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ON THE PROJECTIVE FINSLER METRIZABILITY AND THE INTEGRABILITY OF RAPCSÁK EQUATION

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Abstract. A. Rapcsák obtained necessary and sufficient conditions for the projective Finsler metrizability in terms of a second order partial differential system. In this paper we investigate the integrability of the Rapcsák system and the extended Rapcsák system, by using the Spencer version of the Cartan-Kähler theorem. We also consider the extended Rapcsák system completed with the curvature condition. We prove that in the non-isotropic case there is a nontrivial Spencer cohomology group in the sequences determining the 2-acyclicity of the symbol of the corresponding differential operator. Therefore the system is not integrable and higher order obstruction exists.

Keywords: Euler-Lagrange equation; metrizability; projective metrizability; geodesics; spray; formal integrability

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1. Introduction

Last year we celebrated the 100th anniversary of the birth of András Rapcsák. He was one of the founders of the Finsler geometry research school in Debrecen. His most important results concern the projective Finsler metrizability problem, where one seeks for a Finsler metric whose geodesics are projectively equivalent to the solutions of a given system of second order homogeneous ordinary differential equations (SODE).

The projective Finsler metrizability problem can be considered as a particular case of the inverse problem of the calculus of variations. Rapcsák in [18] obtained necessary and sufficient conditions for the projective Finsler metrizability in terms

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of a second order PDE system, called now Rapcsák equations, see [9], [22], [20]. The coordinate-free formulations of these equations can be found in [14], [22]. Rapcsák's approach is simple and natural: one finds conditions directly on the Finsler function that one seeks for. Recently, several new results appeared about the projective Finsler metrizability problem, see [4], [8], [9], [10], [15], [16]. Various strategies can be chosen to deal with the problem: In [9] the generalized Helmholtz system was considered, in [4] a system in terms of a semi-basic 1-form was investigated and in [10] an approach in terms of 2-forms was formulated. In this paper, in the perspective of the projective metrizability problem, we consider the Rapcsák system, which consists of the homogeneity equation (2.1) and a second order differential equation (3.1), called the Rapcsák equation. We investigate the integrability of the Rapcsák system and the extended Rapcsák system by using the Spencer version of the Cartan-Kähler theorem. The integrability condition can be expressed in terms of the curvature tensor (Proposition 4.5) which is not necessarily fulfilled in the non-isotropic case. Therefore, to solve the projective metrizability problem in this case, one has to consider an enlarged system by including the curvature conditions to the equations. Analysing the system with the classical Cartan-Kähler theory one can show that the Cartan test fails and the symbol of the operator is not involutive. Therefore the system (5.2) is not integrable: higher order integrability conditions exist. Using Spencer technique, the level where these higher order integrability conditions appear can be calculated.

The paper is organized as follows. In Section 2 we give a brief introduction to the Frölicher-Nijenhuis theory and to the canonical structures on the tangent bundle of a manifold. We also introduce the main structures one needs to discuss the geometry of a spray: connection, Jacobi endomorphism, curvature. We also recall the basic tools of Cartan-Kähler theory.

In Section 3 we use the geometric setting presented in Section 2 to show that the Rapcsák system gives a necessary and sufficient condition for the projective metrizability problem. Alternative proofs can be found in [22], [21]. We discuss the integrability of the Rapcsák system by using conditions provided by the Cartan-Kähler theorem. We conclude the section by showing that the obstruction to the formal integrability can be expressed in terms of the nonlinear connection induced by the spray.

In Section 4 we investigate the formal integrability of the extended Rapcsák system composed by the Rapcsák system and its integrability condition found in Section 3. We show that the obstruction to the integrability can be expressed in terms of the curvature tensor of the nonlinear connection induced by the spray. The curvature of flat and of isotropic sprays satisfies this integrability condition. We remark that for these classes of spray manifolds the projective metrizability problem has been discussed in [4], [7], [6], but our approach here is different. As Section 5 shows, this approach can be particularly advantageous from the perspective of further investigations of the cases of non-isotropic curvature.

In Section 5 we consider the case when the curvature is not isotropic. We remark that very few results are known on the inverse problem of the calculus of variations in this situation (see for example [11], [19]). In the non-isotropic case the extended Rapcsák system is not integrable and in order to solve the projective metrizability problem, one has to consider an enlarged system by adding the curvature conditions to the system. We consider here the generic case, when the eigenvalues of the Jacobi endomorphism are pairwise distinct. Analysing the system with the classical Cartan-Kähler theory one can show that the symbol of the operator is not involutive and that the Cartan test fails. Therefore the system (5.2) is not integrable: higher integrability conditions exist. We emphasize that from the non-involutivity of the symbol one cannot obtain information about the level of prolongation where the extra integrability condition arises. However, using Spencer technique, this level (and also the number of the extra integrability conditions) can be calculated. We prove that for this system the $H^{2,2}$ Spencer cohomology group is nontrivial. Hence to solve the projective metrizability problem in the non-isotropic case a third order PDE system (containing the first prolongation of the extended Rapcsák system and the curvature conditions) should be considered instead of the original second order PDE system.

2. Preliminaries

Throughout this paper, M will denote an n-dimensional smooth manifold. $C^\infty(M)$ denotes the ring of real-valued smooth functions, $\mathfrak{X}(M)$ is the $C^\infty(M)$-module of vector fields on M, $\pi\colon TM \to M$ is the tangent bundle of M, $\mathcal{T}M = TM \setminus \{0\}$ is the slit tangent space. We will essentially work on the manifold TM and on its tangent space TTM. When there is no danger of confusion, TTM and T^*TM will simply be denoted by T and T^*, respectively. $VTM = \operatorname{Ker} \pi_*$ is the vertical sub-bundle of T. We denote by $\Lambda^k(M)$, $S^k(M)$ and $\Psi^k(M)$ the $C^\infty(M)$-modules of skew-symmetric, symmetric and vector valued k-forms, respectively. We denote by $\Lambda^k_v(TM)$, $S^k_v(TM)$ and $\Psi^k_v(TM)$ the corresponding semi-basic $C^\infty(TM)$-modules.

The Frölicher-Nijenhuis theory provides a complete description of the derivations of $\Lambda(M)$ with the help of vector-valued differential forms, for details we refer to [12]. The i_* and the d_* type derivations associated to a vector-valued l-form L will be denoted by i_L and d_L. They can be introduced in the following way: if $L \in \Psi^l(M)$, then

$$i_L\omega(X_1, \dots, X_l) = \omega(L(X_1, \dots, X_l)),$$

where $X_1, \dots, X_l \in \mathfrak{X}(M)$, $\omega \in \Lambda^1(M)$. Furthermore, d_L is the commutator of the derivations i_L and d, that is,

$$d_L := [i_L, d] = i_L d - (-1)^{l-1} d i_L.$$

We remark that for $X \in \mathfrak{X}(M)$ we have $d_X = \mathcal{L}_X$ the Lie derivative, and i_X is the substitution operator. The *Frölicher-Nijenhuis bracket* of $K \in \Psi^k(M)$ and $L \in \Psi^l(M)$ is the unique $[K, L] \in \Psi^{k+l}$ form such that

$$[d_K, d_L] = d_{[K,L]}.$$

In the special case, when $K \in \Psi^1(M)$, $X, Y \in \mathfrak{X}(M)$ we have $[K, X] \in \Psi^1(M)$ defined as

$$[K, X](Y) = [KY, X] - K[Y, X].$$

Spray and associated geometric quantities. Let $J\colon TTM \to TTM$ be the vertical endomorphism and $C \in \mathfrak{X}(TM)$ the Liouville vector field. In an induced local coordinate system (x^i, y^i) on TM we have $J = \mathrm{d}x^i \otimes \partial/\partial y^i$, and $C = y^i \partial/\partial y^i$. Euler's theorem for homogeneous functions implies that $L \in C^\infty(TM)$ is a 1-homogeneous function in the $y = (y^1, \dots, y^n)$ variables if and only if

$$y^i \frac{\partial L}{\partial y^i} - L = 0. \tag{2.1}$$

The vertical endomorphism satisfies the following properties: $J^2 = 0$, $\operatorname{Ker} J = \operatorname{Im} J = VTM$ and $[J, C] = J$.

A *spray* is a vector field S on $\mathcal{T}M$ satisfying the relations $JS = C$ and $[C, S] = S$. The coordinate representation of a spray S takes the form

$$S = y^i \frac{\partial}{\partial x^i} + f^i(x, y) \frac{\partial}{\partial y^i},$$

where the functions $f^i(x, y)$ are homogeneous of degree 2 in y. The *geodesics of a spray* are curves $\gamma\colon I \to M$ such that $S \circ \dot\gamma = \ddot\gamma$. Locally, they are the solutions of the equations

$$\ddot x^i = f^i(x, \dot x), \quad i = 1, \dots, n. \tag{2.2}$$

Two sprays S and $\widetilde S$ are called *projective equivalent*, if their geodesics coincide up to an orientation-preserving reparametrization. It is not difficult to show that S and $\widetilde S$ are projective equivalent if and only if they are related by the formula

$$\widetilde S = S - 2\mathcal{P}C, \tag{2.3}$$

where $\mathcal{P} \in C^\infty(TM)$ is a 1-homogeneous function.

To every spray S a *connection* $\Gamma\colon = [J,S]$ can be associated. We have $\Gamma^2 = \mathrm{Id}$. The eigenspace of Γ corresponding to the eigenvalue -1 is the *vertical space* VTM, and the eigenspace corresponding to $+1$ is called the *horizontal space*. For any $x \in TM$ we have $T_xTM = H_xTM \oplus V_xTM$. The horizontal and vertical projectors are denoted by h and v. One has

$$h = \frac{1}{2}(\mathrm{Id} + \Gamma), \quad v = \frac{1}{2}(\mathrm{Id} - \Gamma). \tag{2.4}$$

The *curvature* $R = [h,h]/2$ of the connection is the Nijenhuis torsion of the horizontal projection h. The *Jacobi endomorphism* (or Riemann curvature in [20]) is defined as $\Phi = i_S R$. The Jacobi endomorphism determines the curvature by the formula $R = [J,\Phi]/3$. The spray S is called *flat* if its Jacobi endomorphism has the form $\Phi = \lambda J$ and *isotropic* if $\Phi = \lambda J - \alpha \otimes C$ with some $\lambda \in C^\infty(\mathcal{T}M)$, $\alpha \in \Lambda^1_v(\mathcal{T}M)$.

The vector $X \in H_xTM$, $X \neq 0$ is a *semibasic eigenvector* of Φ if $\Phi X = \lambda JX$. In this situation $\lambda \in \mathbb{R}$ is called the *eigenvalue* corresponding to X. We remark that R is skew symmetric, therefore we have $\Phi(S) = R(S,S) = 0$, hence S is an eigenvector and $\lambda = 0$ is an eigenvalue of Φ corresponding to S.

