

Translation invariant linear spaces of polynomials

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Abstract

A set of polynomials M is called a *submodule* of $\mathbb{C}[x_1, \dots, x_n]$ if M is a translation invariant linear subspace of $\mathbb{C}[x_1, \dots, x_n]$. We present a description of the submodules of $\mathbb{C}[x, y]$ in terms of a special type of submodules. We say that the submodule M of $\mathbb{C}[x, y]$ is an *L-module of order s* if, whenever $F(x, y) = \sum_{n=0}^N f_n(x) \cdot y^n \in M$ is such that $f_0 = \dots = f_{s-1} = 0$, then $F = 0$. We show that the proper submodules of $\mathbb{C}[x, y]$ are the sums $M_d + M$, where $M_d = \{F \in \mathbb{C}[x, y] : \deg_x F < d\}$, and M is an L-module. We give a construction of L-modules parametrized by sequences of complex numbers.

A submodule $M \subset \mathbb{C}[x_1, \dots, x_n]$ is *decomposable* if it is the sum of finitely many proper submodules of M . Otherwise M is *indecomposable*. It is easy to see that every submodule of $\mathbb{C}[x_1, \dots, x_n]$ is the sum of finitely many indecomposable submodules. In $\mathbb{C}[x, y]$ every indecomposable submodule is either an L-module or equals M_d for some d . In the other direction we show that M_d is indecomposable for every d , and so is every L-module of order 1.

Finally, we prove that there exists a submodule of $\mathbb{C}[x, y]$ (in fact, an L-module of order 1) which is not relatively closed in $\mathbb{C}[x, y]$. This answers a problem posed by L. Székelyhidi in 2011.

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1 Introduction and main results

In this note we are concerned with the translation invariant linear subspaces of $\mathbb{C}[x_1, \dots, x_n]$, the ring of polynomials of n variables having complex coefficients. By making use of Taylor's formula it is not difficult to see that a linear subspace of $\mathbb{C}[x_1, \dots, x_n]$ is translation invariant if and only if it is invariant under partial differentiation [2, Lemma 7]. Thus a linear subspace of $\mathbb{C}[x_1, \dots, x_n]$ is translation invariant if and only if it is a module under the ring of partial differential operators. For this reason we say that M is a *submodule* of $\mathbb{C}[x_1, \dots, x_n]$ (or briefly a *module*) if M is a translation invariant linear subspace of $\mathbb{C}[x_1, \dots, x_n]$.

It is easy to check that the only submodules of $\mathbb{C}[x]$ are $\mathbb{C}[x]$ itself and the modules $\{f \in \mathbb{C}[x] : \deg f < d\}$ ($d = 0, 1, \dots$).

Simple examples of submodules of $\mathbb{C}[x, y]$ are $\mathbb{C}[x, y]$ itself, $\mathbb{C}[x]$, $\mathbb{C}[y]$, $\{f(x+y) : f \in \mathbb{C}[x]\}$, $\{f(ax+by) : f \in \mathbb{C}[x]\}$ ($a, b \in \mathbb{C}$), $\{f(x) + g(y) : f, g \in \mathbb{C}[x]\}$, $\{f \in \mathbb{C}[x, y] : \deg f < d\}$ ($d = 0, 1, \dots$), $\{f \in \mathbb{C}[x, y] : \deg_x f < d_1, \deg_y f < d_2\}$ ($d_1, d_2 = 0, 1, \dots$). Here $\deg_x f$ and $\deg_y f$ denote the degree in the variable x (resp. y) of the polynomial $f \in \mathbb{C}[x, y]$.

Each of these modules is relatively closed in $\mathbb{C}[x, y]$ in the following sense: if f_n belongs to the module M in question for every n and $f_n \rightarrow f \in \mathbb{C}[x, y]$ pointwise (or uniformly on compact sets), then $f \in M$.

The investigations of these note were motivated by the following problem posed by L. Székelyhidi [3]: *is it true that every submodule of $\mathbb{C}[x_1, \dots, x_n]$ is relatively closed in $\mathbb{C}[x_1, \dots, x_n]$?* In other words, is every submodule of $\mathbb{C}[x_1, \dots, x_n]$ a variety? In Theorem 25 we show that the answer to Székelyhidi's question is negative. Our example is a special case of a general construction of some submodules of $\mathbb{C}[x, y]$, called L-modules.

We represent the elements of $\mathbb{C}[x, y]$ in the form

$$F(x, y) = \sum_{n=0}^{\infty} f_n(x) \frac{y^n}{n!}, \quad (1)$$

where $f_n \in \mathbb{C}[x]$ for every n , and $f_n = 0$ if n is large enough. We say that the module $M \subset \mathbb{C}[x, y]$ is an *L-module of order s* if, whenever F in (1) belongs

to M and such that $f_n = 0$ for every $n < s$, then $F = 0$. In Section 2 we give a construction of L-modules parametrized by sequences of complex numbers (Theorem 3).

Let M_d denote the module $\{f \in \mathbb{C}[x, y] : \deg_x f < d\}$. In Section 3 we show that every proper submodule of $\mathbb{C}[x, y]$ can be represented in the form $M_d + M$, where M is an L-module (Theorem 8). Under some mild restrictions on M , the representation is unique (see Remark 15). The obstacles in the way of generalizing this result for polynomials of more than two variables are discussed in Remark 24.

We say that a submodule of $\mathbb{C}[x_1, \dots, x_n]$ is indecomposable, if it cannot be written as a finite sum of proper submodules. It is easy to see that every submodule of $\mathbb{C}[x_1, \dots, x_n]$ is the sum of finitely many indecomposable submodules (Proposition 17). It follows from Theorem 8 that every indecomposable submodule of $\mathbb{C}[x, y]$ is either an L-module or equals M_d for some d . In the other direction we prove that M_d is indecomposable for every d , and that all L-modules of order 1 are indecomposable submodules of $\mathbb{C}[x, y]$ (Theorems 21 and 22).

2 L-modules

Let $S = \{f(x + y) : f \in \mathbb{C}[x]\}$. It is clear that S is a submodule of $\mathbb{C}[x, y]$. Since $f(x + y) = \sum_{n=0}^{\infty} f^{(n)}(x) \cdot \frac{y^n}{n!}$ by Taylor's formula, it follows that the elements of S are the polynomials $\sum_{n=0}^{\infty} f_n \cdot \frac{y^n}{n!}$, where $f_n \in \mathbb{C}[x]$ for every n , and $f_n = f'_{n-1}$ for every $n \geq 1$. In particular, S has the property that if $F = \sum_{n=0}^{\infty} f_n \cdot \frac{y^n}{n!} \in S$ and $f_1 = 0$, then $F = 0$.

The module S is the prototype of the submodules we are about to define.

Notation 1. Every polynomial $F \in \mathbb{C}[x, y]$ can be represented uniquely in the form (1), where $f_0, f_1, \dots \in \mathbb{C}[x]$, and $f_n = 0$ if n is large enough. The polynomials f_n will be called *the coordinate polynomials* of F , and will be denoted by $[F]_n$ ($n = 0, 1, \dots$).

