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Short proof of a theorem of Brylawski on the coefficients of the Tutte polynomial☆

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A B S T R A C T

In this short note we show that a system $M = (E, r)$ with a ground set E of size m and (rank) function $r : 2^E \to \mathbb{Z}_{\geq 0}$ satisfying $r(S) \leq \min(r(E), |S|)$ for every set $S \subseteq E$, the Tutte polynomial

$$T_M(x, y) := \sum_{S \subseteq E} (x-1)^{r(E)-r(S)}(y-1)^{|S|-r(S)},$$

written as $T_M(x, y) = \sum_{i,j} t_{ij} x^i y^j$, satisfies that for any integer $h \geq 0$, we have

$$\sum_{i=0}^{h} \sum_{j=0}^{h-i} \binom{h-i}{j} (-1)^j t_{ij} = (-1)^{m-r} \binom{h-r}{h-m},$$

where $r = r(E)$, and we use the convention that when $h < m$, the binomial coefficient $\binom{h-r}{h-m}$ is interpreted as 0.

This generalizes a theorem of Brylawski on matroid rank functions and $h < m$, and a theorem of Gordon for $h \leq m$ with the same assumptions on the rank function.

The proof presented here is significantly shorter than the previous ones. We only use the fact that the Tutte polynomial

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$T_M(x, y)$ simplifies to $(x-1)^{r(E)}y^{|E|}$ along the hyperbola $(x-1)(y-1) = 1$.

1. Introduction

For a graph $G = (V, E)$ with $v(G)$ vertices and $e(G)$ edges, the Tutte polynomial $T_G(x, y)$ is defined as

$$T_G(x, y) = \sum_{A\subseteq E}(x-1)^{k(A)-k(E)}(y-1)^{k(A)+|A|-v(G)},$$

where $k(A)$ denotes the number of connected components of the graph (V, A), see [5]. There are many excellent surveys about the properties of the Tutte polynomial and its applications [1–3,6].

In this paper, we concentrate on Brylawski's identities concerning the Tutte polynomial. Written as a usual bivariate polynomial $T_G(x, y) = \sum_{i,j} t_{ij}x^iy^j$, the coefficients t_{ij} encode the number of certain spanning trees, namely spanning trees with internal activity i and external activity j with respect to a fixed ordering of the edges, for details see [5]. It is not hard to prove that $t_{00} = 0$ and $t_{10} = t_{01}$ if the graph G has at least 2 edges. In general, Brylawski [1] proved that a collection of linear relations hold true between the coefficients of the Tutte polynomial. Namely, he proved that if $0 \leq h < e(G)$, then

$$\sum_{i=0}^{h}\sum_{j=0}^{h-i}\binom{h-i}{j}(-1)^j t_{ij} = 0.$$

In particular, the third relation gives that if $e(G) \geq 3$, then $t_{20}-t_{11}+t_{02} = t_{10}$. Note that Brylawski [1] proved these identities not only for the Tutte polynomial of a graph, but for the Tutte polynomial of an arbitrary matroid M. The Tutte polynomial $T_M(x, y)$ of a matroid $M = (E, r)$ is defined by

$$T_M(x, y) = \sum_{S\subseteq E}(x-1)^{r(E)-r(S)}(y-1)^{|S|-r(S)},$$

where $r(S)$ is the rank of a set $S \subseteq E$. The Tutte polynomial of a graph G simply corresponds to the cycle matroid M of the graph G. Note that the rank function $r : 2^E \to \mathbb{Z}_{\geq 0}$ of a matroid satisfies the following axioms:

(R1) for any $A \subseteq E$ we have $r(A) \leq |A|$,

(R2) (submodularity) for any $A, B \subseteq E$ we have

$$r(A \cap B) + r(A \cup B) \leq r(A) + r(B),$$

(R3) (monotonicity) for any $A \subseteq E$ and $x \in E$ we have

$$r(A) \leq r(A \cup \{x\}) \leq r(A) + 1.$$

Gordon [4] calls a function $r : 2^E \to \mathbb{Z}_{\geq 0}$ a rank function on a ground set E if it satisfies $r(A) \leq \min(r(E), |A|)$ for every set $A \subseteq E$. He showed that for a system $M = (E, r)$ the coefficients of $T_M(x, y)$ satisfy Brylawski's identities if r is a rank function without the assumptions of submodularity and monotonicity. He also extended Brylawski's identities to the case $h = |E|$.

Here we extend the work of Gordon and Brylawski for $h > |E|$, and also simplify the proof significantly. We only use the special form of the polynomial, namely that it simplifies to $(x-1)^{r(E)}y^{|E|}$ along the hyperbola $(x-1)(y-1) = 1$. We use exactly the same assumptions on the function $r : 2^E \to \mathbb{Z}_{\geq 0}$ as Gordon. Our generalized Brylawski's identities are the following.

