ON THE MULTIPLE BORSUK NUMBERS OF SETS

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ABSTRACT. The *Borsuk number* of a set S of diameter $d > 0$ in Euclidean n-space is the smallest value of m such that S can be partitioned into m sets of diameters less than d. Our aim is to generalize this notion in the following way: The *k-fold Borsuk number* of such a set S is the smallest value of m such that there is a k-fold cover of S with m sets of diameters less than d. In this paper we characterize the k-fold Borsuk numbers of sets in the Euclidean plane, give bounds for those of centrally symmetric sets, smooth bodies and convex bodies of constant width, and examine them for finite point sets in the Euclidean 3-space.

1. INTRODUCTION

In 1933, Borsuk [5] made the following conjecture.

Conjecture (Borsuk). Every set of diameter $d > 0$ in the Euclidean n-space $\mathbb{R}^n$ is the union of $n + 1$ sets of diameters less than d.

From the 1930s, this conjecture has attracted a wide interest among geometers. The frequent attempts to prove it led to results in a number of special cases: for sets in $\mathbb{R}^2$ and $\mathbb{R}^3$, for smooth bodies or sets with certain symmetries, etc., but the conjecture in general remained open till 1993, when it was disproved by Kahn and Kalai [17]. Their result did not mean that research on this problem stopped: the investigation of the so-called *Borsuk number* of a bounded set; that is, the minimum number of pieces of smaller diameters that it can be partitioned into, is still one of the fundamental problems of discrete geometry.

Since 1933, a large number of generalizations of Borsuk's problem has been introduced. Without completeness, we list only a few. The *generalized Borsuk problem* asks to find, for a fixed value of $0 < r < 1$, the minimum number m such that any set of diameter one in $\mathbb{R}^n$ can be partitioned into m pieces of diameters at most r (cf., for example, [10]). The *cylindrical Borsuk problem* makes restrictions on the method of partition (cf. [15]). Clearly, the original problem is meaningful for sets in any metric space, for instance, for finite dimensional normed spaces (cf. [3]) or for binary codes equipped with Hamming distance. The latter one is called the *$(0,1)$-Borsuk problem*, and is investigated, for example, in [21], [33] and [22]. For more information on this problem and its generalizations, the reader is referred to the survey [24].

Our aim is to add another generalization to this list. Our main definition is the following.

Definition 1. Let $S \subset \mathbb{R}^n$ be a set of diameter $d > 0$. The smallest positive integer m such that there is a k-fold cover of S, with m sets of diameters strictly less than d, is called the k-*fold Borsuk number* of S. We denote this number by $a_k(S)$.

Recall that a k-fold cover of a set S is a family of sets with the property that any point of S belongs to at least k members of the family. In this definition, we permit some members of the family to coincide. We denote the Borsuk number of a set S by $a(S)$. Clearly, $a_1(S) = a(S)$.

Note that Definition 1 can be naturally adapted to almost any variant of the original Borsuk problem, and thus, raises many open questions that are not examined in this paper. Our goal is to investigate the properties of the k-fold Borsuk numbers of sets in $\mathbb{R}^n$.

We start with three observations. Then in Section 2 we characterize the k-fold Borsuk numbers of planar sets. In Section 3 we give estimates on the k-fold Borsuk numbers of smooth bodies, centrally symmetric sets, and convex bodies of constant width, and determine them for Euclidean balls. In Sections 4 and 5 we examine the k-fold Borsuk numbers of finite point sets in $\mathbb{R}^3$. In particular, in Section 4 we examine the sets with large k-fold Borsuk numbers, and in Section 5 we focus on sets with a nontrivial symmetry group. Finally, in Section 6 we make an additional remark and raise a related open question.

During the investigation, $\mathbf{B}^n$ denotes the closed Euclidean unit ball centered at the origin o, and $\mathbb{S}^{n-1} = \operatorname{bd} \mathbf{B}^n$.

Our first observations are as follows.

Remark 1. The sequence $a_k(S)$ is subadditive for every S. More precisely, for any positive integers k, l, we have $a_{k+l}(S) \leq a_k(S) + a_l(S)$.

Remark 2. For every set $S \subset \mathbb{R}^n$ of diameter $d > 0$ and for every $k \geq 1$, we have $a_k(S) \geq 2k$. Furthermore, for every value of k, if $a(S) = 2$, then $a_k(S) = 2k$, and if $a(S) > 2$, then $a_k(S) > 2k$.

Proof. Without loss of generality, we may assume that S is compact. Let $[p, q]$ be a diameter of S. Since no set of diameter less than d contains both p and q, any k-fold cover of S with sets of smaller diameters has at least $2k$ elements. Furthermore, if $a(S) = 2$, then by Remark 1, $a_k(S) \leq 2k$ for every value of k.

Now assume that $a_k(S) = 2k$ for some value of k. Let $A_1, A_2, \ldots, A_{2k}$ be compact sets of diameters less than d that cover S k-fold. or every k-element subset J of $I = \{1, 2, \ldots, 2k\}$, let $\bar{J} = I \setminus J$, and let S_J be the (compact) set of points in S that are covered by A_i for every $i \in J$. Clearly, the union of the sets S_J is S, when J runs over the k-element subsets of I. We define two sets A and B in the following way: for any pair J and $\bar{J}$, we choose either S_J or $S_{\bar{J}}$ to add to A, and we add the other one to B. Then $S \subseteq A \cup B$.

Note that for any pair of points $p, q \in A$, there is an index i such that $p, q \in A_i$, and thus, $|p - q| \leq \operatorname{diam} A_i < d$. This yields that $\operatorname{diam} A < d$. We may obtain similarly that $\operatorname{diam} B < d$, which implies that $a(S) = 2$. $\square$

Remark 3. Let $S \subset \mathbb{R}^n$ be a set of positive diameter. Then for every value of k, $a_k(S) = a_k(\mathrm{bd}\, S)$.

Proof. Without loss of generality, we may assume that S is compact and that $\mathrm{diam}\, S = 1$. Then, clearly, $a_k(S) \geq a_k(\mathrm{bd}\, S)$ for every k.

