# Boolean-type Retractable State-finite Automata Without Outputs ${ }^{1}$ 

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#### Abstract

An automaton $\mathbf{A}$ is called a retractable automaton if, for every subautomaton $\mathbf{B}$ of $\mathbf{A}$, there is at least one homomorphism of $\mathbf{A}$ onto $\mathbf{B}$ which leaves the elements of $B$ fixed (such homomorphism is called a retract homomorphism of $\mathbf{A}$ onto $\mathbf{B}$ ). We say that a retractable automaton $\mathbf{A}=(\mathrm{A}, \mathrm{X}, \delta)$ is Boolean-type if there exists a family $\left\{\lambda_{B} \mid \mathbf{B}\right.$ is a subautomaton of $\left.\mathbf{A}\right\}$ of retract homomorphisms $\lambda_{B}$ of $\mathbf{A}$ such that, for arbitrary subautomata $\mathbf{B}_{1}$ and $\mathbf{B}_{2}$ of $\mathbf{A}$, the condition $B_{1} \subseteq B_{2}$ implies $\operatorname{Ker} \lambda_{B_{2}} \subseteq \operatorname{Ker} \lambda_{B_{1}}$. In this paper we describe the Boolean-type retractable state-finite automata without outputs.


## 1 Introduction and motivation

Let $\mathbf{A}=(A, X, \delta)$ be an automaton without outputs. A subautomaton $\mathbf{B}$ of $\mathbf{A}$ is called a retract subautomaton if there is a homomorphism of $\mathbf{A}$ onto $\mathbf{B}$ which leaves the elements of $B$ fixed. A homomorphism with this property is called a retract homomorphism of $\mathbf{A}$ onto $\mathbf{B}$.

In [5], A. Nagy introduced the notion of the retractable automaton. An automaton $\mathbf{A}$ (without outputs) is called a retractable automaton if every subautomaton of $\mathbf{A}$ is a retract subautomaton. He proved (in Theorem 3 of [5]) that if the lattice $\mathcal{L}(\mathbf{A})$ of all congruences of an automaton $\mathbf{A}$ is complemented then $\mathbf{A}$ is a retractable automaton. He also defined the notion of the Boolean-type retractable automaton. We say that a retractable automaton $\mathbf{A}=(\mathrm{A}, \mathrm{X}, \delta)$ is Boolean-type if there exists a family $\left\{\lambda_{B} \mid \mathbf{B}\right.$ is a subautomaton of $\left.\mathbf{A}\right\}$ of retract homomorphisms $\lambda_{B}$ of $\mathbf{A}$ such that, for arbitrary subautomata $\mathbf{B}_{1}$ and $\mathbf{B}_{2}$ of $\mathbf{A}$, the condition $B_{1} \subseteq B_{2}$ implies $\operatorname{Ker}_{B_{2}} \subseteq \operatorname{Ker}_{B_{1}}$. He proved (in Theorem 5 of [5]) that if the lattice $\mathcal{L}(\mathbf{A})$ of all congruences of an automaton $\mathbf{A}$ is a Boolean algebra then $\mathbf{A}$ is a Booleantype retractable automaton.

[^0]In [5], A. Nagy investigated the not necessarily state-finite Boolean-type retractable automata containing traps (a state $c$ is called a trap of an automaton $\mathbf{A}=(\mathrm{A}, \mathrm{X}, \delta)$ if $\delta(c, x)=c$ for every $x \in X$ ). He proved that every Booleantype retractable automaton containing traps has a homomorphic image which is a Boolean-type retractable automaton containing exactly one trap. Moreover, he gave a complete description of Boolean-type retractable automata containing exactly one trap.

In [2], the authors defined the notion of the strongly retract extension of automata. They proved that every state finite Boolean-type retractable automaton without outputs is a direct sum of Boolean-type retractable automata whose principal factors form a tree. Moreover, a state-finite automaton $\mathbf{A}$ is a Booleantype retractable automaton whose principal factors form a tree if and only if it is a strongly retract extension of a strongly connected subautomaton of $\mathbf{A}$ by a Boolean-type retractable automaton containing exactly one trap (which is described in [5]).

In [5] and [2], some theorem gives only necessary conditions for special retractable or Boolean-type retractable state-finite automata without outputs. Paper [6] is the first one which gives a complete description of state-finite retractable automata without outputs. Using the results of [6], we give a complete description of Boolean-type retractable state-finite automata without outputs.

## 2 Basic notations

By an automaton without outputs we mean a system $(A, X, \delta)$ where A and X are non-empty sets, and $\delta$ maps from the Cartesian product $A \times X$ to $A$. We will refer to $\mathrm{A}, \mathrm{X}$ and $\delta$ as the state set, the input set and the transition function of $\mathbf{A}$, respectively. An automaton $\mathbf{A}$ is said to be state-finite, if the set A is finite. In this paper by an automaton we always mean a state-finite automaton without outputs. We will follow the definitions and notations of [6].