Finsler structure. A *Finsler function* on a manifold M is a continuous function $F\colon TM \to \mathbb{R}$, which is smooth and positive away from the zero section, homogeneous of degree 1, and strictly convex on each tangent space. The energy function $E\colon TM \to \mathbb{R}$ associated to a Finsler structure F is defined as $E := F^2/2$. The $(0,2)$ tensor field with tensor components

$$g_{ij} := \frac{\partial^2 E}{\partial y^i \partial y^j}$$

is positive definite at any point $(x,y) \in \mathcal{T}M$. The pair (M,F) is called a Finsler manifold. The geodesics of the Finsler manifold (M,F) are the solutions of the Euler-Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial E}{\partial \dot{x}^i} - \frac{\partial E}{\partial x^i} = 0, \quad i = 1,\dots,n. \tag{2.5}$$

It is not difficult to see that for any function $E \in C^\infty(\mathcal{T}M)$ the 1-form

$$\omega_E = i_S dd_J E + d\mathcal{L}_C E - \mathrm{d}E \tag{2.6}$$

is semi-basic, and its coordinate representation takes the form $\omega_E = \omega_i \, \mathrm{d}x^i$, where the coefficients ω_i are the functions appearing in the left-hand side of the Euler-Lagrange equation (2.5). Therefore S corresponds to the geodesic equation of E if and only if the equation

$$\omega_E = 0 \tag{2.7}$$

holds. The spray S is called *Finsler metrizable* if there exists a Finsler function such that for the corresponding energy function (2.7) holds, and S is *projective Finsler metrizable*, if it is projective equivalent to a *Finsler metrizable* spray.

Formal integrability. To investigate the integrability of the Rapcsák system we shall use Spencer's technique of formal integrability in the form explained in [13]. For a detailed account see [2]. We recall here the basic notions in order to fix the terminology.

Let B be a vector bundle over M. If s is a section of B, then $j_k(s)_x$ will denote the kth order jet of s at the point $x \in M$. The bundle of kth order jets of the sections of B is denoted by $J_k B$. In particular $J_k(\mathbb{R}_M)$ will denote the kth order jets of real-valued functions, that is, the sections of the trivial line bundle. Let B_1 and B_2 be vector bundles over M and $P\colon \operatorname{Sec}(B_1) \to \operatorname{Sec}(B_2)$ a differential operator. An $s \in \operatorname{Sec}(B_1)$ is a *solution* to P if $Ps \equiv 0$.

If P is a linear differential operator of order k, then a morphism $p_k(P)\colon J_k(B_1) \to B_2$ can be associated to P. The lth order prolongation $p_{k+l}(P)\colon J_{k+l}(B_1) \to J_l(B_2)$ can be introduced in a natural way by taking the lth order derivatives. $\operatorname{Sol}_{k+l,x}(P) := \operatorname{Ker} p_{k+l,x}(P)$ denotes the set of *formal solutions of order l at $x \in M$*. Obviously, we have

$$Ps \equiv 0 \;\Rightarrow\; j_{l,x}(s) \in \operatorname{Sol}_{l,x}(P),$$

for every $l \geqslant k$ and $x \in M$. The differential operator P is called *formally integrable* if $\operatorname{Sol}_l(P)$ is a vector bundle for all $l \geqslant k$, and the restriction $\overline{\pi}_{l,x}\colon \operatorname{Sol}_{l+1,x}(P) \to \operatorname{Sol}_{l,x}(P)$ of the natural projection is onto for every $l \geqslant k$. In that case any kth order solution or *initial data* can be lifted into an infinite order solution. In the analytic case, formal integrability implies the existence of solutions for arbitrary initial data (see [2], page 397). To prove the formal integrability, one can use the Cartan-Kähler theorem. To present it, we have to introduce some notations.

Let $\sigma_k(P)$ denote the symbol of P determined by the highest order terms of the operator. It can be interpreted as a map $\sigma_k(P)\colon S^kT^*M \otimes B_1 \to B_2$. By $\sigma_{k+l}(P)\colon S^{k+l}T^*M \otimes B_1 \to S^lT^*M \otimes B_2$ we denote the symbol of the lth order prolongation of P. If $\mathcal{E} = \{e_1 \dots e_n\}$ is a basis of T_xM, we set

$$g_{k,x}(P) = \operatorname{Ker} \sigma_{k,x}(P),$$
$$g_{k,x}(P)_{e_1 \dots e_j} = \{A \in g_{k,x}(P)\colon\; i_{e_1}A = \ldots = i_{e_j}A = 0\}, \quad j = 1, \ldots, n.$$

The basis $\mathcal{E}$ is called *quasi-regular* if one has

$$\dim g_{k+1,x}(P) = \dim g_{k,x}(P) + \sum_{j=1}^{n} \dim g_{k,x}(P)_{e_1 \dots e_j}.$$

A symbol is called *involutive*[1] if there exists at any $x \in M$ a quasi-regular basis. The notion of involutivity allows us to check the formal integrability in a simple way by using the following theorem:

Theorem 2.1 (Cartan-Kähler)**.** *Let P be a kth order linear partial differential operator. Suppose that P is regular, that is, $\mathrm{Sol}_{k+1}(P)$ is a vector bundle over $\mathrm{Sol}_k(P)$. If the map $\overline{\pi}_k \colon \mathrm{Sol}_{k+1}(P) \to \mathrm{Sol}_k(P)$ is surjective and the symbol is involutive, then P is formally integrable.*

It can be shown that the condition of the existence of a quasi-regular basis can be replaced by a weaker condition. The obstructions to the higher order successive lift of the kth order solution are contained in some of the cohomological groups of a certain complex called *Spencer complex*. The $H^{m,2}$ *Spencer cohomology group* is defined as

$$H^{m,2} = \frac{\mathrm{Ker}(g_m(P) \otimes \Lambda^2 T^*M \xrightarrow{\delta_2^m} g_{m-1}(P) \otimes \Lambda^3 T^*M)}{\mathrm{Im}(g_{m+1}(P) \otimes T^*M \xrightarrow{\delta_1^m} g_m(P) \otimes \Lambda^2 T^*M)}. \tag{2.8}$$

The symbol of a kth order linear differential operator P is 2-*acyclic* if $H^{m,2} = 0$ for all $m \geqslant k$. Using Spencer cohomology groups, a weaker version of integrability theorem can be stated:

Theorem 2.2 (Goldschmidt)**.** *Let P be a kth order regular linear partial differential operator. If $\overline{\pi}_k \colon \mathrm{Sol}_{k+1}(P) \to \mathrm{Sol}_k(P)$ is onto and the symbol of the operator is 2-acyclic then P is formally integrable.*

Using a classical result in homological algebra, the surjectivity of $\overline{\pi}_{k+1}$ can be verified in the following way (see [13]):

Proposition 2.3. *There exists a morphism $\varphi \colon \mathrm{Sol}_k(P) \to \mathrm{Coker}\,(\sigma_{k+1}(P))$, such that the sequence*

$$\mathrm{Sol}_{k+1}(P) \xrightarrow{\overline{\pi}_k} \mathrm{Sol}_k(P) \xrightarrow{\varphi} \mathrm{Coker}(\sigma_{k+1}(P))$$

is exact. Therefore $\overline{\pi}_k$ is surjective if and only if $\varphi \equiv 0$.

[1] In the works of Cartan, and more generally in the theory of exterior differential systems, "involutivity" means more than the existence of a quasi-regular basis and it refers to "integrability" (cf. [2], pages 107, 140). Here we follow the terminology of Goldschmidt (cf. [2], page 409).

Remark 2.4. The map φ is called *obstruction map* and $\operatorname{Coker}(\sigma_{k+1}(P))$ is called *obstruction space*, because a kth order solution $s \in \operatorname{Sol}_k(P)$ can be prolonged into a $(k+1)$st order solution if and only if $\varphi(s) = 0$. In particular, if $\operatorname{Coker}(\sigma_{k+1}(P)) = \{0\}$ then there is no obstruction to the prolongation.

In practice the map φ and therefore the integrability conditions can be computed as follows:

Remark 2.5. Let $\tau\colon T^* \otimes B_2 \to K$ be a morphism such that $\operatorname{Ker}\tau = \operatorname{Im}\sigma_{k+1}(P)$. Then K is isomorphic to $\operatorname{Coker}(\sigma_{k+1}(P))$. Moreover, if $s_{k,x} = j_k(s)_x$ is a kth order solution, that is, $(Ps)_x = 0$, then

$$\varphi(s_{k,x}) = \tau(\nabla(Ps))_x,$$

where ∇ is an arbitrary linear connection on the bundle B_2.

Let (x^i) be a local coordinate system on M, (x^i, y^i) the associated coordinate system on TM in the neighborhood of $v \in TM$. If $j_k(F)_v \in J_k(\mathbb{R}_{TM})$ is a kth order jet of a real-valued function F on TM, we set the notation

$$F_{i_1\ldots i_a \underline{i_{a+1}\ldots i_l}} := \frac{\partial^l F}{\partial x^{i_1}\ldots\partial x^{i_a}\partial y^{i_{a+1}}\ldots\partial y^{i_l}}(v), \quad 1 \leqslant l \leqslant k,\ 1 \leqslant a \leqslant l. \tag{2.9}$$

3. Differential operator of the projective metrizability: the Rapcsák system

In this section we derive the PDE system describing the necessary and sufficient condition for a spray to be projective Finsler metrizable. We have the following statement:

Proposition 3.1. *A spray S is projective Finsler metrizable if and only if there exists a Finsler function $\widetilde{F}\colon TM \to \mathbb{R}$ (1-homogeneous, continuous, smooth and positive on $\mathcal{T}M$ where $\partial^2\widetilde{F}^2/\partial y^i\partial y^j$ is positive definite), such that*

$$i_S dd_J \widetilde{F} = 0. \tag{3.1}$$

Proof. The spray S is projective Finsler metrizable if and only if there exists a Finsler metrizable spray $\widetilde{S}$ which is projective equivalent to S. Because of the projective equivalence, there exists a function $\mathcal{P}$ such that $\widetilde{S} = S - 2\mathcal{P}C$. Let us denote by $\widetilde{F}$ the Finsler function associated to $\widetilde{S}$. It is well known that $\widetilde{F}$ is invariant by the parallel translation associated to the connection $\widetilde{\Gamma} = [J, \widetilde{S}]$ and therefore we have $d_{\widetilde{h}} \widetilde{F} = 0$. Using the relation

$$\widetilde{h} = h - \mathcal{P}J - d_J\mathcal{P} \otimes C$$

between the horizontal projectors ([5], Chapter 4) and the 1-homogeneity of $\widetilde{F}$, we get

$$0 = d_{\widetilde{h}}\widetilde{F} = d_h\widetilde{F} - d_{\mathcal{P}J}\widetilde{F} - d_J\mathcal{P}C\widetilde{F} = d_h\widetilde{F} - \mathcal{P}d_J\widetilde{F} - \widetilde{F}d_J\mathcal{P} = d_h\widetilde{F} - d_J(\mathcal{P}\widetilde{F}). \tag{3.2}$$

Substituting S into (3.2), using $JS = C$ and the homogeneity of $\widetilde{F}$ and $\mathcal{P}$, we get

$$i_S d_{\widetilde{h}}\widetilde{F} = S\widetilde{F} - C(\mathcal{P}\widetilde{F}) = S\widetilde{F} - 2\mathcal{P}\widetilde{F} = 0,$$

and we find that the projective factor is $\mathcal{P} = (1/2\widetilde{F})S\widetilde{F}$. Replacing $\mathcal{P}$ in (3.2) by the above expression we get

$$d_h\widetilde{F} - d_J\Big(\frac{1}{2\widetilde{F}}(\widetilde{F}\, d_S\widetilde{F})\Big) = d_h\widetilde{F} - \frac{1}{2}d_J(d_S\widetilde{F}) = 0.$$

Using (2.4) and the relation $d_{[J,S]} = d_Jd_S - d_Sd_J$ we obtain

$$\begin{aligned} 0 = d_{\Gamma+I}\widetilde{F} - d_Jd_S\widetilde{F} &= d_{[J,S]}\widetilde{F} + d\widetilde{F} - d_Jd_S\widetilde{F} \\ &= -(i_Sd + di_S)d_J\widetilde{F} + d\widetilde{F} = -i_Sdd_J\widetilde{F} - dC\widetilde{F} + d\widetilde{F} = -i_Sdd_J\widetilde{F}. \end{aligned}$$

□

We note that a coordinate version of the above theorem was proved by Rapcsák in [18], and coordinate free versions were given in [14], [21], [22]. Here we presented a different proof.

