# Flow equation for the scalar model in the large $N$ expansion and its applications 

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#### Abstract

We study the flow equation of the $\mathrm{O}(N) \varphi^{4}$ model in $d$ dimensions at the next-to-leading order (NLO) in the $1 / N$ expansion. Using the Schwinger-Dyson equation, we derive 2 -pt and 4 -pt functions of flowed fields. As the first application of the NLO calculations, we study the running coupling defined from the connected 4 -pt function of flowed fields in the $d+1$ dimensional theory. We show in particular that this running coupling has not only the UV fixed point but also an IR fixed point (Wilson-Fisher fixed point) in the 3 dimensional massless scalar theory. As the second application, we calculate the NLO correction to the induced metric in $d+1$ dimensions with $d=3$ in the massless limit. While the induced metric describes a 4dimensional Euclidean Anti-de-Sitter (AdS) space at the leading order as shown in the previous paper, the NLO corrections make the space asymptotically AdS only in UV and IR limits. Remarkably, while the AdS radius does not receive a NLO correction in the UV limit, the AdS radius decreases at the NLO in the IR limit, which corresponds to the Wilson-Fisher fixed point in the original scalar model in 3 dimensions


## 1 Introduction

In the previous paper [1], the present authors studied the proposal [2] that a $d+1$ dimensional induced metric can be constructed from a $d$ dimensional field theory using gradient flow [3-6], applying the method to the $\mathrm{O}(N) \varphi^{4}$ model. We have shown that in the large $N$ limit the induced metric becomes classical and describes Euclidean Anti-de-Sitter (AdS) space in both ultra-violet (UV) and infra-red (IR) limits of the flow direction. The method proposed in Ref. [2] may provide an alternative way to understand the AdS/CFT (or more generally Gravity/Gauge theory) correspondence [7], and the result in Ref. [1] might be related to the correspondence between $\mathrm{O}(N)$ vector models in $d$-dimensions and (generalized) gravity theories in $d+1$ dimensions [ 8$]$.

To further investigate a possible connection between Ref. [1] and Ref. [8] at the quantum level, one must calculate, for example, the anomalous dimension of the $\mathrm{O}(N)$ invariant operator $\phi^{2}(x)$, which requires the next-to-leading order (NLO) of the $1 / N$ expansion for the flow equation to evaluate necessary quantum corrections. Since the method employed in Refs. [1, 2] is a specific one adopted for the large $N$ limit, some systematic way to solve the flow equation in the $1 / N$ expansion is needed.

In this paper, we employ the Schwinger-Dyson equation (SDE) to solve the flow equation in the $1 / N$ expansion for the $\mathrm{O}(N)$ invariant $\varphi^{4}$ model in $d$ dimensions. Using this method we explicitly calculate the $2-\mathrm{pt}$ and 4 -pt functions at the NLO.

As the first application of the NLO calculations, we define a running coupling from the connected 4-pt function of flowed fields, which runs with the flow time $t$ such that $t=0$ corresponds to the UV limit while $t=\infty$ is the IR limit. This property establishes that the flow equation can be interpreted as a renormalization group transformation. In particular at $d=3$, we show that the running coupling so defined has not only the asymptotic free UV fixed point but also a Wilson-Fisher IR fixed point for the massless case.

As the second application, we investigate the NLO correction to the induced metric in $3+1$ dimensions from the massless scalar model in 3 dimensions. In the massless limit, the whole 4-dimensional space becomes AdS at the leading order, as shown in Ref. [1]. The NLO corrections give a small perturbation to the metric, which makes the space asymptotically AdS in UV $(t=0)$ and IR $(t=\infty)$ limits only. A remarkable thing is that, while the NLO corrections do not change the AdS radius in the UV limit, the AdS radius is reduced by the NLO correction in the IR limit, which corresponds to the Wilson-Fisher IR fixed point of the original theory. In other words, a nontrivial fixed point in the field theory leads to a change of the AdS radius in the geometry at the NLO. The induced metric at NLO
describes a 4-dimensional space connecting one asymptotically AdS space at UV to an other asymptotically AdS space at IR, which have different radii.

This paper is organized as follows. In Sec. 2, we first introduce the $\mathrm{O}(N)$ invariant $\varphi^{4}$ model in $d$ dimensions. We then formulate the Schwinger-Dyson equation (SDE) for the flowed fields, and solve it to derive 2-pt and 4-pt functions of flowed fields at the NLO. In Sec. 3, we define a running coupling from the connected 4-pt function of flowed fields and investigate its behavior as a function of the flow time $t$. In Sec. [4 we study the induced metric from the 3 dimensional massless scalar model at the NLO. We finally give a summary of this paper in Sec. 5. We collect all technical details in appendices. In appendix A. using the SDE, we present results at the NLO in the $1 / N$ expansion of the $d$ dimensional theory necessary for the main text. We also perform the renormalization of the $d$ dimensional theory at the NLO, and explicitly calculate renormalization constants for various $d$. In appendix B. we give detailed derivations of solutions to the SDE for the flow fields at the NLO. We explicitly evaluate 2-pt and 4-pt functions of the flowed field in appendix while we derive the induced metric in appendix D , for the massless scalar theory in 3 dimensions.

## $2 \quad 1 / N$ expansion of the flow equation in $d+1$ dimensions

### 2.1 Model in d dimensions

In this paper, we consider the $N$ component scalar $\varphi^{4}$ model in $d$ dimensions, defined by the action

$$
\begin{equation*}
S\left(\mu^{2}, u\right)=N \int \mathrm{~d}^{d} x\left[\frac{1}{2} \partial^{k} \varphi(x) \cdot \partial_{k} \varphi(x)+\frac{\mu^{2}}{2} \varphi^{2}(x)+\frac{u}{4!}\left(\varphi^{2}(x)\right)^{2}\right] \tag{1}
\end{equation*}
$$

where $\varphi^{a}(x)$ is an $N$ component scalar field, (.) indicates an inner product of $N$ component vectors such that $\varphi^{2}(x) \equiv \varphi(x) \cdot \varphi(x)=\sum_{a=1}^{N} \varphi^{a}(x) \varphi^{a}(x), \mu^{2}$ is the bare scalar mass parameter, and $u$ is the coupling constant of the $\varphi^{4}$ interaction, whose canonical dimension is $4-d$. While it is consistent to take $u$ as $N$ independent, as will be seen later, the mass parameter $\mu^{2}$ is expanded as

$$
\begin{equation*}
\mu^{2}=\mu_{0}^{2}+\frac{1}{N} \mu_{1}^{2}+\cdots \tag{2}
\end{equation*}
$$

where $\mu_{i}^{2}$ is cut-off dependent in order to make the physical mass finite order by order in the $1 / N$ expansion.

This model describes the free massive scalar at $u=0$, while it is equivalent to the nonlinear $\sigma$ model (NLSM) in the $u \rightarrow \infty$ limit, whose action is obtained from eq. (11) as

$$
\begin{equation*}
S(\lambda)=\frac{N}{2 \lambda} \int \mathrm{~d}^{d} x \partial^{k} \sigma(x) \cdot \partial_{k} \sigma(x), \quad \sigma^{2}(x)=1 \tag{3}
\end{equation*}
$$

with the replacement

$$
\begin{equation*}
\sigma^{a}(x)=\sqrt{\lambda} \varphi^{a}(x), \quad \lambda=\lim _{u \rightarrow \infty}-\frac{u}{6 \mu^{2}} \tag{4}
\end{equation*}
$$

Some regularization which preserves $\mathrm{O}(N)$ symmetry is assumed in this paper, so that we can always make formal manipulations without worrying about divergences. 1 Calculations of 2-pt and 4 -pt functions at the next-to-leading order (NLO) of the $1 / N$ expansion in $d$ dimensions will be given in appendix A.

### 2.2 Flow equation in the $1 / N$ expansion

In this paper, we consider the flow equation, given by

$$
\begin{align*}
\frac{\partial}{\partial t} \phi^{a}(t, x) & =-\left.\frac{1}{N} \frac{\delta S\left(\mu_{f}^{2}, u_{f}\right)}{\delta \varphi^{a}(x)}\right|_{\varphi \rightarrow \phi}=\left(\square-\mu_{f}^{2}\right) \phi^{a}(t, x)-\frac{u_{f}}{6} \phi^{a}(t, x) \phi^{2}(t, x),  \tag{5}\\
\phi^{a}(0, x) & =\varphi^{a}(x)
\end{align*}
$$

where $\mu_{f}^{2}$ and $u_{f}$ can be different from $\mu^{2}$ and $u$ in the original $d$ dimensional theory. As in the case of $d$ dimensions, $u_{f}$ is kept fixed and $N$ independent, whereas $\mu_{f}^{2}$ is adjusted as

$$
\begin{equation*}
\mu_{f}^{2}=m_{f}^{2}-\frac{u_{f}}{6} Z\left(m_{f}\right), \quad Z\left(m_{f}\right) \equiv \int \mathrm{D} q \frac{1}{q^{2}+m_{f}^{2}}, \quad \mathrm{D} q \equiv \frac{\mathrm{~d}^{d} q}{(2 \pi)^{d}} \tag{6}
\end{equation*}
$$

where $m_{f}$ is a renormalized mass. The flow with $\mu_{f}=\mu$ and $u_{f}=u$ is called gradient flow, as it is given by the gradient of the original action.

In the case of the free flow $\left(u_{f}=0\right)$, the solution is easily given by

$$
\begin{equation*}
\phi^{a}(t, x)=\mathrm{e}^{t\left(\square-\mu_{f}^{2}\right)} \varphi^{a}(x) \tag{7}
\end{equation*}
$$

We therefore consider the interacting flow $\left(u_{f} \neq 0\right)$ hereafter unless otherwise stated.

[^0]The above flow equation leads to the Schwinger-Dyson equation (SDE) [9] as

$$
\begin{equation*}
\left\langle D_{z}^{f} \phi^{a}(z) \mathcal{O}\right\rangle=-\frac{u_{f}}{6}\left\langle\phi^{a}(z) \phi^{2}(z) \mathcal{O}\right\rangle, \quad D_{z}^{f} \equiv \frac{\partial}{\partial t}-\left(\square-\mu_{f}^{2}\right), \tag{8}
\end{equation*}
$$

where $z=(t, x), \mathcal{O}$ is an arbitrary operator and the expectation value $\langle\mathcal{O}\rangle$ should be calculated in $d$ dimensions as

$$
\begin{equation*}
\langle\mathcal{O}(\varphi)\rangle \equiv \frac{1}{Z} \int[\mathcal{D} \varphi] \mathcal{O}(\varphi) \mathrm{e}^{-S\left(\mu^{2}, u\right)}, \quad Z=\int[\mathcal{D} \varphi] \mathrm{e}^{-S\left(\mu^{2}, u\right)} \tag{9}
\end{equation*}
$$

If we take $\mathcal{O}=\prod_{i=1}^{2 n-1} \phi^{a_{i}}\left(z_{i}\right)$ the SDE becomes

$$
\begin{equation*}
D_{z}^{f} \Gamma_{2 n}^{a a_{1} \cdots a_{2 n-1}}\left(z, z_{1}, \cdots, z_{2 n-1}\right)=-\frac{u_{f}}{6 N^{2}} \sum_{b} \Gamma_{2 n+2}^{a b b a_{1} \cdots a_{2 n-1}}\left(z, z, z, z_{1}, \cdots, z_{2 n-1}\right) \tag{10}
\end{equation*}
$$

where $\Gamma_{n}$ is the $n$-point function, defined by

$$
\begin{equation*}
\Gamma_{n}^{a_{1} \cdots a_{n}}\left(z_{1}, \cdots, z_{n}\right)=N^{n-1}\left\langle\prod_{i=1}^{n} \phi^{a_{i}}\left(z_{i}\right)\right\rangle \equiv \Gamma_{n}[12 \cdots n], \tag{11}
\end{equation*}
$$

which is analogous to the $d$ dimensional counterpart in eq. (A3). We consider only the symmetric phase in this paper, where $\Gamma_{2 n-1}=0$ for all positive integers $n$.

We consider the next-to-leading order of the $1 / N$ expansion, so that the following two SDE's need to be considered.

$$
\begin{align*}
D_{1}^{f} \Gamma_{2}[12] & =-\frac{u_{f}}{6 N^{2}} \sum_{b} \Gamma_{4}[1 b b 2],  \tag{12}\\
D_{1}^{f} \Gamma_{4}[1234] & =-\frac{u_{f}}{6 N^{2}} \sum_{b} \Gamma_{6}[1 b b 234], \tag{13}
\end{align*}
$$

where $z_{b}=z_{1}$, so that the sum over $b$ runs over the $\mathrm{O}(N)$ index only.
The connected part of 4- and 6- pt functions are introduced as

$$
\begin{align*}
\Gamma_{4}[1234] & =K_{4}[1234]+N\left\{\Gamma_{2}[12] \Gamma_{2}[34]+\Gamma_{2}[13] \Gamma_{2}[24]+\Gamma_{2}[14] \Gamma_{2}[23]\right\}  \tag{14}\\
\Gamma_{6}[123456] & =K_{6}[123456]+N\left\{\Gamma_{2}[12] K_{4}[3456]+14 \text { perms. }\right\} \\
& \left.\left.+N^{2}\left\{\Gamma_{2}[12] \Gamma_{2}[34] \Gamma_{2}\right] 56\right]+14 \text { perms. }\right\} \tag{15}
\end{align*}
$$

Furthermore decompositions in $\mathrm{O}(N)$ indices are given by

$$
\begin{align*}
\Gamma_{2}[12] & =\delta^{a_{1} a_{2}} \Gamma\left(z_{1}, z_{2}\right),  \tag{16}\\
K_{4}[1234] & =\delta^{a_{1} a_{2}} \delta^{a_{3} a_{4}} K\left(z_{1}, z_{2} ; z_{3}, z_{4}\right)+2 \text { perms. }  \tag{17}\\
K_{6}[123456] & =\delta^{a_{1} a_{2}} \delta^{a_{3} a_{4}} \delta^{a_{5} a_{6}} H\left(z_{1}, z_{2} ; z_{3}, z_{4} ; z_{5}, z_{6}\right)+14 \text { perms. } \tag{18}
\end{align*}
$$

where $\Gamma\left(z_{1}, z_{2}\right)$, $K\left(z_{1}, z_{2} ; z_{3}, z_{4}\right)$ and $H\left(z_{1}, z_{2} ; z_{3}, z_{4} ; z_{5}, z_{6}\right)$ are invariant under the exchange of arguments such that $z_{2 i-1} \leftrightarrow z_{2 i}$ or $\left(z_{2 i-1}, z_{2 i}\right) \leftrightarrow\left(z_{2 j-1}, z_{2 j}\right)$.

By expanding $\Gamma, K$ and $H$ as

$$
\begin{equation*}
\Gamma=\sum_{i=0}^{\infty} \frac{\Gamma_{i}}{N^{i}}, \quad K=\sum_{i=0}^{\infty} \frac{K_{i}}{N^{i}}, \quad H=\sum_{i=0}^{\infty} \frac{H_{i}}{N^{i}} \tag{19}
\end{equation*}
$$

the above two SDE are reduced to

$$
\begin{equation*}
D_{1}^{f} \Gamma_{0}(12)=-\frac{u_{f}}{6} \Gamma_{0}(12) \Gamma_{0}(11) \tag{20}
\end{equation*}
$$

at the LO of the $1 / N$ expansion, and

$$
\begin{align*}
D_{1}^{f} \Gamma_{1}(12) & =-\frac{u_{f}}{6}\left[K_{0}(12 ; 11)+\Gamma_{0}(12) \Gamma_{1}(11)+\Gamma_{1}(12) \Gamma_{0}(11)+2 \Gamma_{0}(12) \Gamma_{0}(11)\right]  \tag{21}\\
D_{1}^{f} K_{0}(12 ; 34) & =-\frac{u_{f}}{6}\left[\Gamma_{0}(12) K_{0}(11 ; 34)+\Gamma_{0}(11) K_{0}(12 ; 34)+2 \Gamma_{0}(12) \Gamma_{0}(13) \Gamma_{0}(14)\right] \tag{22}
\end{align*}
$$

at the NLO.