Theorem 1.1 (*Generalized Brylawski's Identities*)**.** *Let $M = (E, r)$, where E is a set, and $r : 2^E \to \mathbb{Z}_{\geq 0}$ is a function on the subsets of E satisfying $r(S) \leq \min(r(E), |S|)$ for every set $S \subseteq E$. Let*

$$T_M(x, y) = \sum_{S \subseteq E} (x-1)^{r(E)-r(S)}(y-1)^{|S|-r(S)}$$

be the Tutte polynomial of the system $M = (E, r)$. Let m denote the size of E, and let $r = r(E)$. The coefficients t_{ij} of Tutte polynomial $T_M(x, y) = \sum_{i,j} t_{ij}x^i y^j$ satisfy the following identities. For any integer $h \geq 0$, we have

$$\sum_{i=0}^{h}\sum_{j=0}^{h-i}\binom{h-i}{j}(-1)^j t_{ij} = (-1)^{m-r}\binom{h-r}{h-m},$$

with the convention that when $h < m$, the binomial coefficient $\binom{h-r}{h-m}$ is interpreted as 0.

In particular, by specializing Theorem 1.1 for the cycle matroid of a graph G we get the following.

Theorem 1.2 (*Generalized Brylawski's Identities for Graphs*)**.** *Let G be any graph with n vertices, m edges and c connected components. Let $T_G(x, y) = \sum_{i,j} t_{ij}x^i y^j$ be the Tutte polynomial of the graph G. Then for any integer $h \geq 0$, we have*

$$\sum_{i=0}^{h}\sum_{j=0}^{h-i}\binom{h-i}{j}(-1)^j t_{ij} = (-1)^{m-n+c}\binom{h-n+c}{h-m},$$

with the convention that when $h < m$, the binomial coefficient $\binom{h-n+c}{h-m}$ is interpreted as 0.

2. Proof of Theorem 1.1

This entire section is devoted to the proof of Theorem 1.1.

Let $r = r(E)$ and $m = |E|$. By definition,

$$T_M(x, y) = \sum_{S \subseteq E} (x-1)^{r(E)-r(S)}(y-1)^{|S|-r(S)}.$$

Let us introduce a new variable z, and plug in $x = \frac{z}{z-1}$ and $y = z$. Then

$$T_M\left(\frac{z}{z-1}, z\right) = \sum_{S \subseteq E}(z-1)^{|S|-r} = (z-1)^{-r}z^m = \frac{z^m}{(z-1)^r}.$$

Since $T_M(x, y) = \sum_{i,j} t_{ij}x^i y^j$, we have

$$T_M\left(\frac{z}{z-1}, z\right) = \sum_{i,j} t_{ij}\left(\frac{z}{z-1}\right)^i z^j = \frac{z^m}{(z-1)^r}.$$

Hence

$$\sum_{i,j} t_{i,j}z^{i+j}(z-1)^{r-i} = z^m.$$

Note that if $i > r$, then $t_{ij} = 0$ as $r(S) \geq 0$ for every set S. Hence, both sides are polynomials in z, so we can compare the coefficients of z^k.

$$\sum_{i,j} t_{i,j}(-1)^{r-k+j}\binom{r-i}{k-(i+j)} = \delta_{k,m}, \tag{1}$$

where $\delta_{k,m}$ is 1 if $k = m$, and 0 otherwise. This is not yet exactly Brylawski's identity, but taking appropriate linear combinations of these equations yields Brylawski's identities. Let

$$C_{h,k} = (-1)^k\binom{h-r}{h-k}.$$

Then

$$\sum_{k=0}^{h} C_{h,k}\left(\sum_{i,j} t_{i,j}(-1)^{r-k+j}\binom{r-i}{k-(i+j)}\right) = C_{h,m}.$$

Then

$$\begin{aligned}
C_{h,m} &= \sum_{k=0}^{h} C_{h,k}\left(\sum_{i,j} t_{i,j}(-1)^{r-k+j}\binom{r-i}{k-(i+j)}\right)\\
&= \sum_{k=0}^{h}(-1)^{k}\binom{h-r}{h-k}\left(\sum_{i,j} t_{i,j}(-1)^{r-k+j}\binom{r-i}{k-(i+j)}\right)\\
&= \sum_{i,j} t_{i,j}(-1)^{r+j}\left(\sum_{k=0}^{h}\binom{h-r}{h-k}\binom{r-i}{k-(i+j)}\right)\\
&= \sum_{i,j} t_{i,j}(-1)^{r+j}\binom{h-i}{h-(i+j)}\\
&= \sum_{i,j}\binom{h-i}{j} t_{i,j}(-1)^{r+j}.
\end{aligned}$$

Hence

$$\sum_{i,j}\binom{h-i}{j} t_{i,j}(-1)^{j} = (-1)^{m-r}\binom{h-r}{h-m}.$$

Remark 2.1. Once one conjectures Theorem 1.2, then it can be proved by the deletion–contraction identities via simple induction on h even for matroids. The more general Theorem 1.1 can be proved by certain recursions akin to deletion–contraction too, as was shown by Gordon [4], but seems to be considerably more work than the proof presented in this paper.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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