

CUBE TERM BLOCKERS WITHOUT FINITENESS

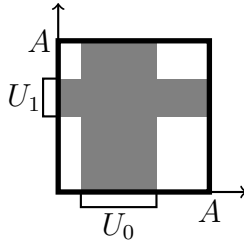
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ABSTRACT. We show that an idempotent variety has a d -dimensional cube term if and only if its free algebra on two generators has no d -ary compatible cross. We employ Hall's Marriage Theorem to show that a variety of finite signature whose fundamental operations have arities n_1, \dots, n_k has a d -dimensional cube term if and only if it has one of dimension $d = 1 + \sum_{i=1}^k (n_i - 1)$. This lower bound on dimension is shown to be sharp. We show that a pure cyclic term variety has a cube term if and only if it contains no 2-element semilattice. We prove that the Maltsev condition "existence of a cube term" is join prime in the lattice of idempotent Maltsev conditions.

1. INTRODUCTION

This note concerns a recently identified Maltsev condition, which promises to be significant. It is called "existence of a cube term".

We begin by discussing relations. A *binary cross* on a set A is a subset of A^2 of the form $(U_0 \times A) \cup (A \times U_1)$, where U_0 and U_1 are nonempty proper subsets of A .



If $U_0 = U_1$, the cross is called *symmetric*. If $|U_0| = |U_1| = 1$, the cross is called *thin*. The sequence (U_0, U_1) is called the *base (sequence)* for the cross. If the cross is symmetric, i.e., if the base has the form (U, U) , then we also refer to U as the base.

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The definition of a cross makes sense for higher arity relations, i.e. a d -ary cross is a subset of A^d of the form

$$(U_0 \times A \times \cdots \times A) \cup (A \times U_1 \times \cdots \times A) \cup \cdots \cup (A \times A \times \cdots \times U_{d-1})$$

where U_0, \dots, U_{d-1} are nonempty proper subsets of A . For $d = 1$ this means that a 1-ary cross is a nonempty proper subset of A . The definitions of symmetric cross, thin cross, and base for a d -ary cross are the expected ones. The arity of a cross is also called its *dimension*.

Our use in this paper of the notion of a cross follows the earlier use of crosses in the 1987 paper [15], which concerns the description of the maximal, locally closed subclones of the clone of all idempotent operations on a given set. In [15], symmetric and asymmetric *thin* crosses play a central role, although they are just called ‘crosses’. In the current paper, we need to consider arbitrary (‘thick’) crosses as well.

Now we turn to cube terms. Let \mathcal{V} be a variety and let $\mathbf{F} = \mathbf{F}_{\mathcal{V}}(x, y)$ be the \mathcal{V} -free algebra generated by the set $\{x, y\}$. Since \mathbf{F} has an automorphism that switches x and y , it follows that exactly one of the following two conditions holds: (i) the set $\{x, y\}^d - \{\mathbf{y}\}$ generates \mathbf{y} , where $\mathbf{y} = (y, y, \dots, y)$ is the constant tuple with range $\{y\}$, or (ii) different subsets of $\{x, y\}^d$ generate different subalgebras of \mathbf{F}^d . For condition (i) to hold, \mathcal{V} must have a term c which applied to elements of $\{x, y\}^d - \{\mathbf{y}\}$ yields \mathbf{y} , i.e.,

$$(1.1) \quad \mathcal{V} \models c(\mathbf{z}_1, \mathbf{z}_2, \dots) = \mathbf{y} \quad \text{with all } \mathbf{z}_i \text{ in } \{x, y\}^d - \{\mathbf{y}\}.$$

This is a vector identity. By considering this single vector identity coordinatewise, this means that \mathcal{V} satisfies d identities of the form

$$(1.2) \quad c(\dots, x, \dots) = y$$

where the only variables that appear in the identity are x and y , and for each place of c there is an identity that has x in that place. For example, if \mathcal{V} has a term c satisfying

$$c\left(\begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} x \\ x \end{bmatrix}, \begin{bmatrix} y \\ x \end{bmatrix}\right) = \begin{bmatrix} y \\ y \end{bmatrix}, \text{ or equivalently, both of } \begin{cases} c(x, x, y) = y \\ c(y, x, x) = y, \end{cases}$$

then c is a term of the desired type for $d = 2$, which is called a *Maltsev term*.

A term c satisfying the condition described in (1.1) is called a *d -dimensional cube term* or just *d -cube term* for \mathcal{V} . Equivalently, c is a *d -cube term* if d identities of the type in (1.2) suffice to establish the condition in (1.1) for each place of c . Clearly, a d -cube term for a variety \mathcal{V} is automatically a d' -cube term for all $d' \geq d$.

Cube terms were introduced in [4] as part of an investigation of finite algebras with few subalgebras of powers. Terms of equal strength, called *parallelogram terms*, were discovered independently and at the same time in the study of finitely related clones, [9]. Cube terms and their equivalents have played roles in [8] in the study of constraint satisfaction problems, in [1, 2, 9] in the study of finitely related clones,

in [10, 13] in natural duality theory, and in [5] concerning the subpower membership problem.

Theorem 2.1 of [12] gives a method for recognizing if a finite idempotent algebra has no cube term. Namely, a finite idempotent algebra fails to have a d -cube term for any d if and only if it has a *cube term blocker*. A cube term blocker of a finite idempotent algebra \mathbf{A} is defined to be a pair (U, B) of nonempty subuniverses of \mathbf{A} , with $U \subsetneq B$, such that U serves as a base for a compatible, symmetric, d -ary cross of \mathbf{B} for every d . It follows that a finite idempotent algebra \mathbf{A} fails to have a d -cube term for any d if and only if some subalgebra $\mathbf{B} \leq \mathbf{A}$ has compatible symmetric crosses of every arity.

The result of [12] does not help if one wants to show that \mathbf{A} has no d -cube term for a fixed d . The result also does not help if \mathbf{A} is infinite. But Lemma 2.8 of [11] shows that an idempotent variety \mathcal{V} fails to have a Maltsev term (i.e. a 2-cube term) if and only if the free \mathcal{V} -algebra on 2 generators has a compatible 2-ary cross. Here \mathcal{V} need not satisfy any finiteness hypothesis, but the cross involved turns out to be asymmetric, while the notion of a cube term blocker involves symmetric crosses only. Furthermore, Lemma 2.8 of [11] is a result about 2-cube terms only.

The current paper may be viewed as establishing a generalization of both Theorem 2.1 of [12] and Lemma 2.8 of [11]. We will prove that an idempotent variety \mathcal{V} fails to have a d -cube term if and only if the free \mathcal{V} -algebra on 2 generators, $\mathbf{F} = \mathbf{F}_{\mathcal{V}}(x, y)$, has a compatible d -ary cross. We will also show that an idempotent variety \mathcal{V} fails to have a d -cube term for every d if and only if \mathbf{F} has a nonempty proper subuniverse U that serves as a base for symmetric crosses of all arities, i.e. (U, F) is a cube term blocker of \mathbf{F} . The message to take away from this is that to avoid cube terms of a fixed dimension one should work with a not-necessarily-symmetric cross of that dimension, but to avoid cube terms of all dimensions it suffices to work with symmetric crosses or cube term blockers.

This note evolved in response to a question we learned from Cliff Bergman, which we state and answer in Section 5. Section 3 contains our proof that $\mathbf{F} = \mathbf{F}_{\mathcal{V}}(x, y)$ has compatible crosses of all arities if and only if \mathbf{F} has a subuniverse U such that (U, F) is a cube term blocker. In Section 4 we develop a tight lower bound on the minimal dimension of a cube term for idempotent varieties of finite signature. In Section 6 we use our results to establish that the Maltsev condition “existence of a cube term” is join prime in the lattice of idempotent Maltsev conditions.

2. CUBE TERMS AND CROSSES

A nonempty subset B of a product $A_0 \times \cdots \times A_{d-1}$ is a *block* if it is a product subset: $B = B_0 \times \cdots \times B_{d-1}$ with $B_i \subseteq A_i$ for all i . A block is *full* in the i -th coordinate if $B_i = A_i$. Thus, for any sequence (U_0, \dots, U_{d-1}) of nonempty proper

subsets of A , the cross

$$\begin{aligned}\mathbf{Cross}(U_0, \dots, U_{d-1}) &= (U_0 \times A \times \dots \times A) \cup \dots \cup (A \times \dots \times A \times U_{d-1}) \\ &= B_0 \cup \dots \cup B_{d-1}\end{aligned}$$

on A is defined to be a subset of A^d that is a union of d blocks B_0, \dots, B_{d-1} where B_i is full in all coordinates except the i -th.

If t is an operation on a set A and $U \subseteq A$, then t is U -absorbing in its i -th variable if

$$t(A, A, \dots, \underbrace{U}_i, \dots, A) \subseteq U.$$

An operation t on a set A is *idempotent* if $t(a, a, \dots, a) = a$ for all $a \in A$.