If $A \subset \mathbb{C}[x, y]$ and s is a positive integer, then we put

$$V_{A,s} = \{([F]_0, \dots, [F]_{s-1}) : F \in A\}.$$

Clearly, if A is a module, then $V_{A,s}$ is a linear subspace of $\mathbb{C}[x]^s$. Note that if $F \in A$, then $([F]_{k-s}, \dots, [F]_{k-1}) \in V_{A,s}$ for every $k \geq s$. This follows from $\frac{\partial^{k-s}}{\partial y^{k-s}} F \in A$. The set $V_{A,s}$ also has the following property: if $(f_0, \dots, f_{s-1}) \in V_{A,s}$, then $(f'_0, \dots, f'_{s-1}) \in V_{A,s}$. This is clear from the fact that if the polynomial in (1) belongs to A , then

$$\frac{\partial}{\partial x} F = \sum_{n=0}^{\infty} f'_n(x) \frac{y^n}{n!} \in A. \quad (2)$$

Definition 2. Let s be a positive integer. We say that $M \subset \mathbb{C}[x, y]$ is an *L-module of order s* if M is a module and, whenever $F \in M$ and $[F]_n = 0$ for every $n < s$, then $F = 0$.

Since the submodules of $\mathbb{C}[x, y]$ are linear spaces, the condition formulated in Definition 2 is equivalent to the following: if $F \in M$, then F is determined by the coordinate polynomials $[F]_0, \dots, [F]_{s-1}$.

It is clear from the definition that if M is an L-module of order s , then it is also an L-module of order t for every $t \geq s$.

Theorem 3. *Let $M \subset \mathbb{C}[x, y]$ be an L-module of order s . Then there exists a linear map $L: \mathbb{C}[x]^s \rightarrow \mathbb{C}[x]$ such that for every $F \in M$ we have*

$$[F]_n = L([F]_{n-s}, \dots, [F]_{n-1}) \quad (3)$$

for every $n \geq s$. More precisely, there are complex numbers $a_{i,j}$ ($i = 1, \dots, s, j = 1, 2, \dots$) such that (3) holds for every $n \geq s$, where

$$L(f_1, \dots, f_s) = \sum_{i=1}^s \sum_{j=1}^{\infty} a_{i,j} f_i^{(j)} \quad (4)$$

for every $f_1, \dots, f_s \in \mathbb{C}[x]$. (Note that the sum in the right hand side of (4) only has a finite number of nonzero terms for every $f_1, \dots, f_s \in \mathbb{C}[x]$).

Proof. If $F \in M$, then we put $L([F]_0, \dots, [F]_{s-1}) = [F]_s$. This definition makes sense, since M is an L-module, and thus $[F]_s$ is uniquely determined by $[F]_0, \dots, [F]_{s-1}$. In this way we defined L on the set $V_{M,s}$. It is clear that L is linear.

Suppose $F \in M$, and let $k \geq s$ be given. Then

$$\frac{\partial^{k-s}}{\partial y^{k-s}} F(x, y) = \sum_{n=0}^{\infty} [F]_{n+k-s}(x) \frac{y^n}{n!} \in M, \quad (5)$$

and thus $[F]_k = L([F]_{k-s}, \dots, [F]_{k-1})$. This proves the first statement of the theorem including (3), except that L is only defined on $V_{M,s}$.

If $F \in M$, then (2) holds, and thus

$$L([F]_0, \dots, [F]_{s-1})' = L([F]'_0, \dots, [F]'_{s-1}),$$

since both sides equal $[F]'_s$. Therefore,

$$L(f_0, \dots, f_{s-1})' = L(f'_0, \dots, f'_{s-1}) \quad (6)$$

holds for every $(f_0, \dots, f_{s-1}) \in V_{M,s}$. Next we prove that

$$\deg L(f_0, \dots, f_{s-1}) \leq \max_{0 \leq i \leq s-1} \deg f_i \quad (7)$$

for every $(f_0, \dots, f_{s-1}) \in V_{M,s}$. Indeed, let $F \in M$, and let $\max_{0 \leq i \leq s-1} \deg [F]_i = d$. Then

$$\frac{\partial^{d+1}}{\partial x^{d+1}} F(x, y) = \sum_{n=0}^{\infty} [F]_n^{(d+1)} \frac{y^n}{n!} \in M.$$

Now we have $[F]_i^{(d+1)} = 0$ for every $i < s$, and thus $[F]_s^{(d+1)} = 0$, since M is an L -module of order s . This proves (7). Then it follows from (3) that if $F \in M$, then $\deg [F]_n \leq \max_{0 \leq i \leq s-1} \deg [F]_i$ for every n . In particular, if the coordinate polynomials $[F]_0, \dots, [F]_{s-1}$ are constants, then $[F]_n$ is constant for every n .

Let W_0 denote the set of s -tuples $(c_0, \dots, c_{s-1}) \in \mathbb{C}^s$ such that $(c_0, \dots, c_{s-1}) \in V_{M,s}$. Clearly, W_0 is a linear subspace of \mathbb{C}^s . Let $\dim W_0 = r$, and let $(c_{i,0}, \dots, c_{i,s-1})$ ($i = 1, \dots, r$) be a basis of W_0 . Let $F_i(y) = \sum_{n=0}^{\infty} c_{i,n} \frac{y^n}{n!} \in M \cap \mathbb{C}[y]$ for every $i = 1, \dots, r$. Then every element of $M \cap \mathbb{C}[y]$ is the linear combination of the functions F_i . Indeed, if

$$F(y) = \sum_{n=0}^{\infty} c_n \frac{y^n}{n!} \in M \cap \mathbb{C}[y],$$

then there is a linear combination \overline{F} of F_1, \dots, F_r such that $\overline{F}(y) = \sum_{n=0}^{\infty} d_n \frac{y^n}{n!}$, where $d_n = c_n$ for every $n \leq s-1$. Since $\overline{F} \in M$ and M is an L -module of order s , it follows that $d_n = c_n$ for every n , and $\overline{F} = F$.

Thus the dimension of $M \cap \mathbb{C}[y]$ is at most r . Since $\dim W_0 = r$, we have $\dim(M \cap \mathbb{C}[y]) = r$. Now $M \cap \mathbb{C}[y]$ is a proper submodule of $\mathbb{C}[y]$, and thus there is a $p \geq 0$ such that $M \cap \mathbb{C}[y] = \{f \in \mathbb{C}[y] : \deg f < p\}$. Clearly, we must have $p = r$.

This implies $p = r \leq s$, and thus $\deg F < s$ for every $F \in M \cap \mathbb{C}[y]$. That is, $c_s = 0$ whenever $\sum_{n=0}^{\infty} c_n \frac{y^n}{n!} \in M \cap \mathbb{C}[y]$. Therefore, we have $L(c_0, \dots, c_{s-1}) = 0$ for every $(c_0, \dots, c_{s-1}) \in W_0$.

We construct the numbers $a_{i,j}$ with the property that, for every $d \geq 1$,

$$L(f_0, \dots, f_{s-1}) = \sum_{i=0}^{s-1} \sum_{j=1}^{d-1} a_{i,j} f_i^{(j)} \quad (8)$$

whenever $(f_0, \dots, f_{s-1}) \in V_{M,s}$ and $\deg f_i < d$ ($i = 0, \dots, s-1$). Note that (8) is true for $d = 1$. Indeed, $\deg f_i < 1$ means that f_i is constant, and thus the left hand side of (8) is zero, and so is the right hand side, since the sums $\sum_{j=1}^{d-1}$ are empty.