Definition 3.2. Let S be a spray on M. The partial differential system composed by the equation (3.1) and the 1-homogeneity condition (2.1) is called the *Rapcsák system.*

According to Proposition 3.1 the projective metrizability leads to the investigation of the Rapcsák system.

Remark 3.3. The Rapcsák system is equivalent to the system composed by the Euler-Lagrange equations (2.5) and the 1-homogeneity condition (2.1).

We remark that the system composed by the Euler-Lagrange equations and the k-homogeneity condition for $k \neq 1$ can be reduced to a first order partial differential system which can be interpreted in terms of the holonomy distribution associated to the spray S. When $k = 2$ (this case corresponds to the Finsler metrizability problem), the computation can be found in [17]. The same reasoning can be applied for other values of k, $k \neq 1$. But this method cannot be used for the value $k = 1$. Nevertheless, in some special situations, the Rapcsák system can also be reduced to a first order PDE system. This is the case for example for the canonical spray of a Lie group, if one seeks for an invariant solution to the projective Finsler metrizability problem. In that case, the Rapcsák system can be reduced to a first order system, and one can show that the invariant Riemann, Finsler and projective Finsler metrizability problems are equivalent, see [3].

Integrability conditions of the Rapcsák system. Let us consider the differential operator P_1 corresponding to the Rapcsák system

$$P_1 = (P_S, P_C), \tag{3.3}$$

where

$$P_S \colon C^\infty(TM) \to \operatorname{Sec} T^*, \quad P_S(F) = i_S dd_J F, \tag{3.4}$$

$$P_C \colon C^\infty(TM) \to C^\infty(TM), \quad P_C(F) = \mathcal{L}_C F - F. \tag{3.5}$$

From the local expression it is clear that P_C is a first and P_S is a second order differential operator. The associated morphisms are defined on the first and second order jet spaces, respectively. Using the coordinate system (2.9) we get

$$p_1(P_C) \colon J_1(\mathbb{R}_{TM}) \to \mathbb{R}, \quad j_1(F) \to y^i F_{\underline{i}} - F,$$
$$p_2(P_S) \colon J_2(\mathbb{R}_{TM}) \to T^*, \quad j_2(F) \to (y^i F_{i\underline{j}} + f^i F_{\underline{i}\underline{j}} - F_j)dx^i - (F_{\underline{i}} + y^j F_{\underline{i}\underline{j}} - F_i)dy^i.$$

The interesting feature of the Rapcsák system is that it is composed by differential operators of different orders. To find the integrability conditions of the system we consider the prolongation of the lower order equation. The morphism associated to this system is

$$p_2(P_1) = p_2(P_S) \times p_2(P_C) \colon J_2(\mathbb{R}_{TM}) \to T^* \times J_1(\mathbb{R}_{TM}).$$

Lemma 3.4. *A* 2nd *order solution $s = j_2(F)_x$ at $x \in \mathcal{T}M$ of the Rapcsák system can be lifted into a* 3rd *order solution if and only if one has*

$$(i_\Gamma dd_J F)_x = 0, \tag{3.6}$$

where $\Gamma = [J, S]$ is the canonical nonlinear connection associated to S.

P r o o f of Lemma 3.4 (first part). The symbols are defined by the highest order part of the operators. For P_C we find

$$\sigma_1(P_C)\colon\ T^* \to \mathbb{R},\quad \sigma_1(P_C)A_1 = A_1(C). \tag{3.7}$$

The symbol of P_S and the prolongation of the symbol of P_C are

$$\sigma_2(P_S)\colon\ S^2T^* \to T^*,\quad (\sigma_2(P_S)A_2)(X) = A_2(S, JX) - A_2(X, C), \tag{3.8}$$

$$\sigma_2(P_C)\colon\ S^2T^* \to T^*,\quad (\sigma_2(P_C)A_2)(X) = A_2(X, C) \tag{3.9}$$

for every $X \in T$, $A_1 \in T^*$, $A_2 \in S^2T^*$. The prolongations of the symbols at third order level are

$$\sigma_3(P_C)\colon\ S^3T^* \to S^2T^*,\ (\sigma_3(P_C)A_3)(X, Y) = A_3(X, Y, C), \tag{3.10}$$

$$\sigma_3(P_S)\colon\ S^3T^* \to T^* \otimes T^*,\ (\sigma_3(P_S)A_3)(X, Y) = A_3(X, S, JY) - A_3(X, Y, C), \tag{3.11}$$

where $X, Y \in T$, $A_3 \in S^3T^*$, and we have

$$\sigma_3(P_1) = (\sigma_3(P_S), \sigma_3(P_C))\colon\ S^3T^* \to (T^* \otimes T^*) \times S^2T^*.$$

Let us consider the map $\tau_1 := (\tau_S^1, \tau_S^2, \tau_C^1, \tau_C^2)$, where

$$\begin{aligned}\tau_S^1(B_S, B_C)(X, Y) = B_S(JX, hY) - B_S(hY, JX)&\\ -B_S(JY, hX) + B_S(hX, JY),&\end{aligned} \tag{3.12}$$

$$\tau_S^2(B_S, B_C)(X) = B_S(X, S), \tag{3.13}$$

$$\tau_C^1(B_S, B_C)(X, Y) = B_S(X, JY) + B_C(X, JY), \tag{3.14}$$

$$\tau_C^2(B_S, B_C)(X, Y) = B_S(C, hX) - B_C(S, JX) + B_C(hX, C) \tag{3.15}$$

for $B_S \in T^* \otimes T^*$, $B_C \in S^2T^*$, $X, Y \in T$.

Remark 3.5. We have $\operatorname{Im}\sigma_3(P_1) = \operatorname{Ker}\tau_1$, that is, if we denote $K_1 = \operatorname{Im}\tau_1$ then the sequence

$$S^3T^* \xrightarrow{\sigma_3(P_1)} (T^* \otimes T^*) \times S^2T^* \xrightarrow{\tau_1} K_1 \to 0 \tag{3.16}$$

is exact.

P r o o f. Since by the definition of K_1 the sequence (3.16) is exact in the third term, we have to check only its exactness in the second term. It is easy to compute that $\tau_1 \circ \sigma_3(P_1) = 0$ and therefore $\operatorname{Im} \sigma_3(P_1) \subset \operatorname{Ker} \tau_1$. Let us compute $\dim \operatorname{Ker} \sigma_3(P_1)$. We consider the basis

$$\mathcal{B} := \{h_1, \dots, h_n, v_1, \dots, v_n\} \subset T_x, \tag{3.17}$$

where h_i are horizontal, $h_n = S$, $Jh_i = v_i$, $i = 1, \dots, n$ (and therefore $v_n = C$). In the sequel we denote the components of a symmetric tensor $A \in S^k T^*$ with respect to (3.17) as

$$A_{i_1 \dots i_j \underline{i_{j+1}} \dots \underline{i_k}} := A(h_{i_1}, \dots, h_{i_j}, v_{i_{j+1}}, \dots, v_{i_k}). \tag{3.18}$$

It is clear that $\operatorname{Ker} \sigma_3(P_1) = \operatorname{Ker} \sigma_3(P_S) \cap \operatorname{Ker} \sigma_3(P_C)$. The symmetric tensor $A \in S^3 T^*$ is in $\operatorname{Ker} \sigma_3(P_C)$ if

$$A_{ij\underline{n}} = A_{i\underline{j}\underline{n}} = A_{\underline{i}\,\underline{j}\underline{n}} = 0, \tag{3.19}$$

and $A \in S^3 T^*$ is an element of $\operatorname{Ker} \sigma_3(P_S)$ if

$$\sigma_3(P_S)(A)(h_i, h_j) = A(h_i, h_n, v_j) - A(h_i, h_j, v_n) = A_{in\underline{j}} - A_{ij\underline{n}} = 0, \tag{3.20}$$

$$\sigma_3(P_S)(A)(h_i, v_j) = -A(h_i, v_j, v_n) = -A_{i\underline{j}\underline{n}} = 0, \tag{3.21}$$

$$\sigma_3(P_S)(A)(v_i, h_j) = A(v_i, h_n, v_j) - A(v_i, h_j, v_n) = A_{\underline{i}n\underline{j}} - A_{\underline{i}\,j\underline{n}} = 0, \tag{3.22}$$

$$\sigma_3(P_S)(A)(v_i, v_j) = -A(v_i, v_j, v_n) = -A_{\underline{i}\,\underline{j}\underline{n}} = 0 \tag{3.23}$$

for $i, j = 1, \dots, n$. Taking into account the symmetry of A we have $2n(n+1)/2 + n^2$ independent equations in (3.19). Moreover, counting the independent equations in (3.20)–(3.23) we get that (3.21) and (3.23) trivially hold because of (3.19). From (3.20) we have only $n^2 - n$ independent equations because for $j = n$ the equations are trivially satisfied, and from (3.22) we have $n(n-1)/2$ independent equations because, again, for $j = n$ they are trivially satisfied. Consequently, we have $2n(n+1)/2 + 2n^2 - n + n(n-1)/2 = 7n^2 - n/2$ independent equations in the system (3.19)–(3.23). Therefore we get

$$\dim(g_3(P_1)) = \dim \operatorname{Ker} \sigma_3(P_1) = \dim S^3 T^* - \frac{7n^2 - n}{2} = \frac{8n^3 - 9n^2 + 7n}{6} \tag{3.24}$$

and

$$\operatorname{rank} \sigma_3(P_1) = \frac{7n^2 - n}{2}. \tag{3.25}$$

On the other hand, let us compute $\dim \operatorname{Ker} \tau_1$. The pivot terms for the equation $\tau_S^1 = 0$ are $B_S(v_i, h_j)$, $i < j < n$. Furthermore, $B_S(v_i, h_n)$, $B_S(h_i, h_n)$, $i = 1, \ldots, n$, are pivot terms for $\tau_S^2 = 0$. Therefore the number of independent equations for $\operatorname{Ker} \tau_S^1$ and $\operatorname{Ker} \tau_S^2$ are $(n-1)(n-2)/2$ and $2n$, respectively. Moreover, the pivot terms for the equations $\tau_C^1 = 0$ and $\tau_C^2 = 0$ are $B_S(h_i, v_j)$, $B_S(v_i, v_j)$, $i, j = 1, \ldots, n$, and $B_S(v_n, h_i)$, $i = 1, \ldots, n-1$, giving in addition $2n^2 + n - 1$ independent equations. Thus,

$$
\begin{aligned}
\text{(3.26)} \qquad \dim \operatorname{Ker} \tau_1 = {} & \dim S^2 T^* + \dim(T^* \otimes T^*) \\
& - \Big[\frac{(n-1)(n-2)}{2} + 2n^2 + 3n - 1 \Big] = \frac{7n^2 - n}{2}.
\end{aligned}
$$

