Balancing of the skateboard with reflex delay

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Summary. A simple mechanical model of the skateboard-skater system is constructed in which a PD controller with time delay is implemented. Equations of motion are derived with the help of Appell-Gibbs method and are linearized around straight uniform motion. The linear stability analysis is carried out analytically using the D-subdivision method. Stability charts are presented for realistic system parameters and the critical time delay is calculated. It is also shown how the control gains have to be varied with respect to the speed of the skateboard in order to stabilize uniform motion.

Introduction

The stability problem of the skateboard at high speed is well known in the community of skaters. However, this phenomenon is not fully explained in the specialist literature (see, for example [1],[2]). The mechanical analysis of the skateboard-skater system is an interesting challenge for researchers due to the complexity of the system. On the one hand, the skateboard is a non-holonomic system due to its rolling wheels, which are steered by the special wheel-suspension system. On the other hand, the skateboard-skater interference turns this non-holonomic system into an enhanced human balancing problem. And, the analysis of human balancing is a complicated task due to the reflex delay in the control loop of the human body.

The aim of this study is the investigation of the effect of the skater reflex delay on the stability of the uniform motion. In order to this, we use a simplified mechanical model of the skateboard-skater system. The balancing effort of the skater is taken into account by a PD controller, which considers the reflex delay too. The linear stability charts are constructed with special attention to the speed of the skateboard and the time delay of the control.

Mechanical model

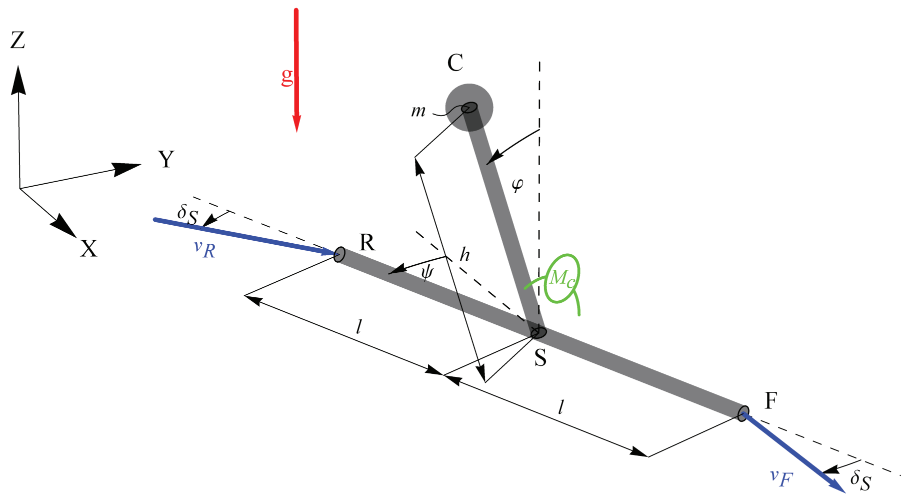

Figure 1: The simplified mechanical model of the skateboard-skater system

The mechanical model (see Figure 1) is based on [1]. Here we use additional reduction in the geometry of the model using a control torque that relates to the balancing effort of the skater. The skateboard is modelled by a massless rod (between the front axle at F and the rear axle at R) while the skater is represented by a massless rod (between the points S and C) with a mass point at C. In this model, the connection between the skater and the board (at S) is assumed to be rigid. The resulting rigid body has zero mass moment of inertia with respect to its centre of gravity at C, which makes the derivation of the equations of motion easier. Thus, the skateboard moves in the three dimensional gravitational field but we have to describe the motion of a mass point only.

