

On integrable boundaries in the 2 dimensional $O(N)$ σ -models

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Abstract

We make an attempt to map the integrable boundary conditions for 2 dimensional non-linear $O(N)$ σ -models. We do it at various levels: classically, by demanding the existence of infinitely many conserved local charges and also by constructing the double row transfer matrix from the Lax connection, which leads to the spectral curve formulation of the problem; at the quantum level, we describe the solutions of the boundary Yang-Baxter equation and derive the Bethe-Yang equations. We then show how to connect the thermodynamic limit of the boundary Bethe-Yang equations to the spectral curve.

1 Introduction

Integrable quantum field theories are useful toy examples of particle physics. Their popularity is due to the fact that many physical quantities can be calculated exactly and, despite their simplicity, they exhibit phenomena relevant for QCD. In particular, 2 dimensional (2D) $O(N)$ σ -models are asymptotically free in perturbation theory and their classical conformal invariance is broken by a dynamically generated mass scale Λ . Massive excitations form the vector multiplet of the $O(N)$ group with factorized scattering [1], which makes possible to calculate the relation between the mass m and the parameter Λ [2].

The $O(N)$ σ -models are also relevant from the AdS/CFT point of view. In a large class of integrable string σ -models strings propagate on the product of an anti-de Sitter space and spheres S^n [3]. Light-cone gauge fixed string theories on the sphere part are described classically by the $O(N)$ σ -models. In the string theory applications we are often interested in open strings, strings ending on some D-brane submanifolds of S^N [4]. This translates to $O(N)$ models with boundaries, and an important question is to classify those boundary conditions which maintain integrability. This is the motivation of our work.

Interestingly, there are not many papers analyzing integrable boundary conditions for $O(N)$ models. Soon after the seminal paper of Ghoshal and Zamolodchikov on integrable boundaries [5] Ghoshal determined the solutions of the boundary Yang-Baxter equation (BYBE) in the $O(N)$ σ -models with diagonal reflections having $O(N)$ and $O(N-1)$ symmetries [6]. These reflection factors correspond to free and fixed boundary conditions for the fundamental fields. Later

Corrigan and Sheng established the classical integrability of the free boundary condition by constructing infinitely many conserved charges via the Lax connection [7]. They also found a new (field dependent) boundary condition in the $O(3)$ model. Using the boundary generalizations of the Goldschmidt-Witten argument, Moriconi and de Martino [8] indicated that free and fixed boundary conditions can be quantum integrable and even extended the result for a mixture of free and fixed boundary conditions (for the boundary value of the fundamental field). Later Moriconi analyzed systematically the boundary conditions of the $O(N)$ models [9, 10]. He identified new types of integrable boundary conditions, which can be implemented by adding a quadratic boundary potential including the time derivative of the fundamental field to the Lagrangian. He managed to transform the $O(3)$ boundary condition found by Corrigan and Sheng to his class. These boundary conditions can be represented by an antisymmetric matrix, which can be brought into a 2 by 2 block-diagonal form. Thus they break the $O(N)$ symmetry to the products of $O(2)$ s. He then searched for the quantum analogues of this new class of boundary conditions and found a few non-diagonal representatives only. Namely only with a single block and Dirichlet boundary conditions, or in $O(2N)$ models with all 2 by 2 block being the same. This classification was confirmed and extended in the $O(4)$ case to a two parameter family of reflection factors in [11]. It was argued in [12] that the non-diagonal boundary conditions does not have a consistent Hamiltonian description, although, as we will show the constraint was not properly implemented in the Lagrangian description. The open boundary integrability in the string theory, relevant for the $O(N)$ models, was analyzed in [13, 14, 15]. It seemed from the investigations that not all combinations of free and fixed boundary conditions are compatible with Lax integrability. Similar conclusion was drawn by investigating integrable boundary conditions in coset theories [16, 17]. The aim of our paper is to investigate the integrability of boundary conditions at various levels: Lagrangian, Lax, quantum, trying to map them as completely as possible and to establish their relations.

The rest of this paper is organized as follows: In section 2 we analyze the integrability at the classical level. We start by reviewing the construction of conserved charges in the periodic case. We use three different descriptions: in the first we use the stereographically projected coordinates on S^{N-1} , in the second we use the embedding coordinates in $\mathbb{R}^N \supset S^{N-1}$, while in the third we consider the sphere S^{N-1} as a coset $\frac{SO(N)}{SO(N-1)}$ theory. In the second part of section 2 we use these descriptions to map the integrable boundary conditions in the model, while also showing the equivalence of the various descriptions. At the end of this section we formulate the boundary integrability at the language of the Lax connection. We construct the double row transfer matrix and use its eigenvalues to define the spectral curve. We analyze the analytical and symmetry properties of this curve, which provides an alternative way to find and classify classical solutions and also helps in quantizing the model. We close the section by explicitly constructing the spectral curve of some rotating string solutions in the $O(4)$ model. Section 3 is devoted to the quantum theory. Integrable boundary conditions at the quantum level are characterized by reflection matrices, which solve the boundary Yang-Baxter equations and satisfy unitarity and boundary crossing unitarity. We systematically describe these reflection matrices and use them to derive the Bethe-Yang equations, which provide the asymptotic spectrum on a large interval. At the end of the section we calculate the classical limit of the spectrum for two boundary conditions in the $O(4)$ model and reproduce the classical spectral curve. We conclude in section 4, while technical details are relegated to two appendices.

2 Classical integrability

In this section we investigate the classical integrability of the $O(N)$ σ -model in the presence of boundaries. We start by introducing three different descriptions in the bulk theory, then following the same descriptions for the theory with boundaries.

2.1 Bulk formulations

Here we recall the bulk formulations using unconstrained and constrained variables on the sphere. We finish by regarding the sphere as a coset.