An automaton $\mathbf{B}=\left(\mathrm{B}, \mathrm{X}, \delta_{B}\right)$ is called a subautomaton of an automaton $\mathbf{A}=(A, X, \delta)$ if B is a subset of A and $\delta_{B}$ is the restriction of $\delta$ to $B \times X$. A subautomaton $\mathbf{B}$ of an automaton $\mathbf{A}$ contained by every subautomaton of $\mathbf{A}$ is called the kernel of $\mathbf{A}$.

By a homomorphism of an automaton $(A, X, \delta)$ into an automaton $(B, X, \gamma)$ we mean a map $\phi$ of the set $A$ into the set $B$ such that $\phi(\delta(a, x))=\gamma(\phi(a), x)$ for all $a \in A$ and $x \in X$.

A congruence of an automaton $(A, X, \delta)$ is an equivalence $\alpha$ of the set $A$ such
that, for all $a, b \in A$ and $x \in X$, the assumption $(a, b) \in \alpha$ implies $(\delta(a, x), \delta(b, x)) \in \alpha$. A congruence class $\alpha$ containing $a \in A$ will be denoted by $[a]_{\alpha}$. The kernel of a homomorphism $\phi:(A, X, \delta) \mapsto(B, X, \gamma)$, which is denoted by $\operatorname{Ker} \phi$, is defined as the following relation of $A: \operatorname{Ker} \phi:=\{(a, b) \in A \times A: \phi(a)=\phi(b)\}$. It is clear that $\operatorname{Ker} \phi$ is a congruence on A.

We will denote the lattice of all congruences of an automaton $\mathbf{A}$ by $\mathcal{L}(A)$. For every $\alpha, \beta \in \mathcal{L}(A), \alpha \wedge \beta:=\alpha \cap \beta$ and $\alpha \vee \beta=(\alpha \cup \beta)^{T}$ where

$$
(\alpha \cup \beta)^{T}=(\alpha \cup \beta) \cup((\alpha \cup \beta) \circ(\alpha \cup \beta)) \cup \ldots
$$

is the transitive closure of $\alpha \cup \beta$ (here $\circ$ denotes the usual operation on the semigroup of all binary relations on $A$ (see [3]) ).

Let $\mathbf{B}=\left(\mathrm{B}, \mathrm{X}, \delta_{B}\right)$ be a subautomaton of an automaton $\mathbf{A}=(A, X, \delta)$. The relation $\varrho_{B}=\left\{\left(b_{1}, b_{2}\right) \in A \times A: b_{1}=b_{2} \quad\right.$ or $\left.\quad b_{1}, b_{2} \in B\right\}$ is a congruence on $\mathbf{A}$. This congruence is called the Rees congruence on $\mathbf{A}$ defined by $\mathbf{B}$. The $\varrho_{B}$-classes of $A$ are $B$ itself and every one-element set $\{a\}$ with $a \in A \backslash B$.

## 3 Retractable automata

Definition $1 A$ subautomaton $\boldsymbol{B}$ of an automaton $\boldsymbol{A}=(A, X, \delta)$ is called a retract subautomaton if there exist a homomorphism $\lambda_{B}$ of $\mathbf{A}$ onto $\mathbf{B}$ which leaves the elements of B fixed. An automaton is called retractable if its every subautomaton is retract. [5]

Theorem 1 A Rees-congruence $\varrho_{B}$ defined by a subautomaton $\boldsymbol{B}=\left(B, X, \delta_{B}\right)$ of an automaton $\boldsymbol{A}=(A, X, \delta)$ has a complement in the lattice $(\mathcal{L}(A), \vee, \wedge)$ if and only if $\boldsymbol{B}$ is a retract subautomaton.

Proof Let $\mathbf{A}=(A, X, \delta)$ be an automaton. Assume that $\mathbf{B}$ is a subautomaton of $\mathbf{A}$ such that the Rees congruence $\varrho_{B}$ has a complement in $\mathcal{L}(A)$. By the proof of Theorem 3 of [5], $\mathbf{B}$ is a retract subautomaton of $\mathbf{A}$. Conversely, assume that $\mathbf{B}$ is a retract subautomaton of $\mathbf{A}$. We will show that the kernel of a retract homomorphism of $\mathbf{A}$ onto $\mathbf{B}$ is the a complement of the Rees congruence $\varrho_{B}$ defined by $\mathbf{B}$. We show this by proving that, for every states $a \neq b$ of $\mathbf{A}$, we have $(a, b) \notin \eta_{B} \wedge \varrho_{B}$ and $(a, b) \in \eta_{B} \vee \varrho_{B}$ (here $\lambda_{B}$ denotes the corresponding retract homomorphism of $\mathbf{A}$ onto $\mathbf{B}$ and $\eta_{B}:=\operatorname{Ker} \lambda_{B}$ ). Let $a, b$ be arbitrary elements in $A$ with the condition $a \neq b$.

- Case $a, b \in B$.

Then $(a, b) \notin \eta_{B} \Rightarrow(a, b) \notin \eta_{B} \cap \varrho_{B}=\eta_{B} \wedge \varrho_{B}$.
Furthermore $a \varrho_{B} b \Rightarrow(a, b) \in \varrho_{B} \cup \eta_{B} \subseteq \varrho_{B} \vee \eta_{B}$.

