mathcal{E}}}\right) e^{\int_{\psi_2}\left(B_{\mathcal{L}_{\mathcal{E}}} + 2\sqrt{R}\Re Z_+\right) + 4\sqrt{R}\Re(\pi_2 - \pi_1) + 2\int_{\eta_2 - \eta_1} B_{\mathcal{L}_{\mathcal{E}}}} \tag{5.45}$$

$$- \frac{v_2}{v_3} \cosh\left(4\sqrt{R}\Re(\pi_2 - \pi_1) + 2\int_{\eta_2 - \eta_1} B_{\mathcal{L}_{\mathcal{E}}}\right) e^{-\int_{\psi_2}\left(B_{\mathcal{L}_{\mathcal{E}}} + 2\sqrt{R}\Re Z_+\right) + 4\sqrt{R}\Re(2\pi_2 - \pi_1 - \pi_3) + 2\int_{2\eta_2 - \eta_1 - \eta_3} B_{\mathcal{L}_{\mathcal{E}}}}, \tag{5.46}$$

and its $(1,2)$-entry $q_2$ agrees with the one of $P_2$ given in (5.40), (5.41). Expanding $2\cosh(w) = e^w + e^{-w}$ and using $\Re(\pi_1 - \pi_2) > 0$, we observe that the first (dominant) term coming from (5.45) is equal to (5.43), and the first term of (5.46) cancels (5.44). This yields

$$p_2 \approx 2 e^{\int_{\psi_2}\left(B_{\mathcal{L}_{\mathcal{E}}} + 2\sqrt{R}\Re Z_+\right)} \tag{5.47}$$

$$- e^{-\int_{\psi_2}\left(B_{\mathcal{L}_{\mathcal{E}}} + 2\sqrt{R}\Re Z_+\right) + 4\sqrt{R}\Re(3\pi_2 - 2\pi_1 - \pi_3) + 2\int_{3\eta_2 - 2\eta_1 - \eta_3} B_{\mathcal{L}_{\mathcal{E}}}}. \tag{5.48}$$

We note the relations

$$\pi_2 - \pi_1 = \int_{t_1}^{t_2} Z_+$$

$$\pi_2 - \int_{\psi_2} Z_+ - \pi_3 = \int_{t_3}^{t_2} Z_+.$$

We now separate cases according to the possible relations of dominance of the terms (5.40), (5.41), (5.47) and (5.48). In case (2a), the term (5.41) dominates (5.40) and (5.48) dominates (5.47). In case (2b), the term (5.41) dominates (5.40) and (5.47) dominates (5.48). In case (2c), the term (5.41) dominates (5.40) and (5.47) dominates (5.48). In case (2a), the term (5.40) dominates (5.41) and (5.47) dominates (5.48). In each case we get the stated result.

The analysis in the case (1) is similar. The $(1,1)$-entry of (5.38) reads again as in (5.39), and its $(1,2)$-entry has the form

$$-\frac{v_2}{v_3}e^{\int_{\psi_2}\left(B_{\mathcal{L}_\mathcal{E}}+2\sqrt{R}\Re Z_+\right)-4\sqrt{R}\Re(\pi_2-\pi_1)-2\int_{\eta_2-\eta_1}B_{\mathcal{L}_\mathcal{E}}} \tag{5.49}$$
$$+\frac{v_2}{v_3}e^{-\int_{\psi_2}\left(B_{\mathcal{L}_\mathcal{E}}+2\sqrt{R}\Re Z_+\right)+4\sqrt{R}\Re(\pi_1-\pi_3)+2\int_{\eta_1-\eta_3}B_{\mathcal{L}_\mathcal{E}}}. \tag{5.50}$$