2.1.1 Unconstrained fields

The $O(N)$ σ -model is a 2D field theory of a variable which lives on an $N - 1$ -dimensional sphere. This sphere is naturally given by the unit sphere in \mathbb{R}^N and can be projected stereographically onto the hyperplane passing through the origin. Let us denote the corresponding projected coordinates by $\vec{\xi} = (\xi^1, \dots, \xi^{N-1})$. The Lagrangian of the model is given by

$$\mathcal{L} = 4 \frac{\partial_\alpha \vec{\xi} \cdot \partial^\alpha \vec{\xi}}{(1 + \xi^2)^2} \quad ; \quad \xi^2 = \vec{\xi} \cdot \vec{\xi}, \quad (1)$$

with the metric being the pullback of the flat metric in \mathbb{R}^N . Variation of the action gives the equations of motion:

$$\partial_\mu \partial^\mu \xi^j - \frac{4 \partial^\mu \xi^j \vec{\xi} \cdot \partial_\mu \vec{\xi} - 2 \xi^j \partial_\mu \vec{\xi} \cdot \partial^\mu \vec{\xi}}{1 + \xi^2} = 0. \quad (2)$$

The model has an explicit $O(N - 1)$ symmetry, as the currents

$$J_\alpha^{ij} = \frac{4}{(1 + \xi^2)^2} (\xi^i \partial_\alpha \xi^j - \xi^j \partial_\alpha \xi^i) \quad (3)$$

are conserved on shell. In fact, the model has an implicit $O(N)$ symmetry. The extra conserved symmetry currents are

$$J_\alpha^{iN} = -J_\alpha^{Ni} = -\frac{2}{(1 + \xi^2)^2} (2 \xi^i \vec{\xi} \cdot \partial_\alpha \vec{\xi} + (1 - \xi^2) \partial_\alpha \xi^i), \quad (4)$$

and correspond to the infinitesimal transformations

$$\xi^j = \xi^j + \epsilon (2 \xi^j \xi^i + (1 - \xi^2) \delta^{ij}), \quad j = 1, \dots, N - 1.$$

These transformations change ξ^2 by $\delta \xi^2 = 2 \epsilon \xi^i (1 + \xi^2)$.

The energy-momentum tensor of the model is traceless, and its lightcone components are

$$T_{++} = \frac{4}{(1 + \xi^2)^2} \sum_i (\partial_+ \xi^i)^2 = - \sum_{I, J=1}^N J_+^{IJ} J_+^{JI},$$

and similarly for T_{--} . Here we introduced light-cone coordinates $\sigma_\pm \equiv \frac{1}{2} (\tau \pm \sigma)$ and $\partial_\pm \equiv \partial_\tau \pm \partial_\sigma$. Conservation of the currents implies that

$$\partial_- T_{++} = 0 \rightarrow T_{++}(\sigma^+) \quad ; \quad \partial_+ T_{--} = 0 \rightarrow T_{--}(\sigma^-).$$

Thus, σ -models are classically conformal and possess infinitely many conservation laws:

$$\partial_- T_{++}^k = 0 \quad ; \quad \partial_+ T_{--}^k = 0 \quad ; \quad k > 2. \quad (5)$$

Indeed, the conserved currents associated to the spin $\pm s$ charges $Q_{\pm s}$ must satisfy

$$\partial_- T^{(s+1)} = \partial_+ \Theta^{(s-1)} \quad ; \quad \partial_- \bar{T}^{(s+1)} = \partial_+ \bar{\Theta}^{(s-1)} \quad (6)$$

Thus choosing $T^{(2s)} = T_{++}^k$, $\Theta^{(2s-2)} = 0$ and $\bar{T}^{(2s)} = T_{--}^k$, $\bar{\Theta}^{(2s-2)} = 0$ leads to conserved higher spin charges

$$Q_{2k-1} = \int T_{++}^k(\sigma, \tau) d\sigma \quad ; \quad Q_{-2k+1} = \int T_{--}^k(\sigma, \tau) d\sigma \quad ; \quad k > 1 \quad (7)$$

which are algebraically independent of the stress tensor (whose charges are given by $k = 1$). In the $O(N)$ σ -models there are higher polynomial expressions of the currents leading to higher spin conserved charges [18]. They are related to Casimirs of the $O(N)$ group and form a Poisson commuting set. In this paper we do not study the Poisson structure and focus only on the charges related to the conformality of the model, i.e. to $T_{\pm\pm}^k$.

2.1.2 Constrained fields

To make the full $O(N)$ symmetry manifest we can use coordinates on \mathbb{R}^N :

$$n^i = \frac{2\xi^i}{1 + \xi^2}, \quad i = 1, \dots, N-1, \quad n^N = \frac{1 - \xi^2}{1 + \xi^2}. \quad (8)$$

They parametrize the unit sphere as $\mathbf{n}^t = (n^1, \dots, n^N) \in \mathbb{R}^N$ with the constraint $\mathbf{n}^t \mathbf{n} = 1$. This constraint has to be included in the Lagrangian to maintain equivalence with eq. (1):

$$\mathcal{L} = \partial_a \mathbf{n}^t \partial^a \mathbf{n} - \lambda(\mathbf{n}^t \mathbf{n} - 1).$$

Variation of the action leads to the equation of motion

$$\partial_a \partial^a \mathbf{n} + \lambda \mathbf{n} = 0.$$

Using the constraint, the Lagrange multiplier can be eliminated leading to

$$\partial_a \partial^a \mathbf{n} + (\partial_a \mathbf{n}^t \partial^a \mathbf{n}) \mathbf{n} = 0,$$

which is equivalent to eq. (2) once the relation (8) is used. The conserved currents take the universal form

$$J_\alpha^{IJ} = n^I \partial_\alpha n^J - n^J \partial_\alpha n^I,$$

and the energy momentum tensor is given by

$$T_{\pm\pm} = \partial_\pm \mathbf{n} \cdot \partial_\pm \mathbf{n}.$$

From the unit vector \mathbf{n} one can define the group element

$$m = \mathbb{I} - 2\mathbf{n}\mathbf{n}^t$$

with $m^{-1} = m$, such that the current one-form reads as

$$J = m dm = -2(d\mathbf{n})\mathbf{n}^t + 2\mathbf{n} d\mathbf{n}^t. \quad (9)$$

In terms of this current one-form, the Lagrangian is simply

$$\mathcal{L} = \text{tr}(J_\alpha J^\alpha) = \text{tr}(J \wedge \star J).$$

The flatness of the current, together with its conservation

$$dJ + J \wedge J = 0 \quad ; \quad d\star J = 0,$$

can be packed into the flatness of a spectral parameter-dependent Lax connection:

$$a(\lambda) := \frac{1}{1 - \lambda^2} J - \frac{\lambda}{1 - \lambda^2} \star J. \quad (10)$$

These formulas together with (9) resemble the formulation of a coset theory. Indeed, S^{N-1} can be represented as an $O(N)/O(N-1)$ coset, which leads to the following description.