Due to the fact that the skateboard is in contact with the ground, which reduces the degrees of freedom by two, one can choose four generalised coordinates to describe the motion: X and Y are the coordinates of the skateboard centre point S in the plane of the ground; ψ describes the direction of the longitudinal axes of the skateboard; and φ is the deflection angle of the body of skater from the vertical direction. The geometrical parameters are the following. The half-height of the skater is denoted by h. The length of the board is $2l$ while m represents the mass of the skater and g is the acceleration due to gravity. A control torque is considered, which can be originated in the ankle of the skater, and acts around the longitudinal axis of the skateboard:

$$M(t) = P\varphi(t-\tau) + D\dot{\varphi}(t-\tau), \tag{1}$$

where τ is the time delay, P and D represent the proportional and the differential control gains, respectively.
Regarding the rolling wheels of the skateboard, kinematic constraint equations can be given for the velocities $\mathbf{v}_\mathrm{F}$ and $\mathbf{v}_\mathrm{R}$ of the points F and R, respectively. The directions of these velocities depend on φ through δ_S, the steering angle (see Figure 1), such that

$$\sin\varphi(t)\tan\kappa = \tan\delta_\mathrm{S}(t), \tag{2}$$

where κ is the complement of the rake angle in the skateboard wheel suspension system (see Figure 2). This formula is straightforward to obtain since, as the skater deflects the board with angle φ, the axles of the wheels turn with angle δ_S. The wheel axle is always parallel to the ground, remaining it is perpendicular to the king pin axis, which rotates together with the board.

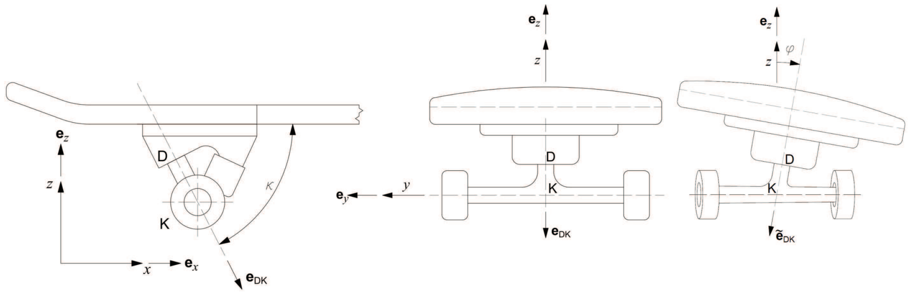

Figure 2: Skateboard wheel suspension system

The equation of motion of the resulting non-holonomic system can be efficiently derived by means of the Appell-Gibbs equation (see in [4]).
We get the governing equation in a compact form of first order ordinary differential equations. In our model, two pseudo velocities are required since the number of the generalised coordinates is two greater than the number of the non-holonomic constraints. An appropriate choice can be the longitudinal speed

$$\sigma_1 = \dot{X}\cos\psi + \dot{Y}\cos\psi \tag{3}$$

of the skateboard and the angular speed

$$\sigma_2 = \dot{\varphi} \tag{4}$$

of the skater around the longitudinal axes of the board. Thus we obtain the equations of the motion

$$\begin{aligned}
\dot{\sigma}_1(t) &= \frac{3h\sin 2\varphi(t)\tan\kappa\,\sigma_1(t)\,\sigma_2(t)}{2\left(l - h\sin^2\varphi(t)\tan\kappa\right)}, \\
\dot{\sigma}_2(t) &= \frac{mgh\sin\varphi(t) - P\varphi(t-\tau) - D\sigma_2(t-\tau)}{mh^2} - \frac{\sin 2\varphi(t)\tan\kappa\left(l - h\sin^2\varphi(t)\tan\kappa\right)}{2hl^2}\sigma_1^2(t), \\
\dot{\varphi}(t) &= \sigma_2(t), \\
\dot{X}(t) &= \sigma_1(t)\cos\psi(t), \\
\dot{Y}(t) &= \sigma_1(t)\sin\psi(t), \\
\dot{\psi}(t) &= -\frac{\tan\kappa}{l}\sigma_1(t)\sin\varphi(t).
\end{aligned} \tag{5}$$

Here X, Y and ψ are cyclic coordinates, so the last three equations are unnecessary for our further investigation. So we have a three dimensional problem.

