Overlaps after quantum quenches in the sine-Gordon model

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Abstract

We present a numerical computation of overlaps in mass quenches in sine-Gordon quantum field theory using truncated conformal space approach (TCSA). To improve the cut-off dependence of the method, we use a novel running coupling definition which has a general applicability in free boson TCSA. The numerical results are used to confirm the validity of a previously proposed analytical Ansatz for the initial state in the sinh-Gordon quench.

1   Introduction

One of the most challenging problems in contemporary physics is the understanding of dynamical and relaxation phenomena in closed quantum systems out of equilibrium. Motivated by both theoretical interest and experimental relevance, recent studies led to a series of interesting discoveries such as the experimental observation of the lack of thermalization in integrable systems [1, 2, 3, 4]. To explain the stationary state of integrable quantum systems, the concept of the generalized Gibbs ensemble (GGE) was proposed [5], and recently experimentally confirmed [6]. It also turned out that the GGE was generally incomplete when only including the well-known local conserved charges [7, 8], and its completion made necessary the inclusion of novel quasi-local charges [9, 10]. Adding to this the unconventional, often ballistic nature of quantum transport [11, 12] or the confinement effects in the spread of correlations in non-integrable systems [13] indeed, a remarkable range of exotic behaviour has emerged in recent years.

A paradigmatic framework for non-equilibrium dynamics is provided by quantum quenches [14], in which the initial state (which is typically the ground state of some pre-quench Hamiltonian H_0) is subject to evolution driven by a post-quench Hamiltonian H, which is obtained from H_0 by instantaneously changing some parameters of the system. For the purpose of computing the time evolution it is useful to know the overlaps, i.e. the amplitudes of the post-quench excitations in the initial state. Indeed, in the case of integrable post-quench

dynamics, knowledge of these overlaps often enables the determination of steady state properties, and even the time evolution [15, 16, 17, 18, 19]. However, the determination of the overlaps is generally a very difficult task. When both the pre-quench and post-quench theories are non-interacting, the overlaps can be determined using the Bogoliubov transformation linking the pre- and post-quench excitation modes, but in genuinely interacting integrable models there are only few cases in which the overlaps are explicitly known. These cases mostly include spin chains and the Lieb-Liniger model [20, 21, 22, 24, 23, 25].

Quantum field theories are known to provide universal descriptions of statistical models and many-body systems, valid at long distances, and therefore quantum quenches in field theories are interesting, especially in the quest for universal characteristics and behaviour under quantum quenches. In massive relativistic integrable quantum field theories there exists a number of efficient approaches to the quench dynamics, which depend on the assumption that the initial state $|\Psi(0)\rangle$ can be written in a squeezed vacuum form in terms of post-quench Zamolodchikov-Faddeev creation operators $Z_a^\dagger(\vartheta)$ for asymptotic particle states and the post-quench vacuum $|0\rangle$

$$|\Psi(0)\rangle = \mathcal{N} \exp \int \frac{d\vartheta}{2\pi} K_{a,b}(\vartheta) Z_a^\dagger(-\vartheta) Z_b^\dagger(\vartheta)|0\rangle \ , \qquad (1.1)$$

which is just the analogue of the Bogoliubov solution for free theories. The above form of the initial state is equivalent to the statement that the multi-particle creation amplitudes factorize into products of independent single pair creation amplitudes. This is obviously reminiscent of the factorisation property of scattering in integrable quantum field theories [26], which justifies calling this class of quenches "integrable". Such a form of the initial state enables the application of methods based on thermodynamic Bethe Ansatz (TBA) [15, 16, 17], form factor based spectral expansions [27, 28] or semi-classical approach [18]. However, even within the class of integrable quenches no exact solutions are known for the overlap functions $K_{a,b}$ apart from non-interacting quantum field theory models.

Recently an Ansatz for the overlaps was proposed for the quench from a massive free boson to an interacting sinh-Gordon model [29, 30], which has already been used to obtain predictions for steady state expectation values [31]. The aim of the present work is to provide a test of this solution from first principles, by comparing their analytical continuation to sine-Gordon theory to a direct evaluation of the overlaps in the framework of the Truncated Conformal Space Approach (TCSA), originally introduced in [32] and extended to the sine-Gordon model in [33].