- Case $a \in A \backslash B, b \in B$.

In this case, it follows that $(a, b) \notin \varrho_{B}$ thus $(a, b) \notin \eta_{B} \cap \varrho_{B}=\eta_{B} \wedge \varrho_{B}$.
Now assume that $\lambda_{B}(a)=\lambda_{B}(b)$. In this case $(a, b) \in \eta_{B}$ is true by definition which implies $(a, b) \in \eta_{B} \cup \varrho_{B} \subseteq \eta_{B} \vee \varrho_{B}$.
Otherwise: $\lambda_{B}(a) \neq \lambda_{B}(b) \Rightarrow \exists c \in B: \lambda_{B}(a)=\lambda_{B}(c)$ because $\lambda_{B}$ maps onto every element B. Thus $(a, c) \in \eta_{B}$ and $(c, b) \in \varrho_{B}$, this implies $(a, b) \in\left(\varrho_{B} \cup \eta_{B}\right)^{T}=$ $\varrho_{B} \vee \eta_{B}$ by definition.

- Case $a, b \in A \backslash B$.
$(a, b) \notin \varrho_{B} \Rightarrow(a, b) \notin \varrho_{B} \cap \eta_{B}=\varrho_{B} \wedge \eta_{B}$. Since $\lambda_{B}$ maps $A$ onto $B$, thus exists such $c$ and $d$ elements of $B$ that $\lambda_{B}(a)=\lambda_{B}(c)$ and $\lambda_{B}(b)=\lambda_{B}(d)$ holds. From $(a, c) \in \eta_{B},(c, d) \in \varrho_{B},(b, d) \in \eta_{B}$ follows $(a, b) \in\left(\varrho_{B} \cup \eta_{B}\right)^{T}=\varrho_{B} \vee \eta_{B}$.


## 4 Boolean-type retractable automata

Definition 2 We say that a retractable automaton $\boldsymbol{A}=(A, X, \delta)$ is Boolean-type if there exists a family $\left\{\lambda_{B} \mid \mathbf{B}\right.$ is a subautomaton of $\left.\mathbf{A}\right\}$ of retract homomorphism $\lambda_{B}$ of $\mathbf{A}$ such that, for arbitrary $\mathbf{B}_{\mathbf{1}}$ and $\mathbf{B}_{\mathbf{2}}$ subautomata of $\mathbf{A}$, the condition $B_{1} \subseteq B_{2}$ implies $\operatorname{Ker}_{B_{2}} \subseteq \operatorname{Ker}_{B_{1}}$.

In the next, if we suppose that $\mathbf{A}$ is a Boolean-type retractable automaton and $\mathbf{C}$ is a subautomaton of $\mathbf{A}$, then $\lambda_{C}$ will denote the retract homomorphism of $\mathbf{A}$ onto $\mathbf{C}$ belonging to a fix family $\left\{\lambda_{B} \mid \mathbf{B}\right.$ is a subautomaton of $\left.\mathbf{A}\right\}$ of retract homomorphisms $\lambda_{B}$ of $\mathbf{A}$ satisfying the conditions of Definition 2.

In this section we shall discuss Boolean-type retractable state-finite automata without outputs. We describe these automata using the concepts and constructions of [6].

Definition 3 We say that an automaton $\mathbf{A}=(A, X, \delta)$ is a direct sum of automata $\left\{\mathbf{A}_{i}=\left(A_{i}, X, \delta_{i}\right)(i \in I)\right\}$ (indexed with the set I) if $A_{i} \cap A_{j}=\emptyset$ for every $i, j \in I$ with $i \neq j$, and moreover $A=\cup \cup_{i \in I} A_{i}$.

Theorem 2 ([6]) For a state-finite automaton $\mathbf{A}=(A, X, \delta)$ the following statements are equivalent:
(i) $\mathbf{A}$ is retractable.
(ii) $\mathbf{A}$ is the direct sum of finitely many state-finite retractable automaton, which contain kernels being isomorphic to eachother.

The next lemma will be used in the proof of Theorem 3 several times.
Lemma 1 If $\mathbf{D} \subseteq \mathbf{B}$ are subautomaton of a Boolean-type retractable automaton A such that $\lambda_{B}(a) \in D$ for some $a \in A$ then $\lambda_{B}(a)=\lambda_{D}(a)$.

Proof. Let $c=\lambda_{B}(a)$. As $c \in D \subseteq B$, we have $\lambda_{B}(c)=c$. Thus $a$ and $c$ are in the same $\operatorname{Ker} \lambda_{B}$-class of $A$. As every $\operatorname{Ker} \lambda_{B}$-class is in a $\operatorname{Ker} \lambda_{D}$-class, we have that $a$ and $c$ are in the same $\lambda_{D}$-class and so $\lambda_{D}(a)=\lambda_{D}(c)$. As $c \in D$, we have $\lambda_{D}(c)=c$ and so $\lambda_{D}(a)=\lambda_{D}(c)=c=\lambda_{B}(a)$.