We then find that the behaviour of the $(1,1)$-entry of (5.42) is given by

$$p_2 \approx \frac{v_2}{2v_3}e^{\int_{\psi_2}\left(B_{\mathcal{L}_\mathcal{E}}+2\sqrt{R}\Re Z_+\right)} \tag{5.51}$$
$$+\frac{v_2}{2v_3}e^{-\int_{\psi_2}\left(B_{\mathcal{L}_\mathcal{E}}+2\sqrt{R}\Re Z_+\right)+4\sqrt{R}\Re(\pi_2-\pi_3)+2\int_{\eta_2-\eta_3}B_{\mathcal{L}_\mathcal{E}}} \tag{5.52}$$
$$-\frac{v_2}{v_3}\cosh\left(4\sqrt{R}\Re(\pi_2-\pi_1)+2\int_{\eta_2-\eta_1}B_{\mathcal{L}_\mathcal{E}}\right)e^{\int_{\psi_2}\left(B_{\mathcal{L}_\mathcal{E}}+2\sqrt{R}\Re Z_+\right)-4\sqrt{R}\Re(\pi_2-\pi_1)-2\int_{\eta_2-\eta_1}B_{\mathcal{L}_\mathcal{E}}} \tag{5.53}$$
$$+\frac{v_2}{v_3}\cosh\left(4\sqrt{R}\Re(\pi_2-\pi_1)+2\int_{\eta_2-\eta_1}B_{\mathcal{L}_\mathcal{E}}\right)e^{-\int_{\psi_2}\left(B_{\mathcal{L}_\mathcal{E}}+2\sqrt{R}\Re Z_+\right)+4\sqrt{R}\Re(\pi_1-\pi_3)+2\int_{\eta_1-\eta_3}B_{\mathcal{L}_\mathcal{E}}}, \tag{5.54}$$

and its $(1,2)$-entry $q_2$ agrees with the one of $P_2$ given in (5.49), (5.50). Now, expanding again $2\cosh(w)=e^w+e^{-w}$ and using $\Re(\pi_2-\pi_1)>0$, we see that the dominant term of (5.53) cancels (5.51), and the dominant term of (5.54) is equal to (5.52). We deduce

$$p_2 \approx \frac{v_2}{v_3}e^{-\int_{\psi_2}\left(B_{\mathcal{L}_\mathcal{E}}+2\sqrt{R}\Re Z_+\right)+4\sqrt{R}\Re(\pi_2-\pi_3)+2\int_{\eta_2-\eta_3}B_{\mathcal{L}_\mathcal{E}}} \tag{5.55}$$
$$-\frac{v_2}{2v_3}e^{\int_{\psi_2}\left(B_{\mathcal{L}_\mathcal{E}}+2\sqrt{R}\Re Z_+\right)-8\sqrt{R}\Re(\pi_2-\pi_1)-4\int_{\eta_2-\eta_1}B_{\mathcal{L}_\mathcal{E}}} \tag{5.56}$$

In case (1a), the dominant term of $p_2$ is (5.56) and the dominant term of $q_2$ is (5.49). In case (1b), the dominant term of $p_2$ is (5.55) and the dominant term of $q_2$ is (5.49). In case (1c), the dominant term of $p_2$ is (5.55) and the dominant term of $q_2$ is (5.49). In case (1d), the dominant term of $p_2$ is (5.55) and the dominant term of $q_2$ is (5.50). In each case, we get the desired result. □

We also state an analogous statement to Propositions 16 and 17 for the complex twist co-ordinate $[p_3 : q_3]$; their proofs being similar to the case of $[p_2 : q_2]$, we omit them.

**Proposition 18.** (1) *If* $\Re(\pi_4-\pi_0)>0$ *then we have*

$$h_4 \approx v_4\begin{pmatrix} 2e^{4\sqrt{R}\Re\pi_0+2\int_{\eta_0}B_{\mathcal{L}_\mathcal{E}}} & -2e^{-4\sqrt{R}\Re(\pi_4-\pi_0)-2\int_{\eta_4-\eta_0}B_{\mathcal{L}_\mathcal{E}}} \\ e^{4\sqrt{R}\Re\pi_4+2\int_{\eta_4}B_{\mathcal{L}_\mathcal{E}}} & 1 \end{pmatrix}$$

*for some* $v_4\in\mathbb{C}^\times$.