2 Overlaps in quantum field theory quenches

2.1 The sinh-Gordon quench overlaps and their continuation to sine-Gordon

The work [30] considered a quench from a massive free bosonic theory with particle mass m_0 to the massive sinh-Gordon theory

$$\mathcal{A} = \int d^2x \left(\frac{1}{2} \partial_\mu \Phi \partial^\mu \Phi - \frac{\mu^2}{g^2} \cosh g\Phi \right) \ , \qquad (2.1)$$

with coupling g and physical particle mass m (which is equal to μ in the classical limit). For these quenches it was argued that the initial state can be cast into the exponential form

$$|\Psi(0)\rangle = \mathcal{N} \exp\left\{ \int \frac{d\vartheta}{2\pi} K(\vartheta) Z^\dagger(-\vartheta) Z^\dagger(\vartheta) \right\} |0\rangle . \tag{2.2}$$

However, properly demonstrating that the initial state has the form (1.1) in terms of post-quench asymptotic particle states is far from straightforward, and has only been possible in the non-interacting case (also in some interacting quenches in spin chains and Bose gases, where the exact overlaps are known and factorize in the thermodynamic limit [20, 21, 22, 24, 23, 25]).

A class of states which has the exponential form is given by so-called integrable boundary states introduced in [34]; however, they cannot be considered physical initial states since they are not normalizable. As shown in [14, 35] it is possible to construct proper initial states in terms of a boundary state $|B\rangle$ using the form

$$|\Psi(0)\rangle = e^{-\sum \tau_i Q_i} |B\rangle$$

where the Q_i are local conserved charges. Assuming that (as usual) the one-particle states are eigenstates of the Q_i, this obviously preserves the exponential form of the state, but it is hard to identify the physical quench (i.e. the pre-quench Hamiltonian H_0) that results in this state for a particular choice of the real parameters τ_i.

For quenches starting from a large initial mass in the sinh-Gordon field theory arguments in favour of the exponential form were advanced in [29], and even the following Ansatz was proposed for the function K:

$$K(\vartheta) = K_{free}(\vartheta) K_D(\vartheta) , \tag{2.3}$$

where $K_{free}(\vartheta)$ is given by

$$K_{free}(\vartheta) = \frac{E_0(\vartheta) - E(\vartheta)}{E_0(\vartheta) + E(\vartheta)}, \quad E(\vartheta) = m \cosh \vartheta \quad , \quad E_0(\vartheta) = \sqrt{m_0^2 + m^2 \sinh^2 \vartheta} ,$$

and is identical (up to a sign) with the Bogoliubov amplitude in a mass quench $m_0 \longrightarrow m$ within a free bosonic model, while $K_D(\vartheta)$ is the amplitude of the Dirichlet boundary ($\Phi = 0$) state in sinh-Gordon theory:[1]

$$K_D(\vartheta) = i \tanh(\vartheta/2) \frac{\cosh(\vartheta/2 - i\pi B/8) \sinh(\vartheta/2 + i\pi(B+2)/8)}{\sinh(\vartheta/2 + i\pi B/8) \cosh(\vartheta/2 - i\pi(B+2)/8)} \quad , \quad B(g) = \frac{2g^2}{8\pi + g^2} . \tag{2.4}$$

In the follow-up work [30] it was shown that provided the initial state contains only multiple particle states composed of pairs with opposite momenta, extensivity of the charges guaranteeing integrability leads to factorisation of multi-pair amplitudes and therefore results in an exponential form of the state, the only undetermined parameter being the pair creation amplitudes $K_{a,b}(\vartheta)$. However, the pair structure itself remains mainly an assumption supported only by some heuristic arguments [30].

Furthermore an infinite integral equation hierarchy was derived that determines (at least in principle) the full form of the initial state in terms of the post-quench multi-particle states for the quenches from a free massive boson to the the sinh-Gordon model, and it was further

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[1] K_D can be obtained by analytic continuation from the first breather boundary amplitude in sine-Gordon theory which was obtained in [36].

shown that the simple Ansatz (2.3) was a very good numerical solution of the lowest member of the hierarchy provided the exponential form of the initial state was assumed. In addition, the next member of the hierarchy was used for a numerical test of the factorisation assumption itself, which worked well within the limitations of the numerics.