On the other hand, assume that some sets $Q_1, Q_2, \ldots, Q_m$ form a k-fold cover of $\mathrm{bd}\, S$ where each Q_i is of diameter less than one. Without loss of generality, we may assume that $Q_i \subset S$ for every i. Let $\varepsilon > 0$ be chosen in such a way that $\mathrm{diam}\, Q_i < 1 - 2\varepsilon$ for all values of i. Then the sets $\bar{Q}_i = (Q_i + \varepsilon \mathbf{B}^n) \cap S$ form a k-fold cover of $(\mathrm{bd}\, S + \varepsilon \mathbf{B}^n) \cap S$ such that $\mathrm{diam}\, \bar{Q}_i < 1$. Let T denote the set $S \backslash (\mathrm{bd}\, S + \varepsilon \mathbf{B}^n)$. Observe that for any point $p \in T$ and $q \in S$, we have $|p - q| \leq 1 - \varepsilon$. Thus, setting $Q'_i = \bar{Q}_i \cup T$ for every i, we have $\mathrm{diam}\, Q'_i \leq \max\{\mathrm{diam}\, \bar{Q}_i, 1 - \varepsilon\} < 1$, and the sets $Q'_1, Q'_2 \ldots, Q'_m$ form a k-fold cover of S. $\square$

By Remark 3, we may imagine the k-fold Borsuk number of a convex body C as a painting of the surface of C, with $a_k(C)$ colors, such that the diameter of each patch is less than $\mathrm{diam}\, C$, and any point on the surface is covered by at least k layers.

Before starting our investigation, we recall two notions from graph theory which we are going to use in the proofs. First, if G is a graph, then the *k-fold chromatic number* $\chi_k(G)$ of G is the smallest integer m with the property that a k-element subset of $\{1, 2, \ldots m\}$ (called *colors*) can be assigned to each vertex of G in such a way that if two vertices are connected by an edge, then the corresponding subsets are disjoint. The second notion is the of the *independence number* $\alpha(G)$ of a graph G: This number is the cardinality of the largest subset of the vertex set $V(G)$ of G in which no two edges are connected by an edge. By the Pigeon-Hole Principle, we clearly have the following inequality.

Remark 4. For any graph G, we have

$$\chi_k(G) \geq \frac{kV(G)}{\alpha(G)}.$$

2. Sets in the Euclidean plane

To formulate our main results, we first recall the well-known fact that for any $S \subset \mathbb{R}^n$ of diameter d, there is a convex body $K \subset \mathbb{R}^n$ of constant width d such that $S \subseteq K$. The following characterization of the Borsuk numbers of plane sets was given by Boltyanskii (cf. [1], or alternatively [2]).

Theorem 1 (Boltyanskii). *Let $S \subset \mathbb{R}^2$ be of diameter $d > 0$. The Borsuk number of S is three if, and only if, there is a unique convex body of constant width d, containing S.*

Now we prove the following.

Theorem 2. *Let $S \subset \mathbb{R}^2$ be a set of diameter $d > 0$ with $a(S) = 3$, and let C be the unique plane convex body of constant width d that contains S. Then for every value of k, we have $a_k(S) = a_k(C)$.*

Our proof is based on the following lemma, used by Boltyanskii (cf. for example, Lemma 8, p. 29, [2])

Lemma 1 (Boltyanskii). *For any point $u \in (\operatorname{bd} C) \setminus S$, there is an open circle arc of radius d in $\operatorname{bd} C$, that contains u, such that the center p of the circle is contained in $\operatorname{bd} C$.*

Proof of Theorem 2. Without loss of generality, let S be compact, and $d = 1$. Clearly, $a_k(C) \geq a_k(S)$. Hence, by Remark 3, it suffices to show that $a_k(\operatorname{bd} C) \leq a_k(S)$.

Assume that $Q_1, Q_2, \ldots, Q_m \subset S$ are sets of diameters less than one that form a k-fold cover of S. Without loss of generality, we may assume that $Q_i \subset C$ for every value of i.

Let $\delta > 0$ be chosen in a way that $\operatorname{diam} Q_i + 2\delta < 1$ for every value of i. Let A_j, where $j = 1, 2, \ldots, t$ denote the connected components of $(\operatorname{bd} C) \setminus S$ longer than 2δ, and let q_j and r_j be the two endpoints of A_j. Clearly, there are finitely many such arcs. First, we note that the sets $\bar{Q}_i = (Q_i + \delta \mathbf{B}^2) \cap C$ form a k-fold cover of $(\operatorname{bd} C) \setminus \left(\sum_{j=1}^{t} A_j \right)$, and that the diameter of any of these sets is less than one. We extend the sets $\bar{Q}_1, \bar{Q}_2, \ldots, \bar{Q}_m$ to cover k-fold all the A_js.

Using Lemma 1, for every value of j, A_j is an open unit circle arc with its center $p_j \in \operatorname{bd} C$. Since C is contained in the intersection of the two unit disks $q_j + \mathbf{B}^2$ and $r_j + \mathbf{B}^2$, we obtain that p_j is not a smooth point of $\operatorname{bd} C$, which yields, by the same lemma, that $p_j \in S$.

Let A_j^q be the set of the points of A_j that are not farther from q_j than from r_j. We define A_j^r analogously. Note that for any $u \in A_j$, the only point of C at distance one from u is p_j. Hence, for any index i such that $q_j \in \bar{Q}_i$, p_j has a neighborhood disjoint from $\bar{Q}_i$. This yields that $\operatorname{diam}(\bar{Q}_i \cup A_j^q) < 1$. We may obtain similarly that if $r_j \in \bar{Q}_i$, then $\operatorname{diam}(\bar{Q}_i \cup A_j^r) < 1$. Thus, we set

$$C_i = \operatorname{conv}\left(\bar{Q}_i \cup \bigcup_{q_j \in \bar{Q}_i} A_j^q \cup \bigcup_{r_j \in \bar{Q}_i} A_j^r \right),$$

and observe that, by induction on j, $\operatorname{diam} C_i < 1$ for every value of i, and that the sets C_i form a k-fold cover of $\operatorname{bd} C$. $\qquad \square$

Now let us recall the notion of *Reuleaux polygons*. These polygons are constant width plane convex bodies bounded by finitely many circle arcs of the same diameter (of radius equal to the width of the body), called the *sides* of the polygon. It is a well-known fact that any such polygon has an odd number of sides (cf. [4] or [7]).