(2) *If* $\Re(\pi_4 - \pi_0) < 0$ *then we have*

$$h_4 \approx v_4 \begin{pmatrix} 2e^{4\sqrt{R}\Re(2\pi_4-\pi_0)+2\int_{2\eta_4-\eta_0} B_{\mathcal{L}_\mathcal{E}}} & -2e^{4\sqrt{R}\Re(\pi_4-\pi_0)+2\int_{\eta_4-\eta_0} B_{\mathcal{L}_\mathcal{E}}} \\ -e^{4\sqrt{R}\Re\pi_4+2\int_{\eta_4} B_{\mathcal{L}_\mathcal{E}}} & 1 \end{pmatrix}$$

*for some* $v_4 \in \mathbb{C}^\times$.

**Proposition 19.** *Fix* $q \in S^3_1$.

(1) *Assume* $\int_{t_0}^{t_4} \Re Z_+ = \Re(\pi_4 - \pi_0) > 0$.

(a) *If*

$$2\int_{t_0}^{t_4} \Re Z_+ < \int_{t_4}^{t_3} \Re Z_+$$

*then we have*

$$\frac{p_3}{q_3} \approx \frac{1}{2} e^{-\int_{t_0}^{t_4}(4\sqrt{R}\Re Z_+ + 2B_{\mathcal{L}_\mathcal{E}})} \to 0$$

(b) *If*

$$\int_{t_0}^{t_4} \Re Z_+ < \int_{t_4}^{t_3} \Re Z_+ < 2\int_{t_0}^{t_4} \Re Z_+$$

*then we have*

$$\frac{p_3}{q_3} \approx -e^{\int_{t_3}^{t_4}(4\sqrt{R}\Re Z_+ + 2B_{\mathcal{L}_\mathcal{E}}) + \int_{t_0}^{t_4}(4\sqrt{R}\Re Z_+ + 2B_{\mathcal{L}_\mathcal{E}})} \to 0$$

(c) *If*

$$0 < \int_{t_4}^{t_3} \Re Z_+ < \int_{t_0}^{t_4} \Re Z_+$$

*then we have*

$$\frac{p_3}{q_3} \approx -e^{\int_{t_3}^{t_4}(4\sqrt{R}\Re Z_+ + 2B_{\mathcal{L}_\mathcal{E}}) + \int_{t_0}^{t_4}(4\sqrt{R}\Re Z_+ + 2B_{\mathcal{L}_\mathcal{E}})} \to \infty$$

(d) *If*

$$\int_{t_4}^{t_3} \Re Z_+ < 0$$

*then we have*

$$\frac{p_3}{q_3} \approx \frac{1}{2} e^{\int_{t_0}^{t_4}(4\sqrt{R}\Re Z_+ + 2B_{\mathcal{L}_\mathcal{E}})} \to \infty$$

*as* $R \to \infty$.

(2) *Similar formulas hold if* $\int_{t_0}^{t_4} \Re Z_+ = \Re(\pi_4 - \pi_0) < 0$, *up to changing the sign of all occurring integrals.*

*In the above formulas, the contour of* $\int_{t_0}^{t_4}$ *is*

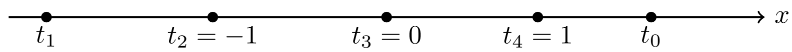
$$\sigma_0 * \eta_0^{-1} * \eta_4 * \sigma_4^{-1},$$

*and the one of* $\int_{t_4}^{t_3}$ *is*

$$\sigma_4 * \eta_4^{-1} * \psi_3^{-1} * \eta_3 * \sigma_3^{-1}.$$

## 6. Proof of Theorem 1

From this point on the choices made in (2.1) will be in effect, namely we set




In addition, we will allow $t_0, t_1$ to vary as needed for our purposes. Choices of continuous parameters do not affect our results on homotopy types of maps, so these are only made to make our arguments more concrete.

### 6.1. Geometry of period integrals.

**Proposition 20.** *There exists a nonempty open set of $V \subset S^3_1$ consisting of quadratic differentials $q$ for which both conditions* (1c) *of Proposition 17 and* (1c) *of Proposition 19 simultaneously hold.*

*Proof.* It is sufficient to prove that there exists an open subset of the Hopf fiber $t^{-1}(-1)$ over $t_2 = -1$ satisfying the required conditions. From now on, we assume $t(q) = -1$.