Continuing to imaginary couplings $g = i\beta$ results in sine-Gordon theory

$$\mathcal{A} = \int d^2x \left(\frac{1}{2} \partial_\mu \Phi \partial^\mu \Phi + \frac{\mu^2}{\beta^2} \cos \beta \Phi \right) , \qquad (2.5)$$

and it is useful to introduce $\xi = \beta^2/(8\pi - \beta^2) = -B/2$. The fundamental excitations are a doublet of soliton/antisoliton of mass M. In the attractive regime ($\xi < 1$) the spectrum also contains breathers B_r (soliton-antisoliton bound states) with masses $m_r = 2M \sin r\pi\xi/2$ with r a positive integer less than ξ^{-1}. Due to integrability, the exact factorized S matrix is also known [26]. Under the analytic continuation to imaginary couplings the sinh-Gordon particle corresponds to the first breather B_1, which can be supported both by perturbation theory and the correspondence between the respective S matrix amplitudes. As a result, form factors of local operators and reflection factors containing only the first breather B_1 are also identical to the corresponding sinh-Gordon quantities under the same analytic continuation, which are all known in the model.

Here we consider sine-Gordon quenches which correspond to abruptly changing the soliton mass $M_0 \to M$ while leaving the interaction strength ξ unaltered in the Hamiltonian H associated with (2.5). Note that under the analytic continuation this is related to a mass quench within sinh-Gordon theory with a fixed coupling g, while the Ansatz (2.3) was obtained for a quench from a free boson to sinh-Gordon theory. However, provided the interaction in the initial Hamiltonian does not play a significant role, we can expect that an analytic continuation

$$K_{B_1 B_1}(\vartheta) = \frac{E_0(\vartheta) - E(\vartheta)}{E_0(\vartheta) + E(\vartheta)} K_D(\vartheta) \qquad (2.6)$$

$$K_D(\vartheta) = i \tanh \left(\frac{\vartheta}{2} \right) \frac{\cosh \left(\frac{\vartheta}{2} + \frac{i\pi\xi}{4} \right)}{\sinh \left(\frac{\vartheta}{2} - \frac{i\pi\xi}{4} \right)} \frac{\sinh \left(\frac{\vartheta}{2} + \frac{i\pi(1-\xi)}{4} \right)}{\cosh \left(\frac{\vartheta}{2} - \frac{i\pi(1-\xi)}{4} \right)}$$

gives a good approximation to the first breather pair creation amplitude in the sine-Gordon mass quench. We shall return to the issue of the initial interaction later when discussing the numerical results. Note that the amplitude depends only on the quench mass ratio, which is the same for each particle species since ξ is fixed, so we substituted the first breather mass by the soliton mass.

2.2 Overlaps in finite volume

In our numerical calculation we consider the system in a finite volume L with periodic boundary conditions, therefore we briefly recall the theory of finite size dependence of boundary state amplitudes, worked out in [37]. To keep the formulas short, we consider only one species of particles as the generalization to more than one species is rather obvious. Denoting the pre-quench ground state by $|B\rangle$, the most general expansion in terms of the post-quench eigenstates is

$$|B\rangle = |0\rangle + \sum_{n=1}^{\infty} \int \prod_{i=1}^{n} \frac{d\,\vartheta_i}{2\pi} K_n(\vartheta_1, ...\vartheta_n)\delta\left(\sum_{i=1}^{n} m \sinh \vartheta_i\right) |\vartheta_1, ..., \vartheta_n\rangle \,, \qquad (2.7)$$

while in finite volume one obtains

$$|B\rangle_L = |0\rangle_L + \sum_{n=1}^{\infty} \sideset{}{'}\sum_{I_1,...I_n} N_n K_n(\vartheta_1^*, ...\vartheta_n^*)|I_1, ..., I_n\rangle_L \,, \qquad (2.8)$$

where $\{\vartheta_i^*\}$ are the solutions of the Bethe-Yang equations, i.e. the system

$$Q_i = mL \sinh \vartheta_i + \sum_{j=1,\neq i}^{n} \delta(\vartheta_i - \vartheta_j) = 2\pi I_i \,, i = 1, ..., n, \qquad (2.9)$$

where $\delta(\vartheta) = -i \ln S(\vartheta)$ is the phase-shift corresponding to the two particle S-matrix $S(\vartheta)$, m is the physical mass of the particle and I_i are the quantum numbers that characterize the finite volume states, with the prime meaning that only zero momentum states are included. In [37] the N_n functions were explicitly determined

$$N_n(\vartheta_1^*, ...\vartheta_n^*) = \frac{\sqrt{\rho_n(\vartheta_1^*, ...\vartheta_n^*)}}{\bar{\rho}_{n-1}(\vartheta_1^*, ...\vartheta_{n-1}^*)} + O(e^{-\mu' L}) \,, \qquad (2.10)$$

where ρ_n is the density of states given by the Bethe-Yang Jacobi determinant [38]

