Density theorems for Riemann's zeta-function near the line Re $s = 1$

by
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To Gábor Halász on the occasion of his 80th birthday

Abstract

We prove a series of density theorems for Riemann's zeta-function for the number of zeros lying near the boundary line Re $s = 1$ of the critical strip. In particular, we improve the constant appearing in the exponent of the Halász–Turán density theorem. The proof uses the relatively recent strong estimate for the zeta-function near the line Re $s = 1$ showed by Heath-Brown. The necessary exponential sums were estimated by Heath-Brown via the new results of Wooley and of Bourgain, Demeter and Guth on Vinogradov's mean value integral.

1 Introduction

In a recent work [Pin2] we gave a relatively simple alternative proof for the density theorem of G. Halász and P. Turán [HT] which was the first time when (about fifty years ago) the density hypothesis

$$N(\sigma, T) := \sum_{\substack{\zeta(\beta+i\gamma)=0 \\ \beta\geq\sigma,\, |\gamma|\leq T}} 1 = O\left(T^{2(1-\sigma)} \log^c T\right) \tag{1.1}$$

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was proved for a fixed half-plane $\sigma > 1 - c_0$, $c_0 > 0$ (absolute constant) of the critical strip.

Their result, in a slightly improved form, presented as Theorem 38.2 of Turán's book [Tur] was

$$N(\sigma, T) \ll T^{A(1-\sigma)^{3/2}} \log^c T \quad \text{with} \quad A = 12{,}000. \tag{1.2}$$

Their method used besides the estimate of Korobov–Vinogradov for the zeta-function Turán's power sum method [Tur] and a simple but ingenious idea of Halász [Hal].

Improved explicit forms of Vinogradov's method by Richert [Ric], Ford [For] and in the last years by Bourgain–Demeter–Guth [BGD] and Heath-Brown [Hea] led to a very substantial improvement of the constant A in (1.2) yielding ([Hea])

$$N(\sigma, T) \ll_\varepsilon T^{6.42(1-\sigma)^{3/2}+\varepsilon} \quad \text{for} \quad 9/10 \le \sigma \le 1 \tag{1.3}$$

and with an unspecified σ_0 very near 1

$$N(\sigma, T) \ll_\varepsilon T^{5.03(1-\sigma)^{3/2}+\varepsilon} \quad \text{for } \sigma > \sigma_0. \tag{1.4}$$

In our recent work [Pin2] our main goal was to provide a simple alternative proof avoiding both Turán's method and the more advanced form of the zero-detection method developed by Montgomery [Mon] using the large sieve [Bom2]. We used also a very restricted part of the theory of complex functions and in particular only some very basic properties of the Riemann zeta-function. Our proof used naturally Vinogradov's estimate for the zeta-function and the mentioned ingenious idea of Halász [Hal].

The original ideas of Halász, Turán, Bombieri [Bom1], [Bom2] and Montgomery [Mon] were followed by new ones of mainly Huxley, Jutila, Ivić and Heath-Brown in the 1970's and 1980's, leading to many density results in the half-plane $\sigma > 3/4$ and extending (1.1) or its analogue form with T^ε in place of $\log^c T$ to the half-plane $\sigma > \sigma_0$ with some $\sigma_0 \in (3/4, 4/5)$. Many of these results are contained in the book of Ivić [Ivi], for example, the above with $\sigma_0 = 11/14 = 0.7857\dots$ (see (11.82) of Theorem 11.4). This last result was improved by Bourgain [Bou2] to $\sigma_0 = 25/32 = 0.78125$. The other direction was to show strong explicit density theorems for $\sigma > \sigma_0$ where σ_0 was near 1, like Theorems 11.2 and 11.3 of [Ivi]. They were written in the form

$$N(\sigma, T) \ll T^{A(\sigma)(1-\sigma)} \log^c T \tag{1.5}$$

(or again with T^ε in place of $\log^c T$) with a possibly small $A(\sigma)$, for example, (11.34) of Theorem 11.3 of [Ivi]:

$$A(\sigma) \leq 35/36 \quad \text{for} \quad 152/155 \leq \sigma \leq 1 \quad (152/155 = 0.9806\ldots). \tag{1.6}$$

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Our present goal is to prove a full series of explicit density theorems for $\sigma \geq 11/12$ with $A(\sigma) \to 0$ as $\sigma \to 1$, where, as the limiting case, we get further improvements of Heath-Brown's results (1.3)–(1.4) with respect to the constants in the exponent. These results (being valid in individual intervals of the form $\sigma_0 \leq \sigma \leq \sigma_1 < 1$) will provide in some sense a bridge between the Halász–Turán density theorem and density theorems similar to (1.5)–(1.6) of Ivić. The main result, making this possible, is a relatively new general result for exponential sums, proved by Heath-Brown [Hea], which is based on the result of Bourgain, Demeter and Guth [BGD]. Apart from this, we will use very limited properties of the zeta-function and we use the method of [Pin1] thereby avoiding both the large sieve and Turán's method. However, we will use the mentioned idea of [Hal] which appears in some form in practically all stronger density theorems in the half-plane $\sigma > 3/4$.