2.1.3 Gauged σ -model point of view

A map between the $O(N)/O(N-1) := \{g \sim gh | g \in O(N), h \in O(N-1)\}$ coset and the sphere $S^{N-1} := \{\mathbf{n} \in \mathbb{R}^N | \mathbf{n}^t \mathbf{n} = 1\}$ can be obtained by choosing a representative point $\mathbf{n}_0^t = \{1, 0, \dots, 0\}$ on S^{N-1} . The $O(N-1)$ subgroup, which leaves \mathbf{n}_0 invariant is the $(N-1) \times (N-1)$ lower right corner of $O(N)$, whose Lie algebra is denoted by \mathfrak{h} . The map between the coset and the sphere is simply

$$\frac{O(N)}{O(N-1)} \rightarrow S^{N-1} \quad ; \quad gh \rightarrow g\mathbf{n}_0.$$

The Maurer-Cartan form,

$$\omega = g^{-1}dg = A + K,$$

can be decomposed w.r.t. the coset structure as $\omega \in \mathfrak{o}(N) = \mathfrak{h} \oplus \mathfrak{f}$ with $A \in \mathfrak{h}, K \in \mathfrak{f}$, where K contains the physical degrees of freedom and A is a gauge field. By definition, this current satisfies the flatness condition:

$$d\omega + \omega \wedge \omega = 0.$$

Using the properties

$$[\mathfrak{h}, \mathfrak{h}] = \mathfrak{h} \quad ; \quad [\mathfrak{h}, \mathfrak{f}] = \mathfrak{f} \quad ; \quad [\mathfrak{f}, \mathfrak{f}] = \mathfrak{h},$$

the flatness condition for ω can be decomposed as

$$dA + A \wedge A + K \wedge K = 0; \tag{11}$$

$$dK + A \wedge K + K \wedge A = 0. \tag{12}$$

One can introduce the operators which project onto the \mathfrak{h} and \mathfrak{f} subspaces as follows:

$$\begin{aligned} \Pi_{\mathfrak{h}} : \mathfrak{o}(N) &\rightarrow \mathfrak{h} & ; & & v &\rightarrow \frac{1}{2}(v + jvj), \\ \Pi_{\mathfrak{f}} : \mathfrak{o}(N) &\rightarrow \mathfrak{f} & ; & & v &\rightarrow \frac{1}{2}(v - jvj), \end{aligned}$$

where $j = \mathbb{I} - 2\mathbf{n}_0\mathbf{n}_0^t$. The gauge invariant Lagrangian takes the form:

$$\mathcal{L} = \text{tr} [\Pi_{\mathfrak{f}}(\omega_a)\Pi_{\mathfrak{f}}(\omega^a)] = \text{tr}(K_a K^a). \tag{13}$$

To obtain the equations of motion one can make the variations

$$g \rightarrow g(1 + \epsilon) \quad ; \quad \omega \rightarrow \omega + d\epsilon + [\omega, \epsilon].$$

where $\epsilon \in \mathfrak{f}$ since $\epsilon \in \mathfrak{h}$ would not change the action. This variation changes the action by

$$\delta\mathcal{L} = 2\text{tr}[(\partial_a\epsilon + [A_a, \epsilon])K^a] = -2\text{tr}[\epsilon(\partial_a K^a + [A_a, K^a])] + 2\partial_a\text{tr}[\epsilon K^a],$$

and leads to the equation of motion

$$d \star K + A \wedge \star K + \star K \wedge A = 0. \tag{14}$$

To make contact with the formulation of the constrained field \mathbf{n} we recall that

$$\mathbf{n} = g\mathbf{n}_0 \quad ; \quad m = jg g^t \quad ; \quad J = g(j\omega j - \omega)g^t = -2g\Pi_{\mathfrak{f}}(\omega)g^t = -2gKg^t \in g\mathfrak{f}g^t. \tag{15}$$

This makes the two formulations completely equivalent. Finally, we note that the equations of motion can be encoded into the flatness of a spectral parameter-dependent Lax connection:

$$L(\lambda) = A + \frac{\lambda^2 + 1}{\lambda^2 - 1}K - \frac{2\lambda}{\lambda^2 - 1} \star K. \tag{16}$$

2.2 Boundary formulations

Let us turn to the formulations of the boundary problem in the same order as they were analyzed in the bulk theory.

2.2.1 Unconstrained fields

Using the unconstrained fields, the boundary theory can be obtained by restricting the space coordinates to an open interval parametrized by $\sigma \in (0, \pi)$

$$S = \int d\tau \int_0^\pi d\sigma \frac{\partial_\alpha \vec{\xi} \cdot \partial^\alpha \vec{\xi}}{(1 + \xi^2)^2}. \quad (17)$$

Now, because of the boundaries, when computing the variations of the action $\xi^k \rightarrow \xi^k + \delta\xi^k$, $\xi^i \rightarrow \xi^i$ ($i \neq k$), the following surface terms arise

$$- \frac{2\delta\xi^k \partial_\sigma \xi^k}{(1 + \xi^2)^2} \Big|_0^\pi. \quad (18)$$

If there is no constraint at the boundary for ξ^k then there is no summation over k . (If there were any constraints, they should be added to the Lagrangian with a Lagrange multiplier). Assuming that there is no long-range interaction between the boundaries, the surface terms must vanish separately, i.e. we find the consistent boundary conditions (b.c.-s)

$$\delta\xi^k \partial_\sigma \xi^k = 0, \quad k = 1, \dots, N-1 \quad (19)$$

on both ends of the interval. If we interpret the conditions $\delta\xi^k|_0 = 0$ (or $\delta\xi^k|_\pi = 0$) as also implying the vanishing of $\partial_\tau \xi^k$, then we conclude that the consistent b.c.-s imply either Neumann or Dirichlet b.c.-s for the fields ξ^i . Let us focus on the integrability of these b.c.-s. According to [5], the bulk conservation laws (6) lead to a boundary conserved quantity if, at the boundary, the difference

$$\left[T^{(s+1)} - \bar{T}^{(s+1)} + \bar{\Theta}^{(s-1)} - \Theta^{(s-1)} \right] \Big| = \frac{d\Sigma}{dt} \quad (20)$$

is a total time-derivative of some quantity Σ . (By the empty vertical line we mean to evaluate the expression at the boundary. When the boundary is not specified, then the statement is true for both boundaries.) In the most general case Σ can even depend on the dynamical fields. In this paper we restrict our analysis only to the cases when $\Sigma = 0$. Clearly, the boundary conditions (19) we found are conformal and guarantee the vanishing of $(T_{++}^k - T_{--}^k) \Big| \propto (T_{++} - T_{--}) \Big| \sim \frac{\partial_\tau \vec{\xi} \cdot \partial_\sigma \vec{\xi}}{(1 + \xi^2)^2} \Big| = 0$ for all k . This ensures the existence of infinitely many conserved higher spin charges, which are independent of the energy. Still it may happen that the infinite number of conserved charges following from the conformality of the boundary condition is not "infinite enough" to ensure integrability. In particular conserved charges should form a commuting family. It was shown in [17] that coset boundary conditions compatible with the bulk coset structure lead to infinite number of conserved charges in involution. Since in the framework of this paper we do not investigate either the Poisson structure or the higher Casimir charges we have no tool to check if conformality implies integrability or not.