Comparing (3.25) and (3.26) we get $\operatorname{Im} \sigma_3(P_1) = \operatorname{Ker} \tau_1$. □

P r o o f of Lemma 3.4 (second part). The morphisms, the symbols and the obstruction map associated to the Rapcsák system can be represented in the following commutative diagram:

$$
\begin{array}{ccccccccc}
g_3(P_1) & \longrightarrow & S^3 T^* & \xrightarrow{\sigma_3(P_1)} & (T^* \otimes T^*) \times S^2 T^* & \xrightarrow{\tau_1} & K_1 & \longrightarrow & 0 \\
\downarrow & & \downarrow \varepsilon & & \downarrow \varepsilon & & & & \\
\mathrm{Sol}_3(P_1) & \xrightarrow{i} & J_3(\mathbb{R}_{TM}) & \xrightarrow{p_3(P_1)} & J_1(T^*) \times J_2(\mathbb{R}_{TM}) & & & & \\
\downarrow \overline{\pi}_2 & & \downarrow \pi_2 & & \downarrow \pi_0 \times \pi_1 & & & & \\
\mathrm{Sol}_2(P_1) & \xrightarrow{i} & J_2(\mathbb{R}_{TM}) & \xrightarrow{p_2(P_1)} & T^* \times J_1(\mathbb{R}_{TM}) & & & &
\end{array}
$$

Let $s = j_2(F)_x \in \mathrm{Sol}_{2,x}(P_1)$ be a second order solution of P_1 at x, that is,

$$
\text{(3.27)} \qquad (i_S dd_J F)_x = 0, \quad (\mathcal{L}_C F - F)_x = 0, \quad (\nabla(\mathcal{L}_C F - F))_x = 0.
$$

The integrability condition can be computed in terms of $\tau_1 = (\tau_S^1, \tau_S^2, \tau_C^1, \tau_C^2)$. According to Remarks 2.4 and 2.5, s can be lifted into a third order solution if and only if $\varphi(s) = 0$, where $\varphi(s) = (\tau_1 \nabla P_1(F))_x$. Computing $\varphi(s)$ we find the following

(1) Using the notation $\omega := i_S dd_J F$ we have $\omega_x = 0$ from (3.27) and

$$
\begin{aligned}
\tau_S^1(\nabla(P_1 F))_x(X, Y) &= \nabla\omega(JX, hY) - \nabla\omega(hY, JX) - \nabla\omega(JY, hX) + \nabla\omega(hX, JY) \\
&= JX\omega(hY) - hY\omega(JX) - JY\omega(hX) + hX\omega(JY) \\
&= i_J d\omega(hX, hY).
\end{aligned}
$$

Moreover, $di_S = -i_S d + d_S$, $i_J d_S = i_{[J,S]} + d_S i_J$ and $d_J d_J = 0$, we obtain that

$$
\begin{aligned}
i_J d\omega(hX, hY)_x &= (i_J d_S dd_J F - i_J i_S ddd_J F)_x(hX, hY) \\
&= (i_{[J,S]} dd_J F + d_S i_J dd_J F)_x(hX, hY) = (i_\Gamma dd_J F)_x(hX, hY).
\end{aligned}
$$

(2) $\tau_S^2(\nabla(P_1F))_x = (\nabla\omega)_x(X,S) = X_x\omega(S) = X_x dd_J(S,S) = 0.$

(3) Using the identity $J[JX,S] = JX$ we have

$$\begin{aligned}\tau_C^1(\nabla(P_1F))_x &= X_x(i_S dd_J F(JY)) + X_x(JY(CF-F))\\ &= X_x(-JY d_J F(S) - d_J F([S,JY])) + X_x(JYCF - JYF)\\ &= -X_x(J[S,JY]F) - X_x(JYF) = X_x(JYF) - X_x(JYF) = 0.\end{aligned}$$

(4) We have $dd_J(CF-F)(S,hX) = S\big(JX(CF-F)\big) - hX\big(C(CF-F)\big)$. Then

$$\begin{aligned}\tau_C^2(\nabla(P_1F))_x &= C(i_S dd_J F(hX)) - S(JX(CF-F)) + hX(C(CF-F))\\ &= d_C dd_J F(S,hX) - dd_J d_C F(S,hX) + dd_J F(S,hX).\end{aligned}$$

Since $d_J d_C - d_C d_J = d_{[J,C]} = d_J$ it follows that

$$d_C dd_J F - dd_J d_C F + dd_J F = d_C dd_J F - dd_C d_J F = d_C dd_J F - d_C dd_J F = 0.$$

From the above computation it follows that $\varphi(s) = (\tau_1 \nabla P_1(F))_x = (i_\Gamma dd_J F_x, 0, 0, 0)$ and therefore the only condition to prolong a second order solution into a third order solution is given by the equation $(i_\Gamma dd_J F)_x = 0$ as Lemma 3.4 stated. □

Lemma 3.6. *The symbol of $P_1 = (P_S, P_C)$ is involutive.*

Proof. Let us consider the basis $\mathcal{B}$ introduced in (3.17). Using the notation (3.18) we have

$$\begin{aligned}g_2(P_1) = \operatorname{Ker}\sigma_2(P_1) &= \{A \in S^2T^*\colon\ A(X,C) = 0,\ A(S,JX) = A(X,C)\}\\ &= \{A \in S^2T^*\colon\ A_{ij} = A_{ji},\ A_{\underline{ij}} = A_{\underline{ji}},\quad A_{i\underline{n}} = A_{n\underline{i}} = A_{\underline{in}} = A_{\underline{ni}} = 0\},\end{aligned}$$

and therefore

$$\dim(g_2(P_1)) = \frac{n(n+1)}{2} + (n-1)^2 + \frac{n(n-1)}{2} = n^2 + (n-1)^2. \tag{3.28}$$

Let us consider the basis $\widetilde{\mathcal{B}} = \{e_i\}_{i=1,\dots,2n}$, where

$$\widetilde{\mathcal{B}} = \Big\{\underbrace{h_1}_{e_1},\dots,\underbrace{h_{n-1}}_{e_{n-1}},\underbrace{h_n + v_1 + \ldots + v_n}_{e_n},\underbrace{v_1}_{e_{n+1}},\dots,\underbrace{v_n}_{e_{2n}}\Big\}. \tag{3.29}$$

Denoting the coefficients of $A \in S^2T^*$ with respect to $\widetilde{\mathcal{B}}$ by $\widetilde{A}_{ij}$, we have

$$\begin{aligned}g_2(P_1)_{e_1\dots e_k} &= \{A \in S^2T^*\colon\ i_{e_1}A = 0,\ \dots,\ i_{e_k}A = 0\}\\ &= \{A \in S^2T^*\colon\ \tilde{A}_{ij} = \tilde{A}_{ji},\ \tilde{A}_{\underline{ij}} = \tilde{A}_{\underline{ji}},\\ &\qquad \tilde{A}_{i\underline{n}} = \tilde{A}_{n\underline{i}} = \tilde{A}_{\underline{in}} = \tilde{A}_{\underline{ni}} = 0,\ \tilde{A}_{lj} = 0,\ \tilde{A}_{l\underline{j}} = 0,\ l \leqslant k\},\end{aligned}$$

therefore

$$\dim\,(g_2(P_1))_{e_1\dots e_k} = \begin{cases} \frac{1}{2}(n-k)(n-k+1)+(n-k)(n-1)+\frac{1}{2}(n-2)(n-1) & \text{if } k\leqslant n, \\ \frac{1}{2}(n-2-k)(n-k-1) & \text{if } k>n, \end{cases}$$

and hence

$$\begin{aligned}\dim g_2(P_1) &+ \sum_{k=1}^{2n} \dim g_2(P_1)_{e_1\dots e_k} \\ &= n^2+(n-1)^2+\sum_{k=1}^{n}\Big(\frac{(n-k)(n-k+1)}{2}+(n-k)(n-1)\Big)+\frac{n(n-2)(n-1)}{2} \\ &\quad+\sum_{k=1}^{n}\frac{(n-2-k)(n-k-1)}{2}=\frac{8n^3-9n^2+7n}{6}\overset{(3.24)}{=}\dim g_3(P_1),\end{aligned}$$

which shows that the basis (3.29) is quasi-regular, and the symbol of P_1 is involutive. □

From Lemmas 3.4 and 3.6 we get the following proposition:

Proposition 3.7. *Let S be a spray on the manifold M. Then the Rapcsák system associated to S is formally integrable if and only if for every second order solution $s=j_2(F)_x$ the equation* (3.6) *is satisfied.*

P r o o f. According to Lemma 3.4, if equation (3.6) is satisfied, then every second order solution can be prolonged into a third order solution. Moreover, the symbol of the differential operator P_1 is involutive and therefore there is no higher order compatibility condition for the operator P_1, hence all third order solutions can be prolonged into an infinite order solution. □

Remark 3.8. For every function $F\in C^\infty(TM)$ the corresponding $i_\Gamma dd_J F$ is a semi-basic 2-form and as such, it is identically zero if the dimension of the manifold M is one. In that case, the Rapcsák system is formally integrable. However, if $\dim M$ is greater than one, the equation (3.6) is not satisfied by all second order solutions and therefore the Rapcsák system is not formally integrable.