$$\rho_n = \det\left\{\frac{\partial Q_k}{\partial \vartheta_j}\right\}_{j,k=1,...,n}$$

whereas $\bar{\rho}_{n-1}$ is the so-called reduced density of states which takes into account momentum conservation and is computed as the Jacobian

$$\bar{\rho}_{n-1} = \det\left\{\frac{\partial \bar{Q}_k}{\partial \vartheta_j}\right\}_{j,k=1,...,n-1}$$

of the constrained Bethe-Yang equations

$$\bar{Q}_i = mL \sinh \vartheta_i + \sum_{j=1,\neq i}^{n-1} \delta(\vartheta_i - \vartheta_j) + \delta(\vartheta_i - \tilde{\vartheta}) = 2\pi I_i \,, i = 1, ..., n-1 \,, \qquad (2.11)$$

$$\tilde{\vartheta} = -\sinh^{-1}(\sum_{i=1}^{n-1} \sinh \vartheta_i)$$

Formula (2.10) is exact to all orders in the inverse volume L^{-1} as indicated by correction terms that decay exponentially with the volume with some characteristic scale μ' (cf. [38]).

For the case when the expansion of the initial state in terms of the post-quench eigenstates only contains paired states

$$|B\rangle = |0\rangle + \sum_{n=1}^{\infty} \int \prod_{i=1}^{n} \frac{d\,\vartheta_i}{2\pi} K_n(\vartheta_1, ...\vartheta_n)|-\vartheta_1, \vartheta_1..., -\vartheta_n, \vartheta_n\rangle \,, \qquad (2.12)$$

the appropriate constrained Bethe-Yang equations are

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$$\bar{Q}_i^p = mL \sinh \vartheta_i + \delta(2\vartheta_i) + \sum_{j \neq i} \delta(\vartheta_i - \vartheta_j) + \delta(\vartheta_i + \vartheta_j) = 2\pi I_i \ , i = 1, ..., n \ , \tag{2.13}$$

with solution $\{\vartheta_i^*\}$, and the finite volume expansion is

$$|B\rangle_L = |0\rangle_L + \sum_{n=1}^{\infty} \sum_{I_1,...I_n} N_n(\vartheta_1^*, ...\vartheta_n^*) K_n(\vartheta_1^*, ...\vartheta_n^*)| - I_1, I_1..., -I_n, I_n\rangle_L \ , \tag{2.14}$$

with the N_n functions [37]

$$N_n(\vartheta_1^*, ...\vartheta_n^*) = \frac{\sqrt{\rho_{2n}(-\vartheta_1^*, \vartheta_1^*... - \vartheta_n^*, \vartheta_n^*)}}{\bar{\rho}_n^p(\vartheta_1^*, ...\vartheta_n^*)} + O(e^{-\mu' L}) \quad , \quad \bar{\rho}_n^p = \det\left\{\frac{\partial \bar{Q}_k^p}{\partial \vartheta_j}\right\}_{j,k=1,...,n} . \tag{2.15}$$

3 Overlaps from TCSA

3.1 TCSA for the sine-Gordon mass quench

We now turn to studying sine-Gordon mass quenches in truncated conformal space approach (TCSA), following the ideas in [39] which applied a similar approach to Ising field theory. For sine-Gordon, TCSA consists of representing the model as a compactified free massless boson conformal field theory (CFT) perturbed by a relevant operator, with the Hamiltonian

$$H = \int dx \frac{1}{2} : (\partial_t \Phi)^2 + (\partial_x \Phi)^2 : -\frac{\lambda}{2} \int dx \, (V_1 + V_{-1}) \tag{3.1}$$
$$V_a =: e^{ia\beta\Phi} :$$

where the semicolon denotes normal ordering in terms of the massless scalar field modes. In a finite volume L, the spectrum of the free boson CFT is discrete and can be truncated to a finite subspace by introducing an upper cut-off e_{cut} in terms of the eigenvalue of the dilatation operator (which gives the energy in conformal units). Physical energies and volumes can be expressed in units of the soliton mass using the relation

$$\lambda = \frac{2\Gamma(\Delta)}{\pi\Gamma(1-\Delta)} \left(\frac{\sqrt{\pi}\Gamma\left(\frac{1}{2-2\Delta}\right) M}{2\Gamma\left(\frac{\Delta}{2-2\Delta}\right)}\right)^{2-2\Delta} \quad , \quad \Delta = \frac{\beta^2}{8\pi} \tag{3.2}$$

so that the dimensionless volume variable and Hamiltonian can be defined as $l = ML$ and $h = H/M$, respectively. For more details on the sine-Gordon TCSA the interested reader is referred to [33].