Let $d \geq 1$, and suppose we have defined the numbers and $a_{i,j}$ ($i = 0, \dots, s-1$, $j = 1, \dots, d-1$) such that (8) holds for every $(f_0, \dots, f_{s-1}) \in V_{M,s}$ and $\deg f_i < d$ ($i = 0, \dots, s-1$).

If $(f_0, \dots, f_{s-1}) \in V_{M,s}$ and $\deg f_i \leq d$ ($i = 0, \dots, s-1$), then, by (6) and (8),

$$(L(f_0, \dots, f_{s-1}))' = L(f'_0, \dots, f'_{s-1}) = \sum_{i=0}^{s-1} \sum_{j=1}^{d-1} a_{i,j} f_i^{(j+1)},$$

and thus

$$L(f_0, \dots, f_{s-1}) = \sum_{i=0}^{s-1} \sum_{j=1}^{d-1} a_{i,j} f_i^{(j)} + C(f_0, \dots, f_{s-1}),$$

where $C(f_0, \dots, f_{s-1})$ is constant. Clearly, the map

$$(f_0, \dots, f_{s-1}) \mapsto C(f_0, \dots, f_{s-1})$$

is linear. Let $f_i = \sum_{\nu=0}^d \alpha_{i,\nu} x^\nu$ ($i \leq s-1$). Then $C(f_0, \dots, f_{s-1})$ only depends on the coefficients $\alpha_{i,d}$. Indeed, if $(g_0, \dots, g_{s-1}) \in V_{M,s}$, where $g_i = \sum_{\nu=0}^d \beta_{i,\nu} x^\nu$ and $\alpha_{i,d} = \beta_{i,d}$ ($i \leq s-1$), then $\deg(f_i - g_i) < d$, and $C(f_0 - g_0, \dots, f_{s-1} - g_{s-1}) = 0$ by (8). Then it follows that there are numbers $b_{i,d}$ ($i = 0, \dots, s-1$) such that

$$C(f_0, \dots, f_{s-1}) = \sum_{i=0}^{s-1} b_{i,d} \cdot \alpha_{i,d} = \sum_{i=0}^{s-1} b_{i,d} \cdot \frac{f_i^{(d)}}{d!}.$$

Putting $a_{i,d} = b_{i,d}/d!$, we obtain (8) with $d+1$ in place of d for every $(f_0, \dots, f_{s-1}) \in V_{M,s}$, $\deg f_i \leq d$ ($i = 0, \dots, s-1$). In this way we obtain the numbers $a_{i,j}$ by induction on j . It is clear that the numbers $a_{i,j}$ defined above satisfy (4) for every $(f_1, \dots, f_s) \in V_{M,s}$. Now, the right hand side of (4) makes sense for every $(f_1, \dots, f_s) \in \mathbb{C}[x]^s$, and defines a linear extension of L to $\mathbb{C}[x]^s$. This completes the proof of the theorem. \square

Remark 4. Let M be an L-module of order s . Then $V_{M,s}$ and L are connected by the following necessary condition: if $(f_0, \dots, f_{s-1}) \in V_{M,s}$ and the sequence of polynomials is defined by $f_n = L(f_{n-s}, \dots, f_{n-1})$ for every $n \geq s$, then $(f_{n-s}, \dots, f_{n-1}) \in V_{M,s}$ for every $n \geq s$. Indeed, let $F(x, y) = \sum_{n=0}^{\infty} g_n(x) \frac{y^n}{n!} \in M$ be such that $g_i = f_i$ for every $i < s$. Since $g_n = L(g_{n-s}, \dots, g_{n-1})$ for every $n \geq s$, it follows that $g_n = f_n$ for every n . For every $k \geq s$ we have (5), hence $(f_{k-s}, \dots, f_{k-1}) \in V_{M,s}$.

In the constructions of L-modules this condition should be taken into account. Consider the following example. Let $s = 2$, $V = \{(f, f) : f \in \mathbb{C}[x]\}$, and let L be the identically zero map from $\mathbb{C}[x]^2$ into $\mathbb{C}[x]$. Then $(f, f) \in V$, $L(f, f) = 0$, but $(f, 0) \notin V$ if $f \neq 0$. Accordingly, the set M of functions of the form (1) such that $(f_0, f_1) \in V$ and $f_n = L(f_{n-2}, f_{n-1})$ for every $n \geq 2$ is not a module. Indeed, $F = f(x) \cdot (1 + y) \in M$ for every $f \in \mathbb{C}[x]$, but $\frac{\partial}{\partial y} F = f(x) \notin M$ if $f \neq 0$.

Note that the necessary condition above is automatically satisfied if $V_{M,s} = \mathbb{C}[x]^s$. Therefore, the following construction always produces L-modules.

Notation 5. Let $\Gamma = \{a_{i,j} : i = 1, \dots, s, j = 1, 2, \dots\}$ be a set of complex numbers, and let M_Γ denote the set of polynomials of the form (1) such that $f_n = L(f_{n-s}, \dots, f_{n-1})$ for every $n \geq s$, where L is defined by (4).

The definition of L implies that

$$\deg f_k < \max_{k-s \leq i \leq k-1} \deg f_i \quad (9)$$

for every $k \geq s$. Therefore, we have $f_n = 0$ for every $n > s + \max_{0 \leq i \leq s-1} \deg f_i$.

Lemma 6. M_Γ is an L -module of order s .

Proof. It is enough to show that M_Γ is a module. Since L is a linear map, M_Γ is a linear subspace of $\mathbb{C}[x, y]$. If $F(x, y)$ is defined by (1), then

$$\frac{\partial}{\partial x} F(x, y) = \sum_{n=0}^{\infty} f'_n(x) \cdot \frac{y^n}{n!}.$$

Since $f'_n = L(f'_{n-s}, \dots, f'_{n-1})$ for every $n \geq s$, we have $\frac{\partial}{\partial x} F \in M_\Gamma$. We also have

$$\frac{\partial}{\partial y} F(x, y) = \sum_{n=1}^{\infty} f_n(x) \cdot \frac{y^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} f_{n+1}(x) \cdot \frac{y^n}{n!}.$$

It is clear that $\frac{\partial}{\partial y} F \in M_\Gamma$, and thus M_Γ is a module. \square

3 A representation of the proper submodules of $\mathbb{C}[x, y]$

By the sum of the sets of polynomials $A, B \subset \mathbb{C}[x_1, \dots, x_n]$ we mean the set $A+B = \{f+g : f \in A, g \in B\}$. It is easy to see that if A, B are submodules of $\mathbb{C}[x_1, \dots, x_n]$, then so is $A+B$.

Notation 7. For every nonnegative integer d we denote by M_d the set of polynomials $F \in \mathbb{C}[x, y]$ such that $\deg [F]_n < d$ for every n .

It is easy to check that M_d is a submodule of $\mathbb{C}[x, y]$ for every nonnegative integer d . Note that $M_0 = \{0\}$ and $M_1 = \mathbb{C}[y]$.

In this section our aim is to prove the following.

Theorem 8. *Let A be a proper submodule of $\mathbb{C}[x, y]$. Then there are integers $d \geq 0$ and $s \geq 1$ such that $A = M_d + M$, where M is an L -module of order s .*

As for the uniqueness of the representation see Remark 15. First we show that the sum of an L -module and M_d is always a proper submodule of $\mathbb{C}[x, y]$.