Theorem 3 For a state-finite automaton $\mathbf{A}=(A, X, \delta)$ the following statements are equivalent:
(i) $\mathbf{A}$ is a Boolean-type retractable automaton.
(ii) $\mathbf{A}$ is the direct sum of finitely many state-finite Boolean-type retractable automata containing kernels being isomorphic to each other.

Proof (i) $\mapsto$ (ii): Let $\mathbf{A}$ be a Boolean-type, retractable, state-finite automaton. Since $\mathbf{A}$ is state-finite and retractable, then by Theorem $2 \mathbf{A}$ is a direct sum of finitely many, state-finite, retractable automata $\mathbf{A}_{\mathbf{i}}(i \in I)$ containing kernels being isomorphic to each other. Let $i_{0} \in I$ be an arbitrary fixed index. Let $\left\{\lambda_{B} \mid \mathbf{B}\right.$ is a subautomaton of $\left.\mathbf{A}\right\}$ be a family of retract homomorphisms such that $B_{1} \subseteq B_{2}$ implies $\operatorname{Ker} \lambda_{B_{2}} \subseteq \operatorname{Ker} \lambda_{B_{1}}$. It is clear that $A_{i_{0}}$ is a subautomaton of $A$. Consider those $\lambda_{C}$ retract homomorphisms which fulfils $C \subseteq A_{i_{0}}$, we shall denote these with $\left\{\Lambda_{C} \mid C \subseteq A_{i_{0}}\right\}$. Since all $\mathbf{C}$ subautomata of $\mathbf{A}$ that has $C \subseteq A_{i_{0}}$ are also subautomata of $\mathbf{A}_{\mathbf{i}_{0}}$, therefore the family $\left\{\Lambda_{C}\right\}$ clearly fulfils the condition $\operatorname{Ker} \Lambda_{C_{2}} \subseteq \operatorname{Ker} \Lambda_{C_{1}}$ for all $C_{1} \subseteq C_{2} \subseteq A_{i_{0}}$.
(ii) $\mapsto$ (i): Assume that the automaton $\mathbf{A}$ is a direct sum of Boolean-type retractable automata $\mathbf{A}_{\mathbf{i}}(i \in I=\{1,2, \ldots, n\})$ whose kernels $\mathbf{T}_{\mathbf{i}}$ are isomorphic to each other. Let $(\cdot) \varphi_{i, i}$ denote the identical mapping of $T_{i}(i=1, \ldots, n)$. For arbitrary $i=1, \ldots n-1$, let $(\cdot) \varphi_{i, i+1}$ denote the corresponding isomorphism of $T_{i}$ onto $T_{i+1}$. For arbitrary $i, j \in I$ with $i<j$, let $(\cdot) \Phi_{i, j}=\varphi_{i, i+1} \circ \cdots \circ \varphi_{j-1, j}$. For
arbitrary $i, j \in I$ with $i>j$, let $(\cdot) \Phi_{i, j}=\varphi_{i-1, i}^{-1} \circ \cdots \circ \varphi_{i, i+1}^{-1}$. It is clear that $\Phi_{i, j}$ is an isomorphism of $T_{i}$ onto $T_{j}$. Moreover, for every $i, j, k \in I, \Phi_{i, j} \circ \Phi_{j, k}=\Phi_{i, k}$.

Let $\mathbf{B}$ be a subautomaton of $\mathbf{A}$. Let $\mathcal{B}$ denote the set of all indexes from $1,2, \ldots, n$ which satisfy $B_{i}=B \cap A_{i} \neq \emptyset$. If $i \in \mathcal{B}$ then $T_{i} \subseteq B_{i}$. Let $i_{B}=\min \mathcal{B}$.

We give a retract homomorphism $\Lambda_{B}$ of $\mathbf{A}$ onto $\mathbf{B}$. If $i \in \mathcal{B}$ then let $\Lambda_{B}(a)=$ $\lambda_{B_{i}}(a)$ for every $a \in A_{i}$. If $i \in I \backslash \mathcal{B}$ (that is, $\left.B_{i}=\emptyset\right)$ then let $\Lambda_{B}(a)=\left(\lambda_{T_{i}}(a)\right) \Phi_{i, i_{B}}$. It is easy to see that $\Lambda_{B}$ is a retract homomorphism of $\mathbf{A}$ onto $\mathbf{B}$.

We show that the set $\left\{\Lambda_{B} \mid \mathbf{B}\right.$ is a subautomaton of $\left.\mathbf{A}\right\}$ satisfies the condition that, for every subautomaton $\mathbf{D} \subseteq \mathbf{B}, \operatorname{Ker} \Lambda_{B} \subseteq \operatorname{Ker} \Lambda_{D}$. Let $\mathbf{D} \subseteq \mathbf{B}$ be arbitrary subautomata of $\mathbf{A}$. We note that $\mathcal{D} \subseteq \mathcal{B}$ and $i_{B} \leq i_{D}$. Assume

$$
\Lambda_{B}(a)=\Lambda_{B}(b)
$$

for some $a \in A_{i}$ and $b \in A_{j}$.
Case 1: $i \in \mathcal{D}$. In this case $i_{D} \leq i$. We have two subcases. If $j \in \mathcal{B}$ then