Since the used version of Vinogradov's mean-value theorem produces results of type with an extra factor T^ε and we are interested in the case σ close to 1 we shall look for estimates of type

$$N(1-\eta, T) \ll_{\varepsilon,\eta} T^{B(\eta)\eta+\varepsilon} \tag{2.1}$$

with $B(\eta)$ as small as possible.

We will deal with the case

$$\eta < 1/12, \tag{2.2}$$

but actually our results will be weaker than that of Bourgain [Bou2] for $\eta \geq 1/24$. We will dissect the interval $(0, 1/12)$ into a two-parameter family of disjoint intervals by the definitions of $k = k(\eta)$ and $\ell = \ell(\eta)$:

$$\begin{aligned} (2.3) \qquad k(\eta) : &= \max_{j\in\mathbb{Z}^+}\{j : j(j-1)\eta < 1\}, \ \ \ell(\eta) := \max_{j\in Z^+}\{2j(j-1)\eta < 1\} \\ I(k,\ell) : &= \{\eta : k = k(\eta), \ell = \ell(\eta)\} \\ &= [1/k(k+1), 1/k(k-1)) \cap [1/2\ell(\ell+1), 1/2\ell(\ell-1)). \end{aligned}$$

In other words we suppose for a given pair of integers $k \geq 4$, $\ell \geq 3$

$$\eta k(k-1) < 1 \leq \eta(k+1)k \quad \text{and} \quad 2\eta\ell(\ell-1) < 1 \leq 2\eta(\ell+1)\ell. \tag{2.4}$$

We remark that for a given $k \geq 4$ there are just a few numbers ℓ with $I(k,\ell) \neq \emptyset$ and vice versa for $\ell \geq 3$. Due to $\eta < 1/12$ we have, as mentioned above,

$$k \geq 4 \quad \text{and} \quad \ell \geq 3. \tag{2.5}$$

Now we can state our first result with the notation (2.1)–(2.5).

Theorem 1. (2.1) *is valid for* $\eta \in I(k,\ell)$ *with*

$$B(\eta) := \max\left\{\frac{3}{\ell(1-2(\ell-1)\eta)}, \frac{4}{k(1-(k-1)\eta)}\right\} \quad \text{for} \quad \eta < \frac{1}{12}. \tag{2.6}$$

Remark 1. This result is for $\eta < 1/24$ stronger than that of Bourgain [Bou1] and for $\eta < \frac{5}{66}$ stronger than all the 12 inequalities of Theorems 11.1–11.5 of [Ivi] referring for various parts of the interval $(3/4, 1)$. We have to mention, however, that several of his results were aimed to be stronger near $3/4$ (extending the range of validity of the density hypothesis) than in the vicinity of 1. The author did not find other results better than (2.6) for $\eta < 1/24$.

Remark 2. The following version follows immediately from (2.6) which is only slightly weaker than (2.6) for small η.

Theorem 1'. (2.1) *is valid for* $\eta \in I(k,\ell)$ *with*

$$B(\eta) := B'(\eta) := \max\left\{\frac{3}{\ell-1}, \frac{4}{k-1}\right\}. \tag{2.7}$$

Thus, (2.6) gives a stronger form of the Halász–Turán density theorem than the improvement (1.4) of Heath-Brown $\big(4 < 3\sqrt{2} < 4.25 = 17/4\big)$:

Theorem 2. *For any* $\varepsilon > 0$ *there exists a* $\sigma_0(\varepsilon) < 1$ *such that for* $\sigma > \sigma_0(\varepsilon)$ *we have*

$$N(\sigma, T) \ll_\varepsilon T^{(3\sqrt{2}+\varepsilon)(1-\sigma)^{3/2}+\varepsilon}.$$

Remark 3. An easy argument shows (for $\ell \geq 15$ "theoretical", for $\ell < 15$ "computational") $B(\eta) = 4/k(1-(k-1)\eta)$ in case of $\eta \leq 1/12$ if and only if $\eta \in I^* = I(6,5) = [1/42, 1/40)$. This means that

$$B(\eta) = 3/\ell(1-2(\ell-1)\eta) \quad \text{for} \quad \eta \in (0, 1/12) \setminus [1/42, 1/40).$$

For completeness we will show the "theoretical" part for $\ell \geq 15$, i.e.