Lemma 2.1. *Let A be a set with nonempty proper subsets U, U_0, \dots, U_{d-1} , and let t be an n -ary idempotent operation on A .*

- (1) *If t is compatible with $\mathbf{Cross}(U_0, \dots, U_{d-1})$ and $\pi \in S_d$ is a permutation, then t is compatible with $\mathbf{Cross}(U_{\pi(0)}, \dots, U_{\pi(d-1)})$. If (i_0, \dots, i_{e-1}) is a subsequence of $(0, \dots, d-1)$, then t is also compatible with $\mathbf{Cross}(U_{i_0}, \dots, U_{i_{e-1}})$.*
- (2) *If t is compatible with $\mathbf{Cross}(U_0, \dots, U_{d-1})$, then each U_i is a subuniverse of (A, t) .*
- (3) *t is compatible with $\mathbf{Cross}(U_0, \dots, U_{d-1})$ if and only if*

$$(*) \text{ for every function}$$

$$m: \{0, \dots, n-1\} \rightarrow \{0, \dots, d-1\}$$

there is some $i \in \text{im}(m)$ such that

$$(2.1) \quad a_j \in U_i \text{ for all } j \in m^{-1}(i) \implies t(a_0, \dots, a_{n-1}) \in U_i.$$

- (4) *If t is compatible with $\mathbf{Cross}(U_0, \dots, U_{d-1})$, and $n \leq d$, then for all except possibly $n-1$ choices of $j < d$ it is the case that t is U_j -absorbing in one of its variables.*
- (5) *The following are equivalent for t :*
 - (i) *t is compatible with the d -ary symmetric cross $\mathbf{Cross}(U, \dots, U)$ for some $d \geq n$.*
 - (ii) *t is U -absorbing in one of its variables.*
 - (iii) *t is compatible with the d -ary symmetric cross $\mathbf{Cross}(U, \dots, U)$ for every $d \geq 1$.*

Proof. For the first statement in (1) observe that $\mathbf{Cross}(U_{\pi(0)}, \dots, U_{\pi(d-1)})$ differs from $\mathbf{Cross}(U_0, \dots, U_{d-1})$ by a permutation of coordinates. Therefore the desired conclusion follows from the fact that if we permute the coordinates of a subuniverse of $(A; t)^d$ we again get a subuniverse of $(A; t)^d$.

For the second statement in (1), choose $a_0 \notin U_0$. Since t is idempotent, the singleton $\{a_0\}$ is a subuniverse of $(A; t)$. Hence

$$\{a_0\} \times \text{Cross}(U_1, \dots, U_{d-1}) = \text{Cross}(U_0, \dots, U_{d-1}) \cap (\{a_0\} \times A^{d-1})$$

is a subuniverse of $(A; t)^d$. It follows that $\text{Cross}(U_1, \dots, U_{d-1})$ is a subuniverse of $(A; t)^{d-1}$. Similarly, $\text{Cross}(U_0, \dots, U_{i-1}, U_{i+1}, \dots, U_{d-1})$ is a subuniverse of $(A; t)^{d-1}$ for all $i < d$. Repeating this procedure for every i not occurring in (i_0, \dots, i_{e-1}) we get that $\text{Cross}(U_{i_0}, \dots, U_{i_{e-1}})$ is a subuniverse of $(A; t)^e$, that is, t is compatible with $\text{Cross}(U_{i_0}, \dots, U_{i_{e-1}})$.

Item (2) is the special case $e = 1$ of the second statement in (1).

For item (3), assume first that $(*)$ fails. Then there is a function

$$m: \{0, \dots, n-1\} \rightarrow \{0, \dots, d-1\}$$

such that for every $i \in \text{im}(m)$ the implication in (2.1) fails. Choose witnessing elements: i.e. choose, for each $i \in \text{im}(m) = \{i_0, \dots, i_{e-1}\}$, elements $a_{i,j} \in A$ ($j < n$) satisfying $a_{i,j} \in U_i$ for all $j \in m^{-1}(i)$ such that $t(a_{i,0}, \dots, a_{i,n-1}) \notin U_i$. The columns of the matrix $[a_{k,\ell}]$ lie in $\text{Cross}(U_{i_0}, \dots, U_{i_{e-1}})$, because every $j < n$ belongs to $m^{-1}(i)$ for some $i \in \text{im}(m) = \{i_0, \dots, i_{e-1}\}$, and hence by construction, the j -th column of $[a_{k,\ell}]$ has i -th entry $a_{i,j} \in U_i$. However, by construction, the column obtained by applying t to the rows of this matrix does not lie in $\text{Cross}(U_{i_0}, \dots, U_{i_{e-1}})$. Thus, t is not compatible with $\text{Cross}(U_{i_0}, \dots, U_{i_{e-1}})$, so it is not compatible with $\text{Cross}(U_0, \dots, U_{d-1})$ either, according to item (1).

Conversely, assume that t is not compatible with $\text{Cross}(U_0, \dots, U_{d-1})$. Then we can select elements from this relation and allow them to serve as columns for a $d \times n$ matrix $[a_{i,j}]$ where (i) for each column j , there is a row index i ($=: m(j)$) such that $a_{i,j} \in U_i$ and (ii) the value obtained by applying t to the i -th row fails to belong to U_i for every i . This yields a function m witnessing that $(*)$ fails.

For item (4), define a bipartite graph with one part $X = \{x_0, \dots, x_{n-1}\}$, other part equal to $\underline{d} := \{0, \dots, d-1\}$, and edge relation containing exactly those pairs (x_i, j) where t is not U_j -absorbing in variable x_i .

Claim 2.2. *There is no matching from X to \underline{d} . (A matching from X to \underline{d} is a subset of the edge set that is an injective function $X \rightarrow \underline{d}$.)*

Proof of Claim 2.2. Assume that there is a matching $\mu: X \rightarrow \underline{d}$. By item (1) we may reorder the U_j 's so that $\mu(x_i) = i$ is the matching. It follows that for every $i < n$, t is not U_i -absorbing in its i -th variable, so there must exist $u_{i,i} \in U_i$ and elements $a_{i,k} \in A$ such that

$$t(a_{i,0}, a_{i,1}, \dots, u_{i,i}, \dots, a_{i,n-1}) \notin U_i.$$

There also exist $a_j \notin U_j$ for every $j < d$. These ingredients yield a situation

$$(2.2) \quad t \left(\begin{bmatrix} u_{0,0} \\ a_{1,0} \\ \vdots \\ a_{n-1,0} \\ a_n \\ \vdots \\ a_{d-1} \end{bmatrix}, \begin{bmatrix} a_{0,1} \\ u_{1,1} \\ \vdots \\ a_{n-1,1} \\ a_n \\ \vdots \\ a_{d-1} \end{bmatrix}, \dots, \begin{bmatrix} a_{0,n-1} \\ a_{1,n-1} \\ \vdots \\ u_{n-1,n-1} \\ a_n \\ \vdots \\ a_{d-1} \end{bmatrix} \right) = \begin{bmatrix} \notin U_0 \\ \notin U_1 \\ \vdots \\ \notin U_{n-1} \\ a_n(\notin U_n) \\ \vdots \\ a_{d-1}(\notin U_{d-1}) \end{bmatrix}.$$

The operands are in $\mathbf{Cross}(U_0, \dots, U_{d-1})$, but the value is not, a contradiction. \diamond

By the Marriage Theorem, there is a subset $Y \subseteq X$ such that the set $K \subseteq \underline{d}$ of elements adjacent to elements of Y satisfies $|K| < |Y| \leq n$. Thus, $Y \neq \emptyset$ and the set K has size at most $n - 1$; moreover, if $j \in \underline{d} - K$ then no element of Y is adjacent to j . Hence t is U_j -absorbing in its variables in Y .

For item (5), the implication (iii) \Rightarrow (i) is a tautology, and the implication (i) \Rightarrow (ii) follows from the statement in (4) we just proved.

To establish the remaining implication (ii) \Rightarrow (iii) assume without loss of generality that t is U -absorbing in its first variable, and consider the d -ary symmetric cross $\mathbf{Cross}(U, \dots, U)$ for some $d \geq 1$. Let $[a_{i,j}]$ be a $d \times n$ matrix of element of A whose columns are in $\mathbf{Cross}(U, \dots, U)$. In particular, the first column lies in $\mathbf{Cross}(U, \dots, U)$, so $a_{i,0} \in U$ for some $i < d$. Since t is U -absorbing in its first variable, we get that $t(a_{i,0}, \dots, a_{i,n-1}) \in U$. Hence the column obtained by applying t to the rows of the matrix $[a_{i,j}]$ lies in $\mathbf{Cross}(U, \dots, U)$. This proves that t is compatible with $\mathbf{Cross}(U, \dots, U)$. \square

Corollary 2.3. *Let \mathbf{A} be an idempotent algebra. If \mathbf{A} has a d -cube term, then \mathbf{A} has no compatible cross of dimension d or larger.*

Proof. Let c be a d -cube term for \mathbf{A} , and assume \mathbf{A} has an e -dimensional compatible cross $\mathbf{Cross}(U_0, \dots, U_{e-1})$ with $d \leq e$. By Lemma 2.1 (4) there exists $j < d$ such that c is U_j -absorbing in one of its variables. This contradicts the cube identities (see (1.1) or (1.2)). \square

Our goal in this section is to characterize idempotent varieties which have no d -cube term (for a fixed $d \geq 2$) or have no cube term (of any dimension). In Theorem 2.4 below the first property will be characterized by the existence of a d -dimensional cross, while the second one will be characterized by the existence of a specific infinite system of crosses, which we call a ‘cross sequence’. A *cross sequence* for an idempotent algebra \mathbf{A} is an ω -sequence (U_0, U_1, \dots) of proper nonempty subsets of A such that $\mathbf{Cross}(U_{i_0}, \dots, U_{i_{k-1}})$ is a compatible relation of \mathbf{A} for every finite subsequence $(U_{i_0}, \dots, U_{i_{k-1}})$ of (U_0, U_1, \dots) . A cross sequence (U_0, U_1, \dots) is *proper* if

$\bigcap_{i < \omega} U_i \neq \emptyset$ and $\bigcup_{i < \omega} U_i \neq A$. It follows from Lemma 2.1(1) that any subsequence and any reordering of a (proper) cross sequence is again a (proper) cross sequence.