The initial state corresponds to the ground state of the same Hamiltonian (3.1) with λ replaced by λ_0 corresponding to M_0. When considering the post-quench evolution in dimensionless volume $l = ML$, implementing the quench means using the ground state computed in the rescaled volume $l_0 = M_0 l/M$ [39].

The cut-off dependence of TCSA can be (partially) eliminated using renormalisation group methods [40, 41, 42]. Here we used a modified version of the running coupling prescription in [43]. We can write an effective Hamiltonian in the form

$$H_{eff} = \int dx \frac{1}{2} : (\partial_t \Phi)^2 + (\partial_x \Phi)^2 : +\lambda_0 \mathbb{I} + \frac{\lambda_1}{2} \int dx \, (V_1 + V_{-1}) + \frac{\lambda_2}{2} \int dx \, (V_2 + V_{-2})$$

where we included counter terms generated at leading order according to the fusion rules $V_a V_b \sim V_{a+b}$. Introducing the dimensionless couplings

$$\tilde{\lambda}_a = \frac{\lambda_a L^{2-2h_a}}{(2\pi)^{1-2h_a}} \quad h_a = \frac{a^2 \beta^2}{8\pi}$$

the running couplings $\tilde{\lambda}_i$ are determined by the RG equations

$$\tilde{\lambda}_c(n) - \tilde{\lambda}_c(n-1) = \frac{1}{2n - d_0(l)} \sum_{a,b} \tilde{\lambda}_a(n)\tilde{\lambda}_b(n) C_{ab}^c \frac{n^{2h_{abc}-2}}{\Gamma(h_{abc})^2} (1 + O(1/n)) \qquad (3.3)$$

where n is the cut-off expressed in conformal levels, C_{ab}^c is the operator product coefficient, $h_{abc} = h_a + h_b - h_c$ and $d_0(l)$ is the vacuum scaling function (cf. [43]). At the lowest order it is only necessary to run the couplings λ_0 and λ_2, from their starting values $\lambda_0 = 0$ and $\lambda_2 = 0$ at $n = \infty$.

The couplings must be run following (3.3) down from $n = \infty$ to the appropriate value of n_{cut} corresponding to the given cut-off e_{cut}. It must be taken into account that the $c = 1$ Hilbert space is spanned by Fock modules $\mathcal{F}_a$ created from the vacuum by V_a and the Hamiltonian is block-diagonal in terms of the Fock modules, symbolically:

$$\begin{pmatrix} H_0 + \mathbb{I} & V_1 & V_2 & & & & \\ V_{-1} & H_0 + \mathbb{I} & V_1 & V_2 & & & \\ V_{-2} & V_{-1} & H_0 + \mathbb{I} & V_1 & V_2 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & V_{-2} & V_{-1} & H_0 + \mathbb{I} & V_1 & V_2 \\ & & & V_{-2} & V_{-1} & H_0 + \mathbb{I} & V_1 \\ & & & & V_{-2} & V_{-1} & H_0 + \mathbb{I} \end{pmatrix}$$

and the eventual value of n_{cut} depends on the block one considers. Namely, when computing the coefficient of the block V_2 between $\mathcal{F}_a$ and $\mathcal{F}_{a+2}$, the intermediate states in the OPE $V_1 V_1 \sim V_2$ are from $\mathcal{F}_{a+1}$ which determines the level n_{cut} appropriate for the given block, and similarly for V_{-2} between $\mathcal{F}_a$ and $\mathcal{F}_{a-2}$ n_{cut} is fixed from $\mathcal{F}_{a-1}$. For the identity term between $\mathcal{F}_a$ and $\mathcal{F}_a$ there are two possible intermediate modules $\mathcal{F}_{a\pm1}$, so the identity coupling must be split into two pieces $\lambda_{0\pm}$, each of them running down to the appropriate n_{cut} determined by the highest level in $\mathcal{F}_{a\pm1}$.

The block-dependent running coupling corresponds to including a non-local counter term. The fact that such counter terms are necessary was noted in [44]; they account for $1/n$ corrections in the running coupling. In the sine-Gordon there is a large $1/n$ effect resulting from the fact that the cut-off level is heavily module dependent, ranging from e_{cut} in Fock module $\mathcal{F}_0$ to 0 for the Fock modules with the largest indices $\mathcal{F}_{\pm a_{max}}$. The consistency of this scheme was verified by numerically checking the cut-off dependence of the 15 lowest-lying levels in the TCSA spectrum, which proved to be negligible with this method.