$$4\ell(1-2(\ell-1)\eta) < 3k(1-(k-1)\eta), \tag{2.8}$$

that is,

$$3k(k-1)\eta - 8\ell(\ell-1)\eta < 3k - 4\ell. \tag{2.9}$$

First we note that by (2.4) we have for $\ell \geq 15$

$$3k(k-1)\eta - 8\ell(\ell-1)\eta \leq 3 - 4\cdot 2\eta\ell(\ell+1)\left(\frac{\ell-1}{\ell+1}\right) \leq 3 - \frac{7}{2}. \tag{2.10}$$

On the other hand we have by (2.4) again

$$k(k+1) \geq 1/\eta > 2\ell(\ell-1) \tag{2.11}$$

which implies $4\ell \leq 3k$ for $\ell \geq 15$. Namely, the indirect supposition $k < (4/3)\ell$ would yield for $\ell \geq 15$

$$k(k+1) \leq \frac{16}{9}\ell^2 + \frac{4}{3}\ell \leq 2\ell^2 - 2\ell, \tag{2.12}$$

since $(2/9)\ell^2 \geq (10/3)\ell$ for $\ell \geq 15$.

Remark 4. The above can also be stated as follows: if

$$\eta < 1/12,\ \eta \notin [1/42, 1/40) \ \text{ and } \ \eta \in \left[\frac{1}{2\ell(\ell+1)}, \frac{1}{2\ell(\ell-1)}\right) \tag{2.13}$$

then the density estimate (2.1) holds with

$$B(\eta) = 3/\ell\big(1-2(\ell-1)\eta\big). \tag{2.14}$$

Now a simple calculation shows in case of $\eta \leq 1/18$

(2.15)

$$\begin{aligned} B(\eta) &= \frac{3}{\ell(1-2(\ell-1)\eta)} \leq \frac{3}{\ell-1} = \frac{3}{(\ell+1/2)-3/2} < \frac{3}{1/\sqrt{2\eta}-3/2} \\ &= \frac{3\sqrt{2\eta}}{1-(3/2)\sqrt{2\eta}} \leq 3\sqrt{2\eta}\big(1+3\sqrt{2\eta}\big) = 3\sqrt{2\eta} + 18\eta. \end{aligned}$$

So we get a slightly weaker but simpler form of Theorem 2 as

Theorem 2'. *For $\eta \le 1/18$ we have*

$$N(1-\eta, T) \ll_{\varepsilon,\eta} T^{3\sqrt{2}\eta^{3/2}+18\eta^2+\varepsilon}. \tag{2.16}$$

Remark 5. As mentioned in Section 1 we use in all our results the estimates on exponential sums which led to (1.3)–(1.4) of [Hea].

To be more explicit we list here the needed Theorems 1 and 5 of [Hea].

Theorem 1 of [Hea]. *Let $k \ge 3$ be an integer, $f(x) : [0, N] \to \mathbb{R}$ having continuous derivatives of order up to k on $(0, N)$ with*

$$0 < \lambda_k \le f^{(k)}(x) \le A\lambda_k, \quad x \in (0, N). \tag{2.17}$$

Then (with $e(x) = e^{2\pi i x}$)

$$\frac{1}{N}\sum_{n\le N} e(f(n)) \ll_{A,k,\varepsilon} N^{\varepsilon}\left(\lambda_k^{1/k(k-1)} + N^{-1/k(k-1)} + N^{-2/k(k-1)}\lambda_k^{-2/k^2(k-1)}\right). \tag{2.18}$$

Theorem 5 of [Hea]. *For $t \ge 1$ and $0 \le \sigma \le 1$ we have*

$$\zeta(\sigma + it) \ll_{\varepsilon} t^{(1-\sigma)^{3/2}/2+\varepsilon}. \tag{2.19}$$

3 Preparation. Auxiliary theorems

In all of the results mentioned below we choose $h(n) := n^{-(1-\xi)-it}$, $t \ge 3$, an integer $r \ge 2$, an interval $I(N)$ with

$$H_\xi(N) := \sum_{n\in I(N)} h(n), \quad I(N) \subset (N, 2N], \quad \xi \le 1/r(r-1) - 6\varepsilon. \tag{3.1}$$

The first corollary follows by partial summation immediately from Theorem 1 of [Hea]. Corollaries 2 and 3 follow from Corollary 1. The value of r will be k or ℓ, ξ will be η_j or $\eta_\nu + \eta_\kappa$ in the proof of Theorem 1.