Theorem 3. *Let C be a constant width convex body in $\mathbb{R}^2$, and k be a positive integer. If C is a Reuleaux-polygon with $2s + 1$ sides, then $a_k(C) = 2k + \left\lceil \frac{k}{s} \right\rceil$, and otherwise $a_k(C) = 2k + 1$.*

Proof. For simplicity, assume that $\operatorname{diam} C = 1$.

First, consider the case that C is a Reuleaux polygon with $2s + 1$ sides. Let us call the common point of two consecutive sides of C a *vertex* of C. Let these vertices be $p_1, p_2, \ldots, p_{2s+1} = p_0$ in counterclockwise order in $\operatorname{bd} C$. Consider the

diameter graph of the vertex set of C: The vertices of this graph are the vertices of C, and two vertices are connected with an edge if, and only if, they are endpoints of a diameter. Clearly, this graph is the cycle C_{2s+1}, of length $(2s+1)$. By [28] (see also [29]), the k-fold chromatic number of C_{2s+1} is $m = 2k + \left\lceil \frac{k}{s} \right\rceil$. This implies that $a_k(C) \geq m$. We note that the inequality $m \geq 2k + \left\lceil \frac{k}{s} \right\rceil$ also follows from Remark 4.

Now we show the existence of some sets $Q_1, Q_2, \ldots, Q_m$, with diameters less than one, that form a k-fold cover of bd C. This, by Remark 3, yields the assertion for Reuleaux-polygons. Let $A_1, A_2, \ldots, A_m$ be sets of diameters less than one that form a k-fold cover of the vertices of C. Let $\widehat{p_i p_{i+1}}$ denote the side of C connecting p_i and p_{i+1}, and let G_i be the union of the points of the two sides $\widehat{p_i p_{i-1}}$ and $\widehat{p_i p_{i+1}}$ that are not farther from p_i than from p_{i-1} and p_{i+1}, respectively. Observe that the sets $Q_j = \bigcup_{p_i \in A_j} G_j$ form a k-fold cover of bd C, and that their diameters are strictly less than one, which readily implies the assertion.

In the remaining part we assume that C is *not* a Reuleaux-polygon. By Remark 2, we have that $a_k(C) \geq 2k + 1$. Hence, it suffices to construct a family of $2k+1$ sets of diameters less than one that form a k-fold cover of C or, by Remark 3, bd C.

First, we carry out this construction for $C = \frac{1}{2} \mathbf{B}^2$. Consider $2k + 1$ distinct diameters of C. Let these be $[p_1, q_1], [p_2, q_2], \ldots, [p_{2k+1}, q_{2k+1}]$, where the notation is chosen in such a way that the points $p_1, p_2, \ldots, p_{2k+1} = p_0$ and $q_1, q_2, \ldots, q_{2k+1} = q_0$ are in counterclockwise order in bd C, and for every i, the shorter arc connecting p_i and p_{i+1} contains exactly one of the q_js (by exclusion, this point is q_{i+k}, cf. Figure 1). Observe that the points p_i have the property that the diameter containing any one of them divides bd C into two open half circles, each of which contains exactly k of the remaining $2k$ points. Let A_i denote the shorter arc in bd C connecting p_i and p_{i+k}. Observe that these sets form a k-fold cover of bd C, and that their diameters are less than one.
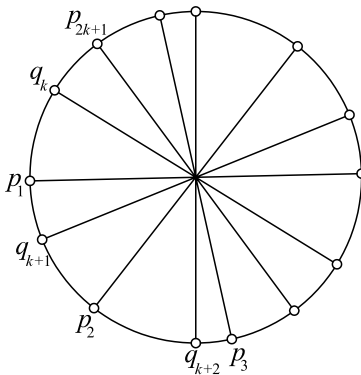

FIGURE 1. Covering a Euclidean disk

In the last step, we show that a similar family can be constructed for any C that is not a Reuleaux-polygon. Before we do that, we recall the following simple property of plane convex bodies of constant width:

- No two diameters of a plane convex body of constant width are disjoint.

For any $p \in \operatorname{bd} C$, let $G(p) \subset \mathbb{S}^1$ be the *Gaussian image* of p; that is, the set of the external unit normal vectors of the lines supporting C at p. Observe that if $G(p)$ is not a singleton, then it is a closed arc in $\mathbb{S}^1$. Furthermore, in this case p is not a smooth point of $\operatorname{bd} C$, and thus, by Lemma 1 in the previous proof, the locus of the other endpoints of the diameters starting at p is a closed unit circle arc in $\operatorname{bd} C$. Apart from the endpoints, the points of this arc are smooth points of $\operatorname{bd} C$, and thus, their Gaussian images are singletons. This yields that if $G(p)$ is not a singleton, then, apart from its endpoints, the points of $-G(p)$ are decomposed into singleton Gaussian images.

Since C is not a Reuleaux-polygon, we may choose $2k+1$ diameters of $\mathbf{B}^2$, say $[p_1, q_1], [p_2, q_2], \ldots, [p_{2k+1}, q_{2k+1}]$, such that the Gaussian image of any point of $\operatorname{bd} C$ intersects at most one of them. Indeed, as $\mathbb{S}^1$ is not covered by the union of finitely many Gaussian images and their antipodal arcs, at least one of the following holds:

Case 1, There are at least $2k+1$ Gaussian images in $\mathbb{S}^1$ that are not singletons: then we may choose $2k+1$ such arcs, and pick one point from each, different from the endpoints of the arc, as an endpoint of one of the chosen diameters.

Case 2, There are less than $2k+1$ Gaussian images that are not singletons. In this case there is an open arc $I \subset \mathbb{S}^1$ such that both I and $-I$ are decomposed into singleton Gaussian images. Thus, we may choose $2k+1$ pairwise distinct diameters with all their endpoints in $I \cup (-I)$.