Theorem 2.4. *Let \mathcal{V} be an idempotent variety, let $\mathbf{F} = \mathbf{F}_{\mathcal{V}}(x, y)$ be the \mathcal{V} -free algebra over the free generating set $\{x, y\}$, and let d be a positive integer.*

- (1) *\mathcal{V} has no d -cube term iff \mathbf{F} has a compatible d -dimensional cross.*
- (2) *\mathcal{V} has no cube term of any dimension iff \mathbf{F} has a proper cross sequence iff \mathbf{F} has a cross sequence.*

We will prove the two statements of Theorem 2.4 simultaneously. In Theorem 2.5 a uniform formulation of these two statements is based on the observation that for any variety \mathcal{V} , the condition ‘ \mathcal{V} has no d -cube term’ for a fixed d is equivalent to

$$\mathcal{V} \text{ has no } e\text{-cube term for any } e < \delta$$

for $\delta = d + 1$, while the condition ‘ \mathcal{V} has no cube term of any dimension’ is equivalent to the same displayed condition for $\delta = \omega$.

For $0 < \delta \leq \omega$, let

$$\delta^- := \begin{cases} \delta - 1 & \text{if } \delta < \omega, \\ \omega & \text{if } \delta = \omega. \end{cases}$$

Theorem 2.5. *Let \mathcal{V} be an idempotent variety, let $\mathbf{F} = \mathbf{F}_{\mathcal{V}}(x, y)$ be the free \mathcal{V} -algebra generated by $\{x, y\}$, and let $2 \leq \delta \leq \omega$. The variety \mathcal{V} fails to have a d -cube for any $d < \delta$ if and only if there is a δ^- -sequence, $\sigma = (U_0, U_1, \dots) = (U_i)_{i < \delta^-}$, of subuniverses of \mathbf{F} such that*

- (1) *$x \in U_i$ and $y \notin U_i$ for every $i < \delta^-$, and*
- (2) *$\text{Cross}(U_{i_0}, \dots, U_{i_{k-1}})$ is a compatible relation of \mathbf{F} for every finite subsequence $(U_{i_0}, \dots, U_{i_{k-1}})$ of σ .*

Proof. The “if” assertion follows from Corollary 2.3: if \mathbf{F} has compatible crosses of every arity $< \delta$, then it cannot have a d -cube term for any $d < \delta$.

For the converse, assume that \mathcal{V} has no d -cube term for any $d < \delta$. This implies, in particular, that \mathcal{V} is nontrivial.

Recursively define the sequence $\sigma = (U_0, U_1, \dots) = (U_i)_{i < \delta^-}$ with $U_i \leq \mathbf{F}$, according to the following rule: for $i < \delta^-$, U_i is chosen to be a subuniverse maximal for the properties that

- (i) $x \in U_i$, and
- (ii) $\mathbf{y} \notin \langle \text{Cross}(U_0, U_1, \dots, U_i, \{x\}, \dots, \{x\}) \rangle_{\mathbf{F}^k}$ for any k with $i < k < \delta$.

It is possible to make these choices, for the following reasons. The fact that \mathcal{V} does not have a d -cube term for any $d < \delta$ means exactly that $\mathbf{y} \notin \langle \text{Cross}(\{x\}, \{x\}, \dots, \{x\}) \rangle_{\mathbf{F}^k}$ for every $k < \delta$. Thus the subuniverse $\{x\}$ satisfies all the properties required of U_0 , except that it need not be maximal among the subuniverses satisfying (i) and (ii) for $i = 0$. But the union of a chain of subuniverses of \mathbf{F} satisfying (i) and (ii) for

a given i again satisfies these conditions for that i , so $\{x\}$ can be extended to a maximal subuniverse U_0 satisfying (i) and (ii) for $i = 0$. Similarly, if $0 < i < \delta^-$ and U_0, \dots, U_{i-1} have been chosen, then condition (ii) for $i - 1$ guarantees that $\{x\}$ satisfies all the properties required of U_i except maximality. Extend $\{x\}$ to a maximal U_i , etc.

Observe that $y \notin U_i$ for any $i < \delta^-$, since otherwise $\{x, y\} \subseteq U_i$, leading to $F = U_i$, leading to $\text{Cross}(U_0, \dots, U_i) = F^{i+1}$, contradicting item (ii) above. Hence item (1) of the theorem statement holds for $\sigma = (U_0, U_1, \dots) = (U_i)_{i < \delta^-}$.

Our remaining task is to show that $\text{Cross}(U_{i_0}, \dots, U_{i_{k-1}})$ is a compatible relation of \mathbf{F} for every finite subsequence $(U_{i_0}, \dots, U_{i_{k-1}})$ of σ . Every finite subsequence of σ is a subsequence of an initial segment of σ , therefore, in view of Lemma 2.1 (1), it suffices to prove that $\text{Cross}(U_0, \dots, U_{d-1})$ is a compatible relation of \mathbf{F} for every $d < \delta$. We will do this simultaneously for every d with an indirect argument. Assume that there is some $d < \delta$ and some element

$$(2.3) \quad \mathbf{p} = (p_0, \dots, p_{d-1}) \in \langle \text{Cross}(U_0, \dots, U_{d-1}) \rangle_{\mathbf{F}^d} - \text{Cross}(U_0, \dots, U_{d-1}).$$

Necessarily $p_i \notin U_i$ for any $i < d$. There is no harm in assuming that, among all possible choices of d and \mathbf{p} , we have chosen those such that $p_i \neq y$ holds in the fewest number of coordinates. That is, we assume that (2.3) holds and also that for no $e < \delta$ do we have $\mathbf{q} \in \langle \text{Cross}(U_0, \dots, U_{e-1}) \rangle_{\mathbf{F}^e} - \text{Cross}(U_0, \dots, U_{e-1})$ with \mathbf{q} differing from \mathbf{p} in strictly fewer coordinates than \mathbf{p} .

Since $\mathbf{p} \in \langle \text{Cross}(U_0, \dots, U_{d-1}) \rangle_{\mathbf{F}^d}$, there exists a term s and there exist elements $\mathbf{b}_0, \dots, \mathbf{b}_{m-1} \in \text{Cross}(U_0, \dots, U_{d-1})$ such that $s(\mathbf{b}_0, \dots, \mathbf{b}_{m-1}) = \mathbf{p}$. Observe that one may lengthen all tuples involved by adding some number of y 's (say g with $d + g < \delta$) to the end in order to obtain

$$(2.4) \quad s \left(\begin{bmatrix} \mathbf{b}_0 \\ y \\ \vdots \\ y \end{bmatrix}, \dots, \begin{bmatrix} \mathbf{b}_{m-1} \\ y \\ \vdots \\ y \end{bmatrix} \right) = \begin{bmatrix} \mathbf{p} \\ y \\ \vdots \\ y \end{bmatrix}.$$

The columns appearing as arguments to s in (2.4) belong to

$$\text{Cross}(U_0, \dots, U_{d-1}, V_d, \dots, V_{d+g-1})$$

for any choice of $(\emptyset \neq) V_i \subset F$.

There must exist some coordinate of \mathbf{p} that is not y , else condition (ii) from the definition of σ is violated when $k = d$. Let i be the first coordinate of \mathbf{p} where $p_i \neq y$; hence $i < d$. Since $p_i \notin U_i$, the subuniverse $U'_i = \langle U_i \cup \{p_i\} \rangle_{\mathbf{F}}$ properly extends U_i . By the maximality of U_i , there must exist some k with $i < k < \delta$ such that $y \in \langle \text{Cross}(U_0, \dots, U'_i, \{x\}, \dots, \{x\}) \rangle_{\mathbf{F}^k}$.

To understand how \mathbf{y} could be generated, observe that since

$$\begin{aligned} \mathbf{Cross}(U_0, \dots, U'_i, \{x\}, \dots, \{x\}) \\ = B_0(U_0) \cup \dots \cup B_{i-1}(U_{i-1}) \cup B_i(U'_i) \cup B_{i+1}(\{x\}) \cup \dots \cup B_{k-1}(\{x\}), \end{aligned}$$

we get that $\langle \mathbf{Cross}(U_0, \dots, U'_i, \{x\}, \dots, \{x\}) \rangle_{\mathbf{F}^k}$ equals the subalgebra join

$$B_0(U_0) \vee \dots \vee B_{i-1}(U_{i-1}) \vee B_i(U'_i) \vee B_{i+1}(\{x\}) \vee \dots \vee B_{k-1}(\{x\}).$$

However $B_i(U'_i) = F^i \times U'_i \times F^{k-i-1}$ is generated by $\{x, y\}^i \times (U_i \cup \{p_i\}) \times \{x, y\}^{k-i-1}$, and all elements of this product set already belong to $\mathbf{Cross}(U_0, \dots, U_i, \{x\}, \dots, \{x\})$ except the tuple $(y, \dots, y, p_i, y, \dots, y)$. Hence $\langle \mathbf{Cross}(U_0, \dots, U'_i, \{x\}, \dots, \{x\}) \rangle_{\mathbf{F}^k}$ is generated by

$$\mathbf{Cross}(U_0, \dots, U_i, \{x\}, \dots, \{x\}) \cup \{(y, \dots, y, p_i, y, \dots, y)\}.$$

Since \mathbf{y} is generated by this set, there is a term t and elements (columns) $\mathbf{c}_i \in \mathbf{Cross}(U_0, \dots, U_i, \{x\}, \dots, \{x\}) \subseteq F^k$ such that

$$(2.5) \quad t \left(\mathbf{c}_0, \dots, \mathbf{c}_{n-1}, \begin{bmatrix} y \\ \vdots \\ y \\ p_i \\ y \\ \vdots \\ y \end{bmatrix} \right) = \begin{bmatrix} y \\ \vdots \\ y \\ y \\ y \\ \vdots \\ y \end{bmatrix} = \mathbf{y}.$$

Lengthen these tuples, by adding y 's at the end, to some length e satisfying $\max\{k, d\} \leq e < \delta$ (hence the columns have length at least d). Equation (2.5) still holds. Write the extension of \mathbf{c}_i as \mathbf{c}'_i , and note that

$$\mathbf{c}'_i \in \mathbf{Cross}(U_0, \dots, U_i, \{x\}, \dots, \{x\}) (\subseteq F^e)$$

where there may be more $\{x\}$'s than before.