Corollary 1. *We have for $r \ge 3$*

$$H_\xi(N) \ll_{r,\varepsilon} N^{\xi-1/(r-1)+3\varepsilon}|t|^{1/r(r-1)} + N^{-3\varepsilon} + N^{\xi+3\varepsilon}|t|^{-2/r^2(r-1)}. \tag{3.2}$$

Corollary 2. *If $N \ll |t|^{2/r}$ then the third term above can be deleted.*

Corollary 3. *If* $N \geq N(r;\xi,T) := T^{1/r(1-(r-1)\xi-6r\varepsilon)}$, $N \ll |t|^{2/r}$, $|t| \leq T$, *then*

$$H_\xi(N) \ll_{\varepsilon,r} N^{-3\varepsilon}. \tag{3.3}$$

The following is a variant of Theorem II.2 of [Mon], Appendix II.

Lemma. *If* $s=\sigma+it$, $\sigma \geq 1-\xi$, $\xi>0$, $0<\theta\leq 1$ *then*

$$\sum_{n=1}^{\infty}\bigl(e^{-n/2N}-e^{-n/N}\bigr)n^{-s} \ll N^{\theta-1+\xi}\max_{\sigma'\geq\theta,\, 1<|t'|\leq 2|t|}\bigl|\zeta(\sigma'+it')\bigr| + N^{\xi}e^{-|t|}. \tag{3.4}$$

The simple proof of the Lemma is essentially the same as that of Theorem II.2 in [Mon]. According to this we will just sketch the proof of it and refer to [Mon] for more details.

Similarly to [Mon] it is sufficient to consider the case $\sigma = 1-\xi$. We write

$$\begin{aligned}\sum_{n=1}^{\infty}&\bigl(e^{-n/2N}-e^{-n/N}\bigr)n^{-(1-\xi)-it}\\ &=\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\zeta(w+1-\xi+it)\Gamma(w)\bigl((2N)^w-N^w\bigr)dw\end{aligned} \tag{3.5}$$

and take the contour to $(\theta-1+\xi)$ making a detour around $w=\xi-it$ if θ is near 1. The residue from the pole of $\zeta(s)$ at $w=\xi-it$ is $\ll N^{\xi}e^{-|t|}$. For $|\mathrm{Im}\ w| \leq t$ the integrand is

$$\ll N^{\theta-1+\xi}e^{-|\mathrm{Im}\ w|}\max_{\sigma'\geq\theta, 1<t'\leq 2|t|}\bigl|\zeta(\sigma'+it')\bigr|, \tag{3.6}$$

while for $|\mathrm{Im}\ w| \geq t$ it is

$$\ll N^{\theta-1+\xi}e^{-|\mathrm{Im}\ w|}|\mathrm{Im}\ w|. \tag{3.7}$$

Remark 6. The role of the general variable r will be played either by k or ℓ, depending whether in the corresponding exponential sum the variable ξ satisfies $\xi \leq \eta$ or $\xi \leq 2\eta$, resp.

Finally we state and prove an easy proposition.

Proposition. *Let* $k=k(\eta)$ *and* $\ell=\ell(\eta)$ *(see* (2.3)*). Then*

$$1/\ell(1-2\eta(\ell-1)) > 1/k(1-\eta(k-1)).$$

Proof. By $2\eta(\ell+1)\ell \geq 1$ and $\eta k(k-1) \leq 1$, $k \geq 4$, $\ell \geq 3$ we have

$$k(1-\eta(k-1)) \geq k-1, \tag{3.8}$$

$$\ell(1-2\eta(\ell-1)) \leq \ell\left(1-\frac{\ell-1}{\ell(\ell+1)}\right) = \ell - \frac{\ell-1}{\ell+1} \leq \ell - \frac{1}{2}. \tag{3.9}$$

Since $1 > 2\ell(\ell-1)\eta \geq \ell(\ell+1)\eta$ and $k(k+1)\eta \geq 1$ we have $k > \ell$ so $k-1 > \ell - 1/2$, hence the Proposition is true. □

4 Proof of Theorem 1

Let us consider the zeros $\varrho_j = \beta_j + i\gamma_j = 1 - \eta_j + i\gamma_j$ with $\beta_j \geq \sigma := 1-\eta$ ($\eta_j \leq \eta$), $2\log T < |\gamma_j| \leq T$ ($j = 1, 2, \ldots, K$), and let us choose with a sufficiently small $\varepsilon > 0$ $\big(\varepsilon < \varepsilon_0(\eta)\big)$, $T > T_0(\varepsilon)$

$$X := T^{\varepsilon/10\ell}, \quad Y := T^{2/k}, \quad Y_1 := Ye^3, \quad \lambda = \log Y, \ \mathcal{L} = \log T, \tag{4.1}$$

$$M_X(s) := \sum_{n \leq X} \frac{\mu(n)}{n^s},$$

$$a_n := \sum_{d|n, d \leq X} \mu(d), \quad N_\ell^*(T) = XN(\ell;\xi;T), \quad N_k^*(T) = XN(k;\xi;T).$$