3.2 The $B_1 - B_1$ pair amplitude

Now we turn to numerical results for the $B_1 - B_1$ pair amplitudes and compare them with the infinite volume prediction (2.6). The first task is to identify states corresponding to $B_1 - B_1$ pairs in the numerical spectrum of the post-quench Hamiltonian. Solving the constrained

Bethe-Yang system (2.13) one obtains the possible rapidities from which the energy levels can be computed. However, apart from the lowest lying levels, the identification is not feasible by merely comparing the TCSA and the Bethe-Yang energies due the density of the TCSA spectrum. This issue can be overcome by supplementing the energy selection procedure with a comparison of the finite volume form factors of the fields V_1 and V_2 obtained using the formalism developed in [38], to the TCSA matrix elements (for an exposition of how this works in sine-Gordon theory cf. [45]). As the form factors depend sensitively on the particle content of the state, the identification can unambiguously be performed.

(a) $R = 2.3$, $M/M_0 = 0.5$, $M_0 L = 55$, $e_{cut} = 24$ (b) $R = 2.3$, $M/M_0 = 0.75$, $M_0 L = 40$, $e_{cut} = 22$

(c) $R = 2.3$, $M/M_0 = 1.5$, $M_0 L = 22$, $e_{cut} = 24$ (d) $R = 2.0$, $M/M_0 = 0.5$, $M_0 L = 50$, $e_{cut} = 24$
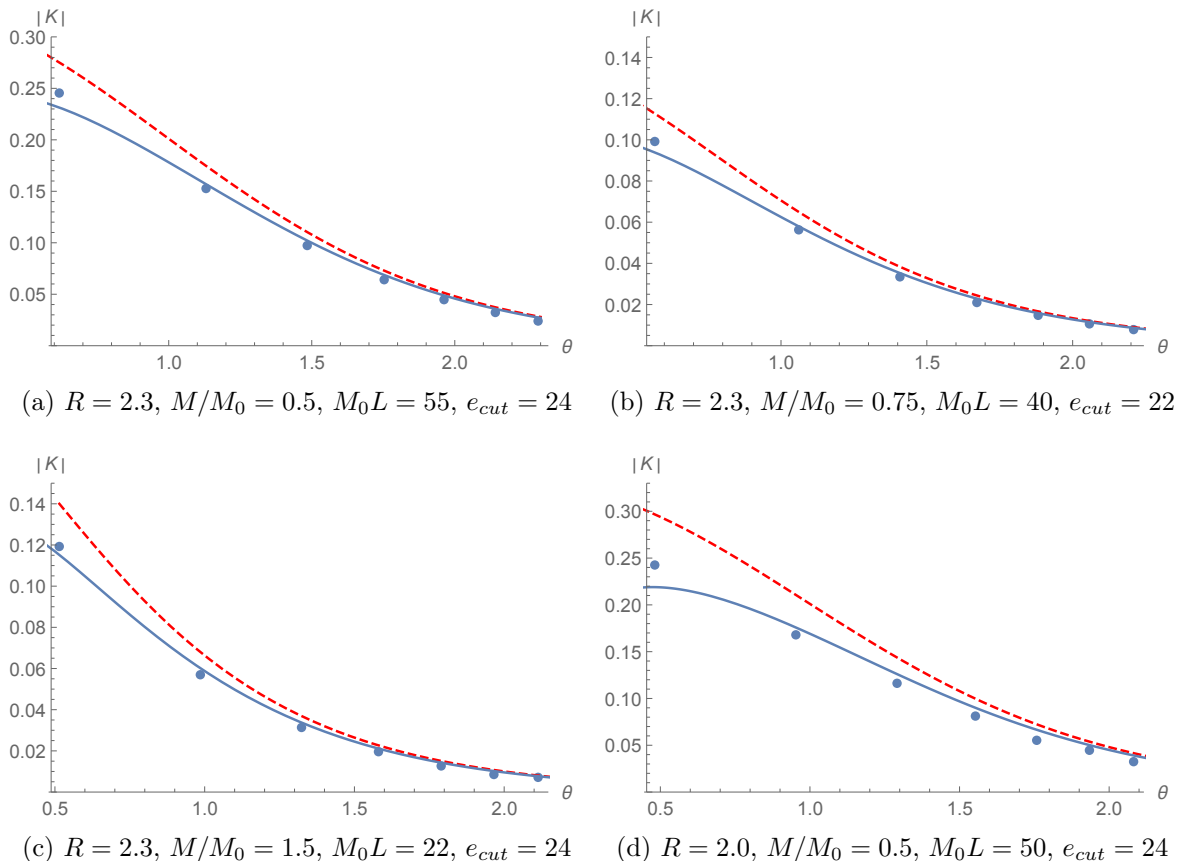
Figure 3.1: The pair amplitude for some mass quenches in sine-Gordon theory. The sine-Gordon coupling β is parametrized as $\beta = \sqrt{4\pi}/R$. The blue (continuous) curves correspond to the sine-Gordon Ansatz (2.6), and the red (dashed) ones to the free theory solutions.