Lemma 9. *If M is an L -module of order s and $d \geq 0$, then $x^d y^s \notin M_d + M$. Consequently, $M_d + M$ is a proper submodule of $\mathbb{C}[x, y]$.*

Proof. Suppose $x^d y^s = F + G$, where $F \in M_d$ and $G \in M$. Then we have (1), where $\deg f_n < d$ for every n . Thus

$$\begin{aligned} -G(x, y) &= F(x, y) - x^d y^s \\ &= \sum_{n=0}^{s-1} f_n(x) \cdot \frac{y^n}{n!} + (f_s(x) - s! \cdot x^d) \cdot \frac{y^s}{s!} + \sum_{n=s+1}^{\infty} f_n(x) \cdot \frac{y^n}{n!}, \end{aligned}$$

and $-G \in M$. By (7) we obtain

$$d = \deg((f_s(x) - s! \cdot x^d)) \leq \max_{0 \leq i \leq s-1} \deg f_i < d,$$

a contradiction. \square

Corollary 10. *If A, B are L -modules and $M_{d_1} + A = M_{d_2} + B$, then $d_1 = d_2$.*

Proof. Suppose $d_1 < d_2$. By Lemma 9, there is an s such that $x^{d_1} y^s \notin M_{d_1} + A = M_{d_2} + B$. However, $x^{d_1} y^s \in M_{d_2} \subset M_{d_2} + B$, which is a contradiction. \square

The rest of the section is devoted to the proof of Theorem 8.

Since $A \subsetneq \mathbb{C}[x, y]$ and A is a linear space, we have $x^m y^s \notin A$ for some $m, s \geq 0$. Let d be the smallest nonnegative integer such that $x^d y^s \notin A$ for some $s \geq 0$. Then we have $M_d \subset A$. Let s be the smallest nonnegative integer such that $x^d y^s \notin A$. In the course of the proof we fix the module A and the nonnegative integers d and s with these properties.

Lemma 11. *For every polynomial $F \in A$ we have*

$$\deg [F]_n < \max(d, \max_{0 \leq i \leq s-1} \deg [F]_i) \quad (10)$$

for every $n \geq s$.

Proof. Let $e = \max_{0 \leq i \leq s-1} \deg [F]_i$, and suppose that $m \geq \max(d, e)$, where $m = \max_{n \geq s} \deg [F]_n$. (Note that $[F]_n = 0$ if n is large enough.)

First we suppose $e \leq d$; then $m \geq d$. Turning to the polynomial $\frac{\partial^{m-d}}{\partial x^{m-d}} F$ we may assume that $m = d$. Let k be the largest index with $\deg [F]_k = d$. Then $k \geq s$, $\deg [F]_n \leq d$ for every n , and $\deg [F]_n < d$ for every $n > k$. Turning to the polynomial $\frac{\partial^{k-s}}{\partial y^{k-s}} F$ we may assume that $k = s$. Then $\deg [F]_n \leq d$ for every n , $\deg [F]_s = d$, and $\deg [F]_n < d$ for every $n > s$. Since $M_d \subset A$ and $x^d y^n \in A$ for every $n < s$ by the choice of s , it follows that $[F]_n \cdot \frac{y^n}{n!} \in A$ for every $n \neq s$, and thus $[F]_s \cdot \frac{y^s}{s!} \in A$. Using $M_d \subset A$ again we find $x^d y^s \in A$, which is impossible.

Next suppose $e > d$; then $m \geq e$. Turning to the polynomial $\frac{\partial^{e-d}}{\partial x^{e-d}} F$ we reduce this case to the case when $e = d$. \square

If $d = 0$, then it follows from Lemma 11 that if $F \in A$ and $[F]_0 = \dots = [F]_{s-1} = 0$, then $F = 0$. That is, if $d = 0$ then A is an L -module of order s . Then $A = M_0 + A$ gives a representation needed. Therefore, we may assume that $d \geq 1$.

If $s = 0$, then it follows from Lemma 11 that if $F \in A$, then $\deg [F]_n < d$ for every n . Thus $A \subset M_d$ and, consequently, we have $A = M_d$. Putting $M = \{0\}$ (which is an L -module of arbitrary order with an arbitrary L), we obtain $A = M_d + M$. Therefore, we may assume $s \geq 1$.

Notation 12. If $\phi \in \mathbb{C}[x]^s$ and $\phi = (f_0, \dots, f_{s-1})$, then we use the notation $\phi' = (f'_0, \dots, f'_{s-1})$. We say that a subset V of $\mathbb{C}[x]^s$ is closed under differentiation, if $\phi \in V$ implies $\phi' \in V$. Note that $V_{A,s}$ is closed under differentiation by (2).

Let $V_k = \{(f_1, \dots, f_s) \in V_{A,s} : \deg f_i < k \ (i = 1, \dots, s)\}$ for every integer k . Note that V_k is also closed under differentiation. We have $V_d = \{(f_1, \dots, f_s) \in \mathbb{C}[x]^s : \deg f_i < d \ (i = 1, \dots, s)\}$, as $M_d \subset A$.

For every polynomial $F \in \mathbb{C}[x, y]$ we denote

$$\Phi(F) = ([F]_0, \dots, [F]_{s-1}) \in \mathbb{C}[x]^s.$$

Clearly, Φ is a linear map from $\mathbb{C}[x, y]$ onto $\mathbb{C}[x]^s$, and maps A onto $V_{A,s}$.

It follows that there exists a linear map $\phi \mapsto F_\phi$ from $V_{A,s}$ into A such

that $\Phi(F_\phi) = \phi$ for every $\phi \in V_{A,s}$.

Lemma 13. *For every integer $k > d$ there is a linear map $L: V_k \rightarrow \mathbb{C}[x]$ with the following properties.*

(i) *For every $\phi \in V_k$ we have*

$$\deg(L(\phi) - [F_\phi]_s) < d. \quad (11)$$

(ii) *$L(\phi') = L(\phi)'$ for every $\phi \in V_k$.*

(iii) *$L(\phi) = 0$ for every $\phi \in V_d$.*

Proof. Let X denote the quotient space of the linear space V_k modulo the linear subspace V_d . (That is, let $X = V_k/V_d$.) Since the linear space V_k is of finite dimension (its dimension is at most k^s), so is X . Let $\phi \mapsto \bar{\phi}$ denote the natural homomorphism from V_k into X . That is, let $\bar{\phi} = \phi + V_d$ for every $\phi \in V_k$.

The derivation $\phi \mapsto \phi'$ maps V_d into itself. Therefore, we can define the derivation on X by $D(\bar{\phi}) = \bar{\phi}'$ ($\phi \in V_k$).

It is clear that D is a nilpotent linear map from X into itself. By [1, §57, Theorem 2, p. 111], there are positive integers r, q_1, \dots, q_r and elements $u_1, \dots, u_r \in X$ such that $D^{q_i}u_i = 0$ for every $i = 1, \dots, r$, and the elements $D^j u_i$ ($i = 1, \dots, r, j = 0, \dots, q_i - 1$) form a basis for X . Let $\psi_1, \dots, \psi_r \in V_k$ be such that $u_i = \bar{\psi}_i$ ($i = 1, \dots, r$). We put

$$\Lambda(D^j u_i) = ([F_{\psi_i}]_s)^{(j)} \quad (12)$$

for every $i = 1, \dots, r$ and $j = 0, \dots, q_i - 1$, and extend Λ linearly to X . We define $L(\phi) = \Lambda(\bar{\phi})$ for every $\phi \in V_k$. Then $L: V_k \rightarrow \mathbb{C}[x]$ is linear. We show that L has properties (i)-(iii).