By assumption we need to consider homogeneous polynomials $q$ of degree 6 such that $t(q) = -1$, i.e. of the form

$$q(z,w) = a(z+1)z(z^2-1)\left(z^2 - \frac{1}{k^2}\right)$$

for some coefficients $a \in S^1$. The parameter $a$ can be identified with $e^{\sqrt{-1}\varphi}$ where $\varphi$ is the parameter of the Hopf fiber. Using this form and (2.11) we get that the corresponding quadratic differential reads as

$$Q(z) = \frac{a(z+1)z(z^2-1)\left(z^2-\frac{1}{k^2}\right)\mathrm{d}z^{\otimes 2}}{z^2(z^2-1)^2\left(z^2-\frac{1}{k^2}\right)^2} = \frac{a\mathrm{d}z^{\otimes 2}}{z(z-1)\left(z^2-\frac{1}{k^2}\right)}. \tag{6.1}$$

The square-root of $Q$ is then given by

$$Z_+ = \sqrt{a}\frac{\mathrm{d}z}{\sqrt{z(z-1)\left(z^2-\frac{1}{k^2}\right)}}.$$

Here, we need to be precise about the determination of the square roots: for $a = 1$ we choose $\sqrt{a} = 1$ and in the denominator we choose

$$\sqrt{z(z-1)\left(z^2-\frac{1}{k^2}\right)} \in \begin{cases} \mathbb{R}_- & \text{if } z < -\frac{1}{k} \\ \sqrt{-1}\mathbb{R}_- & \text{if } -\frac{1}{k} < z < 0 \\ \mathbb{R}_+ & \text{if } 0 < z < 1 \\ \sqrt{-1}\mathbb{R}_+ & \text{if } 1 < z < \frac{1}{k} \\ \mathbb{R}_- & \text{if } \frac{1}{k} < z. \end{cases} \tag{6.2}$$

With these choices and setting $a_0 = 1$, we have

$$\int_{t_4}^{t_3} Z_+ = -\int_0^1 \frac{\mathrm{d}z}{\sqrt{z(z-1)\left(z^2 - \frac{1}{k^2}\right)}} \in \mathbb{R}_-$$
$$\int_{t_2}^{t_3} Z_+ = \int_{-1}^0 \frac{\mathrm{d}z}{\sqrt{z(z-1)\left(z^2 - \frac{1}{k^2}\right)}} \in \sqrt{-1}\mathbb{R}_-$$
$$\int_{t_0}^{t_4} Z_+ = -\int_1^{\frac{1}{k}} \frac{\mathrm{d}z}{\sqrt{z(z-1)\left(z^2 - \frac{1}{k^2}\right)}} \in \sqrt{-1}\mathbb{R}_-$$
$$\int_{t_1}^{t_2} Z_+ = \int_{-\frac{1}{k}}^{-1} \frac{\mathrm{d}z}{\sqrt{z(z-1)\left(z^2 - \frac{1}{k^2}\right)}} \in \sqrt{-1}\mathbb{R}_-$$

**Lemma 2.** *For suitable choices of* $t_0, t_1$ *we have*

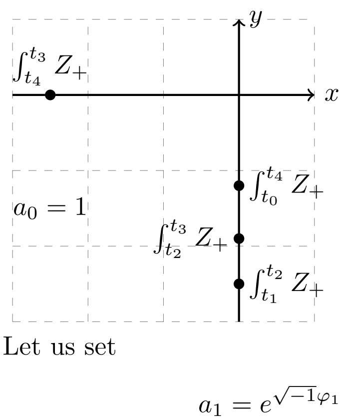
$$\left|\int_{t_0}^{t_4} Z_+\right| < \left|\int_{t_2}^{t_3} Z_+\right| < \left|\int_{t_1}^{t_2} Z_+\right|.$$

*Proof.* We have

$$\lim_{t_1 \to -\infty} \left|\int_{t_1}^{t_2} Z_+\right| = \infty$$
$$\lim_{t_0 \to 1+} \left|\int_{t_0}^{t_4} Z_+\right| = 0.$$

□

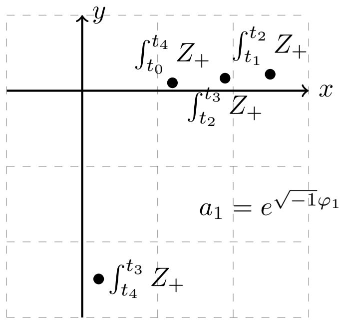
We may then schematically plot these complex numbers on the complex line as follows.