$$
\lambda_{B_{i}}(a)=\Lambda_{B}(a)=\Lambda_{B}(b)=\lambda_{B_{i}}(b)
$$

and so $j=i$. From this it follows that

$$
\lambda_{D_{i}}(a)=\lambda_{D_{i}}(b)
$$

and so

$$
\Lambda_{D}(a)=\lambda_{D_{i}}(a)=\lambda_{D_{i}}(b)=\Lambda_{D}(b)
$$

If $j \in I \backslash \mathcal{B}$ then

$$
\lambda_{B_{i}}(a)=\Lambda_{B}(a)=\Lambda_{B}(b)=\left(\lambda_{T_{j}}(b)\right) \Phi_{j, i_{B}} \in T_{i_{B}} \subseteq D_{i_{B}}
$$

and so $i=i_{B} \leq i_{D}$. This and the above $i_{D} \leq i$ together imply $i=i_{B}=i_{D}$. Then By Lemma 1,

$$
\Lambda_{D}(a)=\lambda_{D_{i}}(a)=\lambda_{B_{i}}(a)
$$

As

$$
\Lambda_{D}(b)=\left(\lambda_{T_{j}}(b)\right) \Phi_{j, i_{D}}
$$

we have

$$
\Lambda_{D}(a)=\lambda_{B_{i}}(a)=\left(\lambda_{T_{j}}(b)\right) \Phi_{j, i_{B}}=\left(\lambda_{T_{j}}(b)\right) \Phi_{j, i_{D}} \Lambda_{D}(b) .
$$

Case 2: $i \notin \mathcal{D}$, but $i \in \mathcal{B}$. If $j \in \mathcal{B}$ then

$$
\lambda_{B_{i}}(a)=\Lambda_{B}(a)=\Lambda_{B}(b)=\lambda_{B_{i}}(b)
$$

and so $j=i$. Then $\Lambda_{D}(a)=\Lambda_{D}(b)$ (see the first subcase of Case 1). If $j \notin \mathcal{B}$ then

$$
\lambda_{B_{i}}(a)=\Lambda_{B}(a)=\Lambda_{B}(b)=\left(\lambda_{T_{j}}(b)\right) \Phi_{j, i_{B}}
$$

and so $i=i_{B}$. Thus $\lambda_{B_{i}}(a) \in T_{i} \subseteq D_{i}$ and so (by Lemma 1)

$$
\lambda_{B_{i}}(a)=\lambda_{D_{i}}(a)=\lambda_{T_{i}}(a) .
$$

If $i_{D}=i_{B}(=i)$ then

$$
\Lambda_{D}(a)=\lambda_{D_{i}}(a)=\lambda_{B_{i}}(a)=\left(\lambda_{T_{j}}(b)\right) \Phi_{j, i_{B}}=\left(\lambda_{T_{j}}(b)\right) \Phi_{j, i_{D}}=\Lambda_{D}(b) .
$$

If $i_{D}>i_{B}(=i)$ then $A_{i} \cap D=\emptyset$ and so

$$
\Lambda_{D}(a)=\left(\lambda_{T_{i}}(a)\right) \Phi_{i, i_{D}}
$$

and

$$
\Lambda_{D}(b)=\left(\lambda_{T_{j}}(b)\right) \Phi_{j, i_{D}}
$$

As

$$
\lambda_{T_{i}}(a)=\lambda_{B_{i}}(a)=\left(\lambda_{T_{j}}(b)\right) \Phi_{j, i_{B}},
$$

we have

$$
\begin{gathered}
\Lambda_{D}(b)=\left(\lambda_{T_{j}}(b)\right) \Phi_{j, i_{D}}=\left(\lambda_{T_{j}}(b)\right)\left(\Phi_{j, i_{B}} \circ \Phi_{i_{B}, i_{D}}\right)= \\
=\left(\left(\lambda_{T_{j}}(b)\right) \Phi_{j, i_{B}}\right) \Phi_{i_{B}, i_{D}}=\left(\lambda_{T_{i}}(a)\right) \Phi_{i_{B}, i_{D}}=\left(\lambda_{T_{i}}(a)\right) \Phi_{i, i_{D}}=\Lambda_{D}(a) .
\end{gathered}
$$

Case 3: $i \notin \mathcal{B}$. If $j \in \mathcal{B}$ then we can prove (as in the second subcases of Case 1 and Case 2) that $\Lambda_{D}(a)=\Lambda_{D}(b)$. Consider the case when $j \notin \mathcal{B}$. Then

$$
\left(\lambda_{T_{i}}(a)\right) \Phi_{i, i_{B}}=\Lambda_{B}(a)=\Lambda_{B}(b)=\left(\lambda_{T_{j}}(b)\right) \Phi_{j, i_{B}} .
$$