### 2.3 Solutions to the flowed SDE at NLO

The solutions to the SDE at NLO are summarized below. Details of calculations can be found in appendix B

At the NLO, the 2-pt function is given by

$$
\begin{equation*}
\left\langle\phi^{a_{1}}\left(z_{1}\right) \phi^{a_{2}}\left(z_{2}\right)\right\rangle=\frac{\delta_{a_{1} a_{2}}}{N} \frac{Z\left(m_{f}\right)}{\sqrt{\zeta\left(t_{1}\right) \zeta\left(t_{2}\right)}} \int \mathrm{D} p \frac{\mathrm{e}^{-p^{2}\left(t_{1}+t_{2}\right)} \mathrm{e}^{i p\left(x_{1}-x_{2}\right)}}{p^{2}+m^{2}}\left[1+\frac{1}{N} G_{1}\left(t_{1}, t_{2} \mid p\right)\right], \tag{23}
\end{equation*}
$$

where $\zeta(t)$ is defined in eq. (B7), and the NLO contribution $G_{1}\left(t_{1}, t_{2} \mid p\right)$ is given in appendix B.3.2. In the continuum limit, $\zeta(t)$ approaches to $\zeta_{0}(t)$ and is finite as long as $t>0$, where

$$
\begin{equation*}
\zeta_{0}(t) \equiv \int \mathrm{D} p \frac{\mathrm{e}^{-2 p^{2} t}}{p^{2}+m^{2}}=\frac{\mathrm{e}^{2 t m^{2}} m^{d-2}}{(4 \pi)^{d / 2}} \Gamma\left(1-d / 2,2 t m^{2}\right) \tag{24}
\end{equation*}
$$

with the incomplete gamma function $\Gamma(a, x)$, while $Z\left(m_{f}\right)$ diverges at $d>1$.
The leading contribution of the connected 4-pt function appearing at the NLO of the $1 / N$ expansion can be obtained as

$$
\begin{equation*}
\left\langle\phi^{a_{1}}\left(z_{1}\right) \phi^{a_{2}}\left(z_{2}\right) \phi^{a_{3}}\left(z_{3}\right) \phi^{a_{4}}\left(z_{4}\right)\right\rangle_{c}=\frac{1}{N^{3}}\left[\delta_{a_{1} a_{2}} \delta_{a_{3} a_{4}} K_{0}(12 ; 34)+2 \text { permutations }\right], \tag{25}
\end{equation*}
$$

where

$$
\begin{align*}
K_{0}(12 ; 34) & =\int \mathrm{d} P_{4} g(12 ; 34 \mid 12 ; 34), \quad \mathrm{d} P_{4} \equiv \prod_{j=1}^{4} \mathrm{D} p_{j} \sqrt{\frac{Z\left(m_{f}\right)}{\zeta\left(t_{j}\right)} \frac{\mathrm{e}^{i p_{j} x_{j}} \mathrm{e}^{-p_{j}^{2} t_{j}}}{p_{j}^{2}+m^{2}},}  \tag{26}\\
g(12 ; 34 \mid 12 ; 34) & =X(23 \mid 12 ; 34)+X(13 \mid 21 ; 34)+X(24 \mid 12 ; 43)+X(14 \mid 21 ; 43) \\
& +Y(2 \mid 12 ; 34)+Y(1 \mid 21 ; 34)+Y(3 \mid 43 ; 12)+Y(4 \mid 34 ; 12) \\
& +Z(\mid 12 ; 34) . \tag{27}
\end{align*}
$$

Here the variables to the left of the vertical line refer to flow times and those to the right refer to momenta. Explicitly we have in the continuum or NLSM limits

$$
\begin{align*}
X\left(t_{1}, t_{2} \mid 12 ; 34\right) & =\hat{\delta}\left(p_{2}^{2}+m^{2}\right)\left(p_{3}^{2}+m^{2}\right) \int_{0}^{t_{1}} \mathrm{~d} s_{1} \int_{0}^{t_{2}} \mathrm{~d} s_{2} \mathrm{e}^{s_{1}\left(p_{2}^{2}-p_{1}^{2}\right)} \mathrm{e}^{s_{2}\left(p_{3}-p_{4}^{2}\right)} \omega\left(s_{1}, s_{2} \mid p_{34}\right)  \tag{28}\\
Y(t \mid 21 ; 34) & =\hat{\delta}\left(p_{1}^{2}+m^{2}\right) \int_{0}^{t} \mathrm{~d} s \mathrm{e}^{s\left(p_{1}^{2}-p_{2}^{2}\right)} \psi(s \mid 34)  \tag{29}\\
Z(\mid 12 ; 34) & =-\hat{\delta} \frac{2}{6 / u+B\left(0 \mid p_{34}\right)} \tag{30}
\end{align*}
$$

where $\hat{\delta} \equiv(2 \pi)^{d} \delta\left(p_{1}+p_{2}+p_{3}+p_{4}\right), p_{34}=p_{3}+p_{4}$,

$$
\begin{equation*}
B(t \mid Q)=\int \mathrm{D} q_{1} \mathrm{D} q_{2} \frac{\mathrm{e}^{-t\left(q_{1}^{2}+q_{2}^{2}\right)}}{\left(q_{1}^{2}+m^{2}\right)\left(q_{2}^{2}+m^{2}\right)}(2 \pi)^{d} \delta\left(q_{12}-Q\right), \quad q_{12}=q_{1}+q_{2} \tag{31}
\end{equation*}
$$

and thus $B(0 \mid Q)=B\left(Q^{2}\right)$, defined in appendix A. Here $\psi$ and $\omega$ satisfy

$$
\begin{align*}
\rho(t \mid 34)+\int_{0}^{t} \mathrm{~d} s K\left(t, s \mid p_{34}\right) \psi(s \mid 34) & =0  \tag{32}\\
\rho\left(t_{1}, t_{2} \mid Q\right)-2 \int_{0}^{t_{1}} \mathrm{~d} s_{1} K\left(t_{1}, s_{1} \mid Q\right) \int_{0}^{t_{2}} \mathrm{~d} s_{2} K\left(t_{2}, s_{2} \mid Q\right) \omega\left(s_{1}, s_{2} \mid Q\right) & =0 \tag{33}
\end{align*}
$$

where

$$
\begin{align*}
K(t, s \mid Q) & =\int \mathrm{D} q_{1} \mathrm{D} q_{2}(2 \pi)^{d} \delta\left(q_{12}-Q\right) \frac{\mathrm{e}^{-(t+s) q_{1}^{2}-(t-s) q_{2}^{2}}}{q_{1}^{2}+m^{2}}  \tag{34}\\
\rho(t \mid 34) & =\mathrm{e}^{-t\left(p_{3}^{2}+p_{4}^{2}\right)}-\frac{B\left(t \mid p_{34}\right)}{6 / u+B\left(0 \mid p_{34}\right)}  \tag{35}\\
\rho\left(t_{1}, t_{2} \mid Q\right) & =B\left(t_{1}+t_{2} \mid Q\right)-\frac{B\left(t_{1} \mid Q\right) B\left(t_{2} \mid Q\right)}{6 / u+B(0 \mid Q)} \tag{36}
\end{align*}
$$

The derivation of these results is given in appendix B.

## 3 Running coupling from flowed fields

### 3.1 Definitions

Using the connected 4-pt functions $g \equiv \hat{\delta} \hat{g}$ for the flow fields given in eq. (25), we define the $t$-dependent dimensionless coupling as

$$
\begin{equation*}
g(t)=-3 \hat{g}\left(t, t ; t, t \mid\{p\}_{\mathrm{sym}}\right) t^{2-d / 2} \tag{37}
\end{equation*}
$$

where $\{p\}_{\text {sym }}$ is given by $p_{i}^{2} t=3 \Delta / 4(i=1 \sim 4)$ and $p_{12}^{2} t=p_{34}^{2} t=\Delta\left(p_{i j}=p_{i}+p_{j}\right)$, which is the symmetric point for $d>2$, and $t^{2-d / 2}$ is introduced to make the coupling dimensionless. Here $\Delta$ is an arbitrary dimensionless constant but we can take $\Delta=1$ without loss of
generality by the rescaling $t \rightarrow \Delta t$. Explicitly we have

$$
\begin{equation*}
\hat{g}\left(t, t ; t, t \mid\{p\}_{\text {sym }}\right)=4 \hat{X}\left(t, t \mid\{p\}_{\text {sym }}\right)+4 \hat{Y}\left(t \mid\{p\}_{\text {sym }}\right)+\hat{Z}\left(\mid\{p\}_{\text {sym }}\right), \tag{38}
\end{equation*}
$$

where we remove $\hat{\delta}$ by defining $\mathcal{O}=\hat{\delta} \hat{\mathcal{O}}$ for $\mathcal{O}=g, X, Y, Z$, and

$$
\begin{align*}
\hat{X}\left(t_{1}, t_{2} \mid 12 ; 34\right) & =\left(p_{2}^{2}+m^{2}\right)\left(p_{3}^{2}+m^{2}\right) \int_{0}^{t_{1}} \mathrm{~d} s_{1} \int_{0}^{t_{2}} \mathrm{~d} s_{2} \mathrm{e}^{s_{1}\left(p_{2}^{2}-p_{1}^{2}\right)} \mathrm{e}^{s_{2}\left(p_{3}^{2}-p_{4}^{2}\right)} \omega\left(s_{1}, s_{2} \mid p_{34}\right)  \tag{39}\\
\hat{Y}(t \mid 12 ; 34) & =\left(p_{2}^{2}+m^{2}\right) \int_{0}^{t} \mathrm{~d} s \mathrm{e}^{s\left(p_{2}^{2}-p_{1}^{2}\right)} \psi(s \mid 34)  \tag{40}\\
\hat{Z}(\mid 12 ; 34) & =-\frac{1}{3} \frac{u}{1+\frac{u}{6} B\left(0 \mid p_{34}\right)} \tag{41}
\end{align*}
$$

### 3.2 Free flow

For simplicity, we first consider the free flow, where $\hat{g}\left(t, t ; t, t \mid\{p\}_{\text {sym }}\right)=\hat{Z}\left(\mid\{p\}_{\text {sym }}\right)$. Taking $\Delta=1$, the running coupling is given by

$$
\begin{equation*}
g(t)=\frac{u t^{2-d / 2}}{1+\frac{u}{6} B(1 / t)} \tag{42}
\end{equation*}
$$

where $B\left(p^{2}\right)=B(0 \mid p)$.

### 3.2.1 $d=2$

In 2-dimensions, we obtain

$$
\begin{equation*}
g(t)=\frac{u t}{1+\frac{u t}{6 \pi \sqrt{1+4 m^{2} t}} \tanh ^{-1}\left(\frac{1}{\sqrt{1+4 m^{2} t}}\right)} \tag{43}
\end{equation*}
$$

which behaves in the UV limit $(t \rightarrow 0)$ and IR limit $(t \rightarrow \infty)$ as

$$
g(t) \simeq\left\{\begin{array}{clc}
\frac{u t}{1-u t \log \left(m^{2} t\right) /(12 \pi)} & \rightarrow 0, & t=0  \tag{44}\\
\frac{u t}{1+u /\left(24 \pi m^{2}\right)} & \rightarrow \infty, \quad t=\infty
\end{array}\right.
$$

In the massless limit $m^{2} \rightarrow 0$, we have

$$
\begin{equation*}
g(t) \simeq-\frac{12 \pi}{\log \left(m^{2} t\right)} \rightarrow 0 \tag{45}
\end{equation*}
$$

### 3.2.2 $d=3$

At $d=3$, the running coupling is given by

$$
\begin{equation*}
g(t)=\frac{u \sqrt{t}}{1+\frac{u \sqrt{t}}{24 \pi} \arctan \left(\frac{1}{\sqrt{4 m^{2} t}}\right)} \tag{46}
\end{equation*}
$$

which behaves as

$$
g(t) \simeq\left\{\begin{array}{cll}
\frac{u \sqrt{t}}{1+u \sqrt{t} / 48} & \rightarrow 0, & t=0  \tag{47}\\
\frac{u \sqrt{t}}{1+u /(48 \pi m)} & \rightarrow \infty, & t=\infty
\end{array} .\right.
$$

In the massless limit, we have

$$
g(t)=\frac{u \sqrt{t}}{1+u \sqrt{t} / 48}=\left\{\begin{array}{lr}
\rightarrow 0, & t \rightarrow 0  \tag{48}\\
\rightarrow 48, & t \rightarrow \infty
\end{array},\right.
$$

which correspond to the asymptotic free UV fixed point and the Wilson-Fisher IR fixed point, respectively.

### 3.2.3 $d \geq 4$

Since $B\left(Q^{2}\right)$ diverges as $\Lambda^{d-4}(\log \Lambda$ at $d=4)$ at $d \geq 4$, the running coupling vanishes as the cut-off is removed $(\Lambda \rightarrow \infty)$. Thus the theory is trivial in the continuum limit at $d \geq 4$.

### 3.3 Interacting flow in the massless limit at $d=3$

### 3.3.1 Massless limit

We next consider the interacting flow case, where we need to evaluate $\hat{X}$ and $\hat{Y}$, which are difficult to calculate in general. We therefore consider the massless limit. 22 In this limit, the kernel function is reduced to

$$
\begin{equation*}
K\left(t, s \mid\{p\}_{\mathrm{sym} .}\right)=D^{d / 2-1} k_{0}(D t, D s) \tag{49}
\end{equation*}
$$

where

$$
\begin{equation*}
k_{0}(w, v)=\frac{\mathrm{e}^{v-w} w^{1-d / 2}}{2^{d-1}(2 \pi)^{d / 2}} \int_{0}^{1} \mathrm{~d} z z^{d / 2-2} \exp \left[\frac{(w-v)^{2} z}{2 w}\right] \tag{50}
\end{equation*}
$$

and we regard $D \equiv Q^{2}=\Delta / t$ as an independent variable. Here the $z$ integral is convergent for $d>2$ while the bubble integral $B(0 \mid Q)$ is finite for $d<4$. We thus concentrate on the $d=3$ case hereafter.

[^1]In this limit, we obtain (see appendix $\mathbb{C}$ for details)

$$
\begin{align*}
\hat{Z}\left(\mid\{p\}_{\text {sym. }}\right) & =-16 \sqrt{D} \frac{\bar{u}(D)}{1+\bar{u}(D)}, \quad \bar{u}(D) \equiv \frac{u}{48 \sqrt{D}},  \tag{51}\\
\hat{Y}\left(t \mid\{p\}_{\text {sym. }}\right) & =\frac{3}{4} \sqrt{D}\left\{\xi_{0}^{(1)}(\Delta)-8 \xi_{0}^{(2)}(\Delta) \frac{\bar{u}(D)}{1+\bar{u}(D)}\right\},  \tag{52}\\
\hat{X}\left(t, t \mid\{p\}_{\text {sym. }}\right) & =\frac{9}{16} \sqrt{D}\left\{\Xi_{0}(\Delta)-4\left\{\xi_{0}^{(2)}(\Delta)\right\}^{2} \frac{\bar{u}(D)}{1+\bar{u}(D)}\right\}, \tag{53}
\end{align*}
$$

where

$$
\begin{align*}
\xi_{0}^{(i)}(\Delta) & =\int_{0}^{\Delta} \mathrm{d} w \phi_{0}^{(i)}(w), \quad i=1,2  \tag{54}\\
\Xi_{0}(\Delta) & =\int_{0}^{\Delta} \mathrm{d} w \int_{0}^{\Delta} \mathrm{d} v \Omega_{0}(w, v) \tag{55}
\end{align*}
$$

and $\phi_{0}^{(i)}$ and $\Omega_{0}$ are solutions to the integral equations

$$
\begin{align*}
\mathrm{e}^{-3 w / 2}+\int_{0}^{w} \mathrm{~d} v k_{0}(w, v) \phi_{0}^{(1)}(v) & =0,  \tag{56}\\
b_{0}(w)+\int_{0}^{w} \mathrm{~d} v k_{0}(w, v) \phi_{0}^{(2)}(v) & =0,  \tag{57}\\
b_{0}(w+v)-2 \int_{0}^{w} \mathrm{~d} x k_{0}(w, x) \int_{0}^{v} \mathrm{~d} y k_{0}(v, y) \Omega_{0}(x, y) & =0, \tag{58}
\end{align*}
$$

where $b_{0}(w)$ is the massless bubble integral given by eq. (C3). These equations can be solved numerically, and at $\Delta=1$, for example, we have $\xi_{0}^{(1)}(1)=-14.8440(1), \xi_{0}^{(2)}(1)=$ $-1.60557(1)$ and $\Xi_{0}(1)=16.6753(1)$.

### 3.3.2 Running coupling and $\beta$ function

Using the above results, the running coupling at $d=3$ is given by

$$
\begin{equation*}
g_{0}(\mu)=G_{1}+G_{2} \frac{\bar{u}(\Delta) \sqrt{t}}{1+\bar{u}(\Delta) \sqrt{t}}, \quad \bar{u}(\Delta)=\frac{u}{48 \sqrt{\Delta}} \tag{59}
\end{equation*}
$$

where $\mu=1 / \sqrt{t}$ and

$$
\begin{equation*}
G_{1}=-9 \sqrt{\Delta}\left[\xi_{0}^{(1)}(\Delta)+\frac{3}{4} \Xi_{0}(\Delta)\right], \quad G_{2}=48 \sqrt{\Delta}\left[1+\frac{3}{4} \xi_{0}^{(2)}(\Delta)\right]^{2} \geq 0 \tag{60}
\end{equation*}
$$

With the numerical values given above we obtain $G_{1}=21.0378(1)$ and $G_{2}=2.00105(1)$ at $\Delta=1$. 3

[^2]We then calculate the $\beta$ function for $g_{0}(\mu)$ as

$$
\begin{equation*}
\beta\left(g_{0}\right) \equiv \mu \frac{\partial}{\partial \mu} g_{0}(\mu)=\frac{\left(g_{0}(\mu)-G_{1}-G_{2}\right)\left(g_{0}(\mu)-G_{1}\right)}{G_{2}}, \tag{61}
\end{equation*}
$$

which becomes zero at $g_{0}(\mu)=G_{1}$ and $g_{0}(\mu)=G_{1}+G_{2}$. The coupling $g_{0}(\mu)$ near $G_{1}$ behaves as

$$
\begin{equation*}
g_{0}(\mu)-G_{1} \simeq C_{U V} \frac{u}{\mu} \rightarrow 0, \quad \mu \rightarrow \infty, \quad C_{U V}=\left[1+\frac{3}{4} \xi_{0}^{(2)}(\Delta)\right]^{2} \tag{62}
\end{equation*}
$$

approaching to the UV fixed point from above, while near $G_{1}+G_{2}$ we have the IR fixed point as

$$
\begin{equation*}
g_{0}(\mu)-G_{1}-G_{2} \simeq-C_{I R} \frac{\mu}{u} \rightarrow 0, \quad \mu \rightarrow 0, \quad C_{I R}=\left\{48 \sqrt{\Delta}\left[1+\frac{3}{4} \xi_{0}^{(2)}(\Delta)\right]\right\}^{2} \tag{63}
\end{equation*}
$$

where the coupling approaches from below to the Wilson-Fisher fixed point in the 3 dimensional scalar theory. Note that the derivative of the $\beta$ function with respect to $g_{0}$ at the fixed point becomes

$$
\beta^{\prime}\left(g_{0}\right) \equiv \frac{d \beta\left(g_{0}\right)}{d g_{0}}= \begin{cases}-1, & g_{0}=G_{1}  \tag{64}\\ 1, & g_{0}=G_{1}+G_{2}\end{cases}
$$

which should be compared with the same quantities calculated for the standard running coupling in the 3 dimensional massless theory in Ref. [10], where $\beta^{\prime}(0)=-1$ (UV) and $\beta^{\prime}(48)=1$ (IR). The derivative of the $\beta$ function at the fixed point gives the anomalous dimension of the operator conjugate to the coupling in the conformal theory at the fixed point, and thus is universal. Our flow coupling indeed satisfies this condition and the derivatives at the two fixed points agree with those for the conventional definition of the coupling. This establishes that our flow coupling gives a good definition of the running coupling of the theory. The scaling dimension $\gamma$ of the operator conjugate to the running coupling $g_{0}$ is given by $\gamma=d+\beta^{\prime}\left(g_{0}\right)$, so that $\gamma_{\mathrm{UV}}=2$ and $\gamma_{I R}=4$ in this model. Interestingly $\gamma_{\mathrm{UV}}=2$ corresponds to the canonical dimension of the $\varphi^{4}$ operator in 3 dimensions, which is the interaction term in our theory.

By the redefinition of the coupling as $g(\mu) \equiv\left(g_{0}(\mu)-G_{1}\right) / G_{2}$, the corresponding $\beta$ function is simplified as

$$
\begin{equation*}
\beta(g) \equiv \mu \frac{\partial}{\partial \mu} g(\mu)=g(\mu)(g(\mu)-1) \tag{65}
\end{equation*}
$$

## 4 NLO corrections to the induced metric

In Ref. [1], the induced metric has been calculated from the flowed scalar field in the large $N$ limit. It has been shown that the metric from the massive scalar field describes a space which becomes the Euclidean AdS space asymptotically both in UV and IR limits, where the radius $R_{\mathrm{IR}}$ in the IR is larger than the radius $R_{\mathrm{UV}}$ in UV as

$$
\begin{equation*}
R_{\mathrm{UV}}^{2}=\frac{d-2}{2} R_{\mathrm{IR}}^{2}<R_{\mathrm{IR}}^{2}, \tag{66}
\end{equation*}
$$

while the metric describes the whole AdS space in the massless limit with the radius $R_{\mathrm{UV}}$. In this section, we consider the NLO correction to the induced metric in the $1 / N$ expansion as another application of the NLO calculation, in particular, in the massless case at $d=3$, in order to see whether the space remains AdS or not and how the radius changes at the NLO.