Let $\varepsilon = (\varepsilon_0, \dots, \varepsilon_{e-1})$ be the endomorphism of \mathbf{F}^e defined coordinatewise as follows:

- (a) $\varepsilon_j: \mathbf{F} \rightarrow \mathbf{F}$ is the identity if $0 \leq j \leq i$ ($\leq d$) or $d \leq j < e$, and
- (b) $\varepsilon_j: \mathbf{F} \rightarrow \mathbf{F}: x \mapsto x, y \mapsto p_j$ if $i < j < d$.

Observe that ε maps $\mathbf{Cross}(U_0, \dots, U_i, \{x\}, \dots, \{x\})$ into itself. Indeed, $\varepsilon(B_j(U_j)) \subseteq B_j(U_j)$ for $j \leq i$ because $\varepsilon_j = id$, while $\varepsilon(B_j(\{x\})) \subseteq B_j(\{x\})$ for $j > i$ because ε_j fixes x .

Applying ε to (2.5) yields

$$(2.6) \quad t \left(\varepsilon(\mathbf{c}'_0), \dots, \varepsilon(\mathbf{c}'_{n-1}), \begin{bmatrix} y \\ \vdots \\ y \\ p_i \\ p_{i+1} \\ \vdots \\ p_{d-1} \\ y \\ \vdots \\ y \end{bmatrix} \right) = \begin{bmatrix} y \\ \vdots \\ y \\ y \\ p_{i+1} \\ \vdots \\ p_{d-1} \\ y \\ \vdots \\ y \end{bmatrix} =: \mathbf{q}.$$

By the choice of i , the last argument of t in (2.6) is the column $(\mathbf{p}, y, \dots, y)$. Therefore, the expression for \mathbf{q} in (2.6) can be rewritten as

$$(2.7) \quad t \left(\varepsilon(\mathbf{c}'_0), \dots, \varepsilon(\mathbf{c}'_{n-1}), s \left(\begin{bmatrix} \mathbf{b}_0 \\ y \\ \vdots \\ y \end{bmatrix}, \dots, \begin{bmatrix} \mathbf{b}_{m-1} \\ y \\ \vdots \\ y \end{bmatrix} \right) \right) = \mathbf{q}$$

using (2.4). The columns, $\varepsilon(\mathbf{c}'_u)$ and $(\mathbf{b}_v, y, \dots, y)$ all belong to $\mathbf{Cross}(U_0, \dots, U_{e-1})$, but $q_j \notin U_j$ for any j . Hence (2.7) asserts that

$$\mathbf{q} \in \langle \mathbf{Cross}(U_0, \dots, U_{e-1}) \rangle_{\mathbf{F}^e} - \mathbf{Cross}(U_0, \dots, U_{e-1})$$

with \mathbf{q} differing from the constant y -tuple in strictly fewer coordinates than \mathbf{p} . This is so because \mathbf{q} differs from \mathbf{p} only in that it may have more y 's at the end and $q_i = y$ while $p_i \neq y$. This conclusion contradicts the choice of \mathbf{p} . This proves that $\mathbf{Cross}(U_0, \dots, U_{d-1})$ is a compatible relation of \mathbf{F} for every $d < \delta$. \square

Proof of Theorem 2.4. For item (1), we assume first that \mathcal{V} has no d -cube term. Hence, \mathcal{V} has no e -cube term for any $e < d + 1$. It follows from Theorem 2.5 (for $\delta = d + 1$) that there is a sequence (U_0, \dots, U_{d-1}) of subuniverses of \mathbf{F} such that $x \in \bigcap_{i < d} U_i$, $y \notin \bigcup_{i < d} U_i$, and $\mathbf{Cross}(U_0, \dots, U_{d-1})$ is a compatible relation of \mathbf{F} .

Conversely, if \mathbf{F} has a compatible d -dimensional cross, then, by Corollary 2.3, \mathbf{F} has no d -cube term. Hence \mathcal{V} has no d -cube term.

For item (2), let us assume first that \mathcal{V} has no cube term. Applying Theorem 2.5 (for $\delta = \omega$) we conclude that there is an ω -sequence, $\sigma = (U_0, U_1, \dots)$, of subuniverses of \mathbf{F} such that σ is a proper cross sequence for \mathbf{F} satisfying $x \in \bigcap_{i < \omega} U_i$ and $y \notin \bigcup_{i < \omega} U_i$.

If \mathbf{F} has a proper cross sequence, then \mathbf{F} has a cross sequence. Finally, if \mathbf{F} has a cross sequence, then \mathbf{F} has compatible crosses of arbitrarily high dimensions.

Therefore, by Corollary 2.3, \mathbf{F} has no cube term of any dimension. Hence \mathcal{V} has no cube term of any dimension either, completing the proof. \square

3. PRODUCING SYMMETRIC CROSSES

We have shown in Theorem 2.4 (2) that an idempotent variety \mathcal{V} fails to have a cube term if and only if its 2-generated free algebra \mathbf{F} has a proper cross sequence. In this section we will use a combinatorial argument to show that this cross sequence can be converted to a constant cross sequence (U, U, U, \dots) . This produces a nonempty proper subuniverse U of \mathbf{F} that is the base of a compatible *symmetric* d -ary cross for every d . In fact, our construction of ‘symmetrizing’ a proper cross sequence works for any idempotent algebra, as the theorem below shows.

Theorem 3.1. *The following are equivalent for an idempotent algebra \mathbf{A} .*

- (1) \mathbf{A} has a proper cross sequence.
- (2) \mathbf{A} has a nonempty proper subuniverse U such that (U, A) is a cube term blocker for \mathbf{A} . (That is, U is a base for compatible symmetric d -ary crosses of \mathbf{A} for all d .)

Proof. The implication (2) \Rightarrow (1) is clear from the definitions: If (U, A) is a cube term blocker of \mathbf{A} , then the d -ary cross $\mathbf{Cross}(U, \dots, U)$ is a compatible relation of \mathbf{A} for every d , so the constant ω -sequence (U, U, \dots) is a proper cross sequence for \mathbf{A} .

To prove the reverse implication (1) \Rightarrow (2), assume that $\sigma = (U_0, U_1, \dots)$ is a proper cross sequence for \mathbf{A} . We shall prove that if $U := \bigcup_{i=0}^{\infty} \left(\bigcap_{j=i}^{\infty} U_j \right)$, then (U, F) is a cube term blocker of \mathbf{A} . Note that

$$\bigcap_{j=0}^{\infty} U_j \subseteq \bigcap_{j=1}^{\infty} U_j \subseteq \dots \subseteq \bigcap_{j=i}^{\infty} U_j \subseteq \bigcap_{j=i+1}^{\infty} U_j \subseteq \dots \subseteq \bigcup_{j=0}^{\infty} U_j,$$

so $U := \bigcup_{i=0}^{\infty} \left(\bigcap_{j=i}^{\infty} U_j \right)$ is the union of an increasing ω -sequence of subuniverses of \mathbf{A} . It follows that U is a subuniverse of \mathbf{A} . Moreover, since the cross sequence $\sigma = (U_0, U_1, \dots)$ is proper, we have $\bigcap_{j=0}^{\infty} U_j \neq \emptyset$ and $\bigcup_{j=0}^{\infty} U_j \neq A$. This implies that U is a nonempty proper subuniverse of \mathbf{A} .

The nontrivial part of the proof, therefore, is the argument that U is a base for symmetric crosses of all arities. To see that this is so, choose an arbitrary term operation $t(x_0, \dots, x_{n-1})$ of \mathbf{A} and as in the proof of Lemma 2.1 (4), use it to define a bipartite graph as follows: the vertices of the two parts are $X = \{x_0, \dots, x_{n-1}\}$, the set of variables of t , and ω , the set indexing the terms of the cross sequence $\sigma = (U_0, U_1, \dots)$; the edges of the graph are the pairs (x_i, j) such that t is not U_j -absorbing in variable x_i .

Claim 3.2. *There is no matching from X to ω .*

Proof of Claim 3.2. Suppose that there is a matching $\mu: X \rightarrow \omega$. Since a reordering of finitely many terms of σ produces a new cross sequence for \mathbf{A} and leaves the set U unchanged, we may assume without loss of generality that $\mu(x_i) = i$ for all $i < n$. This means that for every $i < n$ the term t is not U_i -absorbing in variable x_i , so for every $i < n$ there must exist $u_{i,i} \in U_i$ and elements $a_{i,k} \in A$ such that

$$t(a_{i,0}, a_{i,1}, \dots, u_{i,i}, \dots, a_{i,n-1}) \notin U_i.$$

There also exist $a_j \notin U_j$ for every $j < d$. Now the same calculation (2.2) as in the proof of Lemma 2.1 (4) yields that t is not compatible with $\text{Cross}(U_0, \dots, U_{d-1})$. This contradicts our assumption that (U_0, U_1, \dots) is a cross sequence for \mathbf{A} , and hence proves the claim. \diamond

Since X is finite, the Marriage Theorem holds in this situation. It asserts that, since there is no matching from X to ω , there is a subset $Y \subseteq X$ such that the set $K \subseteq \omega$ of elements adjacent to elements of Y satisfies $|K| < |Y| \leq |X| = n$. Therefore $Y \neq \emptyset$, the set K has size at most $|Y| - 1 \leq n - 1$, and if $x_j \in Y$ then for all $\ell \in \omega - K$ we have that t is U_ℓ -absorbing in variable x_j .