Let us set

$$a_1 = e^{\sqrt{-1}\varphi_1} \in S^1 \quad \text{with} \quad \frac{\pi}{2} < \varphi_1 < \frac{\pi}{2} + \varepsilon$$

for some tiny $\varepsilon > 0$. With such a choice, the quantities under consideration satisfy the required properties, as is visible from the next figure.




□

6.2. **Proof of Theorem 1.** We are in position to prove our main result Theorem 1. We fix any $q \in V$, where $V \subset S^3_1$ is the open set provided by Proposition 20.

Let us first reformulate a few of our results obtained thus far. According to Subsection 2.3, the highest graded piece $\mathrm{Gr}^W_8 H^4(\mathcal{M}_{\mathrm{B}}, \mathbb{C})$ of the MHS on the cohomology of the character variety is spanned by 0-cycles in the union $\tilde{D}^4$ of quadruple intersections of the compactifying divisor components; clearly, $\tilde{D}^4$ is a finite union of points in $\overline{\mathcal{M}}_{\mathrm{B}}$. In addition, these cycles also govern $\mathrm{Gr}^W_{2k} H^k$ for all $0 \leq k \leq 4$. Let us denote by $D_1, D_2, D_3, D_4$ the divisor components of $\overline{\mathcal{M}}_{\mathrm{B}} \setminus \mathcal{M}_{\mathrm{B}}$ given in order by the equations

$$l_2 = \infty, \quad [p_2 \colon q_2] = [1 \colon 0], \quad l_3 = \infty, \quad [p_3 \colon q_3] = [1 \colon 0].$$

Let us denote by

$$Q^* = D_1 \cap D_2 \cap D_3 \cap D_4 \in \tilde{D}^4 \subset \overline{\mathcal{M}}_{\mathrm{B}}$$

their intersection point and fix a punctured neighbourhood $U(Q^*)$ of $Q^*$ in $\mathcal{M}_{\mathrm{B}}$. It follows from Propositions 12, 14, 17(1c), 19(1c) that if $R$ is chosen sufficiently large, then for any $(\mathcal{E}, \theta) \in H^{-1}(Rq)$, the Fenchel–Nielsen co-ordinates

$$l_2(\mathcal{E}, \theta), \quad [p_2(\mathcal{E}, \theta) \colon q_2(\mathcal{E}, \theta)], \quad l_3(\mathcal{E}, \theta), \quad [p_3(\mathcal{E}, \theta) \colon q_3(\mathcal{E}, \theta)]$$

of $\mathrm{RH} \circ \psi(\mathcal{E}, \theta)$ belong to $U(Q^*)$. Fix $R \gg 0$ so that this holds. Moreover, the same results (and the sign assumptions made on the integrals) also imply that the phase factors of the Fenchel–Nielsen co-ordinates of $\mathrm{RH} \circ \psi(\mathcal{E}, \theta)$ defining $D_1, D_2, D_3, D_4$ are in this order given by the following expressions:

$$\begin{aligned} &-\exp\left(2\int_{t_1}^{t_2} B_{\mathcal{L}_{(\mathcal{E},\theta)}}\right), \\ &-\exp\left(2\int_{t_3}^{t_2} B_{\mathcal{L}_{(\mathcal{E},\theta)}} + 2\int_{t_1}^{t_2} B_{\mathcal{L}_{(\mathcal{E},\theta)}}\right), \\ &-\exp\left(2\int_{t_0}^{t_4} B_{\mathcal{L}_{(\mathcal{E},\theta)}}\right), \\ &-\exp\left(2\int_{t_3}^{t_4} B_{\mathcal{L}_{(\mathcal{E},\theta)}} + 2\int_{t_0}^{t_4} B_{\mathcal{L}_{(\mathcal{E},\theta)}}\right), \end{aligned}$$