### 4.1 Induced metric at NLO

The VEV of the induced metric is defined from the normalized flowed field as [1]

$$
\begin{equation*}
g_{\mu \nu}(z)=R_{0}^{2}\left\langle\partial_{\mu} \sigma^{a}(z) \partial_{\nu} \sigma^{a}(z)\right\rangle \tag{67}
\end{equation*}
$$

with some length scale $R_{0}$, where $z=(\tau=2 \sqrt{t}, x)$ and $\mu, \nu=0,1, \cdots, d$. Here $\sigma^{a}(z)$ is the normalized flowed field such that $\left\langle\sigma^{2}(z)\right\rangle=1$, and the corresponding 2-point function is explicitly given at NLO as

$$
\begin{align*}
\left\langle\sigma^{a_{1}}\left(z_{1}\right) \sigma^{a_{2}}\left(z_{2}\right)\right\rangle & =\frac{\delta^{a_{1} a_{2}}}{N} \frac{1}{\sqrt{\zeta_{0}\left(t_{1}\right) \zeta_{0}\left(t_{2}\right)}}\left(1-\frac{\zeta_{1}\left(t_{1}\right)+\zeta_{1}\left(t_{2}\right)}{2 N}\right) \\
& \times \int \mathrm{D} p \frac{\mathrm{e}^{-p^{2}\left(t_{1}+t_{2}\right)} \mathrm{e}^{i p\left(x_{1}-x_{2}\right)}}{p^{2}+m^{2}}\left[1+\frac{G_{1}\left(t_{1}, t_{2} \mid p\right)}{N}\right], \tag{68}
\end{align*}
$$

where

$$
\begin{equation*}
\zeta_{1}(t)=\frac{1}{\zeta_{0}(t)} \mathcal{H}\left[G_{1}(t, t \mid p)\right], \quad \mathcal{H}[f(t \mid p)] \equiv \int \mathrm{D} p \frac{\mathrm{e}^{-2 p^{2} t}}{p^{2}+m^{2}} f(t \mid p) . \tag{69}
\end{equation*}
$$

After some algebra (see appendix (D), we obtain

$$
\begin{equation*}
g_{i j}(\tau)=\delta_{i j} \frac{R_{0}^{2}}{d} A(t), \quad(i, j=1,2, \cdots, d), \quad g_{00}(\tau)=-\frac{R_{0}^{2} t}{2} \partial_{t} A(t), \tag{70}
\end{equation*}
$$

where

$$
\begin{equation*}
A(t)=-\frac{1}{2} \frac{\partial_{t} \zeta_{0}(t)}{\zeta_{0}(t)}+\frac{1}{N} A_{1}(t) \tag{71}
\end{equation*}
$$

and $A_{1}(t)$ in general is a very complicated function given in appendix D.

### 4.2 Induced metric in the massless limit at $d=3$

In the massless limit at $d=3$, the metric at the LO is given by

$$
\begin{equation*}
g_{i j}(\tau)=\delta_{i j} \frac{R_{0}^{2}}{3 \tau^{2}}, \quad g_{00}(\tau)=\frac{R_{0}^{2}}{2 \tau^{2}} \tag{72}
\end{equation*}
$$

which describes the AdS space for all $\tau$.
At the NLO, $A_{1}(t)$ is given by

$$
\begin{align*}
A_{1}(t) & =\frac{1}{2 \sqrt{t}} \int \mathrm{D} Q h_{\mathrm{total}}\left(Q^{2}\right) \frac{\bar{u}\left(Q^{2}\right)}{\left(1+\bar{u}\left(Q^{2}\right) \sqrt{t}\right)^{2}}, \quad \bar{u}\left(Q^{2}\right)=\frac{u}{48 \sqrt{Q^{2}}},  \tag{73}\\
\partial_{t} A_{1}(t) & =-\frac{1}{4 \sqrt{t^{3}}} \int \mathrm{D} Q h_{\mathrm{total}}\left(Q^{2}\right) \frac{\bar{u}\left(Q^{2}\right)\left(1+3 \bar{u}\left(Q^{2}\right) \sqrt{t}\right)}{\left(1+\bar{u}\left(Q^{2}\right) \sqrt{t}\right)^{3}} \tag{74}
\end{align*}
$$

which leads to

$$
\begin{align*}
g_{i j}(\tau) & =\delta_{i j} \frac{R_{0}^{2}}{3 \tau^{2}}\left[1+\frac{\tau}{N} \int \mathrm{D} Q h_{\text {total }}\left(Q^{2}\right) \frac{\bar{u}\left(Q^{2}\right)}{\left(1+\bar{u}\left(Q^{2}\right) \tau / 2\right)^{2}}\right]  \tag{75}\\
g_{00}(\tau) & =\frac{R_{0}^{2}}{2 \tau^{2}}\left[1+\frac{\tau}{2 N} \int \mathrm{D} Q h_{\text {total }}\left(Q^{2}\right) \frac{\bar{u}\left(Q^{2}\right)\left(1+3 \bar{u}\left(Q^{2}\right) \tau / 2\right)}{\left(1+\bar{u}\left(Q^{2}\right) \tau / 2\right)^{3}}\right] \tag{76}
\end{align*}
$$

where $h_{\text {total }}\left(Q^{2}\right)$ is a function given in appendix D .

### 4.3 UV and IR limits

The above expression in the UV limit $(\tau \rightarrow 0)$ leads to

$$
\begin{array}{ll}
g_{i j}(\tau) \simeq \delta_{i j} \frac{R_{0}^{2}}{3 \tau^{2}}\left[1+\frac{\tau}{N} \int \mathrm{D} Q h_{\text {total }}\left(Q^{2}\right) \bar{u}\left(Q^{2}\right)\right], & \tau \rightarrow 0 \\
g_{00}(\tau) \simeq \frac{R_{0}^{2}}{2 \tau^{2}}\left[1+\frac{\tau}{2 N} \int \mathrm{D} Q h_{\mathrm{total}}\left(Q^{2}\right) \bar{u}\left(Q^{2}\right)\right], & \tau \rightarrow 0 \tag{78}
\end{array}
$$

which shows that the NLO correction is less singular than the LO contribution. Therefore the space becomes asymptotically AdS in the UV limit at the NLO whose AdS radius is equal to that at the LO.

We cannot naively take the $\tau \rightarrow \infty$ limit in eqs. (75) and (76), on the other hand, due to the enhancement of the UV contribution of the $Q$ integrals. Careful evaluations of these
$Q$ integrals in appendix $D$ give

$$
\begin{equation*}
g_{i j}(\tau) \simeq \delta_{i j} \frac{R_{0}^{2}}{3 \tau^{2}}\left[1+\frac{r}{N}\right], \quad g_{00}(\tau) \simeq \frac{R_{0}^{2}}{2 \tau^{2}}\left[1+\frac{r}{N}\right], \quad \tau \rightarrow \infty, \tag{79}
\end{equation*}
$$

where $r=-0.41869(1) .4$ Therefore, the space becomes asymptotically AdS again in the IR limit, whose radius, however, is smaller than that in the UV limit 5 The induced metric at the NLO describes a 4 dimensional space which is asymptotically AdS in both UV and IR regions with different radii but non-AdS in-between.

It is clear that the NLO correction to the AdS radius in the IR limit is related to the Wilson-Fisher fixed point in the original 3 dimensional scalar theory, since the eqs. (75) and (76) can be written as

$$
\begin{align*}
g_{i j}(\tau) & =\delta_{i j} \frac{R_{0}^{2}}{3 \tau^{2}}\left[1-\frac{1}{24 N} \int \mathrm{D} Q h_{\text {total }}\left(Q^{2}\right) \beta\left(g\left(48 \mu \sqrt{Q^{2}}\right)\right)\right]  \tag{80}\\
g_{00}(\tau) & =\frac{R_{0}^{2}}{2 \tau^{2}}\left[1-\frac{1}{24 N} \int \mathrm{D} Q h_{\text {total }}\left(Q^{2}\right)\left\{1+\frac{\mu}{2} \frac{\partial}{\partial \mu}\right\} \beta\left(g\left(48 \mu \sqrt{Q^{2}}\right)\right)\right] \tag{81}
\end{align*}
$$

where $\mu=1 / \sqrt{t}=2 / \tau$, and $\beta(g(x))$ is the $\beta$ function for the running coupling $g(x)$ from the free flowed field defined in the previous section with $\Delta=1$ as

$$
\begin{equation*}
\beta(g)=\frac{g(g-48)}{48}, \quad g(x)=48 \frac{u}{x+u} \tag{82}
\end{equation*}
$$

## 5 Summary

In this paper, we studied the flow equation of the $\mathrm{O}(N) \varphi^{4}$ model in $d$ dimensions at the NLO in the $1 / N$ expansion, employing the Schwinger-Dyson equation. We calculated the 2-pt and 4-pt functions at the NLO.

As an application of the NLO calculation, we investigated the running coupling defined from the connected 4-pt function of flowed fields. In particular at $d=3$ in the massless limit, we showed that the running coupling has two fixed points, the asymptotic free one in the UV region and the Wilson-Fisher one in the IR region. We also derived the corresponding $\beta$

[^3]function. Our study suggests that the flow equation can be interpreted as a renormalization group transformation.

We also calculated the NLO correction to the $d+1$ dimensional metric induced from the massless scalar field theory at $d=3$. In the massless limit, the whole 4 -dimensional space becomes AdS at the LO of the $1 / N$ expansion [1]. We found that the NLO corrections give small perturbations to the metric, which make the space only asymptotically AdS in both UV $(t=0)$ and $\operatorname{IR}(t=\infty)$ limits. In addition, while the NLO corrections do not change the AdS radius at the LO in the UV limit, the AdS radius is reduced by the NLO correction in the IR limit, which corresponds to the Wilson-Fisher IR fixed point of the original theory. The nontrivial fixed point in the field theory appears as a change of the AdS radius at the NLO. The induced metric at NLO describes a 4-dimensional space which connects one asymptotically AdS space at UV to the other asymptotically AdS space at IR.

This paper contains two important messages. One is that the flow equation can provide an alternative method to define a renormalization group transformation. The other is that the massless scalar field in $d$ dimensions plus the extra dimension from the RG scale not only generates a $d+1$ dimensional AdS space at LO [1] but also gives a NLO correction, which makes the $d+1$ dimensional space asymptotically AdS only in UV and IR limits at $d=3$. In particular, the AdS radius in the IR limit, which corresponds to the Wilson-Fisher fixed point, becomes smaller than that in the UV limit, which is equal to the radius at the LO. Although the relation found in this paper between the massless scalar field theory and the induced geometry is very suggestive, further studies will be needed to establish an alternative interpretation of AdS/CFT correspondences proposed in Ref. [2] in terms of field theories.

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## References

[1] S. Aoki, J. Balog, T. Onogi and P. Weisz, PTEP 2016 (2016) no.8, 083B04 doi:10.1093/ptep/ptw106 arXiv:1605.02413 [hep-th].
[2] S. Aoki, K. Kikuchi and T. Onogi, PTEP 2015 (2015) no.10, $101 \mathrm{B01}$ doi:10.1093/ptep/ptv131 arXiv:1505.00131 [hep-th]].
[3] R. Narayanan and H. Neuberger, JHEP 0603, 064 (2006) hep-th/0601210.
[4] M. Lüscher, JHEP 1008, 071 (2010) [JHEP 1403, 092 (2014)] arXiv:1006.4518 [hep-lat]].
[5] M. Lüscher, Commun. Math. Phys. 293, 899 (2010) arXiv:0907.5491 [hep-lat]].
[6] M. Lüscher, PoS LATTICE 2013, 016 (2014) arXiv:1308.5598 [hep-lat]].
[7] J. M. Maldacena, Int. J. Theor. Phys. 38, 1113 (1999) [Adv. Theor. Math. Phys. 2, 231 (1998) ] hep-th/9711200.
[8] I. R. Klebanov and A. M. Polyakov, Phys. Lett. B 550 (2002) 213 doi:10.1016/S0370-2693(02)02980-5 hep-th/0210114.
[9] S. Aoki, K. Kikuchi and T. Onogi, JHEP 1504, 156 (2015) arXiv:1412.8249 [hep-th]].
[10] S. Aoki, J. Balog and P. Weisz, JHEP 1409 (2014) 167 Erratum: [JHEP 1507 (2015) 037] doi:10.1007/JHEP07(2015)037, 10.1007/JHEP09(2014)167 arXiv:1407.7079 [hep-lat]].
[11] S. S. Pufu, arXiv:1608.02960 [hep-th].

## A The $1 / N$ expansion in the $d$ dimensional theory

In this appendix, we consider the $1 / N$ expansion in the $d$ dimensional theory.

## A. 1 Schwinger-Dyson equation(SDE)

In order to perform the $1 / N$ expansion, we consider the SDE of this model, which can be written compactly as

$$
\begin{equation*}
\left\langle\delta_{x}^{a} X[\varphi]\right\rangle=\left\langle X[\varphi] \delta_{x}^{a} S\left(\mu^{2}, u\right)\right\rangle \tag{A1}
\end{equation*}
$$

where $\delta_{x}^{a} \varphi^{b}(y)=\delta^{a b} \delta^{(d)}(x-y) \epsilon$ with a small parameter $\epsilon$, so that

$$
\begin{equation*}
\delta_{x}^{a} S\left(\mu^{2}, u\right)=N \epsilon\left[\left(-\square+\mu^{2}\right) \varphi^{a}(x)+\frac{u}{3!} \varphi^{a}(x) \varphi^{2}(x)\right] \tag{A2}
\end{equation*}
$$

Here the vacuum expectation value of an operator $\mathcal{O}$ is defined in eq. (9)).
We define $2 n$-point functions $\Gamma_{2 n}{ }^{6}$ as

$$
\begin{equation*}
\Gamma^{a_{1} a_{2} \cdots a_{2 n}}\left(x_{1}, x_{2}, \cdots, x_{2 n}\right)=N^{2 n-1}\left\langle\prod_{i=1}^{2 n} \varphi^{a_{i}}\left(x_{i}\right)\right\rangle \equiv \Gamma_{2 n}[12 \cdots(2 n)] \tag{A3}
\end{equation*}
$$

which can be written in terms of their connected parts $K_{2 n}$ as

$$
\begin{align*}
\Gamma_{4}[1234] & =K_{4}[1234]+N\left\{\Gamma_{2}[12] \Gamma_{2}[34]+\Gamma_{2}[13] \Gamma_{2}[24]+\Gamma_{2}[14] \Gamma_{2}[23]\right\},  \tag{A4}\\
\Gamma_{6}[123456] & =K_{6}[123456]+N\left\{\Gamma_{2}[12] K_{4}[3456]+14 \text { perms. }\right\} \\
& +N^{2}\left\{\Gamma_{2}[12] \Gamma_{2}[34] \Gamma_{2}[56]+14 \text { perms. }\right\} \tag{A5}
\end{align*}
$$

[^4]and so on. As mentioned in the main text, we assume we are working in a phase where $\mathrm{O}(N)$ symmetry is not broken. We therefore do not add the external source term $h \varphi(x)$ to the action, so that the action has the symmetry under $\varphi \rightarrow-\varphi$, which implies $\Gamma_{2 n-1}=0$ for all positive integers $n$.

In terms of these, the SDE for $X(\varphi)=\varphi^{a_{2}}\left(x_{2}\right)$ becomes

$$
\begin{equation*}
\delta_{12}=\left(-\square+\mu^{2}\right)_{x_{1}} \Gamma_{2}[12]+\frac{u}{3!N^{2}} \sum_{b}\left(K_{4}[b b 12]+N\left\{\Gamma_{2}[b b] \Gamma_{2}[12]+2 \Gamma_{2}[b 1] \Gamma_{2}[b 2]\right\}\right) \tag{A6}
\end{equation*}
$$

where $\delta_{12} \equiv \delta^{a_{1} a_{2}} \delta^{(d)}\left(x_{1}-x_{2}\right)$ and $x_{b}=x_{1}$, so that $b$ in the summation runs over the $\mathrm{O}(N)$ indices only.

For $X(\varphi)=\varphi^{a_{2}}\left(x_{2}\right) \varphi^{a_{3}}\left(x_{3}\right) \varphi^{a_{4}}\left(x_{4}\right)$, on the other hand, we have

$$
\begin{align*}
\delta_{12} \Gamma_{2}[34]+2 \text { perms. } & =\left(-\square+\mu^{2}\right)_{x_{1}} \frac{1}{N}\left(K_{4}[1234]+N\left\{\Gamma_{2}[12] \Gamma_{2}[34]+2 \text { perms. }\right\}\right) \\
& +\frac{u}{3!N^{3}} \sum_{b}\left(K_{6}[b b 1234]+N\left\{\Gamma_{2}[b b] K_{4}[1234]+14 \text { perms. }\right\}\right. \\
& \left.+N^{2}\left\{\Gamma_{2}[b b] \Gamma_{2}[12] \Gamma_{2}[34]+14 \text { perms. }\right\}\right) \tag{A7}
\end{align*}
$$

which can be simplified by using eq. (A6) as

$$
\begin{align*}
0 & =\left(-\square+\mu^{2}\right)_{x_{1}} K_{4}[1234]+\frac{u}{3!N^{2}} \sum_{b}\left(K_{6}[b b 1234]+N \Gamma_{2}[b b] K_{4}[1234]\right. \\
& +2 N\left\{\Gamma_{2}[b 1] K_{4}[b 234]+\Gamma_{2}[b 2] K_{4}[1 b 34]+\Gamma_{2}[b 3] K_{4}[12 b 4]+\Gamma_{2}[b 4] K_{4}[123 b]\right\} \\
& +N\left\{\Gamma_{2}[12] K_{4}[b b 34]+\Gamma_{2}[13] K_{4}[b 2 b 4]+\Gamma_{2}[14] K_{4}[b 23 b]\right\} \\
& +2 N^{2}\left\{\Gamma _ { 2 } [ b 2 ] \left[\Gamma_{2}[b 3] \Gamma_{2}[14]+\Gamma_{2}[b 2]\left[\Gamma_{2}[b 4] \Gamma_{2}[13]+\Gamma_{2}[b 3]\left[\Gamma_{2}[b 4] \Gamma_{2}[12]\right\}\right) .\right.\right. \tag{A8}
\end{align*}
$$

Using the $\mathrm{O}(N)$ symmetry and assuming translational invariance (e.g. infinite volume or periodic boundary condition), we can write

$$
\begin{align*}
\Gamma_{2}[12] & \equiv \delta^{a_{1} a_{2}} \Gamma\left(x_{12}\right), \quad x_{12} \equiv x_{1}-x_{2}  \tag{A9}\\
K_{4}[1234] & \equiv \delta^{a_{1} a_{2}} \delta^{a_{3} a_{4}} K\left(x_{1}, x_{2} ; x_{3}, x_{4}\right)+2 \text { perms. }  \tag{A10}\\
K_{6}[123456] & \equiv \delta^{a_{1} a_{2}} \delta^{a_{3} a_{4}} \delta^{a_{5} a_{6}} H\left(x_{1}, x_{2} ; x_{3}, x_{4} ; x_{5}, x_{6}\right)+14 \text { perms. } \tag{A11}
\end{align*}
$$

where $K\left(x_{1}, x_{2} ; x_{3}, x_{4}\right)$ is invariant under $1 \leftrightarrow 2$ or $3 \leftrightarrow 4$ as well as $(12) \leftrightarrow(34)$, and similar invariances hold for $H\left(x_{1}, x_{2} ; x_{3}, x_{4} ; x_{5}, x_{6}\right)$.