Stability analysis

To investigate the stability of uniform motion of the skateboard-skater system, the equations of motion were linearized around the stationary solution: $\sigma_1 \equiv V$, $\sigma_2 \equiv 0$, $X = Vt$, $Y \equiv 0$, $\psi \equiv 0$ and $\varphi \equiv 0$. The linear equations of the motion can be written in the form:

$$\begin{bmatrix} \dot{\sigma}_1(t) \\ \dot{\sigma}_2(t) \\ \dot{\varphi}(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{g}{h} - \frac{V^2\tan\kappa}{hl} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \sigma_1(t) \\ \sigma_2(t) \\ \varphi(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\frac{D}{mh^2} & -\frac{P}{mh^2} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \sigma_1(t-\tau) \\ \sigma_2(t-\tau) \\ \varphi(t-\tau) \end{bmatrix}. \tag{6}$$

The first row of the two coefficient matrices contains only zeroes, thus σ_1 is also a cyclic coordinate in the linear case. This implies that the characteristic function of the linearized system

$$D(\lambda) = \left(\lambda^2 + \frac{De^{-\lambda\tau}}{mh^2}\lambda - \frac{g}{h} + \frac{Pe^{-\lambda\tau}}{mh^2} + \frac{V^2\tan\kappa}{hl}\right)\lambda\,, \tag{7}$$

where λ is the characteristic exponent, always has a zero root. Hence, uniform motion with longitudinal speed V cannot be asymptotically stable, but it can be Lyapunov stable. Thus, the stability of uniform motion can be stable if and only if the real parts of the other characteristic exponents are less than zero.
The limit of stability corresponds to the case when characteristic exponents are located on the imaginary axis in the λ complex plane for some particular system parameters (e.g.: h, V, P, D, etc.). If both the real and imaginary parts of a characteristic exponent are zero, then a saddle-node (SN) bifurcation can occur. In our case, $D(\lambda = 0) = 0$ leads to

$$\frac{P}{mh^2} + \frac{V^2}{hl}\tan\kappa - \frac{g}{h} = 0\,, \tag{8}$$

which gives a vertical line in the $P - D$ parameter plane:

$$P_{\text{SN}} = mh\left(g - \frac{V^2}{l}\tan\kappa\right)\,. \tag{9}$$

If the imaginary part of the characteristic exponent is not zero, then a Hopf-bifurcation can occur, and the characteristic exponent can be express as $\lambda = \pm \mathbf{i}\omega$, where $\mathbf{i}$ is the imaginary unit and $\omega \in \mathbb{R}$. Using the D-subdivision method, we obtain two equations for P, D, ω, τ, (first is the real part of the characteristic equation, the second one is the imaginary part of it):

$$\begin{aligned} -\frac{D\omega\cos(\omega\tau)}{mh^2} + \frac{P\sin(\omega\tau)}{mh^2} &= 0\,, \\ -\omega^2 - \frac{g}{h} + \frac{V^2\tan\kappa}{hl} + \frac{P\cos(\omega\tau)}{mh^2} + \frac{D\omega\sin(\omega\tau)}{mh^2} &= 0\,. \end{aligned} \tag{10}$$

From these equations, P and D can be obtained as follows

$$\begin{aligned} P &= mh\cos(\omega\tau)\left(g + h\omega^2 - \frac{V^2}{l}\tan\kappa\right)\,, \\ D &= mh\frac{\sin(\omega\tau)}{\omega}\left(g + h\omega^2 - \frac{V^2}{l}\tan\kappa\right)\,. \end{aligned} \tag{11}$$

This leads to a curve in the $P - D$ parameter plane. The vertical line of the SN bifurcation and a Hopf bifurcation curve with the linearly stable (white) and unstable (shaded) domains can be seen in Figure 3 for $V = 1\,\mathrm{m/s}$ and $\tau = 0.2\,\mathrm{s}$. The other, realistically chosen system parameters are shown in Table 1.

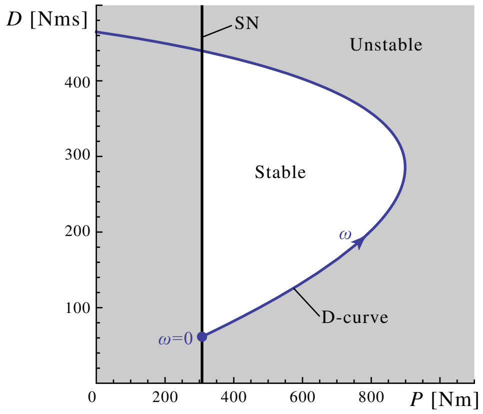

Figure 3: The linear stability chart for $V = 1\,\mathrm{m/s}$ and $\tau = 0.2\,\mathrm{s}$.