Let us label the endpoints of these diameters as in the case that C is a Euclidean disk. That is, assume that the points $p_1, p_2, \ldots, p_{2k+1} = p_0$ are in counterclockwise order in $\operatorname{bd} C$, and for every i, the (counterclockwise) directed arc connecting p_i and p_{i+1} contains exactly q_{i+k} from amongst the q_js. Let $G^{-1}(u)$ denote the (unique) point v of $\operatorname{bd} C$ with the property that $u \in G(v)$. For every i, let F_i denote the closed arc of $\operatorname{bd} C$, with endpoints $G^{-1}(p_i)$ and $G^{-1}(p_{i+k})$, and containing $G^{-1}(p_{i+j})$ for $j = 1, 2, \ldots, k-1$. Observe that since any two diameters of C have a nonempty intersection, no arc F_i contains the endpoints of a diameter, and thus, $\operatorname{diam} F_i < 1$. On the other hand, these arcs form a k-fold cover of $\operatorname{bd} C$, which implies that $a_k(C) = 2k+1$. $\qquad\square$

3. Centrally symmetric sets and smooth bodies

Two of the special cases for which Borsuk's original conjecture is proven, are when the set is centrally symmetric, or is a smooth convex body (cf. [25], [11] and [12]). The proofs in both cases are based on reducing the problem to the Euclidean n-ball $\mathbf{B}^n$, and then to finding $a(\mathbf{B}^n)$. In this section, we investigate the k-fold Borsuk numbers of these sets in the same way.

For preciseness, we first remark that we call a set $S \subset \mathbb{R}^n$ a *smooth body*, if S is homeomorphic to $\mathbf{B}^n$, and its boundary is a C^1-class submanifold of $\mathbb{R}^n$.

Theorem 4. *Let $S \subset \mathbb{R}^n$ be a set of diameter $d > 0$.*

(1) *If S is a smooth body or centrally symmetric, then for every k, we have $a_k(S) \leq a_k(\mathbf{B}^n)$.*

(2) *If S is a convex body of constant width, then for every k, we have $a_k(S) \geq a_k(\mathbf{B}^n)$.*

(3) *For every k, we have $a_k(\mathbf{B}^n) = 2k + n - 1$.*

Clearly, this theorem implies that if S is a smooth convex body of constant width, then for every k, $a_k(S) = 2k + n - 1$.

Proof. Let $d = 1$.

First we examine the case that S is a smooth body. For every $p \in \operatorname{bd} S$, let $G(p)$ denote the Gaussian image of p; that is, the unique external unit normal vector of S at p. Then $G : \operatorname{bd} S \to \mathbb{S}^{n-1}$ is a continuous mapping. Observe that if $[p, q]$ is a diameter of S, then $p, q \in \operatorname{bd} S$, and $G(p) = -G(q) = p - q$. Thus, any k-fold cover of $\mathbb{S}^{n-1}$ by m sets of smaller diameters induces a k-fold cover of $\operatorname{bd} S$, and thus S, by m sets of smaller diameters. This shows that $a_k(S) \leq a_k(\mathbf{B}^n)$.

Now, assume that S is a (not necessarily smooth) convex body of constant width. Then every point of $\operatorname{bd} S$ is an endpoint of some diameter, and thus, a k-fold cover of $\operatorname{bd} S$ induces a k-fold cover of $\mathbb{S}^{n-1}$ like in the previous paragraph. This yields $a_k(S) \geq a_k(\mathbf{B}^n)$ for every k.

Next, let S be symmetric to the origin. Observe that $S \subseteq \frac{1}{2}\mathbf{B}^n$. Then the set $S_d = S \cap \left(\frac{1}{2}\mathbb{S}^{n-1}\right)$ contains all the points of S that are endpoints of some diameter, and thus, the inequality $a_k(S) \leq a_k(\mathbf{B}^n)$ follows by an argument similar to the one used for smooth bodies.

We are left to show that $a_k(\mathbf{B}^n) = 2k + n - 1$, or equivalently, that $a_k(\mathbb{S}^{n-1}) = 2k + n - 1$. To show that $a_k(\mathbb{S}^{n-1}) \geq 2k + n - 1$, we follow the idea of the proof for the usual Borsuk number of $\mathbb{S}^{n-1}$.

Consider a k-fold cover $\mathcal{F} = \{Q_1, Q_2, \ldots, Q_m\}$ of $\mathbb{S}^{n-1}$, with closed sets, such that no element of $\mathcal{F}$ contains a pair of antipodal points. Let us define the function $f : \mathbb{S}^{n-1} \to \mathbb{R}^{n-1}$ as

$$f(x) = (\operatorname{dist}(x, A_1), \operatorname{dist}(x, A_1), \ldots, \operatorname{dist}(x, A_{n-1}))$$

This function is clearly continuous, and hence, by the Borsuk-Ulam Theorem, there is a point $p \in \mathbb{S}^{n-1}$ such that $f(p) = f(-p)$. If some coordinate of $f(p)$ is zero, then both p and $-p$ are elements of one of the sets $A_1, A_2, \ldots, A_{n-1}$; a contradiction. Thus, $f(p)$ has no coordinate equal to zero, which means that neither p nor $-p$ belongs to $A_1 \cup \ldots A_{n-1}$. Since p and $-p$ are antipodal points, there are at least $2k$ elements of $\mathcal{F}$ that contain one of them, which yields that $m \geq 2k + n - 1$. On the other hand, Gale proved (cf. Theorem II' in [8]) the existence of a family of $2k+n-1$ open hemispheres of $\mathbb{S}^{n-1}$ that form a k-fold cover of $\mathbb{S}^{n-1}$. Since contracting these open hemispheres one by one yields a k-fold cover of $\mathbb{S}^{n-1}$ with $2k + n - 1$ closed spherical caps of radii strictly less than $\frac{\pi}{2}$, we obtain that $a_k(\mathbf{B}^n) = 2k + n - 1$. $\square$

4. Multiple Borsuk numbers of finite point sets in Euclidean 3-space

The fact that the (usual) Borsuk numbers of finite sets in 3-space are at most four was first shown by Heppes and Révész in [14], and it also follows from the proof of Vázsonyi's conjecture (cf. [13], [9], [18] or [30]), that stated that in any set $S \subset \mathbb{R}^3$ of cardinality m, $\operatorname{diam} S$ is attained between at most $2m - 2$ pairs of points; or in other words, that the diameter graph of any set of m points in $\mathbb{R}^3$ has at most $2m - 2$ edges. Later we use some of the ideas of these proofs.