4. Extended Rapcsák system

Lemmas 3.4 and 3.6 show that the conditions of Theorem 2.1 are fulfilled if and only if for any initial data $j_2(F)_x$ of P_1 we have also $i_\Gamma dd_J F = 0$. This is true if $\dim M = 1$. However, when $\dim M \geqslant 2$, this condition is not satisfied by every second order solution. Therefore not every second order solution can be lifted into a third order solution. Since the set of initial data is too large (containing some which cannot be prolonged into a higher order solution) we have to reduce it by including the compatibility condition to the system. This leads us to consider the operator (P_S, P_C, P_Γ), where P_Γ is a second order operator defined as

$$P_\Gamma \colon\ C^\infty(TM) \to \mathrm{Sec}(\Lambda^2 T_v^*), \quad P_\Gamma F := i_\Gamma dd_J F.$$

Remark 4.1. If S is a spray and F is a 1-homogeneous Lagrangian, then we have

$$P_S F(X) = i_S dd_J F(X) = dd_J F(S, hX) = \tfrac{1}{2} i_\Gamma dd_J F(S, hX) = P_\Gamma F(S, hX)$$

for every $X \in T$. Consequently, if F is a solution of P_Γ, then it is also a solution of P_S, that is, P_Γ contains in particular the equations of P_S. That lead us to drop the system P_S and consider the *extended Rapcsák* system

$$P_2 = (P_\Gamma, P_C). \tag{4.1}$$

It is clear that a function is a solution to the Rapcsák system if and only if it is a solution to the extended Rapcsák system.

In this chapter we investigate the integrability of the extended Rapcsák system $P_2 = (P_\Gamma, P_C)$. Our method is similar to the one we used in Chapter 3.

Lemma 4.2. *A* 2nd *order solution $s = j_2(F)_x$ of the system $P_2 = (P_\Gamma, P_C)$ at $x \in \mathcal{T}M$ can be prolonged into a* 3rd *order solution if and only if*

$$(i_R dd_J F)_x = 0. \tag{4.2}$$

P r o o f. The symbol of the operator P_Γ and its first prolongation are $\sigma_2(P_\Gamma)\colon S^2T^* \to \Lambda^2 T_v^*$ and $\sigma_3(P_\Gamma)\colon\ S^3T^* \to T^* \otimes \Lambda^2 T_v^*$ with

$$(\sigma_2(P_\Gamma)A_2)(Y, Z) = 2(A_2(hY, JZ) - A_2(hZ, JY)), \tag{4.3}$$

$$(\sigma_3(P_\Gamma)A_3)(X, Y, Z) = 2(A_3(X, hY, JZ) - A_3(X, hZ, JY)), \tag{4.4}$$

where $X, Y, Z \in T$, $A_2 \in S^2T^*$, $A_3 \in S^3T^*$. Let us consider the map

$$\tau_2 := (\tau_\Gamma^1, \tau_\Gamma^2, \tau_{\Gamma C}) \tag{4.5}$$

defined on $(T^* \otimes \Lambda^2 T_v^*) \times S^2T^*$ with

$$\tau_\Gamma^1(B_\Gamma, B_C)(X, Y, Z) = B_\Gamma(hX, Y, Z) + B_\Gamma(hY, Z, X) + B_\Gamma(hZ, X, Y), \tag{4.6}$$
$$\tau_\Gamma^2(B_\Gamma, B_C)(X, Y, Z) = B_\Gamma(JX, Y, Z) + B_\Gamma(JY, Z, X) + B_\Gamma(JZ, X, Y), \tag{4.7}$$
$$\tau_{\Gamma C}(B_\Gamma, B_C)(X, Y) = \tfrac{1}{2} B_\Gamma(C, X, Y) - B_C(hX, JY) + B_C(hY, JX), \tag{4.8}$$

where $B_\Gamma \in T^* \otimes \Lambda^2 T_v^*$, $B_C \in S^2T^*$, $X, Y, Z \in T$. We have the following property:

Property 4.3. *Let K_2 be the image of τ_2. Then the sequence*

$$S^3T^* \overset{\sigma_3(P_2)}{\longrightarrow} (T^* \otimes \Lambda^2 T_v^*) \times S^2T^* \overset{\tau_2}{\longrightarrow} K_2 \longrightarrow 0 \tag{4.9}$$

is exact.

P r o o f. A simple computation shows that $\tau_2 \circ \sigma_3(P_2) = 0$, and therefore $\operatorname{Im} \sigma_3(P_2) \subset \operatorname{Ker} \tau_2$. Let us compute the rank of $\sigma_3(P_2)$. By using the basis (3.17) and the notation (3.18), a symmetric tensor $A \in S^3T^*$ is an element of $\operatorname{Ker} \sigma_3(P_2)$ if in addition to the relations describing the symmetry properties, the equations (3.19) and the equations

$$A_{ij\underline{k}} = A_{ik\underline{j}}, \quad A_{\underline{i}j\underline{k}} = A_{\underline{i}k\underline{j}}, \quad i, j, k = 1, \dots, n, \tag{4.10}$$

hold. We obtain from (4.10) that all of the blocks (3.18) are totally symmetric, and $A_{ij\underline{n}} = A_{i\underline{jn}} = A_{\underline{ijn}} = 0$. That way there are $n(n+1)(n+2)/6$ free components in the block (A_{ijk}) and $(n-1)n(n+1)/6$ free components to choose in each of the blocks $(A_{ij\underline{k}})$, $(A_{i\underline{jk}})$ and $(A_{\underline{ijk}})$. That is,

$$\dim(g_3(P_2)) = \frac{n(n+1)(n+2)}{6} + 3\frac{(n-1)n(n+1)}{6} = \frac{4n^3 + 3n^2 - n}{6}, \tag{4.11}$$

and

$$\operatorname{rank} \sigma_3(P_2) = \dim S^3T^* - \dim(g_3(P_2)) = \frac{4n^3 + 9n^2 + 5n}{6}. \tag{4.12}$$

On the other hand, considering the equations determining $\operatorname{Ker} \tau_2$ we find that the pivot terms for $\tau_\Gamma^1 = 0$ and for $\tau_\Gamma^2 = 0$ are $B_\Gamma(h_i, h_j, h_k)$ and $B_\Gamma(v_i, h_j, h_k)$, $i < j < k$, respectively. Both of them provide $\binom{n}{3}$ independent equations. Furthermore, the terms $B_\Gamma(v_n, h_i, h_j)$, $i < j$, $i, j = 1, \dots, n$, are pivot terms for $\tau_{\Gamma C} = 0$, which gives $n(n-1)/2$ independent equations. Hence

$$
\begin{aligned}
\text{(4.13)} \qquad \dim \operatorname{Ker} \tau_2 &= \dim S^2T^* + \dim(T^* \otimes \Lambda^2 T_v^*) - 2\binom{n}{3} - \frac{n(n-1)}{2} \\
&= \frac{4n^3 + 9n^2 + 5n}{6}.
\end{aligned}
$$

Comparing (4.12) and (4.13) one finds that $\operatorname{rank} \sigma_3(P_2) = \dim \operatorname{Ker} \tau_2$ and that the sequence (4.9) is exact. □

Let us turn our attention to the proof of Lemma 4.2. We have the following commutative diagram:

$$
\begin{array}{ccccccccc}
g_3(P_2) & \longrightarrow & S^3T^* & \xrightarrow{\sigma_3(P_2)} & (T^* \otimes \Lambda^2 T_v^*) \times S^2T^* & \xrightarrow{\tau_2} & K_2 & \longrightarrow & 0 \\
\downarrow & & \downarrow \varepsilon & & \downarrow \varepsilon & & & & \\
\mathrm{Sol}_3(P_2) & \xrightarrow{i} & J_3(\mathbb{R}_{TM}) & \xrightarrow{p_3(P_2)} & J_1(\Lambda^2 T_v^*) \times J_2(\mathbb{R}_{TM}) & & & & \\
\downarrow \overline{\pi}_2 & & \downarrow \pi_2 & & \downarrow & & & & \\
\mathrm{Sol}_2(P_2) & \xrightarrow{i} & J_2(\mathbb{R}_{TM}) & \xrightarrow{p_2(P_2)} & \Lambda^2 T_v^* \times J_1(\mathbb{R}_{TM}) & & & &
\end{array}
$$

Let $s = j_2(F)_x$ be a second order solution of P_2 at a point x, that is, $(P_2F)_x = 0$. We have

$$
\text{(4.14)} \qquad (i_\Gamma dd_J F)_x = 0, \quad (\mathcal{L}_C F - F)_x = 0, \quad (\nabla(\mathcal{L}_C F - F))_x = 0.
$$

The integrability condition can be computed with the help of the map τ_2 (see Proposition 2.3 and Remark 2.5). Indeed, $s \in \mathrm{Sol}_{2,x}(P_2)$ can be prolonged into a third order solution if and only if $\varphi(s) = 0$, where $\varphi(s) = (\tau_2 \nabla P_2(F))_x$. Let us introduce the notation $\Omega = dd_J F$. Using the component maps of τ_2 introduced in (4.5) one finds

$$
\begin{aligned}
\text{(1)} \quad \tau_\Gamma^1(\nabla(P_2F))_x &= d_h(i_\Gamma dd_J F)_x = (d_h i_{2h-I} dd_J F)_x \\
&= (2(d_h i_h dd_J F - d_h dd_J F))_x \\
&= (2d_h(i_h d - di_h) d_J F)_x = (2d_h d_h d_J F)_x \\
&= (d_R d_J F)_x = (i_R \Omega)_x, \\
\text{(2)} \quad \tau_\Gamma^2(\nabla(P_2F))_x &= d_J(i_\Gamma \Omega)_x \overset{(2.4)}{=} (d_J(i_{2h-I}\Omega))_x = (2d_J i_h dd_J F - 2d_J dd_J F)_x \\
&= (-2d_J i_h d_J dF - 2i_J ddd_J F + 2di_J dd_J F)_x \\
&= -(2d_J(d_J i_h dF + d_J dF))_x = 0,
\end{aligned}
$$

where we used $[d, d_J] = 0$, $[i_h, d_J] = d_{Jh} - i_{[h,J]}$ and $[J, h] = 0$,

$$(3)\quad \tau_{\Gamma C}(\nabla P_2(F))_x(X,Y) = \tfrac{1}{2}\nabla i_\Gamma \Omega(C,X,Y) - \nabla P_C F(hX, JY) + \nabla P_C F(hY, JX)$$
$$= \tfrac{1}{2} d_C i_\Gamma \Omega(hX, hY) - \tfrac{1}{2} i_\Gamma d_C d d_J F(hX, hY)$$
$$= \tfrac{1}{2} d_{[C,\Gamma]}\Omega(hX, hY) \overset{[C,\Gamma]=0}{=} 0.$$

The above computation shows that $\varphi(s) = \tau_2(\nabla P_2(F))_x = (i_R\Omega_x, 0, 0)$, which proves Lemma 4.2. □

Lemma 4.4. *The symbol of P_2 is involutive.*

Proof. We consider the basis (3.17) and use the notation (3.18). We have

$$\begin{aligned} g_2(P_2) &= \operatorname{Ker}\sigma_2(P_2) \\ &= \{A \in S^2T^*:\ A(X,C) = 0,\ A(hX, JY) = A(hY, JX)\} \\ &= \{A \in S^2T^*:\ A_{ij} = A_{ji},\ A_{i\underline{j}} = A_{j\underline{i}},\ A_{\underline{ij}} = A_{\underline{ji}},\ A_{i\underline{n}} = 0,\ A_{n\underline{i}} = 0\}. \end{aligned}$$