Let us try to extend the Lagrangian with a boundary potential, which could preserve integrability of the model. Motivated by previous investigations [10], we add also a boundary Lagrangian term $\sum_{I,J} n^I M_{IJ} \dot{\xi}^J$ with an antisymmetric matrix $M_{IJ} = -M_{JI}$ (and where we used $\dot{\ } \equiv \partial_\tau$). Here capital indices I, J run from 1 to N , while lower case indices i, j run from 1 to $N-1$. One readily obtains

$$\sum_{I,J} n^I M_{IJ} \dot{\xi}^J = -\frac{1}{2} \text{tr}(M J_\tau) = \frac{2}{(1 + \xi^2)^2} \left(\sum_{ij} M_{ij} (\xi^i \dot{\xi}^j - \xi^j \dot{\xi}^i) - \sum_i (2\xi^i \vec{\xi} \cdot \dot{\vec{\xi}} + (1 - \xi^2) \dot{\xi}^i) M_{iN} \right).$$

We compute the change of this boundary piece under the $\xi^k \rightarrow \xi^k + \delta\xi^k$ variation, with the understanding that in every term containing $\delta\xi^k$ we integrate by parts and drop the integrated

terms. This way one finds

$$\begin{aligned} \delta \text{tr}(MJ_\tau) = & \frac{16\delta\xi^k}{(1+\xi^2)^2} \left(-\frac{\xi^k}{1+\xi^2} \left(\sum_{ij} M_{ij}(\xi^j \dot{\xi}^i - \xi^i \dot{\xi}^j) + 2 \sum_i M_{iN} \dot{\xi}^i \right) + \right. \\ & \left. \sum_i M_{ik} \dot{\xi}^i + \frac{2\vec{\xi} \cdot \dot{\vec{\xi}}}{1+\xi^2} (M_{kN} + \sum_j M_{kj} \xi^j) \right). \end{aligned}$$

Combining this with the surface terms (18) coming from the variation of the bulk action, one finds the boundary conditions ($k = 1, \dots, N-1$)

$$\begin{aligned} \partial_\sigma \xi^k | = & - \left(-\frac{\xi^k}{1+\xi^2} \left(\sum_{ij} M_{ij}(\xi^j \dot{\xi}^i - \xi^i \dot{\xi}^j) + 2 \sum_i M_{iN} \dot{\xi}^i \right) + \sum_i M_{ik} \dot{\xi}^i + \right. \\ & \left. \frac{2\vec{\xi} \cdot \dot{\vec{\xi}}}{1+\xi^2} (M_{kN} + \sum_j M_{kj} \xi^j) \right) \Big| . \end{aligned} \quad (21)$$

This boundary condition is conformal, as direct calculation guarantees that $\partial_\tau \vec{\xi} \cdot \partial_\sigma \vec{\xi} \propto (T_{++}^k - T_{--}^k) = 0$, providing infinitely many higher spin charges. Let us analyse the same boundary conditions in the alternative formulations.

2.2.2 Constrained fields

The action which corresponds to the theory (17) in terms of the constrained variable \mathbf{n} reads as

$$S = \int d\tau \int_0^\pi d\sigma \left[\partial_a \mathbf{n}^t \partial^a \mathbf{n} - \lambda(\mathbf{n}^t \mathbf{n} - 1) + \delta(\sigma) \lambda_0(\mathbf{n}^t \mathbf{n} - 1) - \delta(\sigma - \pi) \lambda_\pi(\mathbf{n}^t \mathbf{n} - 1) \right]. \quad (22)$$

Observe that we have implemented the constraint $\mathbf{n}^t \mathbf{n}$ also at the boundary. *All previous analysis seemed to miss this term.* Variation now leads to the bulk equation of motion and to the boundary condition:

$$\delta \mathbf{n}^t (\partial_\sigma \mathbf{n} - \lambda | \mathbf{n}) = 0.$$

Thus for any i we can choose either generalized Neumann or Dirichlet boundary conditions. We shall assume that l directions satisfy generalized Neumann boundary condition while $N-l$ directions obey Dirichlet instead.

$$\partial_\sigma n_i |_0 = \lambda_0 n_i |_0 \quad ; \quad i = 1, \dots, l \quad ; \quad \delta n_i |_0 = 0 \quad ; \quad i = l+1, \dots, N.$$

All these boundary conditions are conformal

$$(T_{++} - T_{--}) | \sim \partial_\tau \mathbf{n}^t \cdot \partial_\sigma \mathbf{n} = 0.$$

and conformality also guarantees that

$$(T_{++}^k - T_{--}^k) | \propto (T_{++} - T_{--}) | = 0, \quad (23)$$

thus infinitely many higher spin conserved charges exist.

Without loss of generality, we can choose the Dirichlet directions as

$$n_{l+1} |_0 = \alpha \quad ; \quad n_{l+2} |_0 = \dots = n_N |_0 = 0.$$

This implies that $\sum_{i=1}^l n_i n_i |_0 = 1 - \alpha^2$, which can be used to determine λ_0 and obtain the boundary condition for the generalized Neumann directions:

$$\partial_\sigma n_i |_0 = \frac{n_i}{1 - \alpha^2} \sum_{j=1}^l n_j \partial_\sigma n_j |_0. \quad (24)$$

These boundary conditions are equivalent to Dirichlet and Neumann boundary conditions for appropriately rotated ξ_i variables, see Appendix A for the details.