It follows that if $x_j \in Y$ then t is $(\bigcap_{j=i}^\infty U_j)$ -absorbing in variable x_j for all but finitely many i , and hence that t is $(\bigcup_{i=0}^\infty (\bigcap_{j=i}^\infty U_j))$ -absorbing in variable x_j . This proves that every term operation t of \mathbf{A} has a variable x_j in which t is U -absorbing. By Lemma 2.1 (5), this is equivalent to the statement that all term operations of \mathbf{A} are compatible with all symmetric crosses with base U . \square

In Theorem 2.4 (2) we characterized the idempotent varieties that fail to have cube terms. Using Theorem 3.1 we can strengthen this characterization as follows.

Theorem 3.3. *Let \mathcal{V} be an idempotent variety, and let $\mathbf{F} = \mathbf{F}_{\mathcal{V}}(x, y)$ be the \mathcal{V} -free algebra over the free generating set $\{x, y\}$. The following conditions are equivalent.*

- (1) \mathcal{V} has no cube term.
- (2) \mathbf{F} has a nonempty proper subuniverse U such that (U, F) is a cube term blocker for \mathbf{F} . (That is, U is a base for compatible symmetric d -ary crosses of \mathbf{F} for all d .)

Proof. Combine Theorem 3.1 for $\mathbf{A} = \mathbf{F}$ with Theorem 2.4. \square

As we mentioned in the introduction, cube term blockers were introduced in [12] to prove that a finite idempotent algebra \mathbf{A} fails to have a d -cube term for any d if and only if \mathbf{A} has a cube term blocker (U, B) , or equivalently, some subalgebra \mathbf{B} of \mathbf{A} has compatible symmetric crosses with base U of every arity.

Next we show how to derive this theorem of [12] from the results of our paper.

Lemma 3.4. *In any given signature, the class of idempotent algebras with no cube term blockers is closed under the formation of homomorphic images, subalgebras and finite products.*

Proof. We prove that the nonexistence of cube term blockers is preserved under homomorphic images, subalgebras, and finite products by arguing the contrapositive.

If $\varphi: \mathbf{A} \rightarrow \mathbf{C}$ is a surjective homomorphism and (U, B) is a cube term blocker of \mathbf{C} , then it is easy to see that $(\varphi^{-1}(U), \varphi^{-1}(B))$ is a cube term blocker of \mathbf{A} . This can be done by checking that the inverse image, under φ , of each compatible symmetric cross $\text{Cross}(U, \dots, U)$ of \mathbf{B} is a compatible relation of the subalgebra $\varphi^{-1}(\mathbf{B})$ of \mathbf{A} , and is equal to the symmetric cross $\text{Cross}(\varphi^{-1}(U), \dots, \varphi^{-1}(U))$.

If $\mathbf{C} \leq \mathbf{A}$ and (U, B) is a cube term blocker of \mathbf{C} , then it is also a cube term blocker of \mathbf{A} .

Before turning to products, first note that if (U, B) is a cube term blocker of \mathbf{A} , and V is a subuniverse of \mathbf{A} , then $(U \cap V, B \cap V)$ is a cube term blocker for \mathbf{A} unless $U \cap V = \emptyset$ or $U \cap V = B \cap V$.

Now, to prove the statement for products, we assume that (U, B) is a cube term blocker of $\mathbf{A} \times \mathbf{C}$, and explain how to find a cube term blocker for either \mathbf{A} or \mathbf{C} . Choose $(a, c) \in B - U$, and let $\pi_{\mathbf{A}}, \pi_{\mathbf{C}}$ be the coordinate projections of $\mathbf{A} \times \mathbf{C}$. If $\pi_{\mathbf{A}}(U) \neq \pi_{\mathbf{A}}(B)$, then $(\pi_{\mathbf{A}}(U), \pi_{\mathbf{A}}(B))$ is a cube blocker of \mathbf{A} , and we are done. Otherwise $\pi_{\mathbf{A}}(U) = \pi_{\mathbf{A}}(B)$, hence $a \in \pi_{\mathbf{A}}(B) = \pi_{\mathbf{A}}(U)$, implying the existence of an element $(a, d) \in U$. Now we let $V = \{a\} \times C$ and apply the observation of the previous paragraph: Since $\emptyset \neq U \cap V \neq B \cap V$, the pair $(U \cap V, B \cap V)$ is a cube term blocker for $\mathbf{A} \times \mathbf{C}$. We further have that

$$\emptyset \neq \pi_{\mathbf{C}}(U \cap V) \neq \pi_{\mathbf{C}}(B \cap V),$$

since $d \in \pi_{\mathbf{C}}(U \cap V)$ and $c \in \pi_{\mathbf{C}}(B \cap V) - \pi_{\mathbf{C}}(U \cap V)$. Hence $(\pi_{\mathbf{C}}(U \cap V), \pi_{\mathbf{C}}(B \cap V))$ is a cube term blocker for \mathbf{C} . \square

Corollary 3.5. *If \mathbf{A} is a finite idempotent algebra, then the following are equivalent:*

- (1) \mathbf{A} has no cube term.
- (2) $\mathcal{V}(\mathbf{A})$ has no cube term.
- (3) $\mathbf{F}_{\mathcal{V}(\mathbf{A})}(x, y)$ has a cube term blocker.
- (4) \mathbf{A} has a cube term blocker.

Proof. (1) \Rightarrow (2) is clear, because a cube term for $\mathcal{V}(\mathbf{A})$ would be a cube term for \mathbf{A} . The implication (2) \Rightarrow (3) follows from Theorem 3.3. For (3) \Rightarrow (4) we prove the contrapositive. If \mathbf{A} has no cube term blocker, then by Lemma 3.4 and by the fact that $\mathbf{F}_{\mathcal{V}(\mathbf{A})}(x, y)$ lies in $\mathcal{HSP}_{\text{fin}}(\mathbf{A})$ when \mathbf{A} is finite, we get that $\mathbf{F}_{\mathcal{V}(\mathbf{A})}(x, y)$ has no cube term blocker. Finally, (4) \Rightarrow (1) follows from Corollary 2.3. \square

4. THE INFLUENCE OF FINITE SIGNATURE

By a *signature* we mean a pair $\tau = (\mathcal{O}, \text{arity})$ where \mathcal{O} is a set of operation symbols and $\text{arity}: \mathcal{O} \rightarrow \omega$ is a function assigning arity. We consider only idempotent varieties, so we may and do consider only signatures where $\text{arity}(f) \geq 2$ for all $f \in \mathcal{O}$. We will call such signatures “suitable for idempotent varieties”.

In this section we consider only finite signatures, which are those where $|\mathcal{O}| < \omega$. For such a signature τ we define

$$|\tau| = \max_{f \in \mathcal{O}} (\text{arity}(f)) \quad \text{and} \quad \|\tau\| = 1 + \sum_{f \in \mathcal{O}} (\text{arity}(f) - 1).$$

It is easy to see that $|\tau| = \|\tau\|$ if there is only one operation in the signature, while $|\tau| < \|\tau\|$ if there is more than one.

Now let us consider an idempotent variety \mathcal{V} of finite signature τ , and let $\mathbf{F} = \mathbf{F}_{\mathcal{V}}(x, y)$ be the \mathcal{V} -free algebra generated by $\{x, y\}$. The main results of this section are the following:

- If \mathbf{F} has a compatible cross of arity $\|\tau\|$ or more, then it has compatible crosses of all arities. Equivalently, if \mathcal{V} has a cube term, then it has a $\|\tau\|$ -cube term (Theorem 4.1).
- For any suitable finite signature τ there exists an example for \mathcal{V} where \mathbf{F} has a compatible cross of arity $\|\tau\| - 1$, but no compatible crosses of higher arity. Equivalently, for any suitable finite signature τ there exists an example for \mathcal{V} such that \mathcal{V} has a $\|\tau\|$ -cube term, but has no d -cube term for $d < \|\tau\|$ (Example 4.4).

For symmetric crosses the corresponding statements are as follows:

- If \mathbf{F} has a compatible symmetric cross of arity $|\tau|$ or more, then it has compatible symmetric crosses of all arities (Corollary 4.3).
- For any suitable finite signature there exists an example where \mathbf{F} has a compatible symmetric cross of arity $|\tau| - 1$, but no compatible symmetric crosses of higher arity (Example 4.8).

By adding operations to a signature one can make $\|\tau\|$ large while $|\tau|$ remains small. Thus one can create varieties with cube terms where the least dimension of a cube term is much greater than the arities of the symmetric crosses of \mathbf{F} . These results show that we can't use only symmetric crosses to characterize the existence or nonexistence of cube terms of a fixed dimension.

Theorem 4.1. *Let \mathcal{V} be an idempotent variety of finite signature τ . If \mathcal{V} has no $\|\tau\|$ -cube term, then it has no cube term at all.*

In particular, if the signature of \mathcal{V} consists of a single binary operation symbol, then either \mathcal{V} has a Maltsev term or it has no cube term at all.

Proof. In the second statement, our assumption on the signature τ forces that $\|\tau\| = 2$. Hence the claim is an easy consequence of the first statement of the theorem and the fact that a variety has a Maltsev term if and only if it has a 2-cube term.