Therefore $\dim(g_2(P_2)) = n(n+1)/2 + 2(n-1)n/2$. Let us consider the basis $\widehat{\mathcal{E}} = \{\widehat{e}_i\}_{i=1}^{2n}$, where

$$\begin{aligned} \widehat{e}_i &= h_i + i v_i, \quad i = 1, \ldots, n-1, \\ \widehat{e}_n &= h_n + v_1 + \ldots + v_n, \\ \widehat{e}_{i+n} &= v_i, \quad i = 1, \ldots, n. \end{aligned}$$

In the new basis the components of the block $(\hat{A}_{\underline{ij}}) = (A_{\underline{ij}})$ can be expressed as a combination of the components $\hat{A}_{i\underline{j}}$ as follows: if $i \neq j$, then $\hat{A}_{\underline{ij}} = (i-j)^{-1} \times (\hat{A}_{i\underline{j}} - \hat{A}_{j\underline{i}})$, and if $i = j$, we have $\hat{A}_{\underline{jj}} = \hat{A}_{n\underline{j}} - \sum\limits_{k\neq j}(k-j)^{-1}(\hat{A}_{j\underline{k}} - \hat{A}_{k\underline{j}})$. Then,

$$\dim(g_2(P_2))_{\widehat{e}_1\ldots\widehat{e}_k} = \begin{cases} \frac{1}{2}(n-k+1)(n-k) + (n-1)(n-k) & \text{if } k \leqslant n, \\ 0 & \text{if } k > n, \end{cases}$$

and therefore

$$\begin{aligned} &\dim(g_2(P_2)) + \sum_{k=1}^{2n} \dim(g_2(P_2))_{\widehat{e}_1\ldots\widehat{e}_k} \\ &\quad = \frac{n(n+1)}{2} + 2\frac{(n-1)n}{2} + \sum_{k=1}^{n}\Big(\frac{(n-k+1)(n-k)}{2} + (n-1)(n-k)\Big) \\ &\quad \overset{(4.11)}{=} \dim(g_3(P_2)), \end{aligned}$$

thus the basis $\widehat{\mathcal{E}}$ is quasi-regular. □

From Lemma 4.2 and Lemma 4.4, using the Cartan-Kähler theorem, we get the following proposition.

Proposition 4.5. *The extended Rapcsák system is formally integrable if and only if for every second order solution $j_2(F)_x$ the equation (4.2) is satisfied.*

The proof is similar to that of Proposition 3.7. According to Lemma 4.2, if equation (4.2) is satisfied, then every second order solution can be prolonged into a third order solution. Moreover, the symbol of the differential operator P_2 is involutive and therefore there is no higher order compatibility condition for the extended Rapcsák system and all third order solutions can be prolonged into an infinite order solution. Altogether we obtain that every second order solution can be prolonged into an infinite order solution and the extended system is formally integrable.

Theorem 4.6. *Let S be a spray on a manifold M. The extended Rapcsák system is formally integrable if and only if one of the following conditions is fulfilled:*
(1) $\dim M = 2$;
(2) *the spray S is flat;*
(3) *the spray S is of isotropic curvature.*

Proof. We have the following cases:

(1) If $\dim M = 2$, then the space of semi-basic 3-forms is trivial, that is, $\Lambda^3_v(TM) = \{0\}$. Therefore for every $F \in C^\infty(TM)$ we have $i_R dd_J F \equiv 0$.

(2) If S is flat, that is, $\Phi = \lambda J$, then $R = d_J\lambda \wedge J$. Using the integrability of the vertical distribution we get $i_R dd_J F = d_R d_J F = d_J\lambda d_J^2 F + d_{d_J\lambda} \wedge i_J d_J F = 0$.

(3) If S is of isotropic curvature, then R takes the form $R = \alpha \wedge J + \beta \otimes C$, where $\alpha \in \Lambda^1_v(TM)$, $\beta \in \Lambda^2_v(TM)$. Then

$$i_R dd_J F = i_{\alpha\wedge J+\beta\otimes C} dd_J F = \alpha \wedge i_J dd_J F + \beta \wedge i_C dd_J F = 0.$$

In all three cases the second order solutions $j_2(F)_x$ satisfy the equation (4.2). Using Proposition 4.5 we obtain that in all three cases the extended Rapcsák system is formally integrable. □

Corollary 4.7. *Let S be an analytic spray on an analytic manifold M. If M is a 2-dimensional manifold, or the spray S is flat, or of isotropic curvature, then S is locally projective Finsler metrizable.*

Indeed, in the analytic context, formal integrability implies the existence of solutions for all initial data.

The integrability condition $i_R dd_J F = 0$ also appeared in [1]. It can be shown that this integrability condition is equivalent to the equation $i_\Phi dd_J F = 0$ or $i_W dd_J F = 0$, where Φ is the Jacobi endomorphism and W is the Weyl tensor associated to S.

5. Curvature condition: non-2-acyclicity of the system

As the results of Section 4 show, in the case when the curvature is not isotropic the extended Rapcsák system is not formally integrable. Indeed, the integrability condition can be expressed in terms of the curvature tensor (Proposition 4.5), and in the non-isotropic case the curvature condition (4.2) is not necessarily satisfied by all initial data. Therefore, to solve the projective metrizability problem in this case one has to consider an enlarged system by including the curvature condition to the extended Rapcsák system. We remark that equation (4.2) can be replaced by an equivalent condition

$$i_\Phi dd_J F = 0 \tag{5.1}$$

containing the Jacobi endomorphism instead of the curvature tensor (see [9]). Therefore we introduce the second order differential operator $P_\Phi\colon C^\infty(TM) \to \mathrm{Sec}(\Lambda^2 T_v^*)$, with $P_\Phi(F) = i_\Phi dd_J F$ and consider the system

$$P_3 := (P_\Gamma, P_C, P_\Phi). \tag{5.2}$$

One can remark that from equation (5.1) it follows that Φ has to be self-adjoint with respect the symmetric bilinear form $g = \Omega(J\cdot, \cdot)$, and therefore it must be diagonalizable. We consider here the generic case, when the eigenvalues of Φ are pairwise distinct. We have the following theorem:

Theorem 5.1. *Let S be a non-isotropic spray on an n-dimensional manifold. Then the first nontrivial Spencer cohomology group is $H^{2,2}(P_3)$. Moreover, one has*

$$\dim H^{2,2}(P_3) = \frac{(n-1)(n-2)}{2}.$$

P r o o f. The symbol of the operator P_Φ is defined as

$$\big(\sigma_2(P_\Phi)A\big)(X_1, X_2) = A(\Phi X_1, JX_2) - A(\Phi X_2, JX_1), \tag{5.3}$$

for $X_1, X_2 \in T$, $A \in S^2T^*$ and therefore its $(m-2)$th order prolongation is defined as

$$\begin{aligned} &\sigma_m(P_\Phi)A(X_1, \ldots, X_m) \\ &\quad = A(X_1, \ldots, \Phi X_{m-1}, JX_m) - A(X_1, \ldots, \Phi X_m, JX_{m-1}), \end{aligned} \tag{5.4}$$

($X_i \in T$, $A \in S^mT^*$, $m \geqslant 2$) having an analogous operation as (5.3) in the last two arguments. We denote by $\lambda_1, \dots, \lambda_n$ the distinct eigenvalues of the Jacobi endomorphism. For every spray S we have $\Phi(S) = 0$, and therefore $\lambda_n = 0$ is an eigenvalue of Φ. Let

$$\widehat{\mathcal{B}} := \{e_1, \dots, e_{2n}\} = \{h_1, \dots, h_n, v_1, \dots, v_n\} \subset T_xTM, \tag{5.5}$$

be a basis formed by the horizontal and vertical semibasic eigenvectors of Φ. We have $\Phi h_i = \lambda_i v_i$, $Jh_i = v_i$, $i = 1, \dots, n$, $h_n = S$, $v_n = C$. We have $A \in \operatorname{Ker} \sigma_2(P_\Phi)$ if and only if $A(\Phi h_j, Jh_k) - A(\Phi h_k, Jh_j) = (\lambda_j - \lambda_k) A(v_j, v_k) = 0$, that is, using the notation of (3.18), $A_{\underline{ij}} = 0$ for $j \neq k$. More generally, from (5.4), for an mth order symmetrical tensor $A \in S^mT^*$ we have

$$A \in \operatorname{Ker} \sigma_m(P_\Phi) \Leftrightarrow A(e_{i_1}, \dots, e_{i_{m-2}}, v_j, v_k) = 0 \Leftrightarrow A_{\dots\underline{jk}} = 0, \quad j \neq k, \tag{5.6}$$

where $1 \leqslant i_s \leqslant 2n$, $1 \leqslant s \leqslant m-2$, $1 \leqslant j,\ k \leqslant n$. In particular

$$A \in \operatorname{Ker} \sigma_3(P_\Phi) \Leftrightarrow A_{i\underline{jk}} = A_{\underline{ijk}} = 0, \quad i, j, k = 1, \dots, n,\ j \neq k, \tag{5.7}$$

$$A \in \operatorname{Ker} \sigma_4(P_\Phi) \Leftrightarrow A_{il\underline{jk}} = A_{i\underline{ljk}} = A_{\underline{iljk}} = 0, \quad i, l, j, k = 1, \dots, n,\ j \neq k. \tag{5.8}$$

Let us consider the Spencer sequence corresponding to $m = 2$:

$$0 \longrightarrow g_4(P_3) \xrightarrow{\delta_0^2} T^* \otimes g_3(P_3) \xrightarrow{\delta_1^2} \Lambda^2T^* \otimes g_2(P_3) \xrightarrow{\delta_2^2} \Lambda^3T^* \otimes g_1(P_3) \longrightarrow \dots \tag{5.9}$$

Computation of $\operatorname{Im} \delta_1^2$**.** We remark that $\delta_0^2 = i$ is the canonical inclusion and the symbol of P_3 is 1-acyclic, that is, $H^{m,1} = 0$ for all $m \geqslant 2$. Consequently, (5.9) is exact in the first two terms. We obtain that

$$\operatorname{rank} \delta_1^2 = \dim(T^* \otimes g_3(P_3)) - \dim(g_4(P_3)). \tag{5.10}$$

The symbol of (5.2) and its prolongations are

$$\sigma_m(P_3) = (\sigma_m(P_\Gamma), \sigma_m(P_C), \sigma_m(P_\Phi)). \tag{5.11}$$