Let further similarly to [Pin2]

(4.2)

$$I_j := \frac{1}{2\pi i}\int\limits_{(3)} \sum_{n=1}^{\infty} \frac{a_n}{n^{s+\varrho_j}} \frac{e^{s^2/\lambda+\lambda s}}{s} ds = \frac{1}{2\pi i}\int\limits_{(3)} M_X(s+\varrho_j)\zeta(s+\varrho_j)\frac{e^{s^2/\lambda+\lambda s}}{s} ds$$

$$= \frac{1}{2\pi i}\int\limits_{(-\eta)} M_X(s+\varrho_j)\frac{\zeta(s+\varrho_j)}{s} e^{s^2/\lambda+\lambda s} ds + O\left(\frac{Y^\eta}{|\gamma_j|} e^{-\gamma_j^2/\lambda}\right)$$

$$= \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} M_X(1-\eta-\eta_j+i(\gamma_j+t))\frac{\zeta(1-\eta-\eta_j+i(\gamma_j+t))}{-\eta+it} e^{(-\eta+it)^2/\lambda} Y^{-\eta+it} dt$$

$$+ O(Y^{-3})$$

$$= \frac{1}{2\pi} \int_{-2\lambda}^{2\lambda} M_X(1-\eta-\eta_j+i(\gamma_j+t)) \frac{\zeta(1-\eta-\eta_j+i(\gamma_j+t))}{-\eta+it} e^{(-\eta+it)^2/\lambda} Y^{-\eta+it} dt$$

$$+ O\left(Y^{-3}\right).$$

It is possible to evaluate I_j in a different way, cutting the defining integral I_j into three parts according to the value of n. Using the trivial bound $|a_n| \leq \tau(n) \ll_\delta n^\delta$ for any fixed $\delta > 0$ we can shift first the part of the series $\sum_n a_n n^{-(s+\varrho_j)}$ with $n \geq Y_1$ (i.e. $\lambda - \log n \leq -3$) from Re $s = 3$ to Re $s = \lambda$. Writing $s = \lambda + it$ we obtain

$$\left| \sum_{n \geq Y_1} \frac{a_n}{n^{s+\varrho_j}} \right| \ll \sum_{n \geq Y_1} \frac{n^\delta}{n^{\lambda+1-\eta_j}} = \sum_{n \geq Y_1} \frac{1}{n^{\log Y+1-\eta_j-\delta}} \tag{4.3}$$

$$\ll \frac{1}{Y_1} \frac{1}{Y_1^{\log Y-\eta_j-\delta}} = \frac{1}{(Ye^3)^{\log Y+1-\eta_j-\delta}} \ll e^{-\lambda^2} Y^{-4+\eta_j+\delta}.$$

Then we estimate

(4.4)

$$Y^{-4+\eta_j+\delta} \int_{(\lambda)} e^{-\lambda^2} \frac{\left|e^{(\lambda+it)^2/\lambda+\lambda(\lambda+it)}\right|}{\sqrt{\lambda^2+t^2}} dt = Y^{-4+\eta_j+\delta} \int_{-\infty}^{\infty} \frac{e^{-\lambda^2}}{\sqrt{\lambda^2+t^2}} e^{\lambda-\frac{t^2}{\lambda}+\lambda^2} dt$$

which is equal to

$$Y^{-4+\eta_j+\delta} \int_{-2\lambda}^{2\lambda} \frac{e^{\lambda-\frac{t^2}{\lambda}}}{\sqrt{\lambda^2+t^2}} dt + O\big(Y^{-6}\big). \tag{4.5}$$

This quantity is

$$\ll \frac{Y^{-3+\eta_j+\delta}}{\lambda} \int_{-2\lambda}^{2\lambda} e^{-\frac{t^2}{\lambda}} dt \ll \frac{Y^{-3+\eta_j+\delta}}{\sqrt{\lambda}} \ll Y^{-2}. \tag{4.6}$$

Further, shifting the finite part $n \in (X, Y_1)$ to Re $s = 1/\lambda$, and the term $n = 1$ to Re $s = -4$ we obtain similarly to the estimation of the infinite part above

$$I_j = 1 + \sum_{X<n<Y_1} a_n n^{-\varrho_j} \frac{1}{2\pi i} \int_{1/\lambda-2i\lambda}^{1/\lambda+2i\lambda} \frac{e^{s^2/\lambda+(\lambda-\log n)s}}{s} ds + O\left(\frac{1}{Y^2}\right). \tag{4.7}$$