We finally obtain

$$
\begin{align*}
\delta^{(d)}\left(x_{1}-x_{2}\right) & =\left[\left(-\square+\mu^{2}\right)_{x_{1}}+\frac{u}{3!} \Gamma(0)\right] \Gamma\left(x_{12}\right) \\
& +\frac{u}{3!N}\left[\left(1+\frac{2}{N}\right) K\left(x_{1}, x_{1} ; x_{1}, x_{2}\right)+2 \Gamma(0) \Gamma\left(x_{12}\right)\right] \tag{A12}
\end{align*}
$$

and

$$
\begin{align*}
0 & =\left[\left(-\square+\mu^{2}\right)_{x_{1}}+\frac{u}{3!}\left(1+\frac{2}{N}\right) \Gamma(0)\right] K\left(x_{1}, x_{2} ; x_{3}, x_{4}\right) \\
& +\frac{u}{3!} \Gamma\left(x_{12}\right)\left[2 \Gamma\left(x_{13}\right) \Gamma\left(x_{14}\right)+\left(1+\frac{2}{N}\right) K\left(x_{1}, x_{1} ; x_{3}, x_{4}\right)+\frac{2}{N} K\left(x_{1}, x_{3} ; x_{1}, x_{4}\right)\right] \\
& +\frac{u}{3!N}\left[\left(1+\frac{2}{N}\right) H\left(x_{1}, x_{1} ; x_{1}, x_{2} ; x_{3}, x_{4}\right)+\frac{2}{N} H\left(x_{1}, x_{2} ; x_{1}, x_{3} ; x_{1}, x_{4}\right)\right] \\
& +\frac{2 u}{3!N}\left[\Gamma\left(x_{13}\right) K\left(x_{1}, x_{2} ; x_{1}, x_{4}\right)+\Gamma\left(x_{14}\right) K\left(x_{1}, x_{2} ; x_{3}, x_{1}\right)\right] . \tag{A13}
\end{align*}
$$

## A. 2 The leading order in the $1 / N$ expansion

We introduce the $1 / N$ expansion as

$$
\begin{equation*}
\Gamma\left(x_{12}\right)=\sum_{i=0}^{\infty} N^{-i} \Gamma_{i}\left(x_{12}\right), \quad K\left(x_{1}, x_{2} ; x_{3}, x_{4}\right)=\sum_{i=0}^{\infty} N^{-i} K_{i}\left(x_{1}, x_{2} ; x_{3}, x_{4}\right) \tag{A14}
\end{equation*}
$$

and so on, together with $\mu^{2}=\sum_{i=0}^{\infty} N^{-i} \mu_{i}^{2}$.
At the leading order (LO) of the $1 / N$ expansion, the eq. (A12) in momentum space becomes

$$
\begin{equation*}
1=\left(p^{2}+\mu_{0}^{2}+\frac{u}{6} \int \mathrm{D} q \widetilde{\Gamma}_{0}(q)\right) \widetilde{\Gamma}_{0}(p), \quad \Gamma_{0}(x)=\int D p \widetilde{\Gamma}_{0}(p) \mathrm{e}^{i p x} \tag{A15}
\end{equation*}
$$

which can easily be solved as

$$
\begin{equation*}
\widetilde{\Gamma}_{0}(p)=\frac{1}{p^{2}+m^{2}}, \quad m^{2}=\mu_{0}^{2}+\frac{u}{6} Z(m) \tag{A16}
\end{equation*}
$$

where $m \geq 0$ is the renormalized mass and $Z(m)$ is given in eq. (6). Thus the 2-pt function at the LO becomes

$$
\begin{equation*}
\left\langle\varphi^{a}(x) \varphi^{b}(y)\right\rangle=\frac{\delta^{a b}}{N} \int \mathrm{D} p \frac{\mathrm{e}^{i p(x-y)}}{p^{2}+m^{2}} \tag{A17}
\end{equation*}
$$

Eq. (A13) at the LO leads to

$$
\begin{align*}
\left(-\square+m^{2}\right)_{x_{1}} K_{0}\left(x_{1}, x_{2} ; x_{3}, x_{4}\right) & +\frac{u}{3!} \Gamma_{0}\left(x_{12}\right) K_{0}\left(x_{1}, x_{1} ; x_{3}, x_{4}\right) \\
& =-\frac{2 u}{3!} \Gamma_{0}\left(x_{12}\right) \Gamma_{0}\left(x_{13}\right) \Gamma_{0}\left(x_{14}\right) \tag{A18}
\end{align*}
$$

Introducing a function $G_{0}\left(p_{1}, p_{2}, p_{3}, p_{4}\right)$ to rewrite $K_{0}\left(x_{1}, x_{2}, x_{3}, x_{4}\right)$ as

$$
\begin{equation*}
K_{0}\left(x_{1}, x_{2} ; x_{3}, x_{4}\right)=\left\{\prod_{i=1}^{4} \int \mathrm{D} p_{i} \frac{\mathrm{e}^{i p_{i} x_{i}}}{p_{i}^{2}+m^{2}}\right\} G_{0}\left(p_{1}, p_{2}, p_{3}, p_{4}\right)(2 \pi)^{d} \delta\left(p_{1}+p_{2}+p_{3}+p_{4}\right) \tag{A19}
\end{equation*}
$$

we obtain

$$
\begin{equation*}
G_{0}\left(p_{1}, p_{2}, p_{3}, p_{4}\right)=G_{0}\left(p_{1}+p_{2}\right)=-\frac{2 u}{6+u B\left(p_{12}^{2}\right)} \tag{A20}
\end{equation*}
$$

where $p_{12}=p_{1}+p_{2}$, and

$$
\begin{equation*}
B\left(Q^{2}\right)=\int \mathrm{D} q_{1} \mathrm{D} q_{2} \frac{(2 \pi)^{d} \delta\left(q_{1}+q_{2}-Q\right)}{\left(q_{1}^{2}+m^{2}\right)\left(q_{2}^{2}+m^{2}\right)}=\int_{0}^{1} \mathrm{~d} x \int \mathrm{D} q_{1} \frac{\theta\left(\Lambda^{2}-q_{1}^{2}\right)}{\left(q_{1}^{2}+m^{2}+x(1-x) Q^{2}\right)^{2}} . \tag{A21}
\end{equation*}
$$

This agrees with the previous result obtained by a different method[10]. We here specify the way we introduce the cut-off $\Lambda$ for the case where $B\left(Q^{2}\right)$ diverges.

## A. 3 NLO correction to the 2-pt functions

Let us consider the next-to-leading order (NLO) correction to the 2-pt function $\Gamma_{2}$. At the NLO, eq. (A12) leads to

$$
\begin{align*}
0 & =\left(-\square+m^{2}\right) \Gamma_{1}\left(x_{12}\right)+\left\{\frac{u}{6}\left(2 Z(m)+\gamma_{1}\right)+\mu_{1}^{2}\right\} \Gamma_{0}\left(x_{12}\right)+\frac{u}{6} K_{0}\left(x_{1}, x_{1} ; x_{1}, x_{2}\right),(  \tag{A22}\\
\gamma_{1} & =\int \mathrm{D} q \widetilde{\Gamma}_{1}(q) \tag{A23}
\end{align*}
$$

which can be solved in momentum space as

$$
\begin{equation*}
\widetilde{\Gamma}_{1}(p)=-\frac{1}{\left(p^{2}+m^{2}\right)^{2}}\left(\mu_{1}^{2}+\frac{u}{6} \gamma_{1}+\frac{u}{3} S\left(p^{2}\right)\right) \tag{A24}
\end{equation*}
$$

where

$$
\begin{equation*}
S\left(p^{2}\right)=\int \frac{\mathrm{D} Q}{(p-Q)^{2}+m^{2}} \frac{6}{6+u B\left(Q^{2}\right)}, \tag{A25}
\end{equation*}
$$

and the condition for $\gamma_{1}$ is solved as

$$
\begin{equation*}
\gamma_{1}=-\frac{\mu_{1}^{2} B(0)+C_{2}}{1+\frac{u}{6} B(0)}, \quad C_{2} \equiv-\int \frac{\mathrm{D} Q}{\frac{6}{u}+B\left(Q^{2}\right)} \frac{d}{d m^{2}} B\left(Q^{2}\right) \tag{A26}
\end{equation*}
$$

Substituting eq. (A26) into eq. (A24), we finally obtain

$$
\begin{equation*}
\widetilde{\Gamma}_{1}(p)=-\frac{1}{\left(p^{2}+m^{2}\right)^{2}}\left\{g\left(p^{2}\right)+\tilde{C}\right\} \tag{A27}
\end{equation*}
$$

where

$$
\begin{align*}
g\left(p^{2}\right) & =\int \frac{\mathrm{D} Q}{\frac{6}{u}+B\left(Q^{2}\right)}\left\{\frac{1}{(Q+p)^{2}+m^{2}}+\frac{1}{(Q-p)^{2}+m^{2}}-\frac{2}{Q^{2}+m^{2}}\right\},  \tag{A28}\\
\tilde{C} & =C_{1}+\frac{\mu_{1}^{2}}{1+\frac{u}{6} B(0)}-\frac{C_{2}}{\frac{6}{u}+B(0)}, \quad C_{1}=\int \frac{\mathrm{D} Q}{\frac{6}{u}+B\left(Q^{2}\right)} \frac{2}{\left(Q^{2}+m^{2}\right)}, \tag{A29}
\end{align*}
$$

and $g\left(p^{2}\right)$ can be expanded as

$$
\begin{equation*}
g\left(p^{2}\right)=Z_{1} p^{2}+\tilde{g}\left(p^{2}\right), \quad \tilde{g}\left(p^{2}\right)=O\left(p^{4}\right) \tag{A30}
\end{equation*}
$$

where

$$
\begin{equation*}
Z_{1}=\frac{2}{d} \int \frac{\mathrm{D} Q}{6 / u+B\left(Q^{2}\right)}\left[\frac{4-d}{\left(Q^{2}+m^{2}\right)^{2}}-\frac{4 m^{2}}{\left(Q^{2}+m^{2}\right)^{3}}\right] \tag{A31}
\end{equation*}
$$

## A. 4 Renormalization

Let us now consider the renormalization of the theory.
Our renormalization condition for the renormalized 2 -pt function $\Gamma_{R}$ is given in momentum space as

$$
\begin{equation*}
\widetilde{\Gamma}_{R}^{-1}(p) \simeq p^{2}+m^{2}, \quad p^{2} \simeq 0 \tag{A32}
\end{equation*}
$$

where $m$ is interpreted as the renormalized mass, which is independent of both $N$ and the cut-off. Relating the bare field to the renormalized field by the renormalization constant $Z_{R}$ as $Z_{R}^{1 / 2} \varphi_{R}=\varphi$, we explicitly obtain

$$
\begin{equation*}
Z_{R} \widetilde{\Gamma}_{R}(p)=\widetilde{\Gamma}(p)=\frac{1}{p^{2}+m^{2}+\frac{1}{N} \Sigma_{1}\left(p^{2}\right)}+O\left(\frac{1}{N^{2}}\right) \tag{A33}
\end{equation*}
$$

where

$$
\begin{equation*}
\Sigma_{1}\left(p^{2}\right)=Z_{1} p^{2}+\tilde{C}+\tilde{g}\left(p^{2}\right) . \tag{A34}
\end{equation*}
$$

At the LO of the $1 / N$ expansion, the above condition implies

$$
\begin{equation*}
\mu_{0}^{2}=m^{2}-\frac{u}{6} Z(m), \quad Z_{R}=1 \tag{A35}
\end{equation*}
$$

where $Z(m)$ is potentially divergent at $d>1$. We therefore introduce the momentum cutoff $\Lambda$ to regulate the integral, and $\mu_{0}^{2}$ is tuned to cancel the effect of $Z(m)$ including such divergences, in order to keep the renormalized mass $m$ finite and constant. The lattice regularization or dimensional regularization is more consistent than the momentum cut-off, but
calculations become much more complicated in the lattice regularization or power divergences are difficult to deal with in the dimensional regularization. Since the momentum cut-off is enough to see the leading divergences, we adopt it in this paper.

At the NLO, the renormalization condition implies

$$
\begin{equation*}
Z_{R}=1-\frac{Z_{1}}{N}, \quad \mu_{1}^{2}=\left(1+\frac{u}{6} B(0)\right) Z_{1} m^{2}+\frac{u}{6} C \tag{A36}
\end{equation*}
$$

where

$$
\begin{equation*}
C=-\int \frac{\mathrm{D} Q}{\frac{6}{u}+B\left(Q^{2}\right)}\left[\frac{d B\left(Q^{2}\right)}{d m^{2}}+2 \frac{\frac{6}{u}+B(0)}{Q^{2}+m^{2}}\right] . \tag{A37}
\end{equation*}
$$

The renormalization condition for the coupling, which first appears at the NLO of the $1 / N$ expansion, is given by $G_{0}\left(Q^{2}=s\right)=-u_{r}(s) / 3$, so that $u_{r}(s)$ is regarded as the renormalized coupling at the scale $s$. Eq. (A20) thus leads to

$$
\begin{equation*}
u_{r}(s)=\frac{u}{1+\frac{u}{6} B(s)}, \tag{A38}
\end{equation*}
$$

where $B\left(Q^{2}\right)$ is divergent at $d \geq 4$. Therefore the renormalized coupling goes to zero as

$$
\begin{equation*}
u_{r}(s) \simeq \frac{6}{B(s)} \rightarrow 0, \quad \Lambda \rightarrow \infty \tag{A39}
\end{equation*}
$$

at $d \geq 4$. This indicates the triviality of the $\varphi^{4}$ theory at $d \geq 4$.

## A. 5 Renormalization constants

We here explicitly evaluate the renormalization constants.

## A.5.1 $d=1$

At $d=1, \mu_{0}^{2}$ is finite as

$$
\begin{equation*}
Z(m)=\frac{1}{\pi m} \arctan \left(\frac{\Lambda}{m}\right) \tag{A40}
\end{equation*}
$$

is finite, and the coupling is also finite and nonzero since

$$
\begin{equation*}
B\left(Q^{2}\right)=\frac{1}{m\left(Q^{2}+4 m^{2}\right)} \simeq \frac{1}{m Q^{2}}+\cdots, \quad Q^{2} \rightarrow \infty \tag{A41}
\end{equation*}
$$

has a finite limit as $\Lambda \rightarrow \infty$.

The most divergent part of $Z_{1}$ is given by

$$
Z_{1} \simeq \begin{cases}\int \mathrm{D} Q \frac{u}{\left(Q^{2}+m^{2}\right)^{2}}, & u \neq \infty  \tag{A42}\\ \int \frac{\mathrm{D} Q}{B\left(Q^{2}\right)} \frac{6}{\left(Q^{2}+m^{2}\right)^{2}}, & u=\infty\end{cases}
$$

which shows that $Z_{1}$ is finite for all $u$ including $u=\infty$. Eqs. (A36) and (A37) thus tell us that $\mu_{1}^{2}$ is also finite for all $u$ including $u=\infty$, and therefore, there is no divergence at $d=1$ up to the NLO.

## A.5.2 $d=2$

At $d=2, \mu_{0}^{2}$ is logarithmically divergent as

$$
\begin{equation*}
\mu_{0}^{2}=m^{2}-\frac{u}{6} Z(m), \quad Z(m) \simeq \frac{1}{4 \pi} \log \left(\frac{\Lambda^{2}+m^{2}}{m^{2}}\right) \tag{A43}
\end{equation*}
$$

On the other hand, $B\left(Q^{2}\right)$ is finite as

$$
\begin{align*}
B\left(Q^{2}\right) & =\frac{\tanh ^{-1}\left(\sqrt{\frac{Q^{2}}{Q^{2}+4 m^{2}}}\right)}{\pi \sqrt{Q^{2}\left(Q^{2}+4 m^{2}\right)}} \simeq \frac{1}{2 \pi Q^{2}} \log \frac{Q^{2}}{m^{2}}-\frac{m^{2}}{\pi\left(Q^{2}\right)^{2}}\left(\log \frac{Q^{2}}{m^{2}}-1\right)+\cdots,  \tag{A44}\\
\frac{d B\left(Q^{2}\right)}{d m^{2}} & \simeq-\frac{2 B(0)}{Q^{2}+4 m^{2}}\left[1+\frac{2 m^{2}}{Q^{2}} \log \frac{Q^{2}}{m^{2}}+\cdots\right], \quad B(0)=\frac{1}{4 \pi m^{2}}, \tag{A45}
\end{align*}
$$

so that the renormalized coupling becomes

$$
\begin{equation*}
u_{r}(s)=\frac{6 u}{6+u \frac{\tanh ^{-1}\left(\sqrt{\frac{s}{s+4 m^{2}}}\right)}{\pi \sqrt{s\left(s+4 m^{2}\right)}}} \simeq \frac{12 \pi u s}{12 \pi s+u \log \left(s / m^{2}\right)}, \quad s \rightarrow \infty \tag{A46}
\end{equation*}
$$

The most singular term of $Z_{1}$ for $u \neq \infty$ becomes

$$
\begin{equation*}
Z_{1} \simeq \frac{u}{6} \int \mathrm{D} Q \frac{2}{\left(Q^{2}+m^{2}\right)^{2}} \tag{A47}
\end{equation*}
$$

which is manifestly finite, while at $u=\infty$, we have

$$
\begin{equation*}
Z_{1}=\int \frac{\mathrm{D} Q}{B\left(Q^{2}\right)}\left[\frac{2}{\left(Q^{2}+m^{2}\right)^{2}}-\frac{4 m^{2}}{\left(Q^{2}+m^{2}\right)^{3}}\right] \tag{A48}
\end{equation*}
$$

which diverges as $Z_{1} \simeq \log \left(\log \Lambda^{2}\right)$.
The most divergent part of $\mu_{1}^{2}$ is given by

$$
\mu_{1}^{2} \simeq\left\{\begin{array}{cl}
-\frac{u}{3} Z(m) \delta_{1}, \quad\left(\delta_{1}=1\right), & u \neq \infty  \tag{A49}\\
\frac{u}{12 \pi} \log \left(\frac{\Lambda^{2}+4 m^{2}}{4 m^{2}}\right), & u \rightarrow \infty
\end{array}\right.
$$