A special case is when $\alpha = 0$, i.e. we restrict the boundary field to a sphere of maximal radius. This can be obtained by intersecting the unit sphere with a hyperplane passing through the origin. This boundary condition is given by the $\xi_i = 0$ Dirichlet boundary conditions. Actually, in this case the space-derivative of the constraint $\mathbf{n}^t \mathbf{n} = 1$ implies that

$$\sum_{i=1}^l n_i \partial_\sigma n_i \Big|_0 = 0 \quad \longrightarrow \quad \partial_\sigma n_i \Big|_0 = 0 \quad ; \quad i = 1, \dots, l,$$

and thus the remaining directions satisfy Neumann boundary condition. We analyze the symmetry of this boundary condition in detail in Appendix A. It turns out that the symmetry is $O(l) \times O(N-l)$.

In order to get the most general conformal boundary condition, we could demand that the time and space derivatives of \mathbf{n} are orthogonal at the boundary: $\partial_\tau \mathbf{n}^t \cdot \partial_\sigma \mathbf{n} = 0$. This can be achieved by adding a boundary potential with an antisymmetric matrix M to the boundary Lagrangian:

$$\mathcal{L}_b = \delta(\sigma) \left(\mathbf{n}^t M \partial_\tau \mathbf{n} + \lambda_0 (\mathbf{n}^t \mathbf{n} - 1) \right),$$

where, for definiteness, we added it at the $\sigma = 0$ boundary. Similar terms could be added at $\sigma = \pi$ as well. Here we again emphasize that the constraint had to be added to the boundary piece, as required by consistency. After eliminating the Lagrange multiplier, the boundary condition turns out to be

$$M \partial_\tau \mathbf{n} - (\mathbf{n}^t M \partial_\tau \mathbf{n}) \mathbf{n} = \partial_\sigma \mathbf{n}. \quad (25)$$

Contracting with \mathbf{n}^t on the left we can see that this is indeed consistent with the constraint $\mathbf{n}^t \mathbf{n} = 1$, in contrast to what one can find in the literature, where the b.c. appears without the second term [10, 12]. Using eq. (8), it is straightforward to show that the boundary condition in eq. (21), given in terms of the unconstrained variables, is equivalent to this one in terms of the \mathbf{n} fields. Conformality of the boundary conditions follow from $\partial_\tau \mathbf{n}^t \cdot \partial_\sigma \mathbf{n} = 0$. It is also instructive to rewrite the boundary condition for the current and group elements. In terms of the current, the boundary term reads simply as

$$\mathcal{L}_b = \delta(\sigma) \text{tr} (J_\tau M).$$

Making an infinitesimal variation

$$m \rightarrow m(1 + \epsilon), \quad ; \quad J \rightarrow J + d\epsilon + [J, \epsilon],$$

with ϵ satisfying the constraint (i.e. it is an element of the gfg^t subspace), changes the bulk part of the action as

$$\delta \mathcal{L} = \text{tr} [(\partial_a \epsilon + [J_a, \epsilon]) J^a] = -\text{tr} [\epsilon (\partial_a J^a)] + \partial_a \text{tr} [\epsilon J^a],$$

while the boundary part changes as

$$\delta \mathcal{L}_b = \text{tr} \{ (\partial_\tau \epsilon + [J_\tau, \epsilon]) M \} = \text{tr} (\epsilon [M, J_\tau]) = \text{tr} (\epsilon [g \Pi_{\mathfrak{h}} (g^t M g) g^t, J_\tau]).$$

Therefore, the boundary condition is

$$J_\sigma = [g \Pi_{\mathfrak{h}} (g^t M g) g^t, J_\tau] = \frac{1}{2} ([M, J_\tau] + [m M m, J_\tau]).$$

This particularly nice boundary condition is explicitly conformal, as

$$(T_{++}^k - T_{--}^k) \Big| \sim (T_{++} - T_{--}) \Big| \sim \text{tr} (J_\sigma J_\tau)$$

We show in appendix A that this is equivalent to the boundary condition we got from the ξ variables.

This whole analysis can be also recovered from the gauge theory formulation, which follows.

2.2.3 Gauged σ -model point of view

Boundary conditions in the coset language are geometrical. In particular, in the case of integrable boundary conditions of principal chiral models, the group element is restricted to a coset [16, 17]. Thus, we first analyze the boundary conditions by restricting the boundary field to $S^{N-1} \rightarrow S^{N-k-1}$, where the spheres have radius 1 and there is no extra boundary Lagrangian term in the coset action. In the language of the \mathbf{n} this means

$$n_i = 0, \quad \text{where } i = N - k + 1, \dots, N.$$

We introduce new notation by decomposing \mathbf{n} as: $\mathbf{n} = \tilde{\mathbf{n}} + \hat{\mathbf{n}}$, with

$$\tilde{\mathbf{n}} = (n_1, \dots, n_{N-k}, 0, \dots, 0) \quad , \quad \hat{\mathbf{n}} = (0, \dots, 0, n_{N-k+1}, \dots, n_N),$$

The boundary conditions then become:

$$\partial_1 \tilde{\mathbf{n}} = 0 \quad ; \quad \partial_0 \hat{\mathbf{n}} = 0.$$

Let us derive this boundary condition using the coset language. At the boundary, the only non-zero variables are the $\tilde{\mathbf{n}}$ -s. Introduce an $O(N-k) \times O(k)$ subgroup of $O(N)$, which can be used for the parametrization of the S^{N-k-1} subspace. We will denote it by G_1 , and its Lie algebra by \mathfrak{g}_1 . Let us also denote the little subgroup of G_1 by H_1 and its Lie algebra by \mathfrak{h}_1 . The fields at the boundary can be parametrized by $S^{N-k-1} \equiv G_1/H_1 := \{g_1 \sim g_1 h_1 | g_1 \in G_1, h_1 \in H_1\}$ and $\tilde{\mathbf{n}} = g_1 \mathbf{n}_0$ where $g_1 \in G_1$. We will have to use a decomposition of \mathfrak{h} , \mathfrak{f} and \mathfrak{g}_1 :

$$\begin{aligned} \mathfrak{h} &= \mathfrak{h}_1 \oplus \mathfrak{h}_2, & \mathfrak{f} &= \mathfrak{f}_1 \oplus \mathfrak{f}_2, & \mathfrak{h}_1 \oplus \mathfrak{f}_1 &= \mathfrak{g}_1; \\ [\mathfrak{h}_1, \mathfrak{h}_1] &\subset \mathfrak{h}_1, & [\mathfrak{h}_1, \mathfrak{f}_1] &\subset \mathfrak{f}_1, & [\mathfrak{f}_1, \mathfrak{f}_1] &\subset \mathfrak{h}_1, & [\mathfrak{h}_1, \mathfrak{h}_2] &\subset \mathfrak{h}_2, & [\mathfrak{h}_1, \mathfrak{f}_2] &\subset \mathfrak{f}_2, \\ [\mathfrak{h}_2, \mathfrak{h}_2] &\subset \mathfrak{h}_1, & [\mathfrak{h}_2, \mathfrak{f}_1] &\subset \mathfrak{f}_2, & [\mathfrak{h}_2, \mathfrak{f}_2] &\subset \mathfrak{f}_1, & [\mathfrak{f}_1, \mathfrak{f}_2] &\subset \mathfrak{h}_2, & [\mathfrak{f}_2, \mathfrak{f}_2] &\subset \mathfrak{h}_1. \end{aligned}$$