If $\phi \in V_d$, then $\bar{\phi} = 0$, $L(\phi) = \Lambda(\bar{\phi}) = 0$, and thus (iii) holds.

Next we prove (i). Since L and the map $\phi \mapsto F_\phi$ are both linear, the set of elements $\phi \in V_k$ satisfying (11) is a linear subspace of V_k . Therefore, in order to prove (i) it is enough to check that (11) holds for a set of polynomials

generating V_k . We show that $\Psi = \{\psi_i^{(j)} : i = 1, \dots, r, j = 1, \dots, q_i - 1\} \cup V_d$ is such a set. Indeed, let $\phi \in V_k$. Since $D^j u_i$ is a basis for X , we have

$$\overline{\phi} = \sum_{i=1}^r \sum_{j=1}^{q_i} \lambda_{i,j} D^j u_i \quad (13)$$

with suitable complex coefficients $\lambda_{i,j}$. Now $\overline{\alpha'} = D\overline{\alpha}$ ($\alpha \in V_k$) implies that $\overline{\psi_i^{(j)}} = D^j u_i$ for every i, j , and thus the right hand side of (13) equals the image under the natural homomorphism of a linear combination of the elements $\psi_i^{(j)}$. Thus the difference of ϕ and this linear combination belongs to V_d , showing that Ψ generates V_k .

If $\phi \in V_d$, then $L(\phi) = 0$ by (iii) and $\deg [F_\phi]_s < d$ by Lemma 11, and thus (11) holds.

If $\phi = \psi_i^{(j)}$, then we have $\Phi(F_{\psi_i}) = \psi_i$,

$$\Phi\left(\frac{\partial^j}{\partial x^j} F_{\psi_i}\right) = \psi_i^{(j)} = \Phi(F_{\psi_i^{(j)}}) = \Phi(F_\phi),$$

and thus $\Phi\left(\frac{\partial^j}{\partial x^j} F_{\psi_i} - F_\phi\right) = 0$. In other words, the first s coordinate polynomials of $\frac{\partial^j}{\partial x^j} F_{\psi_i} - F_\phi$ are zero. By Lemma 11 it follows that

$$\begin{aligned} d &> \deg ([\frac{\partial^j}{\partial x^j} F_{\psi_i} - F_\phi]_s) = \deg ([\frac{\partial^j}{\partial x^j} F_{\psi_i}]_s - [F_\phi]_s) \\ &= \deg (([F_{\psi_i}]_s)^{(j)} - [F_\phi]_s) = \deg (\Lambda(D^j u_i) - [F_\phi]_s) \\ &= \deg (L(\phi) - [F_\phi]_s), \end{aligned}$$

which proves (i).

We turn to the proof of (ii). Since L is linear and $L(\phi) = 0$ if $\phi \in V_d$, in order to prove (ii) it is enough to check that $L(\phi') = L(\phi)'$ holds in the cases when $\phi = \psi_i^{(j)}$. Let $1 \leq i \leq r$ and $0 \leq j \leq q_i - 1$ be fixed. If $j < q_i - 1$, then $L(\phi)' = [F_{\psi_i}]_s^{(j+1)} = L(\phi')$, and we are done.

If $j = q_i - 1$, then $\phi' = \psi_i^{(q_i)} = 0$, so we have $\phi \in V_1 \subset V_d$. Then $L(\phi) = 0$ by (iii), and $L(\phi') = L(\phi)' = 0$ follows. \square

Lemma 14. *There exists a linear map $L: V_{A,s} \rightarrow \mathbb{C}[x]$ such that*

(i) (11) holds for every $\phi \in V_{A,s}$,

(ii) $L(\phi') = L(\phi)'$ for every $\phi \in V_{A,s}$, and

(iii) $L(\phi) = 0$ for every $\phi \in V_d$.

Proof. We define

$$L(\phi) = [F_\phi]_s + u_{\phi,d-1}x^{d-1} + \dots + u_{\phi,1}x + u_{\phi,0}, \quad (14)$$

where $u_{\phi,i}$ is an unknown for every $\phi \in V_{A,s}$ and $i = 0, \dots, d-1$. We show that we can assign values to these unknowns in such a way that the resulting map L satisfies the requirements. Since the map $\phi \mapsto F_\phi$ is linear, L will be linear if

$$\begin{aligned} u_{\lambda\phi+\mu\psi,d-1}x^{d-1} + \dots + u_{\lambda\phi+\mu\psi,1}x + u_{\lambda\phi+\mu\psi,0} = \\ \lambda(u_{\phi,d-1}x^{d-1} + \dots + u_{\phi,1}x + u_{\phi,0}) + \\ \mu(u_{\psi,d-1}x^{d-1} + \dots + u_{\psi,1}x + u_{\psi,0}) \end{aligned}$$

holds for every $\phi, \psi \in V_{A,s}$ and $\lambda, \mu \in \mathbb{C}$. It is clear that condition (i) is satisfied with any choice of the unknowns $u_{\phi,i}$. Condition (ii) is satisfied if

$$\begin{aligned} [F_{\phi'}]_s + u_{\phi',d-1}x^{d-1} + \dots + u_{\phi',1}x + u_{\phi',0} = \\ ([F_\phi]_s)' + (d-1)u_{\phi,d-1}x^{d-2} + \dots + u_{\phi,1} \end{aligned}$$

holds for every $\phi \in V_{A,s}$. Finally, (iii) is satisfied if the right hand side of (14) is zero for every $\phi \in V_d$. Summing up: in order that L satisfy the conditions, the unknowns $u_{\phi,i}$ must satisfy a certain infinite system of linear equations S . We have to show that S is solvable. It is well-known that a system S of linear equations is solvable if and only if every finite subsystem of S is solvable. Now a finite subsystem T of S only involves a finite number of elements $\phi \in V_{A,s}$. Then there is a k such that all these elements belong to V_k . As we proved above, there is a map L on V_k satisfying (i)-(iii) on V_k .

Now condition (i) implies that L is of the form (14) with concrete values of the unknowns $u_{\phi,i}$ for every $\phi \in V_k$. These values constitute a solution of the subsystem T , showing that T is solvable. Therefore, S is solvable, proving the existence of L with the required properties. \square

Proof of Theorem 8. Fix a map L as in Lemma 14. We prove that if $\phi = (f_0, \dots, f_{s-1}) \in V_{A,s}$, then the recursion $f_n = L(f_{n-s}, \dots, f_{n-1})$ ($n = s, s +$

$1, \dots$) defines a sequence of polynomials such that $f_n = 0$ for every n large enough, and

$$\deg(f_n - [F_\phi]_n) < d \quad (15)$$

for every n . It is clear that (15) holds for every $n < s$.