Table 1: Parameters of the skater and the board

h [m]	m [kg]	g [m/s^2]	l [m]	κ [^{o}]
0.85	75	9.81	0.3937	63

The resulted stability chart is very similar to stability charts of the human balancing model in [5]. Nevertheless, it can be verified by means of the formulas that the origin of the P-D parameter plane ($P = 0$ and $D = 0$, i.e. without control) is always on the stability boundary if the vertical line (belongs to the SN bifurcation) is on the left side of the plane ($P_{\mathrm{SN}} < 0$). Then, the known critical forward speed (see [3] without torsional spring)

$$V_{\mathrm{cr}} = \sqrt{gl/\tan\kappa} \tag{12}$$

can be obtained, above which uniform motion is linearly stable without any control.

Let us now consider a control loop without time delay. In this case, the stability analysis is easier and we obtain the same conclusion as in [1] with symmetric skateboard skater configuration. The characteristic equation with this simplification reduces to

$$\left(\lambda^2 + \frac{D}{mh^2}\lambda - \frac{g}{h} + \frac{P}{mh^2} + \frac{V^2\tan\kappa}{hl}\right)\lambda = 0\,. \tag{13}$$

The stability of uniform motion can be investigated by means of the Routh-Hurwitz criterion. This leads to the necessary and sufficient conditions:

$$P > mgh - m\frac{h}{l}V^2\tan\kappa \quad \text{and} \quad \frac{D}{mh^2} > 0\,. \tag{14}$$

The first condition is the same as the condition that we obtained for the saddle-node bifurcation. The second one leads to $D > 0$. This problem is similar to the inverted pendulum, with the required spring stiffness, which refers to P in our model, depending on the forward speed V.

Interpretation of the result from the stability analysis

In this section we investigate the shape of the stable parameter domains of the stability charts at different parameters. Stability charts belong to different forward speeds for the same time delay $\tau = 0.7\,\mathrm{sec}$ are shown in Figure 4. The vertical line, the SN bifurcation, moves left in the $P-D$ plane while V increases. But let us concentrate on the behaviour of the Hopf stability curve, which may show more interesting properties.

It is easy to see in Figure 4 that the so-called D-curve intersects itself beyond a certain longitudinal speed, and a loop appears with an enclosed domain. If we cross the D-curve from the inner side to the outer, one complex conjugate pair will go from the left hand side to the right hand side of the complex plane. Hence only the inner side of the loop can be stable.

It can be also observed in Figure 4, that D-curve loop rotates anticlockwise around the origin when velocity increases. It means there are certain velocities for which the stable domain is located on the left, on the bottom, on the right or on the top half plane. It implies there are several parameter sets when negative P and negative D control gains are also appropriate. This is in sharp contrast to the results of the simple human balancing or the controlled inverted pendulum [5].

This property also implies that with fixed $P-D$ parameter pairs uniform motion cannot be linearly stable for any speed. Strictly speaking, only the origin is always at the limit of stability for large enough longitudinal speed (when $P_{\mathrm{SN}} < 0$).