On the other hand the expression on the RHS of (4.2) can be estimated as follows:
(4.8)
$$\left|M_X\big(1-\eta-\eta_j+i(\gamma_j+t)\big)\right| \ll \sum_{n\le X}\frac{1}{n^{1-\eta-\eta_j}} \ll_\eta X^{\eta+\eta_j} \ll X^{2\eta} \ll X^{\frac{1}{6}} = T^{\frac{\varepsilon}{60\ell}}.$$

Next, by (2.19) and $(\gamma_j+t)\ll T$,

$$(4.9)\qquad \left|\zeta\big(1-\eta-\eta_j+i(\gamma_j+t)\big)\right| \ll (\gamma_j+t)^{(\eta+\eta_j)^{3/2}/2+\varepsilon} \ll T^{\sqrt{2}\eta^{\frac{3}{2}}+\varepsilon}.$$

Hence the last member of (4.2) is

$$(4.10)\qquad \ll T^{\sqrt{2}\eta^{\frac{3}{2}}+\varepsilon+\frac{\varepsilon}{60\ell}}Y^{-\eta}\int\limits_{-2\lambda}^{2\lambda}\frac{e^{\frac{\eta^2-t^2}{\lambda}}}{\sqrt{\eta^2+t^2}}dt \ll T^{\sqrt{2}\eta^{\frac{3}{2}}+\varepsilon+\frac{\varepsilon}{60\ell}}Y^{-\eta}.$$

From this we obtain

$$(4.11)\qquad I_j \ll T^{2\varepsilon+\sqrt{2}\eta^{3/2}}\cdot Y^{-\eta} = T^{2\varepsilon+\sqrt{2}\eta^{3/2}-2\eta/k} \ll T^{-\eta^{3/2}/4+2\varepsilon},$$

since by $\eta<1/k(k-1)\le 4/3k^2$ we have $2/k>\sqrt{3\eta}$.

An easy calculation shows that the integral in (4.3) is $\ll \log\lambda$. Hence, the formulae (4.1)–(4.4) imply that with a suitable choice of $\gamma_j^*\in(\gamma_j-2\lambda,\gamma_j+2\lambda)$ we have with $\varrho_j^*=1-(\eta_j-1/\lambda)+i\gamma_j^*$

$$(4.12)\qquad I_j^* := \Big|\sum_{X<n<Y_1}a_n n^{-\varrho_j^*}\Big| > c/\log\lambda \qquad (j=1,2,\dots K),$$

since $I_j=o(1)$ by (4.1)–(4.4). We will select a subset of $\{\varrho_j^*\}$ with $K'\gg K/\lambda\mathcal{L}$ elements and $|\gamma_\nu^*-\gamma_\kappa^*|>2\lambda$. Furthermore, we divide a_n for all $n\in(X,Y_1)$ into two parts with $N_k(T)=N(k;\xi;T)$ (for $N(k,\eta,T)$ see Corollary 3):

$$(4.13)\qquad a_n' = \sum_{\substack{d,m\\ dm=n\\ m>N_k(T)\\ d\le X}}\mu(d),\quad a_n'' = \sum_{\substack{d,m\\ dm=n\\ m\le N_k(T)\\ d\le X}}\mu(d),\quad a_n=a_n'+a_n''.$$

By (3.3) we obtain by $r=k\ge 4$

$$(4.14)\qquad \sum_{X<n<Y_1}a_n' n^{-\varrho_j^*} = \sum_{d\le X}\mu(d)d^{-\varrho_j^*}\sum_{\substack{X/d<m<Y_1/d\\ m>N_k(T)}}m^{-\varrho_j^*}$$

$$\ll_{k,\varepsilon} \lambda X^\eta N_k(T)^{-3\varepsilon} \ll T^{-2\varepsilon/k}.$$

In the above estimation we used a dyadic subdivision of the interval $(X/d, Y_1/d)$ and afterwards (3.3) for the inner sum and the trivial estimate $|d^{-\varrho_j}| \le 1$. Using a dyadic subdivision of (X, Y_1), by (4.12) and (4.14) we have a $U \in (X, N_k^*(T))$ such that for some $I(U) \subset [U, 2U]$ and with a_n'' in (4.13) we have by $T^{-2\varepsilon/k} = o(1/\log\lambda)$

$$\frac{K'}{\lambda\log\lambda} \ll \frac{\sum_{j=1}^{K'}\left(I_j^* - cT^{-2\varepsilon/k}\right)}{\lambda} \le \sum_{\nu=1}^{K'}\Bigl|\sum_{n\in I(U)} a_n'' n^{-\varrho_\nu^*}\Bigr| =: \sum_{\nu=1}^{K'}\Bigl|\sum\nolimits_\nu\Bigr|. \tag{4.15}$$