To prove the first statement, assume that \mathcal{V} is an idempotent variety of finite signature τ , and let $\mathbf{F} = \mathbf{F}_{\mathcal{V}}(x, y)$. We know from Theorem 2.4 (1) that for every $d \geq 1$,

$$\mathcal{V} \text{ has no } d\text{-cube term} \quad \Leftrightarrow \quad \mathbf{F} \text{ has a compatible } d\text{-ary cross.}$$

We also know from Theorem 3.3, from the definition of a cube term blocker, and from Corollary 2.3 that

$$\begin{array}{ccc}
 \mathcal{V} \text{ has no cube term} & \xLeftrightarrow{\text{Thm 3.3}} & \mathbf{F} \text{ has a cube term blocker } (U, F) \text{ where} \\
 & & U \text{ is a nonempty proper subuniverse of } \mathbf{F} \\
 \text{Cor 2.3 } \Uparrow & & \Downarrow \text{ def} \\
 \mathbf{F} \text{ has compatible crosses} & \Leftarrow & U \text{ is the base for compatible symmetric} \\
 \text{of all arities} & & \text{crosses of } \mathbf{F} \text{ of all arities;}
 \end{array}$$

hence all four conditions displayed here are equivalent. Therefore it suffices to prove the first statement of Theorem 4.1 in the following equivalent formulation.

Claim 4.2. *Let \mathcal{V} be an idempotent variety of finite signature τ , and let $\mathbf{F} = \mathbf{F}_{\mathcal{V}}(x, y)$ be the \mathcal{V} -free algebra over the free generating set $\{x, y\}$. If \mathbf{F} has a compatible cross of arity $\geq \|\tau\|$, then \mathbf{F} has a nonempty proper subuniverse U such that (U, F) is a cube term blocker for \mathbf{F} . (That is, U is a base for compatible symmetric crosses of \mathbf{F} of all arities.)*

Proof of Claim 4.2. Assume that \mathbf{F} has a compatible d -ary cross $\text{Cross}(U_0, \dots, U_{d-1})$ where $d \geq \|\tau\|$. Let f_0, \dots, f_{k-1} be the operation symbols of τ , and let $\text{arity}(f_i) = n_i$ ($i < k$). So, $d \geq \|\tau\| = 1 + \sum_{i=0}^{k-1} (n_i - 1)$. Applying Lemma 2.1 (4) to the basic operations f_0, \dots, f_{k-1} of \mathbf{F} we see that for each $i < k$ there exists a subset K_i of $\underline{d} := \{0, \dots, d-1\}$ such that $|K_i| \leq n_i - 1$ and f_i has a U_j -absorbing variable for every $j \in \underline{d} - K_i$. Since $|\underline{d} - \bigcup_{i=0}^{k-1} K_i| \geq d - \sum_{i=0}^{k-1} (n_i - 1) \geq 1$, we obtain that there exists at least one $j \in \underline{d}$ such that every f_i ($i < k$) has a U_j -absorbing variable. It follows from Lemma 2.1 (5) that every f_i ($i < k$) is compatible with the symmetric crosses $\text{Cross}(U_j, \dots, U_j)$ of all arities. Hence, the symmetric crosses $\text{Cross}(U_j, \dots, U_j)$ of all arities are compatible relations of \mathbf{F} . Equivalently, (U_j, F) is a cube term blocker for \mathbf{F} . \diamond

This finishes the proof of Theorem 4.1. \square

The analog of Claim 4.2 for symmetric crosses can be proved similarly.

Corollary 4.3. *Let \mathcal{V} be an idempotent variety of finite signature τ , and let $\mathbf{F} = \mathbf{F}_{\mathcal{V}}(x, y)$ be the \mathcal{V} -free algebra with free generators x, y . If \mathbf{F} has a compatible symmetric cross $\text{Cross}(U, \dots, U)$ of arity $\geq |\tau|$, then (U, F) is a cube term blocker for \mathbf{F} . (That is, U is a base for compatible symmetric crosses of \mathbf{F} of all arities.)*

Proof. Assume that \mathbf{F} has a compatible symmetric d -ary cross $\text{Cross}(U, \dots, U)$ where $d \geq |\tau|$. As before, let f_0, \dots, f_{k-1} be the operation symbols of τ , and let $\text{arity}(f_i) = n_i$ ($i < k$). So, $d \geq n_i$ for all i . It follows from Lemma 2.1 (4) that every basic operation f_0, \dots, f_{k-1} of \mathbf{F} has a U -absorbing variable. Hence, by Lemma 2.1 (5), f_0, \dots, f_{k-1} are compatible with the symmetric crosses $\text{Cross}(U, \dots, U)$ of all arities. Thus, the symmetric crosses $\text{Cross}(U, \dots, U)$ of all arities are compatible relations of \mathbf{F} , or equivalently, (U, F) is a cube term blocker for \mathbf{F} . \square

Example 4.4. Our goal in this example is to show that the bound in Theorem 4.1 is sharp. That theorem shows that if an idempotent variety of finite signature τ has a d -cube term for some d , then it has one for $d = \|\tau\|$. In this example we construct, for any suitable finite signature τ , an idempotent variety that has a $\|\tau\|$ -cube term, but no d -cube term for $d < \|\tau\|$.

If one revisits the definition of “ d -cube term”, one sees that the concept of a 1-cube term is degenerate: the only varieties with 1-cube terms are varieties of 1-element algebras, and for these varieties any term without nullary symbols is a 1-cube term. As noted earlier, a variety has a 2-cube term if and only if it has a Maltsev term. Thus the simplest nondegenerate example to be exhibited is that of a nontrivial variety with a Maltsev term in a signature τ satisfying $\|\tau\| = 2$. If τ is suitable for idempotent varieties, then $\|\tau\| = 2$ implies exactly that τ is a signature with one operation, which is binary. For this signature, take as an example the variety generated by $\langle \mathbb{Z}_3; f(x, y) \rangle$ where $f(x, y) = 2x + 2y$. This variety is nontrivial, has $\|\tau\| = 2$, and has a Maltsev term $m(x, y, z) := f(f(x, z), y) = x + 2y + z$.

The cases where $\|\tau\| > 2$ will be handled by a uniform construction. Suppose that τ has m operation symbols. Set $n_i = \text{arity}(f_i)$, $1 \leq i \leq m$, and set $n := \|\tau\| - 1 = \sum_{i=1}^m (n_i - 1)$. If

$$\begin{aligned} C_1 &= \{1, 2, \dots, (n_1 - 1)\}, \\ C_2 &= \{(n_1 - 1) + 1, (n_1 - 1) + 2, \dots, (n_1 - 1) + (n_2 - 1)\}, \\ &\vdots \\ C_j &= \left\{ \left(\sum_{i=1}^{j-1} (n_i - 1) \right) + 1, \dots, \left(\sum_{i=1}^j (n_i - 1) \right) \right\}, \\ &\vdots \\ C_m &= \left\{ \left(\sum_{i=1}^{m-1} (n_i - 1) \right) + 1, \dots, n \right\}, \end{aligned}$$

then $\{C_1, \dots, C_m\}$ is a partition of $[n] := \{1, \dots, n\}$ whose cells that are in 1–1 correspondence with the operation symbols: $C_i \leftrightarrow f_i$. Moreover, $|C_i| = \text{arity}(f_i) - 1$. The elements of $[n]$ are going to be coordinates in a product algebra. To describe the construction of the algebra we will use the terminology that an element $j \in [n]$ *belongs to* f_i , (or f_i *belongs to* j) if $j \in C_i$.

The universe of the product algebra will be $A = \{0, 1\}^n$. We will explain how to interpret each symbol f_i on this set, by describing its behavior in each coordinate. We need some terminology to do this. Let \vee (join) and \wedge (meet) be the lattice operations on $\{0, 1\}$ for the order $0 < 1$. In any given coordinate we will interpret f_i as either:

(1) the n_i -ary join on $\{0, 1\}$:

$$f_i(x_1, \dots, x_{n_i}) = \bigvee_{1 \leq j \leq n_i} x_j,$$

or

(2) the “canonical” n_i -ary near-unanimity operation on $\{0, 1\}$, namely

$$f_i(x_1, \dots, x_{n_i}) = \bigvee_{1 \leq j < k \leq n_i} (x_j \wedge x_k).$$

(This operation is a near-unanimity operation only when $n_i > 2$, but we shall use the terminology even when $n_i = 2$. In this situation $f_i(x_1, x_2) = x_1 \wedge x_2$.)

We interpret f_i on $A = \{0, 1\}^n$ by stipulating that it acts coordinatewise, and that it acts like the canonical n_i -ary near-unanimity operation in the coordinates that belong to f_i and like the n_i -ary join operation in the coordinates that do not belong to f_i .

The set A equipped with the operations f_1, \dots, f_m just defined is the algebra we call \mathbf{A} . Each f_i is idempotent on \mathbf{A} , since join, meet and near-unanimity are idempotent. Therefore, the set $U_j \subseteq A = \{0, 1\}^n$ consisting of all n -tuples with 1 in the j -th coordinate is a nonempty proper subuniverse of \mathbf{A} for each j between 1 and n .

Claim 4.5. $\text{Cross}(U_1, \dots, U_n)$ is a compatible $(\|\tau\| - 1)$ -ary cross of \mathbf{A} .

Proof of Claim 4.5. It is only the $n > 1$ case of the claim that is interesting, and we are in that case since $\|\tau\| > 2$ and $n = \|\tau\| - 1$.

Elements of $A = \{0, 1\}^n$ will be represented by rows of length n consisting of 0's and 1's. The elements of $\text{Cross}(U_1, \dots, U_n)$ are n -tuples of elements of A , so could be represented by columns of length n , where each entry in the column is a row of length n . But instead of doing this, we drop parentheses and consider elements of $\text{Cross}(U_1, \dots, U_n)$ to be $n \times n$ matrices of 0's and 1's. For such a matrix to belong to this relation we must have the first row in U_1 or the second row in U_2 , etc. Since a row of length n belongs to U_i if and only if it has a 1 in the i -th place, it follows that an $n \times n$ matrix belongs to $\text{Cross}(U_1, \dots, U_n)$ exactly if it has a 1 somewhere on the diagonal.