(along contours as given in the Propositions). Notice that there exists a symplectic basis $A_1, A_2, B_1, B_2$ of $H_1(X_q, \mathbb{Z})$ that is anti-invariant for the involution $\rho^*$ and that satisfies

$$\begin{aligned}(p_q)_* A_1 &= \sigma_1 * \eta_1^{-1} * \eta_2 * \sigma_2^{-1}\\ (p_q)_* B_1 &= \sigma_2 * \eta_2^{-1} * \psi_2 * \eta_3 * \sigma_3^{-1}\\ (p_q)_*(A_1 + A_2) &= \sigma_4 * \eta_4^{-1} * \psi_3^{-1} * \eta_3 * \sigma_3^{-1}\\ (p_q)_* B_2 &= \sigma_0 * \eta_0^{-1} * \eta_4 * \sigma_4^{-1}.\end{aligned}$$

It follows that the above quantities can be rewritten as

$$-\exp\left(\oint_{A_1} B_{\mathcal{L}(\mathcal{E},\theta)}\right), \tag{6.3}$$

$$-\exp\left(\oint_{A_1 - B_1} B_{\mathcal{L}(\mathcal{E},\theta)}\right), \tag{6.4}$$

$$-\exp\left(\oint_{-B_2} B_{\mathcal{L}(\mathcal{E},\theta)}\right), \tag{6.5}$$

$$-\exp\left(\oint_{B_2 - A_1 - A_2} B_{\mathcal{L}(\mathcal{E},\theta)}\right). \tag{6.6}$$

The cycles of the integrals in the arguments of the exponentiation in these formulas generate $H_1(X_q, \mathbb{Z})$ as an Abelian group. It follows from formulas (6.3)–(6.6) that the image of the Hitchin fiber $H^{-1}(Rq)$ under $\mathrm{RH} \circ \psi$ is homotopic to a torus $T^4$ generating $H_4(U(Q^*), \mathbb{Z})$. Now, recall from Subsection 2.7 that we have

$$\mathrm{Gr}_P^{-k-2} H^k(\mathcal{M}_{\mathrm{Dol}}, \mathbb{Q}) \cong \mathrm{Im}(H^k(\mathcal{M}_{\mathrm{Dol}}, \mathbb{Q}) \to H^k(H^{-1}(Y_{-2}), \mathbb{Q}))$$

where $H^{-1}(Y_{-2})$ is the generic Hitchin fiber. We may choose the affine flag so that $Y_{-2} = \{Rq\}$ for $q \in V$. For every $0 \leq k \leq 4$ and any subset $I \subset \{1, 2, 3, 4\}$ with $|I| = 4 - k$ one may define a $k$-dimensional subtorus $T_I^k$ in $H^{-1}(Y_{-2})$ by fixing the phases corresponding to the divisor components $D_i$ with $i \in I$. Let us assume that $T_I^k$ defines a non-trivial homology class in $H_k(\mathcal{M}_{\mathrm{Dol}}, \mathbb{Z})$. Such classes are precisely the ones that generate $\mathrm{Gr}_P^{-k-2} H^k(\mathcal{M}_{\mathrm{Dol}}, \mathbb{Q})$. It is then easy to see that image of $T_I^k$ under $\mathrm{RH} \circ \psi$ is homotopic to a normal torus at the generic point of the intersection

$$\bigcap_{j \in \{1,2,3,4\} \setminus I} D_j$$

of the remaining $k$ divisor components. According to the conventions of Subsection 2.3, $\mathrm{RH} \circ \psi(T_I^k)$ then defines a class in $W_{-2k} H_k(U(Q^*), \mathbb{Z})$ (that is non-trivial by assumption), and the dual cohomology class gives a non-trivial class in $W_{2k} H^k(U(Q^*), \mathbb{Z})$. Since the map

$$H^k(\mathcal{M}_{\mathrm{B}}, \mathbb{C}) \to H^k(U(Q^*), \mathbb{C})$$

preserves $W$ strictly, this finishes the proof.

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