The case $\log U/\log T =: u \ge 1/\ell(1-2\eta(\ell-1)-6\ell\varepsilon)+\varepsilon/10\ell$ would imply by the choice of X that $U > N_\ell^*(T)$ (cf. (4.1)). If $2u > 1/\ell(1-2\eta(\ell-1)-6\ell\varepsilon)+\varepsilon/10\ell$ we raise all Dirichlet sums $\sum_\nu$ $(\nu = 1, \dots, K')$ to the 2^{nd} power. If this condition is not satisfied then we can raise them to a suitable h^{th} power such that $hu \in (1/\ell(1-2\eta(\ell-1)-6\ell\varepsilon),\, 3/2\ell(1-2\eta(\ell-1)-6\ell\varepsilon)+\varepsilon/20\ell) =: J_\ell(\eta)$. Denoting the beginning of the resulting interval by $A_h = U^h$, the endpoint will be $B_h \le (2U)^h$ where $h \le 20/\varepsilon$ by $\ell \le k$,

$$X = T^{\varepsilon/10\ell} \le U \ll Y = T^{2/k}.$$

We further note that in case of $h = 2$, i.e., $2u > 1/\ell(1-2\eta(\ell-1)-6\ell\varepsilon)+\varepsilon/10\ell$, the endpoint after squaring will be $B_2 \le 4\left(N_k^*(T)\right)^2$ by $U \le N_k^*(T)$, while our Proposition in Section 3 shows that the beginning point will be $A_h > N_\ell^*(T)$ for $h \ge 2$. We will denote the resulting coefficients after taking h^{th} power of $\sum_\nu$ (where now $h \in \left[2, \frac{20}{\varepsilon}\right)$) by $a_n^{(h)} = b_n$ and using again a dyadic subdivision we find an interval $I(N) \subset [N, 2N] \subset [A_h, B_h]$ with

$$\sum_{\nu=1}^{K'}\Bigl|\sum_{n\in I(N)}{}_{\nu}^{(h)}\Bigr| := \sum_{\nu=1}^{K'}\Bigl|\sum_{n\in I(N)} b_n n^{-\varrho_\nu^*}\Bigr| \gg_\varepsilon \frac{K'}{\lambda^{h+1}(\log\lambda)^h}. \tag{4.16}$$

From now on the interval $I(N)$ will be fixed. We choose α_ν $(\nu = 1, \dots, K')$ in such a way that

$$\alpha_\nu \sum_{n\in I(N)}{}_{\nu}^{(h)} = \Bigl|\sum_{n\in I(N)}{}_{\nu}^{(h)}\Bigr| \tag{4.17}$$

should hold. Using Halász' idea we square the LHS of (4.16), interchange the order of summation over ν and n, and use a Cauchy–Schwarz inequality for the sum when n runs through elements of $I(N)$ with

$$b_n n^{-\varrho_\nu^*} = d_n e_n^{(\nu)}, \qquad e_n^{(\nu)} = \frac{\left(e^{-n/2N} - e^{-n/N}\right)^{1/2}}{n^{1/2-2\eta-\eta_\nu+i\gamma_\nu^*}}, \tag{4.18}$$

$$d_n = \begin{cases} \dfrac{b_n\left(e^{-n/2N} - e^{-n/N}\right)^{-1/2}}{n^{1/2+2\eta+1/\lambda}} & \text{if } n \in I(N), \\ 0 & \text{if } n \notin I(N). \end{cases}$$

So we obtain from (4.16)–(4.18)

(4.19)

$$\begin{aligned}
\frac{(K')^2}{\mathcal{L}^{4h}} &\ll_{k,h} \left(\sum_{\nu=1}^{K'} \Big| \sum_{n\in I(N)}{}_{\nu}^{(h)} \Big|\right)^2 = \left(\sum_{\nu=1}^{K'} \alpha_\nu \sum_{n\in I(N)}{}_{\nu}^{(h)}\right)^2 \\
&= \left(\sum_{n\in I(N)} b_n \sum_{\nu=1}^{K'} \alpha_\nu n^{-\varrho_\nu^*}\right)^2 = \left(\sum_{n=1}^{\infty} d_n \sum_{\nu=1}^{K'} \alpha_\nu e_n^{(\nu)}\right)^2 \\
&\ll_\varepsilon \left(\sum_{n\in I(N)} \frac{|b_n|^2 (e^{-n/2N} - e^{-n/N})^{-1}}{n^{1+4\eta+2/\lambda}}\right)\left(\sum_{n=1}^{\infty} \sum_{\nu,\kappa=1}^{K'} \frac{\alpha_\nu \overline{\alpha}_\kappa (e^{-n/2N} - e^{-n/N})}{n^{1-\eta_\nu-\eta_\kappa-4\eta+i(\gamma_\nu^*-\gamma_\kappa^*)}}\right) \\
&\ll \mathcal{L}^{C(\varepsilon)}\left(K'(K'-1)T^{-\varepsilon/k} + K' B_h^{2\eta}\right),
\end{aligned}$$

where the last line is obtained as follows (with a $C(\varepsilon)$, a constant $\ll h^2 \ll \varepsilon^{-2}$ by the relation

$$\sum_{n\le x} \tau_h^2(n) = \sum_{n\le x} \tau_{h^2}(n) \le x(\log x)^{h^2},$$

where $\tau_h(n)$ is the generalized divisor function, see e.g. Lemma 3 of [GPY]).