The $\mathfrak{o}(N) = \mathfrak{g}$ algebra w.r.t. to the splitting above becomes:

$$\mathfrak{g} = \begin{pmatrix} 0 & \mathfrak{f}_1 & \mathfrak{f}_2 \\ \mathfrak{f}_1 & \mathfrak{h}_1 & \mathfrak{h}_2 \\ \mathfrak{f}_2 & \mathfrak{h}_2 & \mathfrak{h}_1 \end{pmatrix}.$$

This is a $\mathbb{Z}_2 \times \mathbb{Z}_2$ graded algebra with $\{\mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{f}_1, \mathfrak{f}_2\} \sim \{(0,0), (0,1), (1,0), (1,1)\}$. Thus, the two symmetric cosets G/H and G/G_1 are compatible, where $G = O(N)$.

We can observe that the physical currents live in the $\mathfrak{f}_1 \sim (1,0)$ subspace which is the even part of the G/H and the odd part of the G/G_1 decompositions. Therefore "coset of the bulk" and "coset of the boundary" mean different things.

The decomposition of the current at the boundary is

$$\omega = g^t dg = A^{(1)} + A^{(2)} + K^{(1)} + K^{(2)},$$

where $A^{(i)} \in \mathfrak{h}_1$ and $K^{(i)} \in \mathfrak{f}_1$.

At the boundary, $g \in G_1$, so $A_\tau^{(2)} = K_\tau^{(2)} = 0$. This is equivalent to the boundary conditions

$$[A_\tau, \kappa] = 0 \quad ; \quad [K_\tau, \kappa] = 0,$$

where

$$\kappa = \text{diag}(1, \dots, 1, -1, \dots, -1).$$

When we make the variation of the action (13), we have to use $g \rightarrow g(1 + \epsilon)$ with $\epsilon \in \mathfrak{g}_1$ at the boundary. After the variation we get the $K_\sigma^{(1)} = 0$ boundary conditions, which are equivalent to

$$\{K_\sigma, \kappa\} = 0.$$

Let us now add a gauge-invariant boundary term to the Lagrangian, of the form

$$\mathcal{L}_b = \text{tr}[\Pi_{\mathfrak{f}}(\omega_\tau)\Pi_{\mathfrak{f}}(g^t M g)],$$

where $M \in \mathfrak{o}(n)$ is a constant matrix. After the variation we get:

$$\delta\mathcal{L}_b = \text{tr}\{(\partial_\tau\epsilon + [A_\tau, \epsilon])\Pi_{\mathfrak{f}}(g^t M g) + K_\tau[\Pi_{\mathfrak{h}}(g^t M g), \epsilon]\} = 2\text{tr}(\epsilon[K_\tau, \Pi_{\mathfrak{h}}(g^t M g)]) + \partial_\tau\text{tr}(\epsilon\Pi_{\mathfrak{f}}(g^t M g)).$$

Using also terms from the variation of the bulk part of the action, we arrive at the boundary condition:

$$K_\sigma = [\Pi_{\mathfrak{h}}(g^t M g), K_\tau].$$

This boundary condition is conformal as

$$(T_{++} - T_{--})| \propto \text{tr}(K_\sigma K_\tau) = 0.$$

In terms of components we have

$$A_0^{(2)} = 0 \quad ; \quad K_0^{(2)} = 0 \quad ; \quad K_1^{(1)} = \Pi_{\mathfrak{g}_1}([\Pi_{\mathfrak{h}}(g^t M g), K_0]).$$

This result is consistent with the result obtained in the language of the constrained fields \mathbf{n} :

$$\partial_\tau \hat{\mathbf{n}} = 0 \quad ; \quad \partial_\sigma \tilde{\mathbf{n}} = M \partial_\tau \tilde{\mathbf{n}} - (\tilde{\mathbf{n}}^t M \partial_\tau \tilde{\mathbf{n}}) \tilde{\mathbf{n}}.$$

2.3 Lax connection and spectral curve

In this section we construct Lax matrices for the boundary problem, which lead to its spectral curve formulation.

2.3.1 Bulk transfer matrix

We have already mentioned two different versions of the Lax connections. In the formulation based on the fundamental field it was defined by

$$\mathfrak{a}(\lambda) := \frac{1}{1-\lambda^2} J - \frac{\lambda}{1-\lambda^2} \star J,$$

while in the gauged σ -model formulation we found

$$\mathfrak{L}(\lambda) = \mathfrak{A} + \frac{\lambda^2 + 1}{\lambda^2 - 1} \mathfrak{K} - \frac{2\lambda}{\lambda^2 - 1} \star \mathfrak{K},$$

where $\omega = g^t dg = \mathfrak{A} + \mathfrak{K}$. By recalling the relation

$$J = g(j\omega j - \omega)g^t = -2g\Pi_{\mathfrak{f}}(\omega)g^t = -2g\mathfrak{K}g^t,$$

we can see that the two connections are related by a ‘‘gauge’’ transformation:

$$\mathfrak{L}(\lambda) = \mathfrak{A} + \frac{\lambda^2 + 1}{\lambda^2 - 1} \mathfrak{K} - \frac{2\lambda}{\lambda^2 - 1} \star \mathfrak{K} = \omega - 2\left(\frac{1}{1-\lambda^2} \mathfrak{K} - \frac{\lambda}{1-\lambda^2} \star \mathfrak{K}\right) = g^t dg + g^t \mathfrak{a}(\lambda) g.$$

Consequently, gauge-invariant quantities can be easily expressed in any of these formulations. The usefulness of the Lax connection lies in the fact that one can generate from it an infinite family of conserved charges. One first defines the transport matrix

$$T(b, a, \lambda) = \mathcal{P}^{\overleftarrow{\text{exp}}}\left\{-\int_a^b d\sigma \mathfrak{a}_\sigma(\sigma, \lambda)\right\}, \quad (26)$$

for a path connecting a and b . The transport matrix of the other connection $\mathbf{L}(\lambda)$, can then be expressed as

$$\mathcal{P}\overleftarrow{\text{exp}}\left\{-\int_a^b d\sigma \mathbf{L}_\sigma(\sigma, \lambda)\right\} = g^t(b)T(b, a, \lambda)g(a) \quad (27)$$