Hence

$$
\begin{aligned}
& \Lambda_{D}(a)=\left(\lambda_{T_{i}}(a)\right) \Phi_{i, i_{D}}=\left(\lambda_{T_{i}}(a)\right)\left(\Phi_{i, i_{B}} \circ \Phi_{i_{B}, i_{D}}\right)=\left(\left(\lambda_{T_{i}}(a)\right) \Phi_{i, i_{B}}\right) \Phi_{i_{B}, i_{D}}= \\
= & \left(\left(\lambda_{T_{j}}(b)\right) \Phi_{j, i_{B}}\right) \Phi_{i_{B}, i_{D}}=\left(\lambda_{T_{j}}(b)\right)\left(\Phi_{j, i_{B}} \circ \Phi_{i_{B}, i_{D}}\right)=\left(\lambda_{T_{j}}(b)\right) \Phi_{j, i_{D}}=\Lambda_{D}(b) .
\end{aligned}
$$

In all cases, we have that $\Lambda_{B}(a)=\Lambda_{B}(b)$ implies $\Lambda_{D}(a)=\Lambda_{D}(b)$ for every $a, b \in A$. Consequently

$$
\operatorname{Ker} \Lambda_{B} \subseteq \operatorname{Ker} \Lambda_{D}
$$

Hence $A$ is a Boolean-type retractable automaton.

By Theorem 3, we can focus our attention on a Boolean-type retractable automaton containing a kernel. In our investigation two notions will play important role. These notions are the dilation of automata and the semi-connected automata.

Definition 4 Let $\mathbf{B}$ be an arbitrary subautomaton of an automaton $\mathbf{A}=(A, X, \delta)$. We say that $\mathbf{A}$ is a dilation of $\mathbf{B}$ if there exists a mapping $\phi_{\text {dil }}(\cdot)$ of $A$ onto $B$ that leaves the elements of $B$ fixed, and fulfils $\delta(a, x)=\delta_{B}\left(\phi_{\text {dil }}(a), x\right)$ for all $a \in A$ and $x \in X$. This fact will be denoted by: $\left(A, X, \delta ; B, \phi_{\text {dil }}\right)$. ([5])

If $a$ is an arbitrary element of an $\mathbf{A}$ automaton, then let $R(a)$ denote the subautomaton generated by the element $a$ (the smallest subautomaton containing $a$ ). It is easy to see that

$$
R(a)=\left\{\delta(a, x): x \in X^{*}\right\},
$$

where $X^{*}$ is the free monoid over $X$. Let us define the following relation:

$$
\mathcal{R}:=\{(\mathrm{a}, \mathrm{~b}) \in A \times A: R(a)=R(b)\} .
$$

It is evident that $\mathcal{R}$ is an equivalence relation. The $\mathcal{R}$ class containing a particular $a$ element is denoted by $R_{a}$. The set $R(a) \backslash R_{a}$ is denoted by $R[a]$. It is clear that $R[a]$ is either empty set or a subautomaton of $\mathbf{A} . R\{a\}=R(a) / \rho_{R[a]}$ factor automaton is called a principal factor of $\mathbf{A}$. If $\mathrm{R}[\mathrm{a}]$ is an empty set, then consider $R\{a\}$ as $R(a)$. [6]

An $\mathbf{A}$ automaton is said to be strongly connected if, for any $a, b \in A$ elements, there exist a word $p \in X^{+}$such that $\delta(a, p)=b ;\left(X^{+}\right.$is the free semigroup over $X)$. Remark: for a word $p=x_{1} x_{2} \ldots x_{n}$ and an element $a$ the transition function is defined as the following:

$$
\delta(a, p)=\delta\left(\ldots \delta\left(\delta\left(a, x_{1}\right), x_{2}\right) \ldots x_{n}\right)
$$

An automaton is called strongly trap connected if it contains exactly one trap and, for every $a \in A \backslash\{$ trap $\}$ and $b \in A$, there is a word $p \in X^{+}$such that $\delta(a, p)=b$.

An automaton is said to be semi-connected if its every principal factor is either strongly connected or strongly trap connected. ([6])

Theorem 4 ([6]) A state-finite automaton without outputs is a retractable automaton if and only if it is a dilation of a semi-connected retractable automaton.

The next theorem is the extension of Theorem 4.
Theorem 5 A state-finite automaton without outputs is a Boolean-type retractable automaton if and only if it is a dilation of a semi-connected Boolean-type retractable automaton.

Proof. Let $\mathbf{A}$ be a Boolean-type retractable state- finite automaton without outputs. Then, by Theorem 4, $\mathbf{A}$ is a dilation of the retractable semi-connected automaton $\mathbf{C}$. For a subautomaton $\mathbf{B}$ of $\mathbf{C}$, let $\lambda_{B}^{\prime}$ denote the restriction of $\lambda_{B}$ to $\mathbf{C}$. It is easy to see that $\mathbf{C}$ is a Boolean-type retractable automaton with the family $\left\{\lambda_{B}^{\prime} \mid \mathbf{B}\right.$ is a subautomaton of of $\left.\mathbf{C}\right\}$.
Conversely, let the automaton $\mathbf{A}=\left(\mathrm{A}, \mathrm{X}, \delta ; \mathrm{B}, \phi_{\text {dil }}\right)$ be a dilation of the automaton $\mathbf{B}=\left(\mathrm{B}, \mathrm{X}, \delta_{B}\right)$. Assume that $\mathbf{B}$ is Boolean-type retractable with the family $\left\{\lambda_{C} \mid \mathbf{C}\right.$ is a subautomaton of $\left.\mathbf{B}\right\}$. Since all subautomata of $\mathbf{A}$ are subautomata of $\mathbf{B}$, it is clear that, for every subautomaton $\mathbf{C}$ of $\mathbf{A}, \lambda_{C} \circ \phi_{\text {dil }}$ is a retract homomorphism of $\mathbf{A}$ onto $\mathbf{C}$. Moreover, $\mathbf{A}$ is a Boolean-type retractable automaton with the family $\left\{\lambda_{C} \circ \phi \mid \mathbf{C}\right.$ is a subautomaton of $\left.\mathbf{A}\right\}$.