The operations of \mathbf{A} act coordinatewise on the columns in a relation, and, in a given coordinate, act coordinatewise on rows. Thus, the operations of \mathbf{A} act coordinatewise on matrices. We must show that if f_i is one of the operations of \mathbf{A} and $M_1, \dots, M_{n_i} \in \text{Cross}(U_1, \dots, U_n)$, then $f_i(M_1, \dots, M_{n_i}) \in \text{Cross}(U_1, \dots, U_n)$.

Suppose that some M_k has a 1 on its diagonal in the j, j -th entry, where j does not belong to f_i . Then, as f_i acts as n_i -ary join in the j, j -position, it follows that $f_i(M_1, \dots, M_{n_i})$ has a 1 in its j, j -th entry, so $f_i(M_1, \dots, M_{n_i}) \in \text{Cross}(U_1, \dots, U_n)$. Now suppose that each M_k only has 1's in entries j, j where j does belong to f_i . Since there are n_i arguments of f_i , and only $n_i - 1$ distinct j 's that belong to f_i , it

must be that there are two matrices M_k and M_ℓ , $1 \leq k < \ell \leq n_i$ which both have 1 in the j, j -position for some j belonging to f_i . Since f_i acts like the canonical n_i -ary near unanimity operation in position j, j , it follows that $f_i(M_1, \dots, M_{n_i})$ has a 1 in its j, j -th entry, so $f_i(M_1, \dots, M_{n_i}) \in \mathbf{Cross}(U_1, \dots, U_n)$. These arguments establish the claim. \diamond

Claim 4.6. *\mathbf{A} has a cube term.*

Proof of Claim 4.6. The fact that the operations of \mathbf{A} are defined coordinatewise on $\{0, 1\}^n$ implies that \mathbf{A} is a product of its 2-element coordinate factor algebras. Thus, to prove this claim, it is enough to show that each coordinate factor of \mathbf{A} has a cube term. That this is enough follows from our Corollary 3.5, combined with Lemma 3.4, or from Corollary 2.5 of [12]. Namely, each result implies that if each algebra in a finite family has a cube term, then the product also has a cube term.

But it is easy to see that each coordinate factor of \mathbf{A} has a cube term (in fact, a near-unanimity term). To see this, consider the j -th coordinate factor algebra and suppose that f_i belongs to j . If $\text{arity}(f_i) > 2$, then f_i interprets as the canonical n_i -ary near-unanimity operation in the j -th coordinate and we are done. If $\text{arity}(f_i) = 2$, then f_i interprets as binary meet in the j -th coordinate and f_i belongs to no coordinate other than j . Since $n = \|\tau\| - 1 > 1$, there exists some coordinate different from j . Hence there must exist some $f_k \neq f_i$ belonging to a coordinate other than j . In this case, f_k will interpret as n_k -ary join in the j -th coordinate. With join coming from f_k and meet coming from f_i one can construct a ternary near-unanimity operation in coordinate j . \diamond

Claim 4.6 shows that \mathbf{A} has a d -cube term for some d . Hence, we can use Theorem 4.1 to conclude that \mathbf{A} has a $\|\tau\|$ -cube term. On the other hand, by Corollary 2.3, Claim 4.5 prevents \mathbf{A} from having a d -cube term for any $d < \|\tau\|$. This proves all required properties of \mathbf{A} . \square

Remark 4.7. Example 4.4 was discovered with the help of UACalc, a universal algebra calculator. After including it here we learned that the preprint [6] by Campenella, Conley and Valeriote contains essentially the same example. We are informed that they also discovered the example with the help of UACalc. The purpose of the example in their paper is roughly the same as ours (i.e., lower bounds for dimension estimates), except our application is to cube terms and their application is to near unanimity terms.

Example 4.8. In this example our goal is to show that the bound in Corollary 4.3 is sharp. Accordingly, we want to construct, for any suitable finite signature τ , an idempotent variety \mathcal{V} such that the free algebra $\mathbf{F} = \mathbf{F}_{\mathcal{V}}(x, y)$ has a compatible symmetric cross of arity $|\tau| - 1$, but no compatible symmetric crosses of higher arity.

If τ is suitable for idempotent varieties and $|\tau| = 2$, then τ is a signature with binary operation symbols only, say f_1, \dots, f_m . In this case we can choose \mathcal{V} to be

the variety generated by an algebra $\langle \mathbb{Z}_3; f_1, \dots, f_m \rangle$ where $f_1(x, y) = 2x + 2y$ and for $2 \leq i \leq m$, f_i is interpreted as a projection. Since \mathcal{V} has a Maltsev term, \mathbf{F} has no compatible symmetric crosses of arity ≥ 2 . However, $\text{Cross}(\{x\})$ is a compatible symmetric cross of \mathbf{F} of arity 1.

Let us assume from now on that $|\tau| \geq 3$, and let f_1, \dots, f_m denote the operation symbols in τ where f_i is n_i -ary ($1 \leq i \leq m$). We may assume without loss of generality that $n_1 \geq \dots \geq n_m$, so $|\tau| = n_1 (\geq 3)$. Now consider an algebra $\mathbf{B} = \langle \{0, 1\}; f_1, \dots, f_m \rangle$ where f_1 is interpreted on $\{0, 1\}$ as the “canonical” n_1 -ary near-unanimity operation (see the definition in Example 4.4), and for $2 \leq i \leq m$, f_i is interpreted as a projection. Let \mathcal{V} be the variety generated by \mathbf{B} , and let $\mathbf{F} = \mathbf{F}_{\mathcal{V}}(x, y)$. Since \mathcal{V} has an n_1 -ary near-unanimity operation, we know from Corollary 2.3 that \mathbf{F} has no compatible crosses of arity $\geq n_1 = |\tau|$. On the other hand, since the $(n_1 - 1)$ -ary cross $\text{Cross}(\{1\}, \dots, \{1\}) = \{0, 1\}^{n_1-1} \setminus \{(0, \dots, 0)\}$ is a compatible relation of \mathbf{B} , its inverse image under the homomorphism $\mathbf{F} \rightarrow \mathbf{B}$ sending x to 1 and y to 0 is a compatible symmetric cross of \mathbf{F} of arity $n_1 - 1 = |\tau| - 1$.

5. A FACT ABOUT CYCLIC TERM VARIETIES

This note emerged in response to a question we learned from Cliff Bergman: Suppose that \mathcal{C}_2 is the variety defined with one binary operation and axiomatized by the identities

- (1) $w(x, x) = x$, and
- (2) $w(x, y) = w(y, x)$.

Is it true that every subvariety of \mathcal{C}_2 either contains the 2-element semilattice or is congruence permutable?

Bergman’s question arose out of a certain line of investigation into constraint satisfaction problems. Namely, it is of interest to understand whether the algebras in the *pure cyclic term* varieties have tractable CSP’s. The d -ary pure cyclic term variety \mathcal{C}_d is defined with one d -ary operation satisfying

- (1) $w(x, x, \dots, x) = x$, and
- (2) $w(x_1, x_2, \dots, x_d) = w(x_2, x_3, \dots, x_1)$.

If one could show that each finite algebra in each variety \mathcal{C}_d has tractable associated CSP’s, then one would have solved the Feder–Vardi Conjecture.

Bergman and David Failing showed in [3] that if \mathcal{V} is a subvariety of \mathcal{C}_2 that is the join of a congruence permutable variety and the variety of semilattices, then the finite algebras in \mathcal{V} have tractable associated CSP’s. So, Bergman was really asking whether this theorem applied to every subvariety of \mathcal{C}_2 that is a join of the variety of semilattices and a disjoint subvariety. When Bergman asked the question, he mentioned that an affirmative answer was supported by extensive computer computations performed by Bergman, William DeMeo, and Jiali Li.

Here we explain why the answer to Bergman's question is affirmative, even with 2 replaced by d . That is, we explain why a subvariety of the d -ary pure cyclic term variety either contains a 2-element semilattice or else has a d -cube term. For this, define a *2-element semilattice in \mathcal{C}_d* to be an algebra with 2-element universe, say $\{0, 1\}$ and operation defined by

$$(5.1) \quad w(x_1, \dots, x_d) = \begin{cases} 1 & \text{if } x_1 = x_2 = \dots = x_d = 1; \\ 0 & \text{else.} \end{cases}$$

More generally, a 2-element semilattice for a variety \mathcal{V} is one in which, for every d , each d -ary fundamental operation satisfies (5.1).

Theorem 5.1. *A subvariety of the pure d -ary cyclic term variety either has a d -cube term or contains a 2-element semilattice. A finite algebra in the pure d -ary cyclic term variety either has a d -cube term or has a 2-element semilattice section.*

One should note that a 2-element semilattice has no d -cube term for any d , so the two cases described in the theorem are complementary.

Proof. Let \mathcal{V} be a subvariety of the pure d -ary cyclic term variety. If \mathcal{V} does not have a d -cube term, then by Theorem 2.4 (1) the free algebra $\mathbf{F} = \mathbf{F}_{\mathcal{V}}(x, y)$ has a compatible d -ary cross. But the signature τ of \mathcal{V} satisfies $d = \|\tau\|$, so Claim 4.2 shows that \mathbf{F} has a cube term blocker of the form (U, F) . It follows from Lemma 2.1 (5) that the cyclic term of the variety must be U -absorbing in at least one of its variables, so by cyclicity this term is U -absorbing in all of its variables. This implies that (i) the congruence θ on \mathbf{F} generated by $U \times U$ is the union of $U \times U$ and the equality relation, and (ii) if $t \in F \setminus U$ is chosen arbitrarily, then $S = U \cup \{t\}$ is a subuniverse of \mathbf{F} . The algebra $\mathbf{S}/\theta|_S$ must then be a 2-element semilattice in \mathcal{V} .