Having identified the proper states in the set of numerical eigenstates the numerical overlaps can be obtained from their scalar product with the initial state, divided by the vacuum overlap to eliminate the normalization factor $\mathcal{N}$ in (2.2). As the TCSA matrix elements of the perturbing operator are real numbers, all the numerically computed eigenvectors are also real, corresponding to a specific convention for the phases of the post-quench eigenstates. Therefore the phase of the overlap function $K(\vartheta)$ is absent from the data, so after normalizing the TCSA overlap values with the inverse of (2.15) we compare their modulus to the value obtained from (2.6). This comparison is shown in Fig. 3.1 for a few of the quenches we considered; the conclusion is that it works well except in the low energy range, and that both the free particle and the Dirichlet parts of the analytic formula (2.6) are important.

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Rapidities $\vartheta_1^*, \vartheta_2^*$	BY energy	TCSA energy	Normalized overlap	Factorized prediction
{0.671828, 1.44047}	1.13089	1.13133	0.0255928	0.0244265
{0.651971, 1.72849}	1.34668	1.34742	0.0168602	0.0162507
{1.19428, 1.70726}	1.51794	1.51918	0.0108541	0.0108043
{0.642841, 1.95028}	1.56712	1.56853	0.0117727	0.0113951
{0.637471, 2.1315}	1.73764	1.79245	0.0083998	0.0083603

(a) $R = 2.3$, $M/M_0 = 0.5$, $M_0 L = 55$, $e_{cut} = 24$

Rapidities $\vartheta_1^*, \vartheta_2^*$	BY energy	TCSA energy	Normalized overlap	Factorized prediction
{0.549607, 1.22608}	1.33758	1.33916	0.0296001	0.0278547
{0.524061, 1.51576}	1.56955	1.57217	0.0139780	0.0195048
{1.03577, 1.48932}	1.74274	1.74686	0.0142756	0.0149960
{0.512645, 1.73741}	1.80847	1.81308	0.0137404	0.0141252
{1.00938, 1.72497}	1.98019	1.98813	0.0130837	0.0108970

(b) $R = 2.0$, $M/M_0 = 0.5$, $M_0 L = 50$, $e_{cut} = 24$

Rapidities $\vartheta_1^*, \vartheta_2^*$	BY energy	TCSA energy	Normalized overlap	Factorized prediction
{0.618879, 1.36023}	1.60354	1.60489	0.00454695	0.00344512
{0.599521, 1.64559}	1.89691	1.89943	0.00292898	0.00219769
{1.12225, 1.62472}	2.12314	2.12725	0.00176750	0.00132404
{0.590723, 1.866}	2.19785	2.20287	0.00203194	0.00150384
{1.10091, 1.85633}	2.42300	2.43139	0.00128401	0.00091002

(c) $R = 2.3$, $M/M_0 = 0.75$, $M_0 L = 40$, $e_{cut} = 22$

Table 3.1: Overlaps for 4-B_1 paired states $|-\vartheta_1^*, \vartheta_1^*, -\vartheta_2^*, \vartheta_2^*\rangle$. The sine-Gordon coupling β is parametrized as $\beta = \sqrt{4\pi}/R$. To eliminate differences in phase conventions of energy eigenstates the modulus of the overlaps is reported.

Deviations in the low energy range can be attributed to two sources. First, the initial state is different from the free massive vacuum for which (2.6) (or more precisely, its sinh-Gordon counterpart (2.3)) was obtained. However, the difference is the presence of a relevant perturbation in the pre-quench Hamiltonian, which affects most the low-lying modes due to its relevance. Second, when modelling the finite size effects in Section 2 we used a formalism that neglects exponential corrections in the volume, which normally affect the lower lying states more. Unfortunately, it is not easy to separate these effects, and so we cannot say anything more definite about the low-energy behaviour. However, the analytically continued solution (2.6) definitely provides a good description of the amplitudes in the mid-to-high energy range.