By Theorem 5 and Theorem 3, we can concentrate our attention on semiconnected automata containing kernels.

Definition 5 Let $(T, \leq)$ be a partially ordered set, in which every two element subset has a lower bound, and every non-empty subset of $T$ having an upper bound contains a maximal element. Consider the operation on $T$ which maps a couple $\left(t_{1}, t_{2}\right) \in T \times T$ to the (unique) greatest upper bound of the set $\left\{t_{1}, t_{2}\right\}$. $T$ is a semilattice under this operation. This semilattice is called a tree. It is clear that every finite tree has a least element. ([7])

If a non-trivial state-finite automaton $\mathbf{A}$ contains exactly one trap $a_{0}$ then $A^{0}$ will denote the set $A \backslash a_{0}$. If $A$ is a trivial automaton, then let $A^{0}=A$. On the set $A^{0} \times X$ we consider a partial (transition) function $\delta^{0}$ which is defined only on couples $(a, x)$ for which $\delta(a, x) \in A^{0}$; in this case $\delta^{0}(a, x)=\delta(a, x)$. We shall say that $\left(A^{0}, \mathrm{X}, \delta^{0}\right)$ is the partial automaton derived from the automaton $\mathbf{A}$.

If $\mathbf{A}^{0}$ and $\mathbf{B}^{0}$ are partial automata, then a mapping $\phi$ of $A^{0}$ into $B^{0}$ is called a partial homomorphism of $\mathbf{A}^{\mathbf{0}}$ into $\mathbf{B}^{\mathbf{0}}$ if, for every $a \in A^{0}$ and $x \in X$, the condition $\delta_{A}(a, x) \in A^{0}$ implies $\delta_{B}(\phi(a), x) \in B^{0}$ and $\delta_{B}(\phi(a), x)=\phi(\delta(a, x))$.

Construction ([6]) Let ( $T, \leq$ ) be a finite tree with the least element $i_{0}$. Let $i>j$ ( $i, j \in T$ ) denote the fact that $i \geq j$ and for all $k \in T$, the condition $i \geq k \geq j$ implies $i=k$ or $j=k$. Let $\mathbf{A}_{i}=\left(A_{i}, X, \delta_{i}\right), i \in T$ be a family of pairwise disjoint automata satisfying the following conditions:
(i) $\mathbf{A}_{i_{0}}$ is strongly connected and $\mathbf{A}_{i}$ is strongly trap connected for every $i \in$ $T, i \neq i_{0}$.
(ii) Let $\phi_{i, i}$ denote the identical mapping of $\mathbf{A}_{i}$. Assume that, for every $i, j \in$ $T, i>j$, there exist a homomorphism $\phi_{i, j}$ which maps $\mathbf{A}_{i}^{0}$ into $\mathbf{A}_{j}^{0}$ such that
(iii) for every $i>j$ there exist elements $a \in A_{i}^{0}$ and $x \in X$ such that $\delta_{i}(a, x) \notin A_{i}^{0}$, $\delta_{j}\left(\phi_{i, j}(a), x\right) \in A_{j}^{0}$.

For arbitrary elements $i, j \in T$ with $i \geq j$, we define a partial homomorphism $\Phi_{i, j}$ of $\mathbf{A}_{i}^{0}$ into $\mathbf{A}_{j}^{0}$ as follows: $\Phi_{i, i}=\phi_{i, i}$ and, if $i>j$ such that $i>k_{1}>\ldots k_{n}>j$, then let

$$
\Phi_{i, j}=\phi_{k_{n}, j} \circ \phi_{k_{n-1}, k_{n}} \circ \cdots \circ \phi_{k_{1}, k_{2}} \circ \phi_{i, k_{1}} .
$$

(We note that if $i \geq j \geq k$ are arbitrary elements of $T$, then $\Phi_{i, k}=\Phi_{j, k} \circ \Phi_{i, j}$.)
Let $A=\underset{i \in T}{\cup} A_{i}^{0}$. Define a transition function $\delta^{\prime}: A \times X \mapsto A$ as follows. If $a \in A_{i}^{0}$ and $x \in X$ then let

$$
\delta^{\prime}(a, x)=\delta_{i^{\prime}[a, x]}\left(\Phi_{i, i^{\prime}[a, x]}(a), x\right),
$$

where $i^{\prime}[a, x]$ denotes the greatest element of the set $\left\{j \in T: \delta_{j}\left(\Phi_{i, j}(a), x\right) \in A_{j}^{0}\right\}$. It is clear that $\mathbf{A}=\left(A, X, \delta^{\prime}\right)$ is an automaton which will be denoted by $\left(A_{i}, X, \delta_{i} ; \phi_{i, j}, T\right)$.