## A.5.3 $d=3$

At $d=3, \mu_{0}^{2}$ is linearly divergent as

$$
\begin{equation*}
\mu_{0}^{2}=m^{2}-\frac{u}{6} Z(m), \quad Z(m) \simeq \frac{1}{2 \pi^{2}}\left[\Lambda-m \arctan \left(\frac{\Lambda}{m}\right)\right] \tag{A50}
\end{equation*}
$$

while $B\left(Q^{2}\right)$ is finite as

$$
\begin{align*}
B\left(Q^{2}\right) & =\frac{1}{4 \pi \sqrt{Q^{2}}} \arctan \left(\sqrt{\frac{Q^{2}}{4 m^{2}}}\right) \simeq \frac{1}{8|Q|}-\frac{m}{2 \pi Q^{2}}+\frac{2 m^{3}}{3 \pi\left(Q^{2}\right)^{2}}+\cdots  \tag{A51}\\
\frac{d B\left(Q^{2}\right)}{d m^{2}} & =-\frac{2 B(0)}{Q^{2}+4 m^{2}}, \quad B(0)=\frac{1}{8 \pi m} \tag{A52}
\end{align*}
$$

and the renormalized coupling becomes

$$
\begin{equation*}
u_{r}(s)=\frac{6 u}{6+\frac{u}{4 \pi \sqrt{s}} \arctan \left(\sqrt{\frac{s}{4 m^{2}}}\right)} \simeq \frac{u}{1+\frac{u}{48 \sqrt{s}}}, \quad s \rightarrow \infty \tag{A53}
\end{equation*}
$$

The most singular term of $Z_{1}$ for $u \neq \infty$ becomes

$$
\begin{equation*}
Z_{1} \simeq \frac{u}{9} \int \mathrm{D} Q \frac{1}{\left(Q^{2}+m^{2}\right)^{2}} \tag{A54}
\end{equation*}
$$

which is manifestly finite at $d=3$. On the other hand, at $u=\infty$, we have

$$
\begin{equation*}
Z_{1}=\frac{2}{3} \int \frac{\mathrm{D} Q}{B\left(Q^{2}\right)}\left[\frac{1}{\left(Q^{2}+m^{2}\right)^{2}}-\frac{4 m^{2}}{\left(Q^{2}+m^{2}\right)^{3}}\right] \tag{A55}
\end{equation*}
$$

whose divergent part becomes

$$
\begin{equation*}
Z_{1} \simeq \frac{4}{3 \pi^{2}} \log \Lambda^{2} \tag{A56}
\end{equation*}
$$

The most divergent part of $\mu_{1}^{2}$ becomes

$$
\mu_{1}^{2} \simeq \begin{cases}-\frac{u}{3} Z(m) \delta_{1}, \quad\left(\delta_{1}=1\right), & u \neq \infty  \tag{A57}\\ -m \frac{2 u}{9 \pi^{3}} \log \Lambda^{2}, & u \rightarrow \infty\end{cases}
$$

A.5.4 $d=4$

At $d=4, \mu_{0}^{2}$ is quadratically divergent as

$$
\begin{equation*}
\mu_{0}^{2}=m^{2}-\frac{u}{6} Z(m), \quad Z(m) \simeq \frac{1}{16 \pi^{2}}\left[\Lambda^{2}-m^{2} \log \left(\frac{\Lambda^{2}+m^{2}}{m^{2}}\right)\right] \tag{A58}
\end{equation*}
$$

On the other hand, at $d=4$, we have

$$
\begin{align*}
B\left(Q^{2}\right)= & \frac{1}{(4 \pi)^{2}}\left[\log \left(\frac{\Lambda_{m}^{2}}{m^{2}}\right)+2 \frac{Q^{2}+4 \Lambda_{m}^{2}-2 \Lambda^{2}}{\sqrt{Q^{2}\left(Q^{2}+4 \Lambda_{m}^{2}\right)}} \tanh ^{-1} \sqrt{\frac{Q^{2}}{Q^{2}+4 \Lambda_{m}^{2}}}\right. \\
& \left.-2 \sqrt{\frac{Q^{2}+4 m^{2}}{Q^{2}}} \tanh ^{-1} \sqrt{\frac{Q^{2}}{Q^{2}+4 m^{2}}}\right]  \tag{A59}\\
B(0)= & \frac{1}{(4 \pi)^{2}}\left[\log \frac{\Lambda_{m}^{2}}{m^{2}}-\frac{\Lambda^{2}}{\Lambda_{m}^{2}}\right], \quad \Lambda_{m}^{2} \equiv \Lambda^{2}+m^{2} \tag{A60}
\end{align*}
$$

which diverge logarithmically, so that $u_{r}(s)=0$ as $\Lambda \rightarrow \infty$.
Since $\tanh ^{-1}(x) \simeq_{x \rightarrow 1}-\frac{1}{2} \log \left(\frac{1-x}{2}\right)$, we have

$$
\begin{align*}
B\left(Q^{2}\right)+\frac{6}{u} & =\hat{B}\left(q^{2}, \alpha^{2}\right), \quad q^{2}=\frac{Q^{2}}{\Lambda^{2}}, \quad \alpha=\frac{m}{\Lambda}  \tag{A61}\\
\hat{B}\left(q^{2}, 0\right) & =-c_{0} \log q^{2}+\frac{6}{u}+c_{0} F\left(q^{2}\right), \quad c_{0}=\frac{1}{(4 \pi)^{2}} \tag{A62}
\end{align*}
$$

where

$$
\begin{equation*}
F\left(q^{2}\right)=\frac{2\left(q^{2}+2\right)}{\sqrt{q^{2}\left(q^{2}+4\right)}} \tanh ^{-1} \sqrt{\frac{q^{2}}{q^{2}+4}} \tag{A63}
\end{equation*}
$$

Let us now consider the continuum limit of $Z_{1}$. By rescaling the momentum, we have

$$
\begin{equation*}
Z_{1}=-\frac{\alpha^{2}}{8 \pi^{2}} \int_{0}^{1} \frac{t \mathrm{~d} t}{\hat{B}\left(t, \alpha^{2}\right)\left(t+\alpha^{2}\right)^{3}} \tag{A64}
\end{equation*}
$$

As $\alpha^{2} \rightarrow 0$ in the $\Lambda \rightarrow \infty$ limit, we have

$$
\begin{align*}
\int_{0}^{1} \frac{t \mathrm{~d} t}{\hat{B}\left(t, \alpha^{2}\right)\left(t+\alpha^{2}\right)^{3}} & \simeq \int_{0}^{1} \frac{t \mathrm{~d} t}{\hat{B}(t, 0)\left(t+\alpha^{2}\right)^{3}} \\
& =\int_{0}^{\frac{1}{2}} \frac{t \mathrm{~d} t}{\hat{B}(t, 0)\left(t+\alpha^{2}\right)^{3}}+\int_{\frac{1}{2}}^{1} \frac{t \mathrm{~d} t}{\hat{B}(t, 0)\left(t+\alpha^{2}\right)^{3}} \tag{A65}
\end{align*}
$$

where the second term is finite in this limit, while the first term is bounded from above

$$
\begin{align*}
\int_{0}^{\frac{1}{2}} \frac{t \mathrm{~d} t}{\hat{B}(t, 0)\left(t+\alpha^{2}\right)^{3}} & \leq-\frac{1}{c_{0}} \int_{0}^{\frac{1}{2}} \frac{t \mathrm{~d} t}{\left(t+\alpha^{2}\right)^{3} \log \left(t+\alpha^{2}\right)} \\
& =\frac{1}{c_{0}}\left[\log \left|\log \alpha^{2}\right|+\sum_{r=1}^{\infty} \frac{\left(-\log \alpha^{2}\right)^{r}}{r r!}+(\text { finite terms })\right] \tag{A66}
\end{align*}
$$

so that $Z_{1}$ in eq. (A64) vanishes as $\alpha^{2} \rightarrow 0$.
The most divergent part of $\mu_{1}^{2}$ becomes

$$
\begin{align*}
\mu_{1}^{2} & \simeq-\frac{u}{3} \frac{\Lambda^{2}}{16 \pi^{2}} \delta_{1}, \quad \delta_{1}=\int_{0}^{1} \mathrm{~d} q^{2}\left(\frac{c_{0} T\left(q^{2}\right)-6 / u}{c_{0}\left\{\log q^{2}-F\left(q^{2}\right)\right\}-6 / u}\right),  \tag{A67}\\
T\left(q^{2}\right) & \equiv \log q^{2}+1-\frac{q^{2}}{q^{2}+4}\left(1+\frac{q^{2}+6}{q^{2}+2} F\left(q^{2}\right)\right),
\end{align*}
$$

where $\delta_{1}$ is finite, but is not universal as it depends on how we regulate the integral.
A.5.5 $d>4$

At $d>4, \mu_{0}^{2}$ is $O\left(\Lambda^{d-2}\right)$ as

$$
\begin{equation*}
\mu_{0}^{2}=m^{2}-\frac{u}{6} Z(m), \quad Z(m) \simeq \frac{d}{(4 \pi)^{d / 2}(d-2) \Gamma(1+d / 2)} \Lambda^{d-2} \tag{A68}
\end{equation*}
$$

We also write

$$
\begin{equation*}
B\left(Q^{2}\right)=\frac{d}{(4 \pi)^{d / 2} \Gamma(1+d / 2)} \int_{0}^{1} \mathrm{~d} x \int_{0}^{\Lambda} \frac{p^{d-1} \mathrm{~d} p}{\left[p^{2}+m^{2}+Q^{2} x(1-x)\right]^{2}}, \tag{A69}
\end{equation*}
$$

from which we obtain

$$
\begin{align*}
B\left(Q^{2}\right) & =\Lambda^{d-4} \hat{B}\left(\frac{Q^{2}}{\Lambda^{2}}, \frac{m^{2}}{\Lambda^{2}}\right), \quad \hat{B}(0,0)=\frac{d}{(d-4)} \frac{1}{(4 \pi)^{d / 2} \Gamma(1+d / 2)},  \tag{A70}\\
\frac{d B\left(Q^{2}\right)}{d m^{2}} & =-2 \Lambda^{d-6} \hat{B}_{m}\left(\frac{Q^{2}}{\Lambda^{2}}, \frac{m^{2}}{\Lambda^{2}}\right) \tag{A71}
\end{align*}
$$

where

$$
\begin{align*}
\hat{B}\left(q^{2}, \alpha^{2}\right) & =\frac{d}{(4 \pi)^{d / 2} \Gamma(1+d / 2)} \int_{0}^{1} \mathrm{~d} x \int_{0}^{1} \frac{y^{d-1} \mathrm{~d} y}{\left[y^{2}+\alpha^{2}+q^{2} x(1-x)\right]^{2}},  \tag{A72}\\
\hat{B}_{m}\left(q^{2}, \alpha^{2}\right) & =\frac{d}{(4 \pi)^{d / 2} \Gamma(1+d / 2)} \int_{0}^{1} \mathrm{~d} x \int_{0}^{1} \frac{y^{d-1} \mathrm{~d} y}{\left[y^{2}+\alpha^{2}+q^{2} x(1-x)\right]^{3}} \tag{A73}
\end{align*}
$$

so that $B\left(Q^{2}\right)=O\left(\Lambda^{d-4}\right)$. As in the case at $d=4, u_{r}(s)=0$ in the limit that $\Lambda \rightarrow \infty$.
By the change of variable $Q=\Lambda q$ in eq. (A31) and then taking the limit $\Lambda \rightarrow \infty$, we obtain

$$
\begin{equation*}
Z_{1}=\frac{2(4-d)}{d} \int_{q^{2}<1} \frac{\mathrm{D} q}{\hat{B}\left(q^{2}, 0\right)} \frac{1}{\left(q^{2}\right)^{2}} \tag{A74}
\end{equation*}
$$

The fact that $\hat{B}(0,0) \neq 0$ establishes that $Z_{1}$ is finite at $d>4$.

The most divergent part of $\mu_{1}^{2}$ is given by

$$
\begin{equation*}
\mu_{1}^{2} \simeq-\frac{u}{3} Z(m) \delta_{1} \tag{A75}
\end{equation*}
$$

where

$$
\begin{equation*}
\delta_{1}=(d-2) \int_{0}^{1} \frac{q^{d-1} \mathrm{~d} q}{\hat{B}\left(q^{2}, 0\right)}\left(\frac{\hat{B}(0,0)}{q^{2}}-\hat{B}_{m}\left(q^{2}, 0\right)\right) \tag{A76}
\end{equation*}
$$

with the change of variables as $q^{2}=Q^{2} / \Lambda^{2}$. It is easy to show that $\delta_{1}$ is finite.

## B Solving the SED for the flow equation

In this appendix we explicitly solve the SDE in $d+1$ dimensions, in order to obtain the 2-pt and 4-pt functions for the flow fields at the NLO.

## B. 1 Solution for $\Gamma_{0}$

We first solve the equation at the LO for $\Gamma_{0}$. If we introduce one unknown function $F(t, p)$ as

$$
\begin{equation*}
\Gamma_{0}(12)=\int \mathrm{D} p \frac{F\left(t_{1}, p\right) F\left(t_{2}, p\right)}{p^{2}+m^{2}} \mathrm{e}^{-\left(p^{2}+\mu_{f}^{2}\right)\left(t_{1}+t_{2}\right)} \mathrm{e}^{i p\left(x_{1}-x_{2}\right)} \tag{B1}
\end{equation*}
$$

with the initial condition $F(0, p)=1$, we have

$$
\begin{align*}
D_{1}^{f} \Gamma_{0}(12) & =\int \mathrm{D} p \frac{\dot{F}\left(t_{1}, p\right) F\left(t_{2}, p\right)}{p^{2}+m^{2}} \mathrm{e}^{-\left(p^{2}+\mu_{f}^{2}\right)\left(t_{1}+t_{2}\right)} \mathrm{e}^{i p\left(x_{1}-x_{2}\right)}  \tag{B2}\\
-\frac{u_{f}}{6} \Gamma_{0}(12) \Gamma_{0}(11) & =-\frac{u_{f}}{6} \int \mathrm{D} p \frac{F\left(t_{1}, p\right) F\left(t_{2}, p\right)}{p^{2}+m^{2}} \mathrm{e}^{-\left(p^{2}+\mu_{f}^{2}\right)\left(t_{1}+t_{2}\right)} \mathrm{e}^{i p\left(x_{1}-x_{2}\right)} \Gamma_{0}\left(t_{1}\right),  \tag{B3}\\
\Gamma_{0}\left(t_{1}\right) & =\Gamma_{0}(11)=\int \mathrm{D} p \frac{F^{2}\left(t_{1}, p\right)}{p^{2}+m^{2}} \mathrm{e}^{-2\left(p^{2}+\mu_{f}^{2}\right) t_{1}} \tag{B4}
\end{align*}
$$

where $\dot{F}$ means a $t$-derivative of $F$. Then, the SDE (20) becomes

$$
\begin{equation*}
\frac{\dot{F}(t, p)}{F(t, p)}=-\frac{u_{f}}{6} \Gamma_{0}(t) \tag{B5}
\end{equation*}
$$

which tells us that $F(t, p)$ is independent of $p$, so we put $F(t, p)=F(t)$. The above equation is thus reduced to

$$
\begin{equation*}
\dot{F}(t)=-\frac{u_{f}}{6} F^{3}(t) \mathrm{e}^{-2 \mu_{f}^{2} t} \zeta_{0}(t) \tag{B6}
\end{equation*}
$$

where $\zeta_{0}(t)$ is defined in eq. (24), whose solution is given by

$$
\begin{equation*}
F^{-2}(t)=1+\frac{u_{f}}{3} \int_{0}^{t} \mathrm{~d} s \zeta_{0}(s) \mathrm{e}^{-2 \mu_{f}^{2} s} \equiv \mathrm{e}^{-2 \mu_{f}^{2} t} \frac{\zeta(t)}{Z\left(m_{f}\right)}, \quad \zeta(t)=\zeta_{0}(t)+\Delta(t) \tag{B7}
\end{equation*}
$$

where $m_{f}$ is defined in eq. (6) and

$$
\begin{equation*}
\Delta(t)=\mathrm{e}^{2 t \mu_{f}^{2}}\left(Z\left(m_{f}\right)-Z(m)\right)+\int \mathrm{D} p\left(\frac{p^{2}+m_{f}^{2}}{p^{2}+m^{2}}\right) \frac{\mathrm{e}^{2 t \mu_{f}^{2}}-\mathrm{e}^{-2 t p^{2}}}{p^{2}+\mu_{f}^{2}} \tag{B8}
\end{equation*}
$$

In the case of the interacting flow with $u_{f}>0, \mu_{f}^{2}$ negatively diverges as $Z\left(m_{f}\right) \rightarrow+\infty$ in the continuum limit at $d>1$ or as $u_{f} \rightarrow+\infty$ in the NLSM limit. In these limits, $\Delta(t)$ vanishes as

$$
\begin{equation*}
\lim _{\mu_{f}^{2} \rightarrow-\infty} \Delta(t) \simeq-\frac{m_{f}^{2} \zeta_{0}(t)-\dot{\zeta}_{0}(t) / 2}{\mu_{f}^{2}}+O\left(1 / \mu_{f}^{4}\right) \tag{B9}
\end{equation*}
$$

for $t>0$. In the case of free flow $\left(u_{f}=0\right)$, we simply have $F(t)=1$.
We then obtain

$$
\Gamma_{0}(12)= \begin{cases}\frac{Z\left(m_{f}\right)}{\sqrt{\zeta\left(t_{1}\right) \zeta\left(t_{2}\right)}} \int \mathrm{D} p \frac{\mathrm{e}^{-p^{2}\left(t_{1}+t_{2}\right)} \mathrm{e}^{i p\left(x_{1}-x_{2}\right)}}{p^{2}+m^{2}}, & u_{f} \neq 0  \tag{B10}\\ \int \mathrm{D} p \frac{\mathrm{e}^{-\left(p^{2}+\mu_{f}^{2}\right)\left(t_{1}+t_{2}\right)} \mathrm{e}^{i p\left(x_{1}-x_{2}\right)}}{p^{2}+m^{2}}, & u_{f}=0\end{cases}
$$