In order to compute $g_3(P_3)$ and $g_4(P_3)$ we note that by definition

$$g_m(P_3) = \operatorname{Ker} \sigma_m(P_3) = \operatorname{Ker} \sigma_m(P_\Gamma) \cap \operatorname{Ker} \sigma_m(P_C) \cap \operatorname{Ker} \sigma_m(P_\Phi) \tag{5.12}$$

for every $m \geqslant 2$. Therefore a symmetric tensor $A \in S^3T^*$ is an element of $g_3(P_3)$ if and only if its components satisfy (3.19) and (4.10). We obtain that the block

(A_{ijk}) is totally symmetric and contains $n(n+1)(n+2)/6$ free components. The block $(A_{ij\underline{k}})$ is also totally symmetric and $A_{ij\underline{n}} = 0$. So in the block $(A_{ij\underline{k}})$ we have $(n-1)n(n+1)/6$ free independent components. In each of the blocks $(A_{i\underline{jk}})$, $(A_{\underline{ijk}})$ there are only $n-1$ free components: the $A_{i\underline{ii}}$ and the $A_{\underline{iii}}$, $1 \leqslant i \leqslant n-1$. Adding the number of the independent free components and using the formula for combination $\mathcal{C}_{n,k} = \binom{n}{k}$ and combination with repetition $\mathcal{C}^{\mathrm{rep}}_{n,k} = \binom{n+k-1}{k}$ we can find that

$$\dim g_3(P_3) = \mathcal{C}^{\mathrm{rep}}_{n,3} + \mathcal{C}^{\mathrm{rep}}_{n-1,3} + 2\,\mathcal{C}^{\mathrm{rep}}_{n-1,1}.$$

A completely analogous computation (using components with four indices) shows that

$$\dim g_4(P_3) = \mathcal{C}^{\mathrm{rep}}_{n,4} + \mathcal{C}^{\mathrm{rep}}_{n-1,4} + 3\,\mathcal{C}^{\mathrm{rep}}_{n-1,1}.$$

Therefore, from (5.10) we obtain that

$$\text{(5.13)} \qquad \operatorname{rank} \delta^2_1 = \frac{7n^4}{12} + \frac{2n^3}{3} + \frac{47n^2}{12} - \frac{43n}{6} + 3.$$

Computation of $\operatorname{Ker} \delta^2_2$**.** We remark that for $2 \leqslant m$ we have $g_m(P_3) \cap S^m T^*_v = S^m T^*_v$, that is, there is no restriction on the purely horizontal part of the elements of $g_m(P_3)$. Therefore we can use the canonical exact sequence

$$\text{(5.14)} \qquad 0 \longrightarrow S^4 T^*_v \xrightarrow{\ i\ } T^*_v \otimes S^3 T^*_v \xrightarrow{\delta^2_{1,v}} \Lambda^2 T^*_v \otimes S^2 T^*_v \xrightarrow{\delta^2_{2,v}} \Lambda^3 T^*_v \otimes T^*_v \longrightarrow \dots,$$

where $\delta^2_{1,v}$ and $\delta^2_{1,v}$ denote the restriction of δ^2_1 and δ^2_2 on the corresponding spaces. From the exactness of (5.14) we get that for $m = 2$

$$\dim \operatorname{Ker} \delta^2_2 \Big|_{\Lambda^2 T^*_v \otimes S^2 T^*_v} = \dim(T^*_v \otimes S^3 T^*_v) - \dim(S^4 T^*_v) = n\,\mathcal{C}^{\mathrm{rep}}_{n,3} - \mathcal{C}^{\mathrm{rep}}_{n,4},$$

and the number of independent equations characterizing $\dim \operatorname{Ker} \delta^2_{2,v}$ is

$$\text{(5.15)} \qquad N_0 = \dim(\Lambda^2 T^*_v \otimes S^2 T^*_v) - \dim \operatorname{Ker} \delta^2_{2,v} = (\mathcal{C}_{n,2} + \mathcal{C}^{\mathrm{rep}}_{n,2}) - (n\,\mathcal{C}^{\mathrm{rep}}_{n,3} - \mathcal{C}^{\mathrm{rep}}_{n,4}).$$

Let us consider the equations of $\operatorname{Ker} \delta^2_2$ containing at least one vertical component: we will determine how many independent parameters characterize the mixed part of a tensor D in $\operatorname{Ker} \delta^2_2$. Using the basis (5.5) and the convention (3.18) we have $D \in \Lambda^2 T^* \otimes g_2(P_3)$ if and only if $i_{e_\alpha} i_{e_\beta} D = D(e_\alpha, e_\beta, \cdot, \cdot)$ is an element of $g_2(P_3)$, that is, because of (3.9), (4.3), (5.3) we have

$$\text{(5.16)} \qquad D(e_\alpha, e_\beta, e_\gamma, v_n) = 0, \qquad (\Leftrightarrow D_{\alpha\beta\gamma\underline{n}} = 0)$$

$$\text{(5.17)} \qquad D(e_\alpha, e_\beta, v_k, v_l) = 0, \qquad (\Leftrightarrow D_{\alpha\beta\underline{kl}} = 0) \quad k \neq l$$

$$\text{(5.18)} \qquad D(e_\alpha, e_\beta, h_n, v_k) = 0, \qquad (\Leftrightarrow D_{\alpha\beta n\underline{k}} = 0)$$

$$\text{(5.19)} \qquad D(e_\alpha, e_\beta, h_k, v_l) - D(e_\alpha, e_\beta, h_l, v_k) = 0, \quad (\Leftrightarrow D_{\alpha\beta k\underline{l}} - D_{\alpha\beta l\underline{k}} = 0)$$

where we use Greek letters to denote indices from 1 to $2n$ and Latin letters to denote indices from 1 to n and $e_\alpha = h_\alpha$ if $\alpha \leqslant n$, $e_\alpha = v_{\underline{\alpha - n}}$ if $n < \alpha$. We consider the following notation:

$$\mathcal{E}_{\underline{ijkl}} = \sum_{\underline{ijk}}^{\text{cycl}} D_{\underline{ijkl}}, \quad \mathcal{E}_{i\underline{jkl}} = \sum_{i\underline{jk}}^{\text{cycl}} D_{i\underline{jkl}}, \quad \ldots, \quad \mathcal{E}_{ijk\underline{l}} = \sum_{ijk}^{\text{cycl}} D_{ijk\underline{l}}.$$

Then $D \in \delta_2^2$ if and only if $D \in \delta_{2,v}^2$ and in addition one has

$$\mathcal{E}_{\underline{ijkl}} = \mathcal{E}_{i\underline{jkl}} = \mathcal{E}_{ij\underline{kl}} = \mathcal{E}_{ijk\underline{l}} = \mathcal{E}_{\underline{ijk}l} = \mathcal{E}_{i\underline{jk}l} = \mathcal{E}_{ij\underline{k}l} = 0. \tag{5.20}$$

Using the equations (5.16), (5.17), (5.18) and (5.19) we can observe the following:

Equations $\mathcal{E}_{\underline{ijkl}} = 0$. If $i, j, k, l \neq n$ are pairwise different, then $\mathcal{E}_{\underline{ijkl}} = 0$ trivially holds, that is, all of its terms are zeros. The remaining equations are independent. The pivot terms are

▷ for (i, j), $i < j < n$: $D_{\underline{jnii}}$, $D_{\underline{nijj}}$,
▷ for (i, j, k), $i < j < k < n$: $D_{\underline{ijkk}}$, $D_{\underline{jkii}}$, $D_{\underline{kijj}}$,
and this block gives $N_1 = 2\,\mathcal{C}_{n-1,2} + 3\,\mathcal{C}_{n-1,3}$ independent equations.

Equations $\mathcal{E}_{i\underline{jkl}} = 0$. The equation $\mathcal{E}_{n\underline{i}\,\underline{jk}} = 0$ trivially holds for $i < j$, i, j, k pairwise different. Moreover, for $i < l < n$, $j < k$, i, j, k, l pairwise different we have the following relation:

$$\mathcal{E}_{i\underline{jkl}} = D_{i\underline{jkl}} + D_{\underline{k}i\underline{jl}} + D_{\underline{jk}i\underline{l}} \overset{(5.17),(5.19)}{=} D_{\underline{jk}l\underline{i}} + D_{l\underline{jki}} + D_{\underline{k}l\underline{ji}} = \mathcal{E}_{l\underline{jki}}.$$

The number of the relations is $\mathcal{C}_{n-2,2} \cdot \mathcal{C}_{n-1,2}$. The remaining equations in this block are independent. The pivot terms in the equations are

▷ for i, $i < n$: $D_{\underline{n}n\underline{ii}}, D_{\underline{n}i\underline{ii}}$,
▷ for (i, j), $i < j < n$: $D_{n\underline{ijj}}, D_{n\underline{i}\,\underline{jj}}, D_{j\underline{ijj}}, D_{i\underline{ijj}}, D_{\underline{in}i\underline{j}}, D_{\underline{jn}j\underline{i}}, D_{\underline{in}j\underline{j}}, D_{\underline{jn}i\underline{i}}, D_{\underline{n}i\underline{jj}}, D_{\underline{n}j\underline{ii}}, D_{\underline{j}n\underline{ii}}, D_{\underline{j}i\underline{ii}}$,
▷ for (i, j, k) pairwise different, $i, j, k < n$, $i < j$: $D_{\underline{i}\,\underline{j}i\underline{k}}$, $D_{\underline{i}\,\underline{j}j\underline{k}}$, $D_{\underline{i}\,\underline{j}k\underline{k}}$, $D_{k\underline{i}\,\underline{jj}}$, $D_{\underline{k}k\underline{ii}}$,
▷ for (i, j, k, l) pairwise different, $i < l < n$, $j < k$: $D_{\underline{jk}l\underline{i}}$,
therefore in this block we have $N_2 = 2\,\mathcal{C}_{n-1,1} + 12\,\mathcal{C}_{n-1,2} + 5\,\mathcal{C}_{n-1,2}\,\mathcal{C}_{n-3,1} + \mathcal{C}_{n-1,2}\,\mathcal{C}_{n-2,2}$ independent equations.