In the cylindrical geometry (bulk theory) one can integrate the connection for a spacial non-contractible loop, such as for the path from 0 to 2π , leading to a gauge-invariant quantity, $T(2\pi, 0, \lambda)$, called the monodromy matrix. Using the flatness condition of the connection \mathbf{a}_a : $\partial_\tau \mathbf{a}_\sigma = \partial_\sigma \mathbf{a}_\tau - [\mathbf{a}_\sigma, \mathbf{a}_\tau]$, we can calculate the time derivative of the transport matrix

$$\begin{aligned} \partial_\tau T(b, a, \lambda) &= -\int_a^b d\sigma \partial_\sigma \left(\mathcal{P}\overleftarrow{\text{exp}}\left\{-\int_\sigma^b d\sigma' \mathbf{a}_\sigma(\sigma', \lambda)\right\} \mathbf{a}_\tau(\sigma, \lambda) \mathcal{P}\overleftarrow{\text{exp}}\left\{-\int_a^\sigma d\sigma'' \mathbf{a}_\sigma(\sigma'', \lambda)\right\} \right) \\ &= T(b, a, \lambda) \mathbf{a}_\tau(a, \lambda) - \mathbf{a}_\tau(b, \lambda) T(b, a, \lambda). \end{aligned} \quad (28)$$

Since $\mathbf{a}_\tau(2\pi, \lambda) = \mathbf{a}_\tau(0, \lambda)$ this ensures that the trace of the monodromy matrix, called the transfer matrix, is time-independent

$$\mathbf{T}(\lambda) = \text{tr} \mathcal{P}\overleftarrow{\text{exp}}\left\{-\int_0^{2\pi} d\sigma \mathbf{a}_\sigma(\sigma, \lambda)\right\},$$

and generates infinitely many conserved charges. The bulk theory was thoroughly analysed in [19], in the context of AdS/CFT (see also [20] for the supersymmetric equivalent). Let us now turn to the parallel construction in the presence of boundaries.

2.3.2 Boundary transfer matrix

The monodromy matrix in the boundary case takes a double row type form [13, 14]

$$\Omega(\lambda) = U_0(\lambda) T_R(2\pi, \pi, \lambda) U_\pi(\lambda) T(\pi, 0, \lambda),$$

where $U_{0,\pi}(\lambda)$ are (as of yet λ and time-dependent) $O(N)$ matrices encoding the type of boundary conditions we have at $\sigma = 0, \pi$, respectively, and the matrix $T_R(2\pi, \pi, \lambda)$ is the reflected transport matrix, obtained via a parity transformation $\sigma \rightarrow 2\pi - \sigma$. Taking into account that parity transforms the currents as¹

$$J_\sigma(\sigma) \rightarrow J_\sigma^R(\sigma) = -J_\sigma(2\pi - \sigma) \quad ; \quad J_\tau(\sigma) \rightarrow J_\tau^R(\sigma) = J_\tau(2\pi - \sigma),$$

we can see that

$$\mathbf{a}_\sigma^R(\sigma, \lambda) = -\mathbf{a}_\sigma(2\pi - \sigma, -\lambda).$$

The reflected transport matrix is then given by

$$\begin{aligned} T_R(2\pi, \pi, \lambda) &\equiv \mathcal{P}\overleftarrow{\text{exp}}\left\{-\int_\pi^{2\pi} d\sigma \mathbf{a}_\sigma^R(\sigma, \lambda)\right\} = \mathcal{P}\overleftarrow{\text{exp}}\left\{-\int_\pi^0 d\sigma \mathbf{a}_\sigma(\sigma, -\lambda)\right\} \\ &= T(\pi, 0, -\lambda)^{-1}. \end{aligned} \quad (29)$$

This leads to the following form of the monodromy matrix

$$\Omega(\lambda) = U_0(\lambda) T(-\lambda)^{-1} U_\pi(\lambda) T(\lambda) \quad ; \quad T(\lambda) = T(\pi, 0, \lambda). \quad (30)$$

The most general condition of integrability can be written in terms of $\Omega(\lambda)$ as

$$\partial_\tau \Omega(\lambda) = [N(\lambda), \Omega(\lambda)] \quad (31)$$

¹The minus sign in the reflected current $J_\sigma^R(\sigma)$ comes from the fact that J_σ includes a derivative which under parity transforms as $\partial_\sigma \rightarrow -\partial_\sigma$.

with some appropriate $N(\lambda)$, as this condition guarantees that $\text{tr}(\Omega(\lambda)^k)$ is conserved for any integer k . Thus, expanding the boundary transfer matrix $T(\lambda) = \text{tr}(\Omega(\lambda))$ in the spectral parameter generates infinitely many conserved charges.

Using eq. (28) we can calculate the time derivative of the boundary monodromy matrix as

$$\begin{aligned} \partial_\tau \Omega(\lambda) &= \partial_\tau U_0(\lambda) U_0(\lambda)^{-1} \Omega(\lambda) - U_0(\lambda) \mathfrak{a}_\tau(0, -\lambda) U_0(\lambda)^{-1} \Omega(\lambda) + \Omega(\lambda) \mathfrak{a}_\tau(0, \lambda) + \\ &\quad + U_0(\lambda) T(-\lambda)^{-1} (\partial_\tau U_\pi(\lambda) - U_\pi(\lambda) \mathfrak{a}_\tau(\pi, \lambda) + \mathfrak{a}_\tau(\pi, -\lambda) U_\pi(\lambda)) T(\lambda). \end{aligned}$$

Demanding that

$$\begin{aligned} \partial_\tau U_0(\lambda) &= U_0(\lambda) \mathfrak{a}_\tau(0, -\lambda) - \mathfrak{a}_\tau(0, \lambda) U_0(\lambda); \\ \partial_\tau U_\pi(\lambda) &= U_\pi(\pi) \mathfrak{a}_\tau(\pi, \lambda) - \mathfrak{a}_\tau(\pi, -\lambda) U_\pi(\lambda), \end{aligned} \quad (32)$$

the time evolution of the monodromy matrix is given by

$$\partial_\tau \Omega(\lambda) = [\Omega(\lambda), \mathfrak{a}_\tau(0, \lambda)]. \quad (33)$$