## B. 2 Solution for $K_{0}$

We consider $K_{0}$, which appears at the NLO. The equation for $K_{0}$ in eq. (22) is closed, once $\Gamma_{0}$ is obtained. Using eq. (26), we have

$$
\begin{align*}
D_{1}^{f} K_{0}(12 ; 34) & =\int \mathrm{d} P_{4}\left[\frac{\dot{F}\left(t_{1}\right)}{F\left(t_{1}\right)}+\partial_{t_{1}}\right] g(12 ; 34 \mid 12 ; 34) \\
& =\int \mathrm{d} P_{4}\left[-\frac{u_{f}}{6} F^{2}\left(t_{1}\right) \mathrm{e}^{-2 \mu_{f}^{2} t_{1}} \zeta_{0}\left(t_{1}\right)+\partial_{t_{1}}\right] g(12 ; 34 \mid 12 ; 34),  \tag{B11}\\
\Gamma_{0}(12) \Gamma_{0}(13) \Gamma_{0}(14) & =\int \mathrm{d} P_{4} \hat{\delta}\left(p_{1}^{2}+m^{2}\right) F^{2}\left(t_{1}\right) \mathrm{e}^{-2 \mu_{f}^{2} t_{1}} \mathrm{e}^{\left(p_{1}^{2}-p_{2}^{2}-p_{3}^{2}-p_{4}^{2}\right) t_{1}},  \tag{B12}\\
\Gamma_{0}(11) K_{0}(12 ; 34) & =F^{2}\left(t_{1}\right) \mathrm{e}^{-2 \mu_{f}^{2} t_{1}} \zeta_{0}\left(t_{1}\right) \int \mathrm{d} P_{4} g(12 ; 34 \mid 12 ; 34),  \tag{B13}\\
\Gamma_{0}(12) K_{0}(11 ; 34) & =F^{2}\left(t_{1}\right) \mathrm{e}^{-2 \mu_{f}^{2} t_{1}} \int \mathrm{~d} P_{4} \hat{\delta}\left(p_{1}^{2}+m^{2}\right) \mathrm{e}^{t_{1}\left(p_{1}^{2}-p_{2}^{2}\right)} \\
& \times \int \mathrm{D} q_{1} \mathrm{D} q_{2} \frac{\mathrm{e}^{-t_{1}\left(q_{1}^{2}+q_{2}^{2}\right)}}{\left(q_{1}^{2}+m^{2}\right)\left(q_{2}^{2}+m^{2}\right)} g\left(11 ; 34 \mid q_{1} q_{2} ; 34\right), \tag{B14}
\end{align*}
$$

so that the SDE leads to

$$
\begin{align*}
\partial_{t_{1}} g(12 ; 34 \mid 12 ; 34) & =-\frac{u_{f}}{6} F\left(t_{1}\right)^{2} \mathrm{e}^{-2 \mu_{f}^{2} t_{1}}\left(p_{1}^{2}+m^{2}\right) \mathrm{e}^{t_{1}\left(p_{1}^{2}-p_{2}^{2}\right)} \hat{\delta}\left[2 \mathrm{e}^{-t_{1}\left(p_{3}^{2}+p_{4}^{2}\right)}\right. \\
& \left.+\int \mathrm{D} q_{1} \mathrm{D} q_{2} \frac{\mathrm{e}^{-t_{1}\left(q_{1}^{2}+q_{2}^{2}\right)}}{\left(q_{1}^{2}+m^{2}\right)\left(q_{2}^{2}+m^{2}\right)} g\left(11 ; 34 \mid q_{1} q_{2} ; 34\right)\right] \tag{B15}
\end{align*}
$$

From eq. (B15), one can easily see $\partial_{t_{2}} \partial_{t_{1}} g(12 ; 34 \mid 12 ; 34)=0$, which implies

$$
\begin{align*}
g(12 ; 34 \mid 12 ; 34) & =X(23 \mid 12 ; 34)+X(13 \mid 21 ; 34)+X(24 \mid 12 ; 43)+X(14 \mid 21 ; 43) \\
& +Y(2 \mid 12 ; 34)+Y(1 \mid 21 ; 34)+Y(3 \mid 43 ; 12)+Y(4 \mid 34 ; 12) \\
& +Z(\mid 12 ; 34) \tag{B16}
\end{align*}
$$

where we require that $X$ and $Y$ satisfy

$$
\begin{align*}
X\left(\tau, \tau^{\prime} \mid 12 ; 34\right) & =X\left(\tau^{\prime}, \tau \mid 43 ; 21\right), \quad X(\tau, 0 \mid 12 ; 34)=0  \tag{B17}\\
Y(\tau \mid 12 ; 34) & =Y(\tau \mid 12 ; 43), \quad Y(0 \mid 12 ; 34)=0 \tag{B18}
\end{align*}
$$

Since $g(12 ; 34 \mid 12 ; 34)$ agrees with the amputated connected 4 -pt function in the $d$ dimensional theory at $\tau_{i}=0(i=1,2,3,4)$, we obtain

$$
\begin{equation*}
Z\left(\mid p_{1}, p_{2}, p_{3}, p_{4}\right)=-\hat{\delta} \frac{2}{6 / u+B\left(0 \mid p_{34}\right)} \tag{B19}
\end{equation*}
$$

where $B(t \mid Q)$ is defined in eq. (31). Then one can easily check that $g$ satisfies the required symmetries

$$
\begin{equation*}
g(12 ; 34 \mid 12 ; 34)=g(21 ; 34 \mid 21 ; 34)=g(12 ; 43 \mid 12 ; 43)=g(34 ; 12 \mid 34 ; 12) \tag{B20}
\end{equation*}
$$

## B.2.1 Solution for $Y$

Terms which depend only on $t_{1}$ in eq. (B15) can be written as

$$
\begin{align*}
\partial_{t} Y(t \mid 21 ; 34) & =-\frac{u_{f}}{3} F^{2}(t) \mathrm{e}^{-2 t \mu_{f}^{2}}\left(p_{1}^{2}+m^{2}\right) \mathrm{e}^{t\left(p_{1}^{2}-p_{2}^{2}\right)} \hat{\delta} \\
& \times\left[\rho(t \mid 34)+\int \mathrm{D} q_{1} \mathrm{D} q_{2} \frac{\mathrm{e}^{-t\left(q_{1}^{2}+q_{2}^{2}\right)} Y\left(t \mid q_{1}, q_{2} ; 34\right)}{\left(q_{1}^{2}+m^{2}\right)\left(q_{2}^{2}+m^{2}\right)}\right] \tag{B21}
\end{align*}
$$

where $\rho(t \mid 34)$ is defined in eq. (35). To solve this equation, we set

$$
\begin{equation*}
Y(t \mid 21 ; 34)=\hat{\delta}\left(p_{1}^{2}+m^{2}\right) \int_{0}^{t} \mathrm{~d} s \mathrm{e}^{s\left(p_{1}^{2}-p_{2}^{2}\right)} \psi(s \mid 34) \tag{B22}
\end{equation*}
$$

satisfying eq. (B18). Eq. (B21) is reduced to

$$
\begin{equation*}
\psi(t \mid 34)=-\frac{u_{f}}{3} F^{2}(t) \mathrm{e}^{-2 t \mu_{f}^{2}}\left[\rho(t \mid 34)+\int_{0}^{t} \mathrm{~d} s K\left(t, s \mid p_{34}\right) \psi(s \mid 34)\right] \tag{B23}
\end{equation*}
$$

which shows $\psi$ does not depend on $p_{1}, p_{2}$, where $K$ is defined in eq. (34). Since $u_{f} F^{2}(t) \mathrm{e}^{-2 t \mu_{f}^{2}}=u_{f} Z\left(m_{f}\right) / \zeta(t)$ goes to infinity in the continuum limit at $t>0$ and $d>1$ or in the NLSM limit $u_{f} \rightarrow \infty$, eq. (32) must hold in either of the two limits.

## B.2.2 Solution for $X$

We next consider the solution for $X$. Terms depending on both $t_{1}$ and $t_{3}$ in eq. (B15), and thereafter replacing $t_{3}$ by $t_{2}$ and interchanging $p_{1} \leftrightarrow p_{2}$, gives

$$
\begin{align*}
\partial_{t_{1}} X\left(t_{1}, t_{2} \mid 12 ; 34\right) & =-\frac{u_{f}}{6} F^{2}\left(t_{1}\right) \mathrm{e}^{-2 t_{1} \mu_{f}^{2}}\left(p_{2}^{2}+m^{2}\right) \mathrm{e}^{t_{1}\left(p_{2}^{2}-p_{1}^{2}\right)} \hat{\delta} \int \mathrm{D} q_{1} \mathrm{D} q_{2} \\
& \times \frac{\mathrm{e}^{-t_{1}\left(q_{1}^{2}+q_{2}^{2}\right)}}{\left(q_{1}^{2}+m^{2}\right)\left(q_{2}^{2}+m^{2}\right)}\left\{2 X\left(t_{1}, t_{2} \mid q_{1}, q_{2} ; 34\right)+Y\left(t_{2} \mid 43 ; q_{1}, q_{2}\right)\right\},( \tag{B24}
\end{align*}
$$

where

$$
\begin{equation*}
Y\left(t \mid 43 ; q_{1}, q_{2}\right)=(2 \pi)^{d} \delta\left(p_{34}+q_{12}\right)\left(p_{3}^{2}+m^{2}\right) \int_{0}^{t} \mathrm{~d} s \mathrm{e}^{s\left(p_{3}^{2}-p_{4}^{3}\right)} \psi\left(s \mid q_{1}, q_{2}\right) \tag{B25}
\end{equation*}
$$

We define

$$
\begin{equation*}
\partial_{t_{2}} \partial_{t_{1}} X\left(t_{1}, t_{2} \mid 12 ; 34\right)=\hat{\delta}\left(p_{2}^{2}+m^{2}\right)\left(p_{3}^{2}+m^{2}\right) \mathrm{e}^{t_{1}\left(p_{2}^{2}-p_{1}^{2}\right)} \mathrm{e}^{t_{2}\left(p_{3}^{2}-p_{4}^{2}\right)} \beta\left(t_{1}, t_{2} \mid 12 ; 34\right), \tag{B26}
\end{equation*}
$$

where properties of $X$ imply $\beta\left(t_{1}, t_{2} \mid 12 ; 34\right)=\beta\left(t_{2}, t_{1} \mid 43 ; 21\right)$ and $\beta(t, 0 \mid 12 ; 34)=$ $\beta(0, t \mid 12 ; 34)=0$. Then the above equation becomes

$$
\begin{align*}
\beta\left(t_{1}, t_{2} \mid 12 ; 34\right) & =-\frac{u_{f}}{6} F^{2}\left(t_{1}\right) \mathrm{e}^{-2 t_{1} \mu_{f}^{2}}\left[g\left(t_{1}, t_{2} \mid p_{34}\right)+2 \int_{0}^{t_{1}} \mathrm{~d} s_{1} \int \mathrm{D} q_{1} \mathrm{D} q_{2}(2 \pi)^{d} \delta\left(q_{12}+p_{34}\right)\right. \\
& \left.\times \frac{\mathrm{e}^{-\left(t_{1}+s_{1}\right) q_{1}^{2}-\left(t_{1}-s_{1}\right) q_{2}^{2}}}{q_{1}^{2}+m^{2}} \beta\left(s_{1}, t_{2} \mid q_{1}, q_{2} ; 34\right)\right] \tag{B27}
\end{align*}
$$

where

$$
\begin{equation*}
g\left(t_{1}, t_{2} \mid Q\right)=\int \mathrm{D} q_{1} \mathrm{D} q_{2}(2 \pi)^{d} \delta\left(q_{12}+Q\right) \frac{\mathrm{e}^{-t_{1}\left(q_{1}^{2}+q_{2}^{2}\right)}}{\left(q_{1}^{2}+m^{2}\right)\left(q_{2}^{2}+m^{2}\right)} \psi\left(t_{2} \mid q_{1}, q_{2}\right) \tag{B28}
\end{equation*}
$$

Since the above expression tells us that $\beta$ depends only on $p_{34}$, we can write

$$
\begin{equation*}
\beta\left(t_{1}, t_{2} \mid 12 ; 34\right)=\omega\left(t_{1}, t_{2} \mid p_{34}\right)=\omega\left(t_{1}, t_{2} \mid-p_{34}\right) \tag{B29}
\end{equation*}
$$

so that we have
$\omega\left(t_{1}, t_{2} \mid p_{34}\right)=-\frac{u_{f}}{6} F^{2}\left(t_{1}\right) \mathrm{e}^{-2 t_{1} \mu_{f}^{2}}\left[g\left(t_{1}, t_{2} \mid p_{34}\right)+2 \int_{0}^{t_{1}} \mathrm{~d} s_{1} K\left(t_{1}, s_{1} \mid p_{34}\right) \omega\left(s_{1}, t_{2} \mid p_{34}\right)\right]$,
which is reduced to

$$
\begin{equation*}
g\left(t_{1}, t_{2} \mid Q\right)+2 \int_{0}^{t_{1}} \mathrm{~d} s_{1} K\left(t_{1}, s_{1} \mid Q\right) \omega\left(s_{1}, t_{2} \mid Q\right)=0 \tag{B31}
\end{equation*}
$$

in the continuum limit or NLSM limit. Eq. (B31) leads to eq. (33) in the main text, since

$$
\begin{equation*}
\int_{0}^{t_{2}} d s_{2} K\left(t_{2}, s_{2} \mid Q\right) g\left(t_{1}, s_{2} \mid Q\right)=-\rho\left(t_{1}, t_{2} \mid Q\right) \tag{B32}
\end{equation*}
$$

## B. 3 Solution for $\Gamma_{1}$

## B.3.1 SDE at NLO

The $\operatorname{SDE}$ for $\Gamma_{1}$ is a little modified as
$D_{1}^{f} \Gamma_{1}(12)+\mu_{1, f}^{2} \Gamma_{0}(12)=-\frac{u_{f}}{6}\left[K_{0}(12 ; 11)+\Gamma_{0}(12) \Gamma_{1}(11)+\Gamma_{1}(12) \Gamma_{0}(11)+2 \Gamma_{0}(12) \Gamma_{0}(11)\right]$,
where we replace $\mu_{f}^{2}$ by $\mu_{f}^{2}+\frac{\mu_{1, f}^{2}}{N}$, so that $D_{1}^{f} \rightarrow D_{1}^{f}+\frac{1}{N} \mu_{1, f}^{2}$. Here $u_{1, f}^{2}$ is given by eq. (A36) with the replacement $u, m \rightarrow u_{f}, m_{f}$.

We parametrize $\Gamma_{1}$ as

$$
\begin{equation*}
\Gamma_{1}(12)=F\left(t_{1}\right) F\left(t_{2}\right) \int \mathrm{D} p \frac{\mathrm{e}^{-\left(p^{2}+\mu_{f}^{2}\right)\left(t_{1}+t_{2}\right)} \mathrm{e}^{i p\left(x_{1}-x_{2}\right)}}{p^{2}+m^{2}} G_{1}\left(t_{1}, t_{2} \mid p\right) \tag{B34}
\end{equation*}
$$

with the boundary condition

$$
\begin{equation*}
G_{1}(0,0 \mid p) \equiv b(p)=-\frac{\Sigma_{1}(p)}{p^{2}+m^{2}} \tag{B35}
\end{equation*}
$$

where $\Sigma_{1}(p)$ is the self-energy at the NLO in the $d$ dimensional theory.
The NLO SDE becomes

$$
\begin{equation*}
\partial_{t_{1}} G_{1}\left(t_{1}, t_{2} \mid p_{1}\right)+\mu_{1, f}^{2}=-\frac{u_{f}}{6} F^{2}\left(t_{1}\right) \mathrm{e}^{-2 t_{1} \mu_{f}^{2}} \mathcal{H}\left[G_{1}\left(t_{1}, t_{1} \mid p\right)\right]+\lambda\left(t_{1}, t_{2} \mid p_{1}\right) \tag{B36}
\end{equation*}
$$

where $\mathcal{H}$ is defined in eq. (69) and

$$
\begin{align*}
\lambda\left(t_{1}, t_{2} \mid p_{1}\right) & \equiv-\frac{u_{f}}{6} F^{2}\left(t_{1}\right) \mathrm{e}^{-2 t_{1} \mu_{f}^{2}} \Delta\left(t_{1}, t_{2} \mid p_{1}\right)  \tag{B37}\\
\Delta\left(t_{1}, t_{2} \mid p_{1}\right) & =2 \zeta_{0}\left(t_{1}\right)+\mathrm{e}^{t_{1} p_{1}^{2}} \int \prod_{i=2}^{4} \frac{\mathrm{D} p_{i} \mathrm{e}^{-t_{1} p_{i}^{2}}}{p_{i}^{2}+m^{2}}\{Z(\mid 21 ; 34)+2 Y(1 \mid 34 ; 21) \\
& +2 X(11 \mid 12 ; 34)+Y(1 \mid 12 ; 34)+2 X(21 \mid 21 ; 34)+Y(2 \mid 21 ; 34)\} . \tag{B38}
\end{align*}
$$

Using solutions $X$ and $Y$, we have in the continuum limit

$$
\begin{align*}
\lambda\left(t_{1}, t_{2} \mid p_{1}\right) & =\int \mathrm{d} p_{2} \frac{\mathrm{e}^{t_{1}\left(p_{1}^{2}-p_{2}^{2}\right)}}{p_{2}^{2}+m^{2}}\left[\psi\left(t_{1} \mid 12\right)+\left(p_{2}^{2}+m^{2}\right) \int_{0}^{t_{1}} \mathrm{~d} s \mathrm{e}^{s\left(p_{2}^{2}-p_{1}^{2}\right)} \omega\left(t_{1}, s \mid p_{12}\right)\right. \\
& \left.+\left(p_{1}^{2}+m^{2}\right) \int_{0}^{t_{2}} \mathrm{~d} s \mathrm{e}^{s\left(p_{1}^{2}-p_{2}^{2}\right)} \omega\left(t_{1}, s \mid p_{12}\right)\right] \tag{B39}
\end{align*}
$$

Since the right-hand side of eq. (B39) is finite, $\Delta\left(t_{1}, t_{2} \mid p\right) \rightarrow 0$ in the continuum limit.