A complete characterization of the Borsuk numbers of finite sets in $\mathbb{R}^3$, even of those with $a(S) = 4$ looks hopeless: indeed, by (2) of Theorem 4, if S is the vertex

set of a Reuleaux-polytope in $\mathbb{R}^3$, then $a(S) = 4$, and a result of Sallee [27] yields that the family of Reuleaux-polytopes in $\mathbb{R}^3$ is an everywhere dense subfamily of the family of convex bodies of constant width in $\mathbb{R}^3$. Thus, unlike in Section 2, in this and the next sections we restrict our investigation to point sets with some special properties.

The main goal of this section is to find the finite sets $S \subset \mathbb{R}^3$ with $a_k(S) = 4k$ for every value of k. We observe that for a finite set $S \subset \mathbb{R}^3$, Remark 1 readily implies that $a_k(S) \leq 4k$ for every value of k, where we have equality, for example, for regular tetrahedra.

Using the next, naturally arising concept, we may rephrase our question in a different form.

Definition 2. Let $S \subset \mathbb{R}^n$ be of diameter $d > 0$. Then the quantity

$$a_{frac}(S) = \inf \left\{ \frac{a_k(S)}{k} : k = 1, 2, 3 \ldots \right\}$$

is called the *fractional Borsuk number* of S.

Clearly, $a_{frac}(S) \leq a(S)$ for every set S.

Problem 1. Prove or disprove that if $S \subset \mathbb{R}^3$ is a finite point set with $a_{frac}(S) = 4$, then its diameter graph contains K_4 as a subgraph.

We give only a partial answer to this problem. During the investigation, we denote the diameter graph of S by G_S, and recall that the *girth* of a graph G is the length of a shortest cycle in G. We denote this quantity by $g(G)$, and note that for any finite set $S \subset \mathbb{R}^3$, we have $a_k(S) = \chi_k(G_S)$, where $\chi_k(G_S)$ denotes the k-fold chromatic number of G_S.

Our main result is the following.

Theorem 5. *Let $S \subset \mathbb{R}^3$ be a finite set with $g(G_S) > 3$. Then $a_k(S) < 4k$ for some value of k.*

This theorem may be rephrased in the following form: For any finite set $S \subset \mathbb{R}^3$ with $a_{frac}(S) = 4$, G_S contains K_3 as a subgraph. The proof is based on Lemma 2.

Lemma 2. *There is an m-fold $(2m + 1)$-coloring of the $(2m + 1)$-cycle C with the property that any two nonconsecutive vertices have a common color.*

Proof of Lemma 2. Let the vertices of C be $v_1, v_2, \ldots, v_{2m+1} = v_0$ in counterclockwise order. Let the colors be $1, 2, \ldots, 2m+1$. We define a 3-coloring of C as follows with the colors $t, t+1, 2m+1$, where $t \in \{1, 3, \ldots, 2m-1\}$: only v_t is colored with $2m + 1$, and the vertices $v_{t+1}, v_{t+2}, \ldots$ are colored with t and $t + 1$, alternately. It is easy to see that the union of these m 3-colorings is an m-fold $(2m + 1)$-coloring of C.

Consider any two vertices v_i and v_j with $|i - j| \geq 2$. Since C is an odd cycle, exactly one of the two connected components of $C \backslash \{v_i, v_j\}$ contains an even number of vertices. If this component does not contain a vertex colored with the color $2m+1$, then v_i and v_j are v_{2m-1} and v_1, both of which are colored with $2m + 1$. If the even component contains the vertex a_t colored with $2m + 1$, then both v_i and v_j are colored either with t or with $t + 1$. $\square$

Proof of Theorem 5. Let C be a shortest odd cycle in G_S, of length $2m + 1 \geq 5$. We show that $\chi_m(G_S) \leq 4m - 1$. Note that if G_S contains no odd cycle, then it is bipartite, and thus, the statement follows from $\chi(G_S) = 2$.

By [6], any odd cycle of G_S intersects C, or in other words, $G_S \setminus C$ is a bipartite graph. Let the two parts of $V(G_S \setminus C)$ in this partition be V_1 and V_2. Clearly, since G_S contains no triangle, no vertex of V_1 is connected to two consecutive vertices of C. Furthermore, no vertex of V_1 is connected to more than two vertices of C. Indeed, if a vertex v is connected to the distinct vertices $v_1, v_2, v_3 \in C$, then there is a path in C, of odd length at most $2m - 3$, that connects two of v_1, v_2 and v_3. This yields that C is not a shortest odd cycle of G_S; a contradiction (for this argument, cf. also [6]).

Now we define an m-fold $(4m - 1)$-coloring of G_S. We color each vertex of V_2 with the colors $3k, 3k + 1, \ldots, 4k - 1$, and use only the remaining colors for $C \cup V_1$. We color C in the way described in Lemma 2, using only the colors $1, 2, \ldots, 2m + 1$. We color the vertices of V_1, using $1, 2, \ldots, 3m - 1$, in the following way. Consider a vertex $v \in V_1$. Then v is connected to at most two vertices of C, which are not consecutive. Hence, by Lemma 2, there are at most $2m - 1$ colors used for coloring them. Thus, there are at least $(3m - 1) - (2m - 1) = m$ colors, from amongst $1, 2, \ldots, 3m - 1$, that do not color any neighbor of v in C. We color v with m such colors. $\square$

In the remaining part we show that the statement of Problem 1 holds for any set S with $\operatorname{card} S \leq 7$. We start with finding the 4-critical subsets of diameter graphs of sets in $\mathbb{R}^3$ of at most seven points. Recall that a graph G is m-critical if $\chi(G) = m$, and for any proper subgraph H of G, $\chi(H) < m$.