The estimate $\mathcal{L}^{C(\varepsilon)} B_h^{2\eta}$ for an arbitrary diagonal term follows if we use that the total contribution of the terms with $n > N \log^2 N \asymp B_h \log^2 B_h$ are negligible, the first factor is $O\big(\mathcal{L}^{C(\varepsilon)} B_h^{-4\eta}\big)$, the second $O\big(B_h^{6\eta}\big)$. To show the required estimate $\mathcal{L}^{C(\varepsilon)} T^{-\varepsilon/k}$ for an arbitrary non-diagonal term with indices ν and κ we have to distinguish two cases according to the size $t = \left|\gamma_\nu^* - \gamma_k^*\right| > 2\lambda$ compared with $N \asymp B_h$. If t is "small" we use the Lemma together with the deep estimate (2.19) of Heath-Brown for $\zeta(s)$ (see (4.20)). If t is "large" then for the interval $N \in \left[A_h, B_h \log^2 B_h\right]$ we might use Corollary 3 since the power h was defined in such a way that $A_h \gg N(\ell, 2\eta, T)$ should hold and thereby all conditions of Corollary 3 are satisfied (for the result see

(4.20)). Additionally, for small values of n the decay of the kernel function $e^{-n/2N} - e^{-n/N}$ for decreasing $n \to 1$ (see (4.22)) and Corollary 2 (with $r = \ell$, $\xi = 2\eta$, $n \in [N_j, 2N_j]$, $N_j < N$) ensure the same estimate (4.21) as for $I(N) \subset \big[A_h, B_h \log^2 B_h\big]$.

According to the above the crucial estimate $T^{-\varepsilon/k}$ follows in case of $B_h \gg t^{1.9/\ell}$ from the Lemma with the choice $\xi = 6\eta$, $\theta = 1 - 6\eta$ since $\eta < 1/\ell^2$ and so by (2.19)

$$N^{-4\eta} t^{6\sqrt{6}\eta^{3/2}/2} \ll T^{(3\sqrt{6}-7.6)\eta^{3/2}} \ll T^{-\eta^{3/2}/4} \tag{4.20}$$

as by $2\eta < 1/\ell(\ell-1) < 2/\ell^2$ we have $2/\ell > 2\sqrt{\eta}$ and $N \gg B_h$. On the other hand, if $B_h \ll t^{1.9/\ell}$ then we obtain from Corollary 3 that for any $I(N) \subset \big[A_h, B_h \log^2 B_h\big]$

$$\sum_{n \in I(N)} \frac{1}{n^{1-\eta_\nu-\eta_\kappa+i(\gamma_\nu^*-\gamma_k^*)}} \ll B_h^{-3\varepsilon} \ll T^{-3\varepsilon/\ell}. \tag{4.21}$$

(The second term in (3.4) can be neglected since $N^\xi e^{-|t|} \ll (Y^2)^{6\eta} e^{-2\lambda} \ll 1/Y$.)

If we have an interval $I(N_j)$ with $N_j < A_h$, then by Corollary 2 and

$$e^{-n/2N} - e^{-n/N} \ll n/N \ll N_j/N \tag{4.22}$$

we obtain

$$\sum_{n \in I(N_j)} \frac{e^{-n/2N} - e^{-n/N}}{n^{1-\eta_\nu-\eta_k+i(\gamma_\nu^*-\gamma_\kappa^*)}} \ll \frac{N_j}{N} \cdot N_j^{-3\varepsilon} \left(1 + \frac{|t|^{1/r(r-1)}}{N_j^{1/r}}\right) \tag{4.23}$$

which is monotonically increasing in N_j.

So the same estimate as in (4.21) follows too. Hence, using $B_2 \leq 4\big(N_k^*(T)\big)^2$ and $hu \in J_\ell(\eta)$ we obtain

$$K' \ll \mathcal{L}^{C'(\varepsilon,\eta)} B_h^{2\eta} \ll T^{\eta(\max(3/\ell(1-2\eta(\ell-1)-6\ell\varepsilon),4/k(1-(k-1)\eta)-6k\varepsilon))}. \tag{4.24}$$

Consequently,

$$K \ll T^{B(\eta)\eta+\varepsilon}. \qquad \square \tag{4.25}$$

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