## B.3.2 Solution to the SDE

Let us define

$$
\begin{equation*}
G_{1}\left(t_{1}, t_{2} \mid p\right) \equiv b(p)+\kappa\left(t_{1}, t_{2} \mid p\right)+H\left(t_{1}\right)+H\left(t_{2}\right) \tag{B40}
\end{equation*}
$$

with $\kappa\left(t_{1}, t_{2} \mid p\right)=\kappa\left(t_{2}, t_{1} \mid p\right)$ and $\kappa(0,0 \mid p)=H(0)=0$, where

$$
\begin{align*}
\partial_{t_{1}} \kappa\left(t_{1}, t_{2} \mid p\right) & =\lambda\left(t_{1}, t_{2} \mid p\right),  \tag{B41}\\
\frac{d H(t)}{d t} & =-\frac{u_{f}}{6} F^{2}(t) \mathrm{e}^{-2 t \mu_{f}^{2}}\left[\mathcal{H}\left[G_{1}(t, t \mid p)\right]-2 \zeta(t) \delta_{1}\right] . \tag{B42}
\end{align*}
$$

The second equation (B42) can be rewritten as

$$
\begin{equation*}
\frac{d H(t)}{d t}=-\frac{u_{f}}{6} F^{2}(t) \mathrm{e}^{-2 t \mu_{f}^{2}}\left[2 \zeta_{0}(t) H(t)+b_{0}(t)+\kappa_{0}(t)-2 \zeta(t) \delta_{1}\right] \tag{B43}
\end{equation*}
$$

so that we have in the continuum limit

$$
\begin{equation*}
H(t)=-\frac{b_{0}(t)+\kappa_{0}(t)}{2 \zeta_{0}(t)}+\delta_{1} \tag{B44}
\end{equation*}
$$

where we define $b_{0}(t)=\mathcal{H}[b(p)]$ and $\kappa_{0}(t)=\mathcal{H}[\kappa(t, t \mid p)]$.
The first equation (B41) can be solved as

$$
\begin{equation*}
\kappa\left(t_{1}, t_{2} \mid p\right)=k_{2}\left(t_{1}, t_{2} \mid p\right)+k_{1}\left(t_{1} \mid p\right)+k_{1}\left(t_{2} \mid p\right) \tag{B45}
\end{equation*}
$$

where

$$
\begin{align*}
k_{1}(t \mid p) & =\int_{0}^{t} \mathrm{~d} s \lambda_{1}(s \mid p)  \tag{B46}\\
\lambda_{1}(t \mid p) & =\int \mathrm{D} q \frac{\mathrm{e}^{\left(p^{2}-q^{2}\right) t}}{q^{2}+m^{2}} \psi(t \mid p, q)+\int_{0}^{t} \mathrm{~d} s \int \mathrm{D} q \mathrm{e}^{\left(p^{2}-q^{2}\right)(t-s)} \omega(t, s \mid Q),  \tag{B47}\\
k_{2}\left(t_{1}, t_{2} \mid p\right) & =\int_{0}^{t_{1}} \mathrm{~d} s_{1} \int_{0}^{t_{2}} \mathrm{~d} s_{2} \int \mathrm{D} q \frac{p^{2}+m^{2}}{q^{2}+m^{2}} \mathrm{e}^{\left(p^{2}-q^{2}\right)\left(s_{1}+s_{2}\right)} \omega\left(s_{1}, s_{2} \mid Q\right) \tag{B48}
\end{align*}
$$

with $Q=p+q$.

## C Calculations in the massless limit at $d=3$

It can be shown that the flow bubble integral can be represented as

$$
B\left(t \mid\{p\}_{\text {sym. }}\right)=-2 \int_{0}^{t} \mathrm{~d} s K\left(s, 0 \mid\{p\}_{\text {sym. }}\right)+B\left(0 \mid\{p\}_{\text {sym. }}\right), \quad B\left(0 \mid\{p\}_{\text {sym. }}\right)=\frac{1}{8 \sqrt{D}},
$$

which can be rescaled as

$$
\begin{equation*}
B\left(t \mid\{p\}_{\mathrm{sym} .}\right)=\frac{1}{\sqrt{D}} b_{0}(D t), \tag{C2}
\end{equation*}
$$

where

$$
\begin{equation*}
b_{0}(w)=\frac{1}{8}-\frac{\sqrt{w}}{2(2 \pi)^{3 / 2}} \int_{0}^{1} \frac{\mathrm{~d} x}{\sqrt{x}} \mathrm{e}^{-w x} \int_{0}^{1} \frac{\mathrm{~d} z}{\sqrt{z}} \mathrm{e}^{w z x / 2} . \tag{C3}
\end{equation*}
$$

Rescaling

$$
\rho\left(t \mid\{p\}_{\text {sym. }}\right)=R_{0}(D t, D), \quad \psi\left(t \mid\{p\}_{\text {sym. }}\right)=\sqrt{D} \phi_{0}(D t, D),
$$

the integral equation for $\psi$ in the massless limit is written as

$$
\begin{equation*}
R_{0}(w, D)+\int_{0}^{w} \mathrm{~d} v k_{0}(w, v) \phi_{0}(v, D)=0, \tag{C5}
\end{equation*}
$$

where

$$
\begin{equation*}
R_{0}(w, D)=\mathrm{e}^{-3 w / 2}-8 b_{0}(w) \frac{\bar{u}(D)}{1+\bar{u}(D)}, \quad \bar{u}(D)=\frac{u}{48 \sqrt{D}} . \tag{C6}
\end{equation*}
$$

Since the problem is linear, we can write

$$
\phi_{0}(w, D)=\phi_{0}^{(1)}(w)-8 \phi_{0}^{(2)}(w) \frac{\bar{u}(D)}{1+\bar{u}(D)},
$$

where $\phi_{0}^{(i)}, i=1,2$ solve the momentum-independent equations (56) and (57). We thus finally obtain eq. (52).

As the source term can be rescaled as

$$
\rho\left(t, s \mid\{p\}_{\text {sym. }}\right)=\frac{1}{\sqrt{D}}\left[b_{0}(D(t+s))-8 b_{0}(D t) b_{0}(D s) \frac{\bar{u}(D)}{1+\bar{u}(D)}\right],
$$

the equation for $\omega$ in the massless limit is written for $\omega\left(t, s \mid\{p\}_{\text {sym. }}\right)=\sqrt{D} W_{0}(D t, D s, D)$ as

$$
\begin{align*}
& b_{0}(D(t+s))-8 b_{0}(D t) b_{0}(D s) \frac{\bar{u}(D)}{1+\bar{u}(D)} \\
= & 2 \int_{0}^{D t} \mathrm{~d} u k_{0}(D t, u) \int_{0}^{D s} \mathrm{~d} v k_{0}(D s, v) W_{0}(u, v, D), \tag{C9}
\end{align*}
$$

which can be solved as

$$
\begin{equation*}
W_{0}(w, v, D)=\Omega_{0}(w, v)-4 \phi_{0}^{(2)}(w) \phi_{0}^{(2)}(v) \frac{\bar{u}(D)}{1+\bar{u}(D)}, \tag{C10}
\end{equation*}
$$

where $\Omega_{0}$ solves the momentum ( $D$ ) independent equation (58). We thus obtain eq. (53).

## D Induced metric in the massless limit at $d=3$

## D. 1 Induced metric

The space component of the induced metric is given by

$$
\begin{equation*}
g_{i j}(z)=\delta_{i j} \frac{R_{0}^{2}}{d \zeta_{0}(t)}\left(1-\frac{\zeta_{1}(t)}{N}\right) \mathcal{H}\left[p^{2}\left(1+\frac{G_{1}(t, t \mid p)}{N}\right)\right] . \tag{D1}
\end{equation*}
$$

We then evaluate

$$
\begin{align*}
\zeta_{1}(t) & =\frac{1}{\zeta_{0}(t)} \mathcal{H}\left[G_{1}(t, t \mid p)\right]=2 \delta_{1}, \quad \mathcal{H}[1]=\zeta_{0}(t), \quad \mathcal{H}\left[p^{2}\right]=-\frac{\partial_{t} \zeta_{0}(t)}{2},  \tag{D2}\\
\mathcal{H}\left[p^{2} G_{1}(t, t \mid p)\right] & =\mathcal{H}[\lambda(t, t \mid p)]+\zeta_{0}(t) \partial_{t} H(t)-\partial_{t} \zeta_{0}(t) \delta_{1}=\zeta_{0}(t) \partial_{t} H(t)-\partial_{t} \zeta_{0}(t) \delta_{1}, \tag{D3}
\end{align*}
$$

where in the last equation we use $\mathcal{H}[\lambda(t, t \mid p)]=0$. Altogether we obtain

$$
\begin{equation*}
g_{i j}(z)=\delta_{i j} R_{0}^{2}\left[g^{(0)}(t)+\frac{1}{N} g^{(1)}(t)\right], \quad g^{(0)}(t)=-\frac{\partial_{t} \zeta_{0}(t)}{2 d \zeta_{0}(t)}, g^{(1)}(t)=\frac{\partial_{t} H(t)}{d} . \tag{D4}
\end{equation*}
$$

The time component is evaluated as

$$
\begin{align*}
g_{00}(t) & =t \partial_{t_{1}} \partial_{t_{2}}\left[\frac{R_{0}^{2}}{\sqrt{\zeta_{0}\left(t_{1}\right) \zeta_{0}\left(t_{2}\right)}} \int \mathrm{D} p \frac{\mathrm{e}^{-p^{2}\left(t_{1}+t_{2}\right)}}{p^{2}+m^{2}}\left(1+\frac{\widetilde{G}_{1}\left(t_{1}, t_{2} \mid p\right)}{N}\right)\right]_{t_{1}=t_{2}=t}  \tag{D5}\\
& =R_{0}^{2}\left\{g_{00}^{(0)}(t)+\frac{1}{N} g_{00}^{(1)}(t)\right\} \tag{D6}
\end{align*}
$$

where

$$
\begin{equation*}
\widetilde{G}_{1}\left(t_{1}, t_{2} \mid p\right)=-2 \delta_{1}+G_{1}\left(t_{1}, t_{2} \mid p\right) \tag{D7}
\end{equation*}
$$

The leading term is

$$
\begin{equation*}
g_{00}^{(0)}(t)=\frac{t}{4} \partial_{t}^{2}\left[\log \zeta_{0}(t)\right] \tag{D8}
\end{equation*}
$$

and for the NLO term we have

$$
\begin{equation*}
\frac{1}{t} g_{00}^{(1)}(t)=\left.\partial_{t_{1}} \partial_{t_{2}} \frac{I\left(t_{1}, t_{2}\right)}{\sqrt{\zeta_{0}\left(t_{1}\right) \zeta_{0}\left(t_{2}\right)}}\right|_{t_{1}=t_{2}=t}, \tag{D9}
\end{equation*}
$$

where

$$
\begin{equation*}
I\left(t_{1}, t_{2}\right)=\int \mathrm{D} p \frac{\mathrm{e}^{-p^{2}\left(t_{1}+t_{2}\right)}}{p^{2}+m^{2}} \widetilde{G}_{1}\left(t_{1}, t_{2} \mid p\right) \tag{D10}
\end{equation*}
$$

With this notation

$$
\begin{equation*}
\frac{1}{t} g_{00}^{(1)}(t)=\frac{1}{4} \frac{\left(\partial_{t} \zeta_{0}(t)\right)^{2}}{\zeta_{0}^{3}(t)} I(t, t)-\frac{1}{2} \frac{\partial_{t} \zeta_{0}(t)}{\zeta_{0}^{2}(t)} \partial_{t} I(t, t)+\left.\frac{1}{\zeta_{0}(t)} \partial_{t_{1}} \partial_{t_{2}} I\left(t_{1}, t_{2}\right)\right|_{t_{1}=t_{2}=t} \tag{D11}
\end{equation*}
$$

Since

$$
\begin{equation*}
I(t, t)=\mathcal{H}\left[\widetilde{G}_{1}(t, t \mid p)\right]=0, \tag{D12}
\end{equation*}
$$

the first two terms vanish. Further,

$$
\begin{equation*}
\left.\partial_{t_{1}} \partial_{t_{2}} I\left(t_{1}, t_{2}\right)\right|_{t_{1}=t_{2}=t}=\mathcal{H}\left[\left(p^{2}\right)^{2} \widetilde{G}_{1}(t, t \mid p)-2 p^{2} \lambda(t, t \mid p)+\left.\partial_{t_{2}} \lambda\left(t, t_{2} \mid p\right)\right|_{t_{2}=t}\right]+\partial_{t} H(t) \partial_{t} \zeta_{0}(t) . \tag{D13}
\end{equation*}
$$

Using the identities

$$
\begin{equation*}
\mathcal{H}[\lambda(t, t \mid p)]=0 ; \quad \mathcal{H}\left[p^{2} \widetilde{G}_{1}(t, t \mid p)\right]=\zeta_{0}(t) \partial_{t} H(t) \tag{D14}
\end{equation*}
$$

and their derivatives this can be further simplified:

$$
\begin{equation*}
\left.\partial_{t_{1}} \partial_{t_{2}} I\left(t_{1}, t_{2}\right)\right|_{t_{1}=t_{2}=t}=-\frac{1}{2} \zeta_{0}(t) \partial_{t}^{2} H(t)+\mathcal{H}\left[\left.\partial_{t_{2}} \lambda\left(t, t_{2} \mid p\right)\right|_{t_{2}=t}-\partial_{t} \lambda(t, t \mid p) / 2\right] . \tag{D15}
\end{equation*}
$$

Here the second term vanishes and we finally obtain

$$
\begin{equation*}
g_{00}^{(1)}(t)=-\frac{t}{2} \partial_{t}^{2} H(t) . \tag{D16}
\end{equation*}
$$

D. 2 Calculation of $H(t)$ in the massless limit

We recall the definition of $H(t)$ as

$$
\begin{equation*}
H(t)=-\frac{b_{0}(t)+\kappa_{0}(t)}{2 \zeta_{0}(t)}+\delta_{1} \tag{D17}
\end{equation*}
$$

where

$$
\begin{equation*}
b_{0}(t)=\mathcal{H}[b(p)], \quad \kappa_{0}(t)=\mathcal{H}[\kappa(t, t \mid p)], \tag{D18}
\end{equation*}
$$

with

$$
\begin{equation*}
b(p)=-\frac{\Sigma_{1}(p)}{p^{2}+m^{2}}, \quad \kappa(t, t \mid p)=k_{2}(t, t \mid p)+2 k_{1}(t \mid p) \tag{D19}
\end{equation*}
$$

Here $k_{1}$ and $k_{2}$ are given in eqs. (B46), (B47) and (B48).
Hereafter we consider the massless limit at $d=3$, where we have $\zeta_{0}(t)^{-1}=2(2 \pi)^{3 / 2} \sqrt{t}$.

## D.2. 1 Calculation of $b_{0}(t)$

We first calculate $b_{0}(t)$. In the massless limit, we have

$$
\begin{equation*}
H_{b}(t) \equiv-\frac{b_{0}(t)}{2 \zeta_{0}(t)}=\frac{1}{2 \zeta_{0}(t)} \int \mathrm{D} p \frac{\mathrm{e}^{-2 p^{2} t}}{\left(p^{2}\right)^{2}} g\left(p^{2}\right) \tag{D20}
\end{equation*}
$$

since $\tilde{C}=Z_{1} m^{2}=0$ and

$$
\begin{equation*}
g\left(p^{2}\right)=\frac{u}{3} \int \frac{\mathrm{D} Q}{1+\bar{u}\left(Q^{2}\right)}\left\{\frac{1}{(Q+p)^{2}}-\frac{1}{Q^{2}}\right\} . \tag{D21}
\end{equation*}
$$

After rescaling, we obtain

$$
\begin{equation*}
H_{b}(t)=\int \mathrm{D} Q h_{b}\left(Q^{2}\right) \frac{\bar{u}\left(Q^{2}\right) \sqrt{t}}{1+\bar{u}\left(Q^{2}\right) \sqrt{t}}, \tag{D22}
\end{equation*}
$$

where

$$
\begin{equation*}
h_{b}\left(Q^{2}\right)=32 \sqrt{2} \sqrt{\pi^{3}} \sqrt{Q^{2}} \int \mathrm{D} p \frac{\mathrm{e}^{-2 p^{2}}}{\left(p^{2}\right)^{2}}\left\{\frac{1}{(Q+p)^{2}}-\frac{1}{Q^{2}}\right\} . \tag{D23}
\end{equation*}
$$

## D.2.2 Calculation of $\kappa_{0}(t)$

For this we need $\psi$ and $\omega$ in the massless limit, which can be obtained as

$$
\begin{align*}
\psi_{0}(t \mid p, q) & =\sqrt{Q^{2}}\left[\varphi_{0}\left(Q^{2} t, z\right)-8 \phi_{0}^{(2)}\left(Q^{2} t\right) \frac{\bar{u}\left(Q^{2}\right)}{1+\bar{u}\left(Q^{2}\right)}\right]  \tag{D24}\\
\omega_{0}(t, s \mid Q) & =\sqrt{Q^{2}}\left[\Omega_{0}\left(Q^{2} t, Q^{2} s\right)-4 \phi_{0}^{(2)}\left(Q^{2} t\right) \phi_{0}^{(2)}\left(Q^{2} s\right) \frac{\bar{u}\left(Q^{2}\right)}{1+\bar{u}\left(Q^{2}\right)}\right] \tag{D25}
\end{align*}
$$

with $z=\left(p^{2}+q^{2}\right) / Q^{2}$, where $\phi_{0}^{(2)}$ and $\Omega_{0}$ are already obtained in section 3, while $\varphi_{0}$ satisfies

$$
\begin{equation*}
\mathrm{e}^{-z w}+\int_{0}^{w} \mathrm{~d} x k_{0}(w, x) \varphi_{0}(x, z)=0 \tag{D26}
\end{equation*}
$$

instead of eq. (56) and thus $\varphi_{0}(x, 3 / 2)=\phi_{0}^{(1)}(x)$.
Using these, we first calculate

$$
\begin{align*}
H_{\kappa}^{(1)}(t) & \equiv-\frac{1}{\zeta_{0}(t)} \int \mathrm{D} p \frac{\mathrm{e}^{-2 p^{2} t}}{p^{2}} \int_{0}^{t} \mathrm{~d} s \int \mathrm{D} q \frac{\mathrm{e}^{\left(p^{2}-q^{2}\right) s}}{q^{2}} \psi_{0}(s \mid p, q) \\
& =H_{\kappa}^{(1)}(0)+\int \mathrm{D} Q \int_{0}^{1} \mathrm{~d} x \phi_{0}^{(2)}\left(Q^{2} x\right) h_{11}\left(x, Q^{2}\right) \frac{\bar{u}\left(Q^{2}\right) \sqrt{t}}{1+\bar{u}\left(Q^{2}\right) \sqrt{t}} \tag{D27}
\end{align*}
$$

where $H_{\kappa}^{(1)}(0)$ is some constant and

$$
\begin{equation*}
h_{11}\left(x, Q^{2}\right)=32 \sqrt{2} \sqrt{\pi^{3}} \sqrt{Q^{2}} \int \mathrm{D} p \mathrm{D} q(2 \pi)^{3} \delta(q+p-Q) \frac{\mathrm{e}^{-(2-x) p^{2}-x q^{2}}}{p^{2} q^{2}} \tag{D28}
\end{equation*}
$$