Let $k \geq s$, and suppose we have defined f_n for every $n < k$ such that (15) holds for every $n < k$. Let $\psi = (f_{k-s}, \dots, f_{k-1})$. We have $F_\phi \in A$ and $G = \frac{\partial^{k-s}}{\partial y^{k-s}} F_\phi \in A$. Since $\deg(f_{k-s+i} - [F_\phi]_{k-s+i}) < d$ and $[G]_i = [F_\phi]_{k-s+i}$ for every $i < s$, we have $\Phi(G) - \psi \in V_d$. Since $V_d \subset V_{A,s}$ and $\Phi(G) \in V_{A,s}$, we obtain $\psi \in V_{A,s}$. Therefore, $L(f_{k-s}, \dots, f_{k-1})$ is defined. Let $f_k = L(f_{k-s}, \dots, f_{k-1})$. By (i) of Lemma 14, we have $\deg(f_k - [F_\psi]_s) < d$.

Now $G - F_\psi \in A$ and $\Phi(G - F_\psi) \in V_d$. By Lemma 11, this implies $\deg[G - F_\psi]_s < d$. Since $[G]_s = [F_\phi]_k$, we obtain $\deg([F_\phi]_k - [F_\psi]_s) < d$ and $\deg(f_k - [F_\phi]_k) < d$. This proves that the recursion $f_n = L(f_{n-s}, \dots, f_{n-1})$ defines f_n for every n such that (15) holds for every n .

Since $F_\phi \in A$, there is an N such that $[F_\phi]_n = 0$ for every $n \geq N$. Then $\deg f_n < d$ for every $n \geq N$. If $n > N + s$, then $(f_{n-s}, \dots, f_{n-1}) \in V_d$ by (15), and thus $f_n = L(f_{n-s}, \dots, f_{n-1}) = 0$ by (iii) of Lemma 14. Therefore $f_n = 0$ for every n large enough. Let H_ϕ denote the polynomial $\sum_{n=0}^{\infty} f_n \cdot \frac{y^n}{n!}$. Then (15) implies that $H_\phi - F_\phi \in M_d$. Since $M_d \subset A$ and $F_\phi \in A$, it follows that $H_\phi \in A$.

Let M be the set of polynomials H_ϕ , where $\phi \in V_{A,s}$. Then we have $M \subset A$. It is easy to see that the map $\phi \mapsto H_\phi$ is linear, and thus M is a linear subspace of A . It is also easy to check that $F \in M$ implies $\frac{\partial}{\partial y} F \in M$. Now (ii) of Lemma 14 implies that $\frac{\partial}{\partial x} F_\phi = H_{\phi'} \in M$ for every $\phi \in V_{A,s}$. Thus M is also closed under partial differentiation w.r.t. x . Consequently, M is a module. It is clear that M is an L-module of order s .

If $F \in A$, then $\phi = \Phi(F) \in V_{A,s}$. Now $F - F_\phi \in M_d$ by Lemma 11, and $H_\phi - F_\phi \in M_d$ by (15). Thus

$$F = ((F - F_\phi) - (H_\phi - F_\phi)) + H_\phi \in M_d + M,$$

which proves $A = M_d + M$. □

Remark 15. In the representation $A = M_d + M$, where M is an L-module,

the value of d is unique (see Corollary 10). However, the term M is not unique in general, as the following example shows.

Let $M_{d,s}$ denote the set of polynomials of the form (1), where $\deg f_n < d$ for every $n < s$, and $f_n = 0$ for every $n \geq s$. It is clear that $M_{d,s}$ is an L-module of order s . Since $M_{d,s} \subset M_d$, we have $M_d = M_d + \{0\} = M_d + M_{d,s}$.

We can make the representations unique if we restrict the L-module terms. Note that the proof of Theorem 8 produces L-modules with a linear map L such that $L(f_1, \dots, f_s) = 0$ whenever $\deg f_i < d$ ($i = 1, \dots, s$); see (iii) of Lemma 14. We may also assume that $M_{d,s} \subset M$, since otherwise we replace M by $M + M_{d,s}$. Now, it is easy to check that the representation $A = M_d + M$ is unique, if we require that the L-module M should satisfy both $M_{d,s} \subset M$ and $L(f_1, \dots, f_s) = 0$ whenever $\deg f_i < d$ ($i = 1, \dots, s$).

4 Indecomposable submodules

Definition 16. We say that a submodule M of $\mathbb{C}[x_1, \dots, x_n]$ is *decomposable*, if M can be represented as the sum of finitely many proper submodules of M . Otherwise the submodule M is *indecomposable*.

Proposition 17. *Every submodule of $\mathbb{C}[x_1, \dots, x_n]$ is the sum of finitely many indecomposable submodules.*

Proof. The family of submodules of $\mathbb{C}[x_1, \dots, x_n]$ has the minimal condition; that is, if $M_1 \supset M_2 \supset \dots$ are submodules of $\mathbb{C}[x_1, \dots, x_n]$, then there is a positive integer K such that $M_k = M_K$ for every $k \geq K$ (see [2, Lemma 8]). Therefore, if the statement of the proposition is not true, then there is a minimal counterexample M . Then M must be decomposable. If $M = A_1 + \dots + A_k$, where A_1, \dots, A_k are proper submodules of M then, by the minimality of M , each A_i is the sum of finitely many indecomposable submodules. Then the same is true for M , which is impossible. \square

It is not clear if the representation of a module as the sum of indecomposable submodules containing a minimal number of terms is unique or not.

In the following we confine ourselves to the submodules of $\mathbb{C}[x, y]$ (except in Remark 24). It follows from Theorem 8 that *if M is an indecomposable*

submodule of $\mathbb{C}[x, y]$, then either $M = M_d$ for some d or M is an L -module of order s for some s .

Our next aim is to show that M_d is indecomposable for every d , and so is every L -module of order 1.

Lemma 18. *The system of translation invariant linear subspaces of $\mathbb{C}[x]^s$ has the minimal condition.*

Proof. We prove the statement by induction on s . Since every translation invariant linear subspace of $\mathbb{C}[x]$ equals $\mathbb{C}[x]$ or $\{f \in \mathbb{C}[x] : \deg f < d\}$ for some $d \geq 0$, it easily follows that the statement is true for $s = 1$.

Let $s \geq 1$, suppose that the statement is true for s , and let $V_1 \supset V_2 \supset \dots$ be translation invariant linear subspaces of $\mathbb{C}[x]^{s+1}$. We have to show that $V_n = V_{n+1} = \dots$ if n is large enough.

Put $A_n = \{f \in \mathbb{C}[x] : (0, \dots, 0, f) \in V_n\}$ for every n . Since A_n is a translation invariant linear subspace of $\mathbb{C}[x]$ and $A_1 \supset A_2 \supset \dots$, there is an N_1 such that $A_n = A_{N_1}$ for every $n \geq N_1$. Let

$$B_n = \{(f_1, \dots, f_s) \in \mathbb{C}[x]^s : \exists f, (f_1, \dots, f_s, f) \in V_n\}.$$

Then B_n is a translation invariant linear subspace of $\mathbb{C}[x]^s$ and $B_1 \supset B_2 \supset \dots$. By the induction hypothesis it follows that there is an N_2 such that $B_n = B_{N_2}$ for every $n \geq N_2$. Let $N = \max(N_1, N_2)$; we prove that $V_n = V_N$ for every $n \geq N$. Let $n \geq N$ and $(f_1, \dots, f_{s+1}) \in V_N$ be given; we prove $(f_1, \dots, f_{s+1}) \in V_n$.