Lemma 3. *If $S \subset \mathbb{R}^3$ with $\operatorname{card} S \leq 7$ and $a(S) = 4$, and H is a 4-critical subgraph of G_S, then H is either K_4, or the wheel graph W_6, or the Mycielskian $\mu(C_3)$ of the 3-cycle C_3 (cf. Figure 2).*
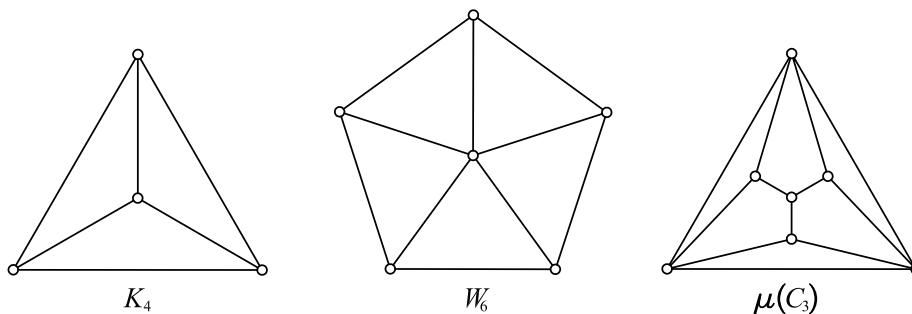

FIGURE 2. 4-critical subgraphs of diameter graphs

Proof. By the proof of Vázsonyi's conjecture, G_S has at most $2 \operatorname{card} S - 2$ edges, and by [6], any two odd cycles of G_S intersect. Clearly, these properties hold also for all the subgraphs of G_S. Thus, it suffices to prove the following, slightly more general statement: If G is a 4-critical graph with at most $m \leq 7$ vertices and at

most $2m - 2$ edges such that any two odd cycles of G intersect, then G is either K_4, or W_6 or $\mu(C_3)$.

Answering a question of Toft [32] it was proven in [16] that if a 4-critical graph has at least one vertex of degree 3, then the graph contains a *fully odd subdivision* of K_4 as a subgraph, where a fully odd subdivision of a graph H is a graph, obtained from H in a way that the edges of H are replaced by paths with odd numbers of edges.

Since our graph G has at most $2m - 2$ edges and, being 4-critical, the degree of any vertex is at least 3, G has at least four vertices of degree 3, and hence, by [16], it contains a fully odd subdivision of K_4. If G is not K_4, then, as G is 4-critical, this subdivision does not coincide with K_4, and thus, $m \geq 6$.

We leave it to the reader to show that the only 4-critical graph with six vertices and satisfying our conditions is W_6. We deal only with the case $m = 7$. For the proof we use the notations in Figure 3. Since G has at most 12 edges, there are at most four edges not shown in Figure 3. Note that as G contains no disjoint triangles and the degree of every vertex is at least 3, a_7 is connected to exactly one of a_5 or a_6. By symmetry, we may assume that $a_5 a_7$ is an edge, and $a_6 a_7$ is not. This implies also that the degrees of a_5, a_6 and a_7 are 3, and that G has exactly 12 edges.
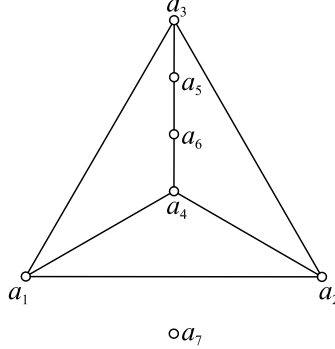

FIGURE 3. An illustration for the proof of Lemma 3

By a similar argument, we may obtain that exactly one of $a_1 a_6$ and $a_2 a_6$ is an edge, say $a_1 a_6$. Then the two additional edges of G connect a_7 to two of a_1, a_2, a_3 and a_4. It is an elementary exercise to check that if these edges are not $a_2 a_7$ and $a_4 a_7$, then G is 3-colorable or contains disjoint triangles. But if they are $a_2 a_7$ and $a_4 a_7$, then $G = \mu(C_3)$, which finishes the proof. □

Theorem 6. *If $S \subset \mathbb{R}^3$ with* card $S \leq 7$*, then either $a_k(S) < 4k$ for every $k > 1$, or G_S contains K_4 as a subgraph.*

Proof. If $a_1(S) \leq 3$ or $a_2(S) \leq 7$, then the assertion readily follows from Remark 1. Thus, we may assume that card $S \leq 7$ and that $a_2(S) = 2a_1(S) = 8$, or equivalently, that $2\chi(G_S) = \chi_2(G_S) = 8$. As a consequence of Lemma 3, G_S contains K_4, W_6 or $\mu(C_3)$ as a subgraph.

If G_S contains K_4, we are done. If G_S contains $\mu(C_3)$ as a subgraph, then $G_S = \mu(C_3)$, since $\mu(C_3)$ has 7 vertices and 12 edges, and by Vázsonyi's problem G_S has no more than 12 edges. It is easy to see that $\chi_2(\mu(C_3)) = 7$, which immediately implies the assertion.

Now we deal with the case that G_S contains W_6 as a subgraph. If card $S = 6$, then by Vázsonyi's problem $G_S = W_6$, and thus, $a_2(S) = \chi_2(W_6) = 7$. Assume that card $S = 7$. Then there are at most two edges of G_S not contained in the subgraph W_6. We may assume that W_6 is an induced subgraph of G_S, since if an additional edge of G_S connects two vertices of W_6, then G_S contains K_4 as a subgraph. Thus, G_S is obtained from W_6 by adding an additional vertex, and connecting it to at most two vertices. Depending on the choice of these vertices, it is an elementary exercise to find a 2-fold 7-coloring of G_S in each case. $\square$

5. The Borsuk numbers of symmetric finite point sets in $\mathbb{R}^3$

In this section, our aim is to examine the Borsuk numbers of finite point sets in $\mathbb{R}^3$ with their Borsuk numbers equal to four, and with a nontrivial symmetry group. Our project is motivated by a result of Rogers [26], who proved Borsuk's conjecture for n-dimensional sets with symmetry groups containing that of a regular n-dimensional simplex.

In our investigation, for a finite set $S \subset \mathbb{R}^3$, we denote the symmetry group of S by $\mathrm{Sym}(S)$, the symmetry group of a regular tetrahedron by $\mathcal{T}_4$, and that of a regular k-gon by $\mathcal{D}_k$.