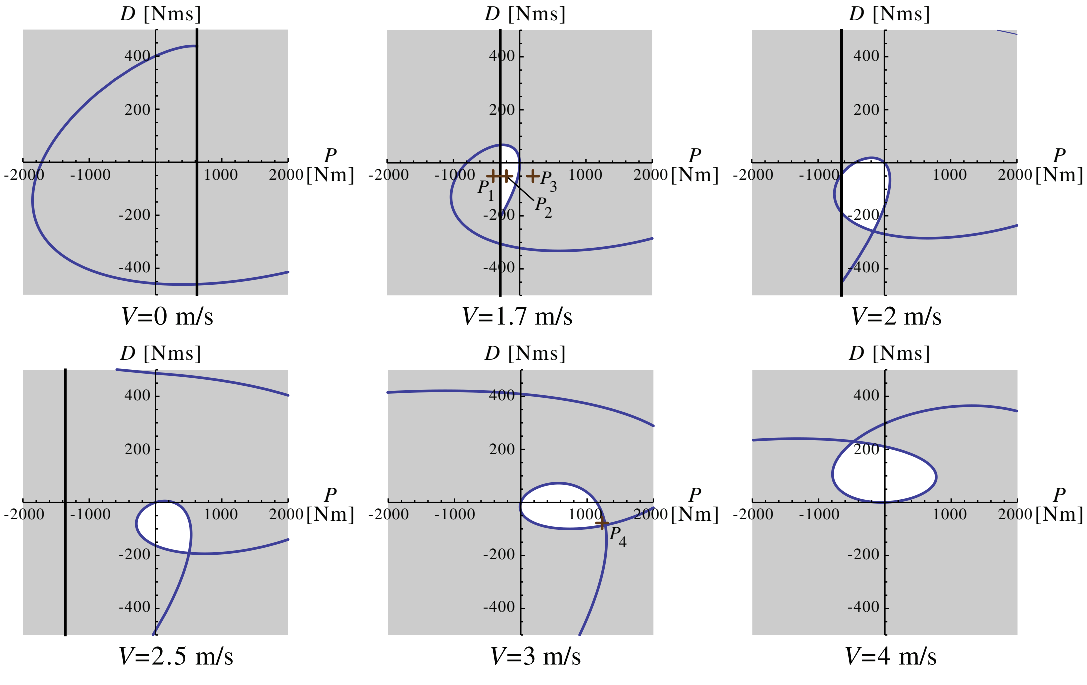

Figure 4: The linear stability charts for different reflex delays at zero speed

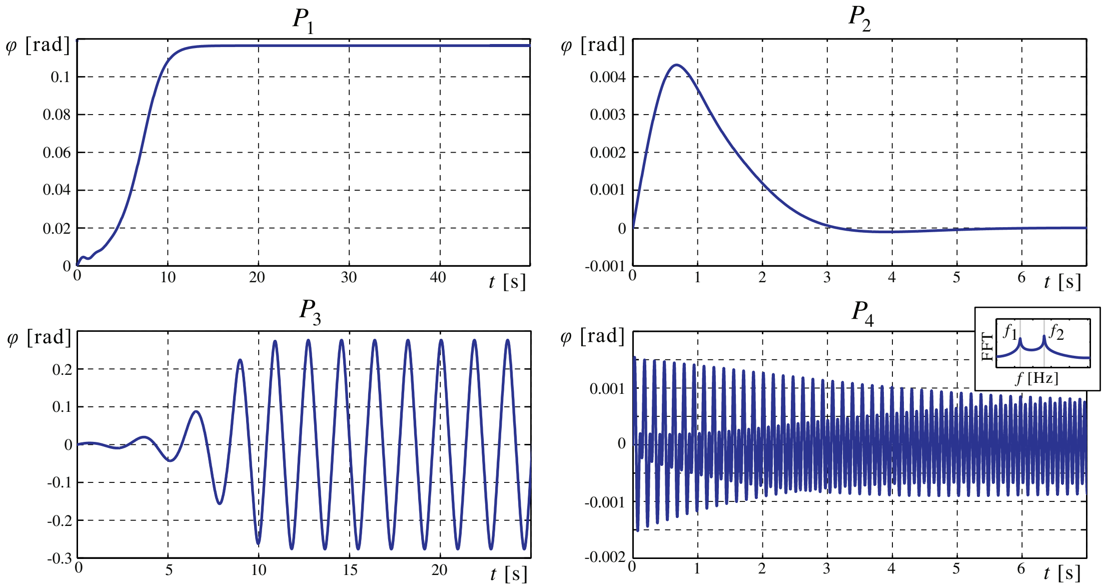

Figure 5: Time functions from numerical simulations

If the skater does not change P and D parameters then motion will not be stable at high speed even it was at lower speed. Thus, skaters have to tune their control gains (P and D) with respect to their speed, which means that an adaptive control strategy is required. It can be seen in Figure 4, that when the speed changes from 1.7 m/s to 4 m/s, the stable domain rotates significantly around the origin. Hence, small variation of the velocity strongly influences the optimal control gains. To check our analytical results, numerical simulations were done. The crosses in Figure 4 refer to parameter points $\mathrm{P}_1 - \mathrm{P}_4$, where the simulations were carried out. The time histories are shown in Figure 5 for an impact-like initial conditions. The simulations have good agreement with the analytical results. The time history at the parameter point P_1 shows linearly unstable steady rectilinear motion. The simulation diverges exponentially from the rectilinear motion and it converges to a non-zero solution, which relates to the turning maneuver of the skateboard with a constant radius. Simulation at P_2 confirms that an appropriate choice of the P and D control gains can lead to stable uniform motion. In case of the parameter point P_3, the skateboard starts to vibrate with constant amplitude, which corresponds to the existence of a stable limit cycle around the unstable rectilinear motion.

An interesting point of the results is the intersection of the D-curve, where double Hopf bifurcation points can occur. At these points, there are two different angular frequencies in the emerging motion. This is also confirmed by numerical simulation at the parameter point P_4. See Figure 5, where both the time history and its spectrum are plotted, and two different frequencies can be clearly identified.