3.3 Amplitudes for 4 B_1 particles and factorization

Once the amplitude $K(\vartheta)$ is pinned down, all higher overlaps are determined by the exponential form of the state. This entails the factorisation property which states that states which do not have an exclusive pair structure in terms of particles have zero overlap, and for paired states the overlap is just equal to the product of individual pair state overlaps.

Another prediction from factorization is that the overlaps for paired 4-B_1 states is the

product of pair overlaps. This is also consistent with the TCSA data as shown in Table 3.1. For the quenches in sub-tables (a) and (b), the overlaps are large enough so that one can observe a quantitative agreement between the predictions of (2.6) and the TCSA results. For the example in sub-table (c), the overlap is too small to be measured and the agreement is only qualitative.

The question is whether this constitutes a non-trivial test of overlap factorization? Note that when the quench is small factorization is expected to be valid to a very good approximation. A small quench means that the average energy density $\mathcal{E}$ after the quench satisfies

$$\mathcal{E} = \frac{1}{L}\left(\langle\Psi(0)|H|\Psi(0)\rangle - \langle 0|H|0\rangle\right) \ll m_1^2$$

with respect to the mass of the lightest particle m_1. In such a case the density of even the lightest pairs is so small that the average distance d between pairs is much larger than the correlation length m_1^{-1}. Since the interactions are suppressed by the distance as $e^{-m_1 d}$, the multi-pair amplitudes are expected to factorize irrespective of integrability when the quench is small.

We evaluated $\mathcal{E}$ for all the quenches for which we could produce reliable TCSA data and found that d was at least an order of magnitude larger than m_1^{-1}, therefore all observed deviations from factorization are expected to be TCSA related (either truncation errors or unmodeled finite size effects). Indeed, when testing the overlaps for non-paired 4-B_1 states, they proved to be an order of magnitude smaller than the overlaps for paired states, and were of the same order as the deviations between the prediction (2.6) and the measured two-particle overlap, which is consistent with factorization.

4   Conclusions

In this paper we studied mass quenches in the sine-Gordon integrable quantum field theory in the attractive regime, in particular, we numerically determined the two-particle overlaps for the B_1 breathers in the finite volume theory with a periodic boundary condition. The main results of our paper is the verification an Ansatz (2.3) and the exponential form of the initial state (1.1) proposed in [29, 30] for quenches from the free bosonic theory to the interacting sinh-Gordon theory. Based on the well-known analytic continuation between the sine- and sinh-Gordon theories, the numerical overlaps were compared with the Ansatz. The numerical data points and the theoretical curve were found to match very well in the middle and high energy range, with some quantitative deviations in the low energy part which can be attributed to initial state interactions and finite size effects. These results confirm the validity of the sinh-Gordon Ansatz, whose original derivation relied on the assumption that the initial state contains only multiple particle states composed of pairs with opposite momenta, which lacks rigorous justification at this moment.

For the numerical determination of the overlaps the truncated conformal space approach was used. To improve upon the usual renormalization group treatment of TCSA [40, 41, 42], we added non-local counter terms that dominate the next order corrections in the inverse energy cut-off. Whereas in general, the construction of such non-local terms is difficult, in sine-Gordon TCSA it is easy to implement this correction and the accuracy of the numerical spectrum can be substantially improved.

As a closing remark, it has to be mentioned that the quantum sine-Gordon theory is a very interesting model in its own right, attracting a lot of attention due to its theoret-

ical tractability and experimental relevance. Quenches in sine-Gordon theory have recently become realizable in experimental setups, describing the evolution of the relative phase of trapped and coupled condensates of cold atoms [46]. The knowledge of the pair amplitudes in (1.1) is crucial for the computation of steady state one- and two-point functions by currently available techniques [18, 28], and the method presented in this paper provides a direct way to the numerical determination of the pair overlaps. Indeed, in addition to the B_1 overlaps presented here, our method can be used to extract pair amplitudes for higher breathers and soliton-antisoliton pairs. In this work we refrained from reporting the corresponding numerical data, since at present we have no theoretical description for them. The understanding of these overlaps, which is important for a full description of sine-Gordon quenches, is relegated to future works.

Acknowledgements

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