Equations $\mathcal{E}_{ij\underline{kl}} = 0$. For i, j, k pairwise different, $i, k \neq n$, $i < k$ we have the relation $\mathcal{E}_{in\underline{jk}} = \mathcal{E}_{kn\underline{ji}}$ and for $i < j < l < n$, $i, j, k \neq l$ we have $\mathcal{E}_{ij\underline{kl}} = \mathcal{E}_{il\underline{kj}} - \mathcal{E}_{jl\underline{ki}}$. Therefore we have $\mathcal{C}_{n-1,2}\mathcal{C}_{n-2,1} + \mathcal{C}_{n-1,3}\mathcal{C}_{n-3,1}$ relations between the equations of this block. The remaining equations are independent. The pivot terms are:

▷ for i, $i < n$: $D_{n\underline{n}\,\underline{ii}}$, $D_{in\underline{ii}}$,

▷ for (i,j), $i<j<n$: $D_{n\underline{i}i\underline{j}}$, $D_{n\underline{j}j\underline{i}}$, $D_{n\underline{i}j\underline{j}}$, $D_{n\underline{j}i\underline{i}}$, $D_{j\underline{n}i\underline{i}}$, $D_{\underline{n}ij\underline{j}}$, $D_{in\underline{jj}}$, $D_{jn\underline{ii}}$, $D_{\underline{i}ij\underline{j}}$, $D_{\underline{j}ij\underline{i}}$, $D_{ij\underline{jj}}$, $D_{ij\underline{ii}}$, $D_{\underline{n}ni\underline{j}}$,
▷ for (i,j,k) pairwise different, $i,j,k<n$, $i<j$: $D_{j\underline{k}i\underline{i}}$, $D_{\underline{k}ij\underline{j}}$, $D_{ij\underline{kk}}$, $D_{\underline{i}ij\underline{k}}$, $D_{\underline{j}ij\underline{k}}$, $D_{n\underline{k}j\underline{i}}$,
▷ for (i,j,k,l) pairwise different, $i<j<k<n$: $D_{\underline{l}jk\underline{i}}$, $D_{\underline{l}ik\underline{j}}$,
therefore in this block we have $N_3 = 2\,\mathcal{C}_{n-1,1} + 13\,\mathcal{C}_{n-1,2} + 6\,\mathcal{C}_{n-1,2}\,\mathcal{C}_{n-3,1} + 2\,\mathcal{C}_{n-1,3}\,\mathcal{C}_{n-3,1}$ independent equations.

Equations $\mathcal{E}_{ijk\underline{l}} = 0$. The following relations hold between the equations of this block: if $i<j<k<n$, then $\mathcal{E}_{ijn\underline{k}} = \mathcal{E}_{ikn\underline{j}} + \mathcal{E}_{kjn\underline{i}}$, and $\mathcal{E}_{ijk\underline{l}} = \mathcal{E}_{ijl\underline{k}} + \mathcal{E}_{jkl\underline{i}} - \mathcal{E}_{ikl\underline{j}}$ for $k,l \neq n$, $i<j<k<l$. The number of these relations is $\mathcal{C}_{n,3} + \mathcal{C}_{n-1,4}$. The remaining equations are independent. The pivot terms are:
▷ for (i,j), $i<j<n$: $D_{jni\underline{i}}$, $D_{nij\underline{j}}$,
▷ for (i,j,k) where $i<j<k<n$: $D_{jki\underline{i}}$, $D_{kij\underline{j}}$, $D_{ijk\underline{k}}$, $D_{nik\underline{j}}$, $D_{nkj\underline{i}}$,
▷ for (i,j,k,l) where $i<j<k<l<n$: $D_{ijl\underline{k}}$, $D_{jkl\underline{i}}$, $D_{ijk\underline{l}}$,
therefore in this block we have $N_4 = 2\,\mathcal{C}_{n-1,2} + 5\,\mathcal{C}_{n-1,3} + 3\,\mathcal{C}_{n-1,3}$ independent equations.

Equations $\mathcal{E}_{ij\underline{k}l} = 0$. We have the relations $\mathcal{E}_{ij\underline{l}k} = \mathcal{E}_{ik\underline{l}j} - \mathcal{E}_{jk\underline{l}i} + \mathcal{E}_{ijk\underline{l}}$ for $i<j<k$. The remaining equations are independent. The pivot terms are:
▷ for (i,j,k) where $i<j$: $D_{j\underline{k}ii}$, $D_{\underline{k}ijj}$,
▷ for (i,j,k,l) where $i<j<k$: $D_{\underline{l}ikj}$, $D_{\underline{l}jki}$,
therefore this block adds $N_5 = 2\,\mathcal{C}_{n,2}\,\mathcal{C}_{n,1} + 2\,\mathcal{C}_{n,3}\,\mathcal{C}_{n,1}$ independent equations.

Equations $\mathcal{E}_{i\underline{jk}l} = 0$. The following relations hold between the equations of the type $\mathcal{E}_{i\underline{j}\underline{k}l} = 0$ and $\mathcal{E}_{ij\underline{kl}} = 0$: for $j<k$ and $i \neq l$ we have $\mathcal{E}_{i\underline{jk}l} = \mathcal{E}_{l\underline{jk}i} + \mathcal{E}_{il\underline{kj}} - \mathcal{E}_{il\underline{jk}}$. The pivot terms are
▷ for (i,j,k) where $i<j$: $D_{\underline{jk}ii}$,
▷ for (i,j,k,l) where $i<j$, $k<l$: $D_{\underline{kl}ij}$,
therefore this block adds $N_6 = \mathcal{C}_{n,2}\,\mathcal{C}_{n,1} + \mathcal{C}_{n,2}\,\mathcal{C}_{n,2}$ independent equations to the previous.

Equations $\mathcal{E}_{\underline{i}\underline{j}\underline{k}l} = 0$. These equations can be expressed with the equations $\mathcal{E}_{i\underline{j}\underline{kl}} = 0$ since we have $\mathcal{E}_{\underline{ijk}l} = \mathcal{E}_{l\underline{ijk}} + \mathcal{E}_{l\underline{kij}} + \mathcal{E}_{l\underline{jki}}$.

The above calculation shows that the system (5.20) contains

$$N = \sum_{i=1}^{6} N_i = \frac{31}{24}n^4 + \frac{7}{12}n^3 - \frac{175}{24}n^2 + \frac{113}{12}n - 4 \tag{5.21}$$

independent equations and taking into consideration (5.15) and (5.21) we obtain that

$$\dim \operatorname{Ker}\delta_2^2 = \dim(\Lambda^2 T^* \otimes g_2(P_3)) - (N_0 + N) = \frac{7}{12}n^4 + \frac{2}{3}n^3 + \frac{53}{12}n^2 - \frac{26}{3}n + 4. \tag{5.22}$$

Comparing (5.13) and (5.22) we have $\operatorname{rank} \delta_1^2 < \dim \operatorname{Ker} \delta_2^2$. More precisely we have

$$\dim H^{2,2}(P_3) = \dim \left(\operatorname{Ker} \delta_2^2 / \operatorname{Im} \delta_1^2 \right) = \frac{(n-1)(n-2)}{2},$$

which proves Theorem 5.1. □

Remark 5.2. In order to solve the projective metrizability problem in the non-isotropic case, one has to consider the second order partial differential system P_3 containing the extended Rapcsák system and the curvature conditions. As Theorem 5.1 shows, this system is not 2-acyclic, which means that the integrability condition of the prolonged system is not the prolongation of the integrability conditions. More precisely, there are $\frac{1}{2}(n-1)(n-2)$ extra obstructions to lift a third order solution into a fourth order solution. We can expect the same phenomenon for the system enlarged with the integrability condition of P_3 too: second, third and possibly higher order integrability conditions may arise. The Spencer generalization of the Cartan-Kähler integrability theory is particularly well adapted to deal with such systems, therefore it can be the proper tool of further investigation.

References

[1] *S. Bácsó, Z. Szilasi*: On the projective theory of sprays. Acta Math. Acad. Paedagog. Nyházi. (N.S.) (electronic only) *26* (2010), 171–207. zbl MR

[2] *R. L. Bryant, S. S. Chern, R. B. Gardner, H. L. Goldschmidt, P. A. Griffiths*: Exterior Differential Systems. Mathematical Sciences Research Institute Publications 18, Springer, New York, 1991. zbl MR doi

[3] *I. Bucataru, T. Milkovszki, Z. Muzsnay*: Invariant metrizability and projective metrizability on Lie groups and homogeneous spaces. Mediterr. J. Math. (2016), 4567–4580. zbl MR doi

[4] *I. Bucataru, Z. Muzsnay*: Projective metrizability and formal integrability. SIGMA, Symmetry Integrability Geom. Methods Appl. (electronic only) *7* (2011), Paper 114, 22 pages. zbl MR doi

[5] *I. Bucataru, Z. Muzsnay*: Projective and Finsler metrizability: parameterization-rigidity of the geodesics. Int. J. Math. *23* (2012), 1250099, 15 pages. zbl MR doi

[6] *M. Crampin*: Isotropic and R-flat sprays. Houston J. Math. *33* (2007), 451–459. zbl MR

[7] *M. Crampin*: On the inverse problem for sprays. Publ. Math. *70* (2007), 319–335. zbl MR

[8] *M. Crampin*: Some remarks on the Finslerian version of Hilbert's fourth problem. Houston J. Math. *37* (2011), 369–391. zbl MR

[9] *M. Crampin, T. Mestdag, D. J. Saunders*: The multiplier approach to the projective Finsler metrizability problem. Differ. Geom. Appl. *30* (2012), 604–621. zbl MR doi

[10] *M. Crampin, T. Mestdag, D. J. Saunders*: Hilbert forms for a Finsler metrizable projective class of sprays. Differ. Geom. Appl. *31* (2013), 63–79. zbl MR doi

[11] *T. Do, G. Prince*: New progress in the inverse problem in the calculus of variations. Differ. Geom. Appl. *45* (2016), 148–179. zbl MR doi

[12] *A. Frölicher, A. Nijenhuis*: Theory of vector-valued differential forms. I: Derivations in the graded ring of differential forms. Nederl. Akad. Wet., Proc., Ser. A. *59* (1956), 338–359. zbl MR

[13] *J. Grifone, Z. Muzsnay*: Variational Principles for Second-Order Differential Equations. Application of the Spencer Theory to Characterize Variational Sprays. World Scientific Publishing, Singapore, 2000. zbl MR doi
[14] *J. Klein, A. Voutier*: Formes extérieures génératrices de sprays. Ann. Inst. Fourier *18* (1968), 241–260. (In French.) zbl MR doi
[15] *V. S. Matveev*: On projective equivalence and pointwise projective relation of Randers metrics. Int. J. Math. *23* (2012), 1250093, 14 pages. zbl MR doi
[16] *T. Mestdag*: Finsler geodesics of Lagrangian systems through Routh reduction. Mediterr. J. Math. *13* (2016), 825–839. zbl MR doi
[17] *Z. Muzsnay*: The Euler-Lagrange PDE and Finsler metrizability. Houston J. Math. *32* (2006), 79–98. zbl MR
[18] *A. Rapcsák*: Über die bahntreuen Abbildungen metrischer Räume. Publ. Math. *8* (1961), 285–290. (In German.) zbl MR
[19] *W. Sarlet, G. Thompson, G. E. Prince*: The inverse problem of the calculus of variations: The use of geometrical calculus in Douglas's analysis. Trans. Am. Math. Soc. *354* (2002), 2897–2919. zbl MR doi
[20] *Z. Shen*: Differential Geometry of Spray and Finsler Spaces. Kluwer Academic Publishers, Dordrecht, 2001. zbl MR doi
[21] *J. Szilasi, R. L. Lovas, D. C. Kertész*: Connections, Sprays and Finsler Structures. World Scientific Publishing, Hackensack, 2014. zbl MR
[22] *J. Szilasi, S. Vattamány*: On the Finsler-metrizabilities of spray manifolds. Period. Math. Hung. *44* (2002), 81–100. zbl MR doi

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