Another important parameter is the reflex time of the skater, the time delay τ. Its effect is illustrated in Figure 6, where charts have been created with the parameters of Table 1 at steady state position ($V = 0$). Here the stability charts correspond to different time delays. The left half plane cannot be stable due to the saddle-node bifurcation, so the D-curve must depart to the right from the SN bifurcation line in order to have a stable domain. This condition can be fulfilled if and only if

$$\tau < \tau_{cr} = \sqrt{\frac{2hl}{(gl - V^2 \mathrm{tg}\kappa)}}\,. \tag{15}$$

When the skater's reflex delay is large then uniform motion cannot be stable. This critical time delay τ_{cr} increases together with the forward speed, so the higher the forward speed the easier the stabilization. The critical time delay τ_{cr} is around

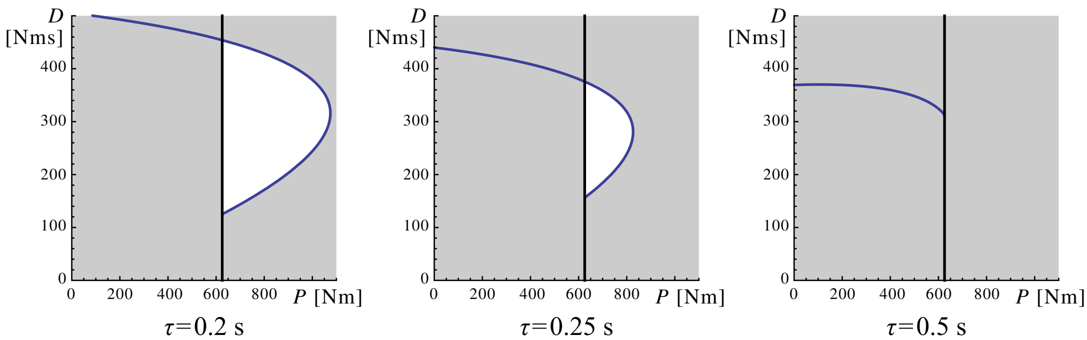

Figure 6: The linear stability charts for different reflex delays at zero speed

0.42 sec at zero forward speed (for the given system parameters), what is in a same order of magnitude of the human reflex delay (see for example [6]).

Conclusions

We have shown that, the reflex delay of the skater strongly influences the stability of uniform motion and requires speed varying control gain parameters to keep the straight motion linearly stable, although greater time delay is adequate at higher speeds. Moreover, in some cases, the skater have to use negative control gains to stabilize the rectilinear motion of the skateboard, which obviously demonstrates the difficulties of skateboarding. Skaters have to use such control gains sometimes that clearly override all the well-studied control strategies of the human balancing. Another possibility for the skater is to switch off the control and let the skateboard balance itself. This cannot be realized when the skater tries to follow a specific trajectory, which requires a more complicate control strategy that could be the objective of future studies.

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