Theorem 7. *Let $S \subset \mathbb{R}^3$ be a finite set with $\mathcal{T}_4 \subseteq \mathrm{Sym}(S)$. If $g(G_S) = 3$, then G_S contains K_4 as a subgraph.*

Proof. Without loss of generality, let diam $S = 1$, and let the regular tetrahedron, with symmetry group $\mathcal{T}_4$ and with unit edge length, be T.

Let $a, b, c \subseteq S$ be the vertices of a regular triangle of unit edge length. Let M be any plane reflection contained in $\mathrm{Sym}(S)$. Then the points $M(a), M(b), M(c)$ are contained in S. By [6] or [30], any two odd cycles of G_S intersect, and thus the sets $\{a, b, c\}$ and $\{M(a), M(b), M(c)\}$ are not disjoint. If a point is the reflecion of another one, say $b = M(a)$, then, clearly, $a = M(b)$, and then c is on the reflection plane; that is, $M(c) = c$. If a point is its own reflection, then the point is on the reflection plane. Thus, we have shown that each reflection plane of $\mathrm{Sym}(S)$ contains at least one vertex from any 3-cycle in G_S.

We leave it to the reader to show that since diam $S = 1$, then $\{a, b, c\}$ does not contain the center of T. Thus, there is an axis of rotation in $\mathcal{T}_4$ that is disjoint from $\{a, b, c\}$. Let R be a rotation with angle $\frac{2\pi}{3}$ around this axis. Then the triples $\{a, b, c\}$ and $\{R(a), R(b), R(c)\}$ have a point in common; say, $b = R(a)$ (note that no point is the rotated copy of itself). In this case $\{a, b, R(b)\}$ is a 3-cycle in G_S which is invariant under R. As any 3-cycle has a point on each reflection plane, it implies that the vertices of this cycle are on the other three axes of rotation. Applying the symmetries of $\mathcal{T}_4$ to these vertices we obtain the vertices of a regular tetrahedron of unit edge length, which readily implies the assertion. $\square$

Remark 5. Combining Theorems 7 and 5, we have that if for some finite set $S \subset \mathbb{R}^3$ we have $\mathcal{T}_4 \subseteq \mathrm{Sym}(S)$, and $a_k(S) = 4k$ for every k, then G_S contains K_4 as a subgraph.

We note that by [30], for every finite set $S \subset \mathbb{R}^3$, G_S can be embedded in the projective plane. On the other hand, an example in [30] shows that not all these graphs are planar.

Remark 6. It is known that the chromatic number of every triangle-free planar graph is at most three. Thus, Theorem 7 yields that if G_S is planar and $\mathcal{T}_4 \subseteq \mathrm{Sym}(S)$, then S contains K_4 as a subgraph.

Problem 2. Prove or disprove that if $S \subset \mathbb{R}^3$ is a finite set with $\mathcal{T}_4 \subseteq \mathrm{Sym}(S)$ and with $a(S) = 4$, then G_S contains K_4 as a subgraph.

Our next aim is to examine sets S, with $a(S) = 4$ and with $\mathcal{D}_{2k+1} \subseteq \mathrm{Sym}\, S$ for some integer $k \geq 1$. We construct a family of sets satisfying these conditions.

In the construction we use the notion of the *p-Mycielskian* of a graph G (cf. [31]), denoted by $\mu_p(G)$. We regard the *wheel graph* W_{2k+2} as the 0th Mycielskian of the odd cycle C_{2k+1}.

Theorem 8. *For any $p \geq 0$, and $k > 0$, $\mu_p(C_{2k+1})$ is the diameter graph of a finite set $S \subset \mathbb{R}^3$.*

Proof. Let $p_1, p_2, \ldots, p_{2k+1} = p_0$ be the vertices of a regular $(2k + 1)$-gon in the (x, y)-plane, centered at the origin. Assume that the diameter of the point set is $r \leq 1$. Consider the points $q = (0, 0, \sqrt{1 - r^2})$ and $r = (0, 0, \sqrt{1 - r^2} - 1)$. Note that for every i, $||p_i - q|| = 1$ and $||p_i - r|| < 1$.

Let v_i denote the inner unit normal vector of the supporting plane of the pyramid $\mathrm{conv}\{p_1, \ldots, p_{2k+1}, q\}$ passing through the points $p_{i\pm k}$ and q. An elementary computation shows that $\langle v_i, p_j - q \rangle \geq 0$ for every i and j, and if $j \neq i, i + 1$, then we have strict inequality. Thus, for every i, we may choose a point q_i such that the points q_i are the vertices of a regular $(2k + 1)$-gon (and have equal z-coordinates), $|p_j - q_i| \leq 1$ with equality if and only if $j = i \pm k$. Furthermore, if the points q_i are sufficiently close to q, then for the point r' on the negative half of the z-axis that satisfies $||q_i - r|| = 1$, we have $||p_i - r'|| < 1$.

Now, to obtain the required p-Mycielskian, we start with the wheel graph W_{2k+2}. This can be realized as the vertex set V_1 of a pyramid, with a regular $(2k + 1)$-gon of diameter one as its base, and with the property that the distance of its apex from any other vertex is one. To obtain a p-Mycielskian, we may apply the procedure described in the first two paragraphs $(p - 1)$ times. $\qquad \square$

Remark 7. Let S be a point set with $G_S = \mu_p(C_{2k+1})$. Then $a(S) = 4$ for every p and k. Hence, since for $k \geq 2$ and $p > 1$ $\mu_p(C_{2k+1})$ is triangle-free, it is not a planar graph. On the other hand, it is easy to see that the number of edges in G_S is equal to $2 \,\mathrm{card}\, S - 2$. Thus, these sets form an infinite family of nonplanar Vázsonyi-critical graphs.