Thus integrability requires the existence of matrices U_0 and U_π which satisfy eq. (32). These matrices are not gauge invariant. Indeed, using eq. (27) we can see that the boundary matrix U_i in the \mathbf{L} formulation is $\tilde{U}_i = g U_i g^t$ such that

$$[A_0, \tilde{U}] = 0 \quad ; \quad [K_0, \tilde{U}] = 0 \quad ; \quad \{K_1, \tilde{U}\} = 0. \quad (34)$$

This actually shows that constant U matrices in one description lead to time-dependent matrices in the other. Classification of all U matrices satisfying eq. (32) means the classification of the integrable boundary conditions in the Lax language. Let us start with the investigations of time-independent U -s. In terms of the currents, the time-independent restrictions become

$$\begin{aligned} U_0(\lambda) (J_\tau - \lambda J_\sigma) - (J_\tau + \lambda J_\sigma) U_0(\lambda)|_{\sigma=0} &= 0; \\ U_\pi(\lambda) (J_\tau + \lambda J_\sigma) - (J_\tau - \lambda J_\sigma) U_\pi(\lambda)|_{\sigma=\pi} &= 0. \end{aligned}$$

Assuming a Taylor expansion for the matrices $U_i(\lambda) \in O(N)$:

$$U_i(\lambda) = \sum_{n=0}^{+\infty} U_i^{(n)} \lambda^n, \quad (35)$$

we can solve the above restrictions order by order in powers of λ . We easily obtain

$$\begin{aligned} [U_i^{(0)}, J_\tau] \Big|_{\sigma=i} &= 0, \quad i = 0, \pi; \\ [U_0^{(k)}, J_\tau] \Big|_{\sigma=0} &= \{U_0^{(k-1)}, J_\sigma\} \Big|_{\sigma=0} \quad ; \quad [U_\pi^{(k)}, J_\tau] \Big|_{\sigma=\pi} = - \{U_\pi^{(k-1)}, J_\sigma\} \Big|_{\sigma=\pi}, \quad k \geq 1. \end{aligned} \quad (36)$$

In the principal chiral model we found λ dependent U - s, but in the $O(N)$ σ -model we could manage to find only constant ones. In this case, the requirements become

$$[U_i, J_\tau] \Big|_{\sigma=i} = \{U_i, J_\sigma\} \Big|_{\sigma=i}, \quad i = 0, \pi. \quad (37)$$

Using the definition of the currents J_\pm , it is easy to check that at the boundary we have

$$J_\pm \Big|_{\sigma=i} = U_i^{-1} J_\mp \Big|_{\sigma=i} U_i, \quad i = 0, \pi. \quad (38)$$

and that $[J_\pm, U_i^2] \Big|_{\sigma=i} = 0$, $i = 0, \pi$. Naturally, this condition is satisfied with the condition $U_i^2 = \mathbf{1}$. This actually means that $J_\pm \rightarrow \alpha(J_\pm) = U J_\pm U^{-1}$ defines an automorphism of the Lie algebra leaving the energy momentum tensor invariant: $T_{++} = \text{tr}(J_+^2) = T_{--} = \text{tr}(J_-^2)$. Automorphisms, which satisfy eq. (37), are well-known for Lie algebras and are related to their symmetric space decompositions [21]:

$$\mathfrak{o}(N) = \mathfrak{h} + \mathfrak{m} \quad ; \quad \alpha(\mathfrak{h}) = \mathfrak{h} \quad ; \quad \alpha(\mathfrak{m}) = -\mathfrak{m},$$

such that

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h} \quad , \quad [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m} \quad ; \quad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h} .$$

The various components of the currents at the boundary have to be in different subspaces

$$J_\tau| \in \mathfrak{h} \quad ; \quad J_\sigma| \in \mathfrak{m} ,$$

Choosing the decomposition

$$\mathfrak{h} = \mathfrak{so}(k) \oplus \mathfrak{so}(N-k) \quad ; \quad \mathfrak{m} = \frac{\mathfrak{so}(N)}{\mathfrak{so}(N) \oplus \mathfrak{so}(N-k)}$$

leads to the matrix

$$U = \text{diag}(\underbrace{1, \dots, 1}_k, \underbrace{-1, \dots, -1}_{N-k}) \quad (39)$$

The corresponding boundary condition is exactly what we can describe in all cases as

$$\partial_\sigma n_i| = 0 \quad ; \quad i = 1, \dots, k \quad ; \quad n_i| = 0 \quad ; \quad i = k+1, \dots, N ,$$

i.e. Neumann for the first k components and vanishing Dirichlet for the remaining $N-k$. This also means that the group element m is restricted to the coset $\frac{O(N)}{O(k) \times O(N-k)}$ and $[U, m] = 0$.

In the following we analyze the analytic properties of the eigenvalues of the monodromy matrix for this case (39).

2.4 Analytic properties of the boundary transfer matrix

In the following we assume that $U_0 = U_\pi = U$ is a λ independent constant such that $U^2 = 1$. By definition, the transport matrix is an element of the $O(N)$ group. Since an open path can be contracted to a point, it has to be in the identity component of $O(N)$, i.e. in $SO(N)$. Taking a generic complex λ , the transport matrix sits in $SO(N, \mathbb{C})$, and so will also be the the monodromy matrix. Then, by diagonalization we can bring it into the form

$$\Omega_{\text{diag}}(\lambda) = \begin{cases} \text{diag}(e^{iq_1(\lambda)}, e^{-iq_1(\lambda)}, \dots, e^{iq_{[N/2]}(\lambda)}, e^{-iq_{[N/2]}(\lambda)}) & , \text{ for } N \text{ even} \\ \text{diag}(e^{iq_1(\lambda)}, e^{-iq_1(\lambda)}, \dots, e^{iq_{[N/2]}(\lambda)}, e^{-iq_{[N/2]}(\lambda)}, 1) & , \text{ for } N \text{ odd} \end{cases} \quad (40)$$

The eigenvalues come in pairs, and the $q_i(\lambda)$ are the so-called quasi-momenta. They parametrize a multi-sheeted Riemann surface, which has a specific structure for each classical solution. Common features for all are the pole singularities at $\lambda = \pm 1$ and branch cuts, starting whenever two quasi-momenta coincide. Let us denote by \mathcal{C}_τ the collection of such branch cuts. The conserved quantities are related to traces of powers of the monodromy matrix, thus