We have $(f_1, \dots, f_s) \in B_N = B_n$, and thus there is a g such that $(f_1, \dots, f_s, g) \in V_n \subset V_N$. From $(f_1, \dots, f_{s+1}) \in V_N$ we obtain $(0, \dots, 0, f_{s+1} - g) \in V_N$, $f_{s+1} - g \in A_N = A_n$ and $(0, \dots, 0, f_{s+1} - g) \in V_n$. Thus

$$(f_1, \dots, f_{s+1}) = (f_1, \dots, f_s, g) + (0, \dots, 0, f_{s+1} - g) \in V_n,$$

and the proof is complete. \square

Theorem 19. *If $A, B \subset \mathbb{C}[x, y]$ are L -modules, then so is $A + B$.*

Proof. Suppose A is of order s_1 and B is of order s_2 . If $s = \max(s_1, s_2)$, then A, B are both of order s . For every $k > s$ we denote by Z_k the set of s -tuples

$(f_1, \dots, f_s) \in V_{A,s} \cap V_{B,s}$ such that if $F \in A$, $G \in B$ and $[F]_n = [G]_n = f_n$ for every $n < s$, then $[F]_n = [G]_n$ for every $n < k$.

It is easy to check that Z_k is a translation invariant linear subspace of $\mathbb{C}[x]^s$, and $Z_{s+1} \supset Z_{s+2} \supset \dots$. By Lemma 18, there is a $K > s$ such that $Z_k = Z_K$ for every $k \geq K$. We prove that $A + B$ is an L-module of order K .

Let $S \in A + B$ such that $[S]_n = 0$ for every $n < K$. We show that $[S]_K = 0$. Let $S = F + G$, where $F \in A$ and $G \in B$. Then $[F]_n + [G]_n = 0$; that is, $[F]_n = -[G]_n$ for every $n < K$. Since $-G \in B$, it follows that $([F]_0, \dots, [F]_{s-1}) \in V_{A,s} \cap V_{B,s}$. We prove $([F]_0, \dots, [F]_{s-1}) \in Z_K$.

Suppose $H \in A$ and $[H]_n = [F]_n$ for every $n < s$. Since $F, H \in A$ and A is an L-module of order s , it follows that $F = H$, and thus $[H]_n = [F]_n$ for every $n < K$. Similarly, if $P \in B$ and $[P]_n = [F]_n = -[G]_n$ for every $n < s$, then $P = -G$, and thus $[P]_n = -[G]_n = [F]_n$ for every $n < K$, proving $([F]_0, \dots, [F]_{s-1}) \in Z_K$.

Since $Z_K = Z_{K+1}$, we find $([F]_0, \dots, [F]_{s-1}) \in Z_{K+1}$. This implies $[F]_K = -[G]_K$; that is, $[S]_K = 0$. \square

Remark 20. The proof above does not give any estimate of the order of $A + B$. We do not know if the order of $A + B$ is bounded from above by, say, the sum of the order of A and of B .

Theorem 21. M_d is indecomposable for every d .

Proof. Suppose this is not true, and let $M_d = A_1 + \dots + A_k$, where A_1, \dots, A_k are proper submodules of M_d . By Theorem 8, we have $A_i = M_{d_i} + B_i$, where B_i is an L-module for every i . By Theorem 19 we find that $B = B_1 + \dots + B_k$ is an L-module. It is clear that $M_{d_1} + \dots + M_{d_k} = M_e$, where $e = \max_{1 \leq i \leq k} d_i$. Therefore, $M_d = A_1 + \dots + A_k$ gives $M_d = M_e + B$, where B is an L-module. By Corollary 10, we have $e = d$, and thus $d_i = d$ for a suitable $1 \leq i \leq k$. Then $M_d = M_{d_i} \subset A_i$, which is impossible, since A_i is a proper submodule of M_d . \square

Theorem 22. Every L-module of order 1 is indecomposable.

Proof. Let M be an L-module of order 1, and suppose $M = A_1 + \dots + A_k$, where A_1, \dots, A_k are proper submodules of M . Each of the linear spaces

$V_{M,1}$ and $V_{A_i,1}$ ($i = 1, \dots, k$) equals one of $\mathbb{C}[x]$ or $\{f \in \mathbb{C}[x] : \deg f < d\}$ for some $d \geq 0$. Since

$$V_{M,1} = V_{A_1,1} + \dots + V_{A_k,1},$$

it follows that $V_{M,1} = V_{A_i,1}$ for a suitable i . Then we have $M = A_i$ by the definition of L-modules, which is impossible. \square

Remarks 23. (i) There are decomposable L-modules: if A, B are L-modules, $A \subsetneq B$ and $B \subsetneq A$, then $A + B$ is a decomposable L-module.

(ii) There are indecomposable L-modules of order > 1 . Indeed, let

$$A^* = \{f(y, x) : f(x, y) \in A\}$$

for every $A \subset \mathbb{C}[x, y]$. It is clear that if M is a module, then so is M^* , and if M is indecomposable then so is M^* . Thus M_2^* is indecomposable. On the other hand, $M_2^* = \{f(x) + g(x)y : f, g \in \mathbb{C}[x]\}$. It is clear that M_2^* is an L-module of order 2. Similarly, M_d^* is an indecomposable L-module of order d for every d .

(iii) It follows from the definition that a submodule of an L-module is also an L-module. Also, we have $M_d \not\subset M$ for every $d \geq 1$ and for every L-module M . Indeed, if M is an L-module of order s , then $y^s \notin M$ and $y^s \in M_1 \subset M_d$. From these observations it follows that *the representation of the submodules of $\mathbb{C}[x, y]$ as sums of indecomposable submodules containing a minimal number of terms is unique if and only if this is true for L-modules.*

Remark 24. We show that Theorem 8 does not have a straightforward generalization to $\mathbb{C}[x, y, z]$. Such a generalization would operate with modules defined as follows. Generalizations of the modules M_d could be defined as the set of polynomials

$$F(x, y, z) = \sum_{n=0}^{\infty} f_n(x, y) \frac{z^n}{n!} \quad (16)$$

such that the degrees of the polynomials $f_n(x, y) \in \mathbb{C}[x, y]$ satisfy some prescribed inequalities. Let these modules be called bounded modules. We call M an L-module of order s if, whenever the polynomial in (16) belongs to M and $f_n = 0$ for every $n < s$, then $F = 0$.

Now suppose Theorem 8 had a generalization to $\mathbb{C}[x, y, z]$. It would claim that every proper submodule of $\mathbb{C}[x, y, z]$ is the sum of bounded modules and

L-modules. Then it would follow that whenever M is an indecomposable submodule of $\mathbb{C}[x, y, z]$, then either M is bounded, or M is an L-module.

We show that this is false, no matter how we define bounded modules. Let M be the set of polynomials

$$\sum_{n=0}^{\infty} (a_n(x+y) + b_n) \cdot \frac{z^n}{n!}, \quad (17)$$

where a_n and b_n are complex numbers and $a_n = b_n = 0$ if n is large enough. It is clear that M is a module. Suppose $M = A_1 + \dots + A_k$, where A_1, \dots, A_k are submodules of M . Then there is an i such that A_i has the following property: for every N there is a polynomial of the form (17) belonging to A_i and such that $a_n \neq 0$ for at least one index $n > N$. It is easy to check that this condition implies $A_i = M$, and thus M is indecomposable.