Remark 8. Clearly, if $p = 0$, then we have $a_k(S) = k + \chi_k(C_{2m+1}) = 3k + \lceil \frac{k}{m} \rceil$. By [19], for $p = 1$, we have

$$a_k(S) = \begin{cases} 4 & \text{if } k = 1, \\ \frac{5k}{2} + 1 & \text{if } k \text{ is even}, \\ 2k + \frac{k+3}{2}, & \text{if } k \text{ is odd and } k \le m \le \frac{3k+3}{2}, \text{ and} \\ 2k + \frac{k+5}{2}, & \text{if } k \text{ is odd and } m \ge \frac{3k+5}{2}. \end{cases}$$

Theorem 9. *Let $S \subset \mathbb{R}^3$ be a finite set with $\mathcal{D}_{2m+1} \subseteq \mathrm{Sym}(S)$ for some $m \ge 2$. If $a(S) = 4$ and $g(G_S) = 3$, then G_S contains a topological wheel graph W_{2m+2} as a subgraph.*

In the proof, we use the following lemma.

Lemma 4. *If $S \subset \mathbb{R}^3$ is a finite set such that $\mathrm{Sym}(S)$ contains a reflection about the plane H, then every odd cycle of G_S has a vertex on H.*

Proof. Swanepoel (cf. Theorem 2 of [30]) showed that for any $S \subset \mathbb{R}^3$, G_S has a bipartite double cover, with a centrally symmetric drawing on $\mathbb{S}^2$: in this drawing, any point p is represented by a pair of antipodal points p_b and $p_r = -p_b$ that are colored differently, and a diameter of S, connecting p and q, corresponds to the two edges $p_b q_r$ and $p_r q_b$. In his construction, the point p_r representing p is an arbitrary relative interior point of the conic hull of the diameters of S starting at p. Using the geometric properties of the conic hulls of the diameters, he concluded that any two odd cycles of G_S, which are represented by centrally symmetric closed curves on $\mathbb{S}^2$, have a common vertex.

Now consider the plane H', parallel to H and containing o. We apply the construction of Swanepoel with a special choice of points. For any $p \in S$, let CH_p denote the conic hull of the diameters of S, starting at p. Then we choose $p_r \in \mathbb{S}^2$ as the projection of the center of gravity of $\mathbf{B}^3 \cap CH_p$ on $\mathbb{S}^2$ from o. Clearly, p_r is on H' if, and only if p is on H.

Consider an odd cycle C in G_S. If its vertex set is symmetric about H, then H contains one of the vertices. Assume that C is not symmetric about H, and let C' denote its reflected copy about H. Clearly, the curves representing C and C' on $\mathbb{S}^2$ are symmetric about H', and thus, they intersect on H'. By Lemmas 1 and 2 of [30], these common points belong to common vertices of C and C', which yields that both cycles have a vertex on H. $\square$

Proof of Theorem 9. Assume that $\mathrm{diam}\, S = 1$.

By Lemma 4, any odd cycle, and in particular any triangle, of G_S contains a point on each plane of symmetry in $\mathrm{Sym}(S)$. Since $\mathrm{Sym}(S)$ contains at least $2m + 1 \ge 5$ symmetry planes, any triangle T of G_S has a vertex on the axis L of the rotations of $\mathcal{D}_{2m+1}$. Clearly, this triangle T has at most two vertices on L. If T has exactly two vertices on L, then the diameter of the union of the rotated copies of T is strictly greater than one; a contradiction. Thus, we have that T has exactly one vertex on L, which we denote by a. Let the remaining two vertices of T be b and c. Let $b = b_1, b_2, \dots, b_{2m+1}$, and $c = c_1, c_2, \dots, c_{2m+1}$ denote the rotated copies of b and c, respectively, about L.

First, consider the case that the points b_i and c_j are pairwise distinct. Let C be a shortest odd cycle that does not contain a. Such a cycle exists, as otherwise

$G \setminus \{a\}$ contains no odd cycle, and $\chi(G) = 3$. Since any two odd cycles intersect, C contains at least one point from each pair $\{b_i, c_i\}$. Thus, the required subgraph is defined as the union of C, a, and for each i an edge connecting a to either b_i or c_i on C.

Finally, assume that from amongst the b_is and the c_js there are coinciding vertices. Note that since they are not on L, we have that $b_i = c_j$ for some i and j. But then $\{b_1, b_2, \ldots, b_{2m+1}\} = \{c_1, c_2, \ldots, c_{2m+1}\}$, and these vertices, and a, are the vertices of a subgraph W_{2m+2}. $\square$

Corollary 1. *Let $S \subset \mathbb{R}^3$ be a finite set with $\mathcal{D}_{2m+1} \subseteq \mathrm{Sym}(S)$ for some $m \geq 2$. If $a(S) = 4$ and G_S is a plane graph, then G_S contains a topological wheel graph W_{2m+2} as a subgraph.*

Problem 3. Is it true that if $S \subset \mathbb{R}^3$ is a finite set with $\mathcal{D}_{2m+1} \subseteq \mathrm{Sym}(S)$ for some $m \geq 2$, and with $a(S) = 4$, then G_S contains $\mu_p(C_{2t+1})$ as a subgraph, for some p and t satisfying $(2m+1)|(2t+1)$? If the answer is negative, is it true for Vázsonyi-critical graphs?

6. An additional remark

Let $a_k(n)$ denote the maximum of the Borsuk numbers of n-dimensional sets of positive diameter, and let $a(n) = a_1(n)$. One of the fundamental questions regarding Borsuk's problem is to determine the asymptotic behavior of $a(n)$.

Presently, the best known asymptotic lower bound for $a(n)$ is due to Raigorodskii [21], who proved that for sufficiently large values of n,

$$a(n) \geq \left(\left(\frac{2}{\sqrt{3}} \right)^{\sqrt{2}} \right)^{\sqrt{n}} = (1.225...)^{\sqrt{n}} :$$

he constructed a finite n-dimensional set S with the property that the independence number of its diameter graph is not greater than $\frac{\mathrm{card}\, S}{1.225^{\sqrt{n}}}$, if n is sufficiently large. Clearly, by Remark 4, this property implies not only that $a(S) \geq 1.225^{\sqrt{n}}$ for large values of n, but also that $a_{frac}(S) \geq 1.225^{\sqrt{n}}$. Thus, we have the following.

Remark 9. If n is sufficiently large, then for every value of k, we have $a_k(n) \geq k1.225^{\sqrt{n}}$.

Problem 4. Is it true that for every value of k and n, we have $a_k(n) = ka(n)$? If not, do the two sides have the same magnitude?

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