Similarly we have

$$
\begin{align*}
H_{\kappa}^{(2)}(t) & \equiv-\frac{1}{\zeta_{0}(t)} \int \mathrm{D} p \frac{\mathrm{e}^{-2 p^{2} t}}{p^{2}} \int_{0}^{t} \mathrm{~d} s \int \mathrm{D} q \mathrm{e}^{\left(p^{2}-q^{2}\right) s} \int_{0}^{s} \mathrm{~d} r \mathrm{e}^{\left(q^{2}-p^{2}\right) r} \omega_{0}(s, r \mid Q) \\
& =H_{\kappa}^{(2)}(0)+2 \int \mathrm{D} Q \int_{0}^{1} \mathrm{~d} x \phi_{0}^{(2)}\left(Q^{2} x\right) \int_{0}^{x} \mathrm{~d} y \phi_{0}^{(2)}\left(Q^{2} y\right) h_{10}\left(x-y, Q^{2}\right) \\
& \times \frac{\bar{u}\left(Q^{2}\right) \sqrt{t}}{1+\bar{u}\left(Q^{2}\right) \sqrt{t}}, \tag{D29}
\end{align*}
$$

where

$$
\begin{equation*}
h_{10}\left(z, Q^{2}\right)=8 \sqrt{2} \sqrt{\pi^{3}} \sqrt{Q^{2}} \int \mathrm{D} p \mathrm{D} q(2 \pi)^{3} \delta(q+p-Q) \frac{\mathrm{e}^{-(2-z) p^{2}-z q^{2}}}{p^{2}} \tag{D30}
\end{equation*}
$$

The last contribution becomes

$$
\begin{align*}
H_{\kappa}^{(3)}(t) & \equiv-\frac{1}{2 \zeta_{0}(t)} \int \mathrm{D} p \mathrm{e}^{-2 p^{2} t} \int_{0}^{t} \mathrm{~d} s \int \mathrm{D} q \frac{\mathrm{e}^{\left(p^{2}-q^{2}\right) s}}{q^{2}} \int_{0}^{t} \mathrm{~d} r \mathrm{e}^{\left(p^{2}-q^{2}\right) r} \omega_{0}(s, r \mid Q) \\
& =H_{\kappa}^{(3)}(0)+\int \mathrm{D} Q \int_{0}^{1} \mathrm{~d} x \phi_{0}^{(2)}\left(Q^{2} x\right) \int_{0}^{1} \mathrm{~d} y \phi_{0}^{(2)}\left(Q^{2} y\right) h_{10}\left(2-x-y, Q^{2}\right) \\
& \times \frac{\bar{u}\left(Q^{2}\right) \sqrt{t}}{1+\bar{u}\left(Q^{2}\right) \sqrt{t}} . \tag{D31}
\end{align*}
$$

## D. 3 Total contributions

We thus obtain the $H(t)$ a: $\square^{7}$

$$
\begin{equation*}
H(t)=H(0)+\int \mathrm{D} Q h_{\text {total }}\left(Q^{2}\right) \frac{\bar{u}\left(Q^{2}\right) \sqrt{t}}{1+\bar{u}\left(Q^{2}\right) \sqrt{t}} \tag{D32}
\end{equation*}
$$

where

$$
\begin{align*}
H(0) & =H_{\kappa}^{(1)}(0)+H_{\kappa}^{(2)}(0)+H_{\kappa}^{(3)}(0)+\delta_{1}  \tag{D33}\\
h_{\text {total }}\left(Q^{2}\right) & =h_{b}\left(Q^{2}\right)+\int_{0}^{1} \mathrm{~d} x \phi_{0}^{(2)}\left(Q^{2} x\right)\left\{h_{11}\left(x, Q^{2}\right)+2 \int_{0}^{x} \mathrm{~d} y \phi_{0}^{(2)}\left(Q^{2} y\right) h_{10}\left(x-y, Q^{2}\right)\right. \\
& \left.+\int_{0}^{1} \mathrm{~d} y \phi_{0}^{(2)}\left(Q^{2} y\right) h_{10}\left(2-x-y, Q^{2}\right)\right\} \tag{D34}
\end{align*}
$$

which leads to eqs. (73) and (74) by $A_{1}(t) \equiv \partial_{t} H(t)$ and $\partial_{t} A_{1}(t) \equiv \partial_{t}^{2} H(t)$.

[^5]
## D. 4 IR behaviors

## D.4.1 Some definitions

We write the NLO induced metric as

$$
\begin{equation*}
g_{i j}(\tau)=\delta_{i j}\left\{\frac{R_{0}^{2}}{12 t}\left[1+\frac{R(t)}{N}\right]\right\}, \quad g_{00}(\tau)=-t \partial_{t}\left\{\frac{R_{0}^{2}}{8 t}\left[1+\frac{R(t)}{N}\right]\right\} \tag{D35}
\end{equation*}
$$

where the relative correction is a sum of four contributions,

$$
\begin{equation*}
R(t)=R_{b}(t)+\sum_{i=1}^{3} R_{\kappa}^{(i)}(t), \quad R_{b}(t) \equiv 4 t \partial_{t} H_{b}(t), \quad R_{\kappa}^{(i)}(t) \equiv 4 t \partial_{t} H_{\kappa}^{(i)}(t) \tag{D36}
\end{equation*}
$$

We also introduce $G(v)$ by

$$
\begin{equation*}
\phi_{0}^{(2)}(v)=-\frac{(2 \pi)^{3 / 2}}{\sqrt{v}} G(v), \quad G(0)=1 / 8, \quad G(v) \sim \exp (-v / 2), v \rightarrow \infty \tag{D37}
\end{equation*}
$$

and use the time variable $T=u \sqrt{t} / 48$.
In the following we will use the fact that a double 3-dimensional integral of any function depending only on the absolute values $p, q$ and $|Q|$, where $Q=p+q$, can be written

$$
\begin{equation*}
\int \mathrm{D} p \int \mathrm{D} q f\left(p, q, Q^{2}\right)=\frac{1}{(2 \pi)^{4}} \int_{0}^{\infty} p \mathrm{~d} p \int_{0}^{\infty} q \mathrm{~d} q \int_{(q-p)^{2}}^{(q+p)^{2}} \mathrm{~d} Q^{2} f\left(p, q, Q^{2}\right) \tag{D38}
\end{equation*}
$$

## D.4.2 The $R_{b}$ contribution

Here we can do the angular part of the $Q^{2}$ integral analytically and find

$$
\begin{equation*}
R_{b}(t)=\frac{32 T}{\sqrt{2 \pi}^{5}} \int_{0}^{\infty} \frac{q \mathrm{~d} q}{(q+T)^{2}} \rho_{b}(q) \tag{D39}
\end{equation*}
$$

where

$$
\begin{equation*}
\rho_{b}(q)=q^{2} \int_{0}^{\infty} \frac{\mathrm{d} p}{p^{3}} \mathrm{e}^{-2 p^{2}}\left\{\ln \frac{(p+q)^{2}}{(p-q)^{2}}-\frac{4 p}{q}\right\} \tag{D40}
\end{equation*}
$$

which behaves as $\rho_{b}(q)=O(q)$ for small $q$, while

$$
\begin{equation*}
\rho_{b}(q) \sim \frac{\sqrt{2 \pi}}{3 q} \tag{D41}
\end{equation*}
$$

for large $q$. Thus we can establish that $R_{b}(t)=O(T)$ for small $t$, while for large $t$

$$
\begin{equation*}
r_{b} \equiv R_{b}(\infty)=\frac{8}{3 \pi^{2}}=0.27019 \tag{D42}
\end{equation*}
$$

D.4.3 The $R_{\kappa}^{(1)}$ contribution

We have

$$
\begin{equation*}
R_{\kappa}^{(1)}(t)=-32(2 \pi)^{3} \int \mathrm{D} p \int \mathrm{D} q \int_{0}^{1} \frac{\mathrm{~d} x}{\sqrt{x}} \frac{\mathrm{e}^{-p^{2}(2-x)-q^{2} x}}{p^{2} q^{2}} \frac{|Q| T}{(T+|Q|)^{2}} G\left(Q^{2} x\right) \tag{D43}
\end{equation*}
$$

Doing the $q^{2}$ integral first and introducing $x=y^{2}$ we can rewrite it as

$$
\begin{equation*}
-\frac{32}{\pi} \int_{0}^{\infty} \frac{\mathrm{d} p}{p} \mathrm{e}^{-2 p^{2}} \int_{0}^{\infty} \frac{Q^{2} T}{(T+Q)^{2}} \mathrm{~d} Q \int_{0}^{1} \mathrm{~d} y G\left(Q^{2} y^{2}\right) \int_{(Q-p)^{2}}^{(Q+p)^{2}} \frac{\mathrm{e}^{\left(p^{2}-q^{2}\right) y^{2}}}{q^{2}} \mathrm{~d} q^{2} \tag{D44}
\end{equation*}
$$

After some further rescaling we get

$$
\begin{equation*}
R_{\kappa}^{(1)}(t)=-\frac{64}{\pi} \int_{0}^{\infty} \mathrm{d} Q \frac{Q T}{(T+Q)^{2}} \rho_{\kappa}^{(1)}(Q) \tag{D45}
\end{equation*}
$$

where

$$
\begin{gather*}
\rho_{\kappa}^{(1)}(Q)=\int_{0}^{\infty} \frac{\mathrm{d} p}{p} \mathrm{e}^{-2 p^{2}} \int_{0}^{Q} \mathrm{~d} z G\left(z^{2}\right) Y\left(\frac{p}{Q}, z\right)  \tag{D46}\\
Y(\varepsilon, z)=\int_{|1-\varepsilon|}^{1+\varepsilon} \frac{\mathrm{d} \xi}{\xi} \mathrm{e}^{\left(\varepsilon^{2}-\xi^{2}\right) z^{2}}=2 \varepsilon \mathrm{e}^{-z^{2}}+O\left(\varepsilon^{2}\right) \tag{D47}
\end{gather*}
$$

From this we see that $\rho_{\kappa}^{(1)}(Q)=O(Q)$ for small $Q$, while

$$
\begin{equation*}
\rho_{\kappa}^{(1)}(Q) \sim \frac{2}{Q} \int_{0}^{\infty} \mathrm{d} p \mathrm{e}^{-2 p^{2}} \int_{0}^{\infty} \mathrm{d} z G\left(z^{2}\right) \mathrm{e}^{-z^{2}}=\frac{1}{Q} \sqrt{\frac{\pi}{2}} \int_{0}^{\infty} \mathrm{d} z G\left(z^{2}\right) \mathrm{e}^{-z^{2}} \tag{D48}
\end{equation*}
$$

for large $Q$, so that we numerically obtain

$$
\begin{equation*}
r_{\kappa}^{(1)} \equiv R_{\kappa}^{(1)}(\infty)=-\frac{64}{\sqrt{2 \pi}} \int_{0}^{\infty} \mathrm{d} z G\left(z^{2}\right) \mathrm{e}^{-z^{2}}=-1.14734 \tag{D49}
\end{equation*}
$$

D.4.4 The $R_{\kappa}^{(2)}$ contribution

Similarly

$$
\begin{equation*}
R_{\kappa}^{(2)}(t)=16 \sqrt{2 \pi}^{9} \int \mathrm{D} p \int \mathrm{D} q \int_{0}^{1} \frac{\mathrm{~d} x}{\sqrt{x}} \int_{0}^{x} \frac{\mathrm{~d} y}{\sqrt{y}} \frac{T}{(T+|Q|)^{2}} G\left(Q^{2} x\right) G\left(Q^{2} y\right) \frac{\mathrm{e}^{-2 p^{2}}}{p^{2}} \mathrm{e}^{\left(p^{2}-q^{2}\right)(x-y)} \tag{D50}
\end{equation*}
$$

Doing the $q^{2}$ integrations first, we have

$$
\begin{align*}
R_{\kappa}^{(2)}(t)=64 \sqrt{2 \pi} & \int_{0}^{\infty} \mathrm{d} Q \frac{Q T}{(T+Q)^{2}} \int_{0}^{\infty} \frac{\mathrm{d} p}{p} \mathrm{e}^{-2 p^{2}} \\
& \times \int_{0}^{1} \mathrm{~d} x \int_{0}^{x} \mathrm{~d} y G\left(Q^{2} x^{2}\right) G\left(Q^{2} y^{2}\right) \int_{(Q-p)^{2}}^{(Q+p)^{2}} \mathrm{e}^{\left(p^{2}-q^{2}\right)\left(x^{2}-y^{2}\right)} \mathrm{d} q^{2} . \tag{D51}
\end{align*}
$$

The $q^{2}$ integral can be done analytically and we find

$$
\begin{equation*}
R_{\kappa}^{(2)}(t)=128 \sqrt{2 \pi} \int_{0}^{\infty} \mathrm{d} Q \frac{Q T}{(T+Q)^{2}} \rho_{\kappa}^{(2)}(Q), \tag{D52}
\end{equation*}
$$

where

$$
\begin{equation*}
\rho_{\kappa}^{(2)}(Q)=\int_{0}^{\infty} \frac{\mathrm{d} p}{p} \mathrm{e}^{-2 p^{2}} \int_{0}^{Q} \mathrm{~d} z \int_{0}^{z} \mathrm{~d} w G\left(z^{2}\right) G\left(w^{2}\right) \frac{\mathrm{e}^{w^{2}-z^{2}}}{z^{2}-w^{2}} \sinh \frac{2 p}{Q}\left(z^{2}-w^{2}\right) \tag{D53}
\end{equation*}
$$

Thus $\rho_{\kappa}^{(2)}=O(Q)$ for small $Q$, while

$$
\begin{equation*}
\rho_{\kappa}^{(2)}(Q) \sim \frac{1}{Q} \int_{-\infty}^{\infty} \mathrm{d} p \mathrm{e}^{-2 p^{2}} \int_{0}^{\infty} \mathrm{d} z \int_{0}^{z} \mathrm{~d} w G\left(z^{2}\right) G\left(w^{2}\right) \mathrm{e}^{w^{2}-z^{2}} \tag{D54}
\end{equation*}
$$

for large $Q$, and

$$
\begin{equation*}
r_{\kappa}^{(2)} \equiv R_{\kappa}^{(2)}(\infty)=128 \pi \int_{0}^{\infty} \mathrm{d} z \int_{0}^{z} \mathrm{~d} w G\left(z^{2}\right) G\left(w^{2}\right) \mathrm{e}^{w^{2}-z^{2}}=0.45846 \tag{D55}
\end{equation*}
$$

D.4.5 The $R_{\kappa}^{(3)}$ contribution

For $R_{\kappa}^{(3)}$ we find

$$
\begin{equation*}
R_{\kappa}^{(3)}(t)=32 \sqrt{2 \pi} \int_{0}^{\infty} \mathrm{d} Q \frac{Q T}{(T+Q)^{2}} \rho_{\kappa}^{(3)}(Q) \tag{D56}
\end{equation*}
$$

with

$$
\begin{equation*}
\rho_{\kappa}^{(3)}(Q)=\int_{0}^{1} \mathrm{~d} x \int_{0}^{1} \mathrm{~d} y \int_{0}^{\infty} p \mathrm{~d} p \mathrm{e}^{-2 p^{2}+p^{2}\left(x^{2}+y^{2}\right)} G\left(Q^{2} x^{2}\right) G\left(Q^{2} y^{2}\right) \int_{(Q-p)^{2}}^{(Q+p)^{2}} \frac{\mathrm{e}^{-q^{2}\left(x^{2}+y^{2}\right)}}{q^{2}} \mathrm{~d} q^{2} . \tag{D57}
\end{equation*}
$$

After rescaling

$$
\begin{equation*}
\rho_{\kappa}^{(3)}(Q)=\frac{1}{Q^{2}} \int_{0}^{Q} \mathrm{~d} z \int_{0}^{Q} \mathrm{~d} w G\left(z^{2}\right) G\left(w^{2}\right) \int_{0}^{\infty} p \mathrm{~d} p \mathrm{e}^{-2 p^{2}} Z\left(\frac{p}{Q}, z^{2}+w^{2}\right) \tag{D58}
\end{equation*}
$$

where

$$
\begin{equation*}
Z(\varepsilon, A)=2 \mathrm{e}^{A \varepsilon^{2}} \int_{|1-\varepsilon|}^{1+\varepsilon} \frac{\mathrm{e}^{-A \xi^{2}}}{\xi} \mathrm{~d} \xi \approx 4 \varepsilon \mathrm{e}^{-A}, \quad \varepsilon \rightarrow 0 \tag{D59}
\end{equation*}
$$

Thus $\rho_{\kappa}^{(3)}(Q)=O(Q)$ for small $Q$, while

$$
\begin{align*}
\rho^{(3)}(Q) \sim \frac{4}{Q^{3}} \int_{0}^{\infty} \mathrm{d} z & \int_{0}^{\infty} \mathrm{d} w G\left(z^{2}\right) G\left(w^{2}\right) \int_{0}^{\infty} p^{2} \mathrm{~d} p \mathrm{e}^{-2 p^{2}-z^{2}-w^{2}} \\
& =\sqrt{\frac{\pi}{8}}\left(\int_{0}^{\infty} \mathrm{d} z G\left(z^{2}\right) \mathrm{e}^{-z^{2}}\right)^{2} \frac{1}{Q^{3}}, \tag{D60}
\end{align*}
$$

for large $Q$, which leads to

$$
\begin{equation*}
r_{\kappa}^{(3)} \equiv R_{\kappa}^{(3)}(\infty)=0 \tag{D61}
\end{equation*}
$$

Thus the total relative correction is negative:

$$
\begin{equation*}
r=r_{b}+r_{\kappa}^{(1)}+r_{\kappa}^{(2)}+r_{\kappa}^{(3)}=-0.41869 \tag{D62}
\end{equation*}
$$


[^0]:    ${ }^{1}$ We will call the infinite cutoff $(\Lambda \rightarrow \infty)$ limit the 'continuum limit'.

[^1]:    ${ }^{2}$ We will indicate the massless limit by a subscript 0 .

[^2]:    ${ }^{3}$ It turns out that $G_{2}(\Delta)$ has only one zero at $\Delta=0.36228(1)$.

[^3]:    ${ }^{4}$ This is independent of $u_{f} \neq 0$ (the interacting flow). In the case of free flow $\left(u_{f}=0\right)$, however, $r=\frac{8}{3 \pi^{2}} \simeq$ 0.27019 .
    ${ }^{5}$ It is interesting and also suggestive to see that the F-coefficient of the 3 dimensional $O(N)$ scalar model is given by $F_{\mathrm{IR}}=F_{U V}-\zeta(3) /\left(8 \pi^{2}\right)+O(1 / N)$, where $F_{\mathrm{UV}}=N F_{S}$ with $F_{S} \simeq 0.0638$ as an example of a conjecture, the so-called "the F-theorem", which claims that the F-coefficient monotonically decreases along a RG trajectory connecting two 3 dimensional CFTs. Furthermore, in the holographic dual picture, the F-coefficient is proportional to the AdS radius squared. (See Ref. [11] and references therein.)

[^4]:    ${ }^{6}$ Note that we use the same notation $\Gamma_{2 n}$ for the $2 n$-point functions in both $d$ and $d+1$ dimensions, since no confusion may occur.

[^5]:    ${ }^{7}$ Here $H(0)$ is potentially divergent but it does not contribute to the metric.