Now M is not a bounded module, since no matter how we prescribe the inequalities satisfied by the elements of M , the polynomial $x + 2y$ would also satisfy these conditions, but $x + 2y \notin M$. It is also clear that M is not an L-module, since the coordinate polynomials $a_n(x+y) + b_n$ can be chosen independently. This shows that no generalization of the form described above is possible.

5 Construction of a submodule of $\mathbb{C}[x, y]$ which is not closed

We equip $\mathbb{C}[x_1, \dots, x_n]$ with the topology of uniform convergence on compact sets. The closure of a set $M \subset \mathbb{C}[x_1, \dots, x_n]$ w.r.t. this topology is denoted by $\text{cl } M$.

Theorem 25. *There exists a module $M \subset \mathbb{C}[x, y]$ such that $x \in \text{cl } M$ but $x \notin M$.*

Proof. We use the notation $e(x) = e^x$, $e_2(x) = e(e(x))$ and $e_3(x) = e(e_2(x))$. Then we have

$$e_3(n-1)^{e(n)} / e_3(n) \rightarrow 0 \quad (n \rightarrow \infty). \quad (18)$$

Indeed, we have $n + e(n-1) - e(n) \rightarrow -\infty$, hence $e(n) \cdot e_2(n-1)/e_2(n) \rightarrow 0$, hence $e(n) \cdot e_2(n-1) - e_2(n) \rightarrow -\infty$, hence $e_3(n-1)^{e(n)}/e_3(n) \rightarrow 0$ as $n \rightarrow \infty$.

Let $a_1 = 1$ and $a_n = -e_3(n)$ for every $n \geq 2$. We put $L(f) = a_1 f' + a_2 f'' + \dots$ for every $f \in \mathbb{C}[x]$. Note that the number of nonzero terms in the sum is finite for every $f \in \mathbb{C}[x]$. Let M denote the set of polynomials

$$\sum_{n=0}^{\infty} L^n(f)(x) \cdot \frac{y^n}{n!}, \quad (19)$$

where $f \in \mathbb{C}[x]$ is arbitrary. By Lemma 6, M is a submodule of $\mathbb{C}[x, y]$. Now we have $x \notin M$. Indeed, if $F(x, y)$ is defined by (1) and $F(x, y) = x$, then $f_0(x) = x$ and $f_1 = 0$. However, we have $f_1 = L(f_0) = L(x) = 1$, a contradiction. (In fact, the same argument gives $f \notin M$ for every $f \in \mathbb{C}[x]$ with $\deg f \geq 1$.)

We put $\varepsilon_n = |a_n \cdot n!|^{-1}$, $g_n(x) = x + \varepsilon_n x^n$ and $G_n(x, y) = \sum_{k=0}^n L^k(g_n) \cdot \frac{y^k}{k!}$ for every n . We show that the sequence of polynomials G_n converges to x locally uniformly on \mathbb{C}^2 . Since $G_n \in M$ for every n , this will prove that $x \in \text{cl } M$.

If $f \in \mathbb{C}[x]$, $f = \sum_{i=0}^n c_i x^i$, then we put $\|f\| = \max_{0 \leq i \leq n} |c_i|$. Clearly, $\|\cdot\|$ is a norm on $\mathbb{C}[x]$. If $\deg f \leq n$ and $|x| \leq e^n$, then

$$\begin{aligned} |f(x)| &\leq \|f\| \cdot (1 + e(n) + e(2n) + \dots + e(n^2)) < (n+1) \cdot e(n^2) \cdot \|f\| \\ &< e_2(n) \cdot \|f\| < e_3(n-1) \cdot \|f\| \end{aligned} \quad (20)$$

if $n > n_0$. If $f \in \mathbb{C}[x]$ and $\deg f \leq n$, then $\|f'\| \leq n \cdot \|f\|$. Therefore,

$$\begin{aligned} \|Lf\| &\leq \sum_{i=1}^n |a_i| \cdot \|f^{(i)}\| \leq n! \sum_{i=1}^n |a_i| \cdot \|f\| \\ &\leq n! \cdot n \cdot e_3(n) \cdot \|f\| \\ &\leq e_2(n) \cdot e_3(n) < e_3(n)^2 \cdot \|f\| \end{aligned} \quad (21)$$

for every $f \in \mathbb{C}[x]$ with $\deg f \leq n$. (We used the trivial estimate $n! \cdot n \leq n^n < e(n^2) < e_2(n)$.) Let $n \geq 2$ be fixed. If $|x| \leq e^n$, then

$$|g_n(x) - x| \leq \varepsilon_n \cdot e(n^2) < e(n^2)/e_3(n) = e(n^2 - e_2(n)). \quad (22)$$

Now we have

$$\begin{aligned}
L(g_n) &= g'_n + \sum_{i=2}^{n-1} a_i g_n^{(i)} + a_n g_n^{(n)} \\
&= 1 + n \cdot \varepsilon_n x^{n-1} + \sum_{i=2}^{n-1} a_i \cdot n(n-1) \cdots (n-i+1) \varepsilon_n x^{n-i} + a_n \cdot n! \cdot \varepsilon_n \\
&= n \cdot \varepsilon_n x^{n-1} + \sum_{i=2}^{n-1} a_i \cdot n(n-1) \cdots (n-i+1) \varepsilon_n x^{n-i},
\end{aligned}$$

and thus

$$\|L(g_n)\| \leq \varepsilon_n \cdot \max_{1 \leq i \leq n-1} |a_i| \cdot n! < \varepsilon_n \cdot e_3(n-1) \cdot n! = e_3(n-1)/e_3(n).$$

Since $\deg L(g_n) = n-1$, (21) gives

$$\|L^k(g_n)\| \leq e_3(n-1)^{2k-2} \cdot \|Lg_n\| < e_3(n-1)^{2k-1}/e_3(n).$$

for every $k \geq 2$. Then we find, by (20), that if $2 \leq k \leq n$ and $|x| \leq e^n$, then

$$|L^k(g_n)(x)| \leq e_3(n-1)^{2n}/e_3(n). \quad (23)$$

If $|x| \leq e^n$ and $|y| \leq e^n$, then it follows from (22) and (23) that

$$|G_n(x, y) - x| < e(n^2 - e_2(n)) + n \cdot e(n^2) \cdot e_3(n-1)^{2n}/e_3(n)$$

if $n > n_0$. Since $e(n^2 - e_2(n)) \rightarrow 0$ and

$$e(n^2) \cdot e_3(n-1)^{2n} < e_3(n-1)^{2n+1} < e_3(n-1)^{e(n)},$$

it follows from (18) that $G_n(x, y) \rightarrow x$ locally uniformly on \mathbb{C}^2 . \square

Remarks 26. (i) It is easy to see that $\mathbb{C}[x_1, x_2]$ is a closed submodule of $\mathbb{C}[x_1, \dots, x_n]$ for every $n \geq 2$. Therefore, Theorem 25 implies that *for every $n \geq 2$ there exists a submodule of $\mathbb{C}[x_1, \dots, x_n]$ which is not closed.*

(ii) Using an elaborate version of the proof of Theorem 25 one can show that *there are L -modules of order 1 which are everywhere dense in $\mathbb{C}[x, y]$ w.r.t. the topology of uniform convergence on compact sets.*

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