Theorem 6 ([6]) A state-finite automaton without outputs is a semi connected retractable automaton containing a kernel if and only if it is isomorphic to an automaton $\left(A_{i}, X, \delta_{i} ; \phi_{i, j}, T\right)$ defined in the Construction.

Remark 1 By the proof of Theorem 7 of [6] if $\mathbf{R}$ is a subautomaton of an automaton $\left(A_{i}, X, \delta_{i} ; \phi_{i, j}, T\right)$ constructed as above, then there is an ideal $\Gamma \subseteq T$ such that $R=\cup_{j \in \Gamma} A_{j}^{0}$. As T is a tree

$$
\pi: i \mapsto \max \{\gamma \in \Gamma: \gamma \leq i\}
$$

is a well defined mapping of T onto $\Gamma$ which leaves the elements of $\Gamma$ fixed. $\lambda_{R}$ defined by $\lambda_{R}(a)=\Phi_{i, \pi(i)}(a)\left(a \in A_{i}^{0}\right)$ is a retract homomorphism of $\mathbf{A}$ onto R. ([6]) This fact will be used in the proof of the next Theorem.

Theorem 7 A state-finite automaton without outputs is a semi-connected Booleantype retractable automaton containing a kernel if and only if it is isomorphic to an automaton $\left(A_{i}, X, \delta_{i} ; \phi_{i, j}, T\right)$ defined in the Construction.

Proof Let $\mathbf{A}$ be a state-finite automaton without outputs which contains a kernel. Assume that $\mathbf{A}$ is also semi-connected and Boolean-type retractable. Then, by Theorem 6, $\mathbf{A}$ is isomorphic to an automaton $\mathbf{A}=\left(A_{i}, X, \delta_{i} ; \phi_{i, j}, T\right)$ which is defined in the Construction.

The main part of the proof is to show that every automaton $\mathbf{A}=\left(A_{i}, X, \delta_{i} ; \phi_{i, j}, T\right)$ constructed as above is Boolean-type retractable. According to Theorem 6 the automaton $\mathbf{A}=\left(A_{i}, X, \delta_{i} ; \phi_{i, j}, T\right)$ is retractable. Let $\mathbf{B}$ be a subautomaton of $\mathbf{A}$. By Remark 1 there is an ideal $\Gamma \subseteq T$ such that $B=\cup_{j \in \Gamma} A_{j}^{0}$. Let $\pi_{B}: i \mapsto\{\gamma \in \Gamma: \gamma \leq i\}$. For every $a \in A_{j}(j \in T)$ let $\lambda_{B}(a):=\Phi_{j, \pi(j)}(a)$. Using also Remark 1 , it is easy to see that $\lambda_{B}$ is a retract homomorphism of $\mathbf{A}$ onto $\mathbf{B}$. Let $B_{1}$ and $B_{2}$ be arbitrary subautomata with $B_{1} \subseteq B_{2}$. We will show that $\operatorname{Ker} \lambda_{B_{2}} \subseteq \operatorname{Ker} \lambda_{B_{1}}$. Assume $\lambda_{2}(a)=$ $\lambda_{2}(b)$ for some $a, b \in A$. According to Remark 1, $\lambda_{B_{1}}=\Phi_{\pi_{B_{2}}(j), \pi_{B_{1}}(j)} \circ \Phi_{j, \pi_{B_{2}}(j)}$. Thus

$$
\begin{aligned}
& \lambda_{B_{1}}(a)=\left(\Phi_{\pi_{B_{2}}(i), \pi_{B_{1}}(i)} \circ \Phi_{j, \pi_{B_{2}}(i)}\right)(a)=\left(\Phi_{\pi_{B_{2}}(i), \pi_{B_{1}}(i)} \circ \lambda_{B_{2}}\right)(a)= \\
& =\left(\Phi_{\pi_{B_{2}}(i), \pi_{B_{1}}(i)} \circ \lambda_{B_{2}}\right)(b)=\left(\Phi_{\pi_{B_{2}}(i), \pi_{B_{1}}(i)} \circ \Phi_{j, \pi_{B_{2}}(i)}\right)(b)=\lambda_{B_{1}}(b) .
\end{aligned}
$$

Consequently $\operatorname{Ker} \lambda_{B_{2}} \subseteq \operatorname{Ker} \lambda_{B_{1}}$. Hence $\mathbf{A}=\left(A_{i}, X, \delta_{i} ; \phi_{i, j}, T\right)$ is a Boolean-type retractable automaton with the family $\left\{\lambda_{B} \mid \mathbf{B}\right.$ is a subautomaton of $\left.\mathbf{A}\right\}$.

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