For the second statement, assume that \mathbf{A} is a finite algebra in the pure d -ary cyclic term variety and that \mathbf{A} does not have a d -cube term. Then $\mathcal{V}(\mathbf{A})$ cannot have a d -cube term, so by Theorem 4.1, $\mathcal{V}(\mathbf{A})$ cannot have a cube term at all. Therefore Corollary 3.5 guarantees that \mathbf{A} has a cube term blocker, say (U, B) . Now construct a 2-element semilattice from this blocker in the same manner one was constructed from the blocker (U, F) in the preceding paragraph. It will be a section of \mathbf{A} . \square

What matters to us in Theorem 5.1 is that the d in “ d -ary cyclic term” agrees with the d in “ d -cube term”; that is, the theorem establishes the existence of a cube term under some condition, *and bounds its index*. If one is not concerned with such a bound, then one can establish a result about varieties with many cyclic fundamental operations, namely:

Theorem 5.2. *Let \mathcal{V} be an idempotent variety whose fundamental operations each satisfy cyclic identities. That is, for each fundamental operation $w(x_1, \dots, x_n)$ it is the case that $\mathcal{V} \models w(x_1, x_2, \dots, x_n) = w(x_2, x_3, \dots, x_1)$. Then \mathcal{V} either has a cube term or contains a 2-element semilattice.*

Proof. Let $\mathbf{F} = \mathbf{F}_{\mathcal{V}}(x, y)$. If \mathcal{V} has no cube term, then by Theorem 3.3 there is a cube term blocker (U, F) for \mathbf{F} . By repeating the argument in the first paragraph of the proof of Theorem 5.1 we find that \mathcal{V} contains a 2-element semilattice. \square

6. GENERIC CROSSES

In this section we focus on idempotent varieties of finite type. We show that if a nontrivial member of such a variety has a compatible d -ary cross, then some countably infinite algebra \mathbf{A} in the variety has a ‘generic’ compatible d -ary cross. By a ‘generic cross’ we mean a cross $\text{Cross}(U_1, \dots, U_n)$ where the sets U_1, \dots, U_n are as independent as possible. Specifically, when $\text{Cross}(U_1, \dots, U_n)$ is a cross on a countably infinite set A , we call $\text{Cross}(U_1, \dots, U_n)$ *generic* if every nonzero Boolean combination of the sets U_1, \dots, U_n is countably infinite.

Theorem 6.1. *If \mathcal{X} is an idempotent variety of finite type and some member of \mathcal{X} has a compatible d -ary cross, then some countably infinite member of \mathcal{X} has a compatible d -ary generic cross.*

Proof. If some member of \mathcal{X} has a compatible d -ary cross, then by Corollary 2.3, \mathcal{X} cannot have a d -cube term. Hence Theorem 3.3 implies that the algebra $\mathbf{F} = \mathbf{F}_{\mathcal{X}}(x, y)$ must have a compatible d -ary cross, say

$$\text{Cross}(U_1, \dots, U_d) = B_1 \cup \dots \cup B_d$$

where $B_i = F \times \dots \times F \times U_i \times F \times \dots \times F$ is full in all coordinates except the i th. Here \mathbf{F} need not be infinite, and this cross need not be generic, so we modify the situation as follows.

Let $\mathbf{A} \in \mathcal{X}$ be a countably infinite algebra. Now define

$$\begin{aligned} \mathcal{F} &= \mathbf{F}^d \times \mathbf{A}, \\ \mathcal{U}_1 &= B_1 \times \mathbf{A}, \\ &\vdots \\ \mathcal{U}_d &= B_d \times \mathbf{A}. \end{aligned}$$

It is easy to see that \mathcal{F} is countably infinite and that each \mathcal{U}_i is a nonempty proper subuniverse of \mathcal{F} .

Claim 6.2. *$\text{Cross}(\mathcal{U}_1, \dots, \mathcal{U}_d)$ is a compatible generic d -ary cross of \mathcal{F} .*

Proof of Claim 6.2. We first argue that $\text{Cross}(\mathcal{U}_1, \dots, \mathcal{U}_d)$ is compatible, i.e. a subuniverse of \mathcal{F}^d . For this we consider $\mathcal{F} = \mathbf{F}^d \times \mathbf{A}$ to have $d + 1$ coordinates, so

$$\mathcal{F}^d = \mathbf{F}^d \times \mathbf{A} \times \mathbf{F}^d \times \mathbf{A} \times \dots \times \mathbf{F}^d \times \mathbf{A}$$

has $d(d + 1)$ coordinates. Notice that all coordinate algebras in this direct representation of \mathcal{F}^d are \mathbf{F} except those whose coordinates lie in the arithmetical progression $d + 1, 2(d + 1), \dots, d(d + 1)$, in which case the coordinate algebras are \mathbf{A} .

There is a projection homomorphism $\pi: \mathcal{F}^d \rightarrow \mathbf{F}^d$ which projects onto the coordinates in the arithmetic progression $1, (d+1)+2, 2(d+1)+3, \dots, (d-1)(d+1)+d$, which projects onto the first coordinate of the first block of $d+1$ factors of \mathcal{F}^d , the second factor of the second block of $d+1$ factors, etc. It is not hard to verify that

$$\text{Cross}(\mathcal{U}_1, \dots, \mathcal{U}_d) = \pi^{-1}(\text{Cross}(U_1, \dots, U_d)),$$

thereby establishing that $\text{Cross}(\mathcal{U}_1, \dots, \mathcal{U}_d)$ is compatible.

To show that $\text{Cross}(\mathcal{U}_1, \dots, \mathcal{U}_d)$ is generic, it suffices to show that any intersection $\mathcal{U}_1^{\varepsilon_1} \cap \dots \cap \mathcal{U}_d^{\varepsilon_d}$ contains infinitely many elements, where $\varepsilon_i = \pm 1$ for each i and $\mathcal{U}_1^{+1} = \mathcal{U}_1$ while $\mathcal{U}_1^{-1} = \mathcal{F} \setminus \mathcal{U}_1$. Observe that a $(d+1)$ -tuple (u_1, \dots, u_d, a) belongs to the set $\mathcal{U}_1^{\varepsilon_1} \cap \dots \cap \mathcal{U}_d^{\varepsilon_d}$ exactly when $u_i \in \mathcal{U}_i$ if $\varepsilon_i = +1$, $u_i \in \mathcal{F} \setminus \mathcal{U}_i$ if $\varepsilon_i = -1$, and $a \in A$. Such choices are possible since \mathcal{U}_i is a nonempty proper subuniverse of \mathcal{F} for each i and A is nonempty. If we fix the u_i 's and let the last coordinate a range over the infinite set A we obtain infinitely many elements in $\mathcal{U}_1^{\varepsilon_1} \cap \dots \cap \mathcal{U}_d^{\varepsilon_d}$. \diamond

This completes the proof of Theorem 6.1. \square

Corollary 6.3. *The class of varieties having a d -cube term represents a join-prime filter in the lattice of idempotent Maltsev conditions.*

Proof. If not, then there are idempotent varieties \mathcal{X} and \mathcal{Y} that do not have a d -cube term, but their coproduct $\mathcal{X} \sqcup \mathcal{Y}$ does have a d -cube term. But if $\mathcal{X} \sqcup \mathcal{Y}$ has a d -cube term, then so must $\mathcal{X}' \sqcup \mathcal{Y}'$ for some finitely presentable varieties \mathcal{X}' interpretable in \mathcal{X} and \mathcal{Y}' interpretable in \mathcal{Y} . Replacing \mathcal{X} and \mathcal{Y} by \mathcal{X}' and \mathcal{Y}' we may assume that \mathcal{X} and \mathcal{Y} are finitely presentable, in particular of finite type.

We prove the corollary by arguing that if \mathcal{X} and \mathcal{Y} have finite type and no d -cube term, then $\mathcal{X} \sqcup \mathcal{Y}$ has no d -cube term.

By Theorem 6.1 there exist countably infinite algebras $\mathbf{A} \in \mathcal{X}$ and $\mathbf{B} \in \mathcal{Y}$ which have generic compatible d -ary crosses, say $\text{Cross}(U_1, \dots, U_d)$ and $\text{Cross}(V_1, \dots, V_d)$. By genericity, it is possible to find a bijection $\alpha: A \rightarrow B$ such that $\alpha(U_i) = V_i$ for all i . There is a unique \mathcal{X} -structure \mathbf{B}' on B that makes $\alpha: \mathbf{A} \rightarrow \mathbf{B}'$ an isomorphism. Thus $\text{Cross}(V_1, \dots, V_d) = \alpha(\text{Cross}(U_1, \dots, U_d))$ is a compatible cross of \mathbf{B}' . Since it is also a compatible cross of $\mathbf{B} \in \mathcal{Y}$, the algebra on B obtained by merging \mathbf{B} and \mathbf{B}' is a model of the identities of $\mathcal{X} \sqcup \mathcal{Y}$ which has a compatible d -ary cross. This is enough to show that $\mathcal{X} \sqcup \mathcal{Y}$ has no d -cube term. \square

Remark 6.4. The results in this section were discovered during the 2016 ‘Algebra and Algorithms’ workshop after hearing a talk by Matthew Moore on the join-primeness among *idempotent linear* Maltsev conditions of the condition expressing the existence of a cube term. Later, Jakub Opršal pointed us to his preprint [14] where he proves our Corollary 6.3 (among other things). Opršal told us that he learned of our characterization of cube terms in terms of crosses from a talk of Szendrei at the AAA90 conference in Novi Sad in 2015, and then developed a similar

characterization of his own which allowed him to prove Corollary 6.3. His discovery of Corollary 6.3 predates ours. His argument depends on an analogue of Theorem 6.1, which he proves for varieties in arbitrary languages. Our proof also works for arbitrary languages, but we decided not to change ours after learning about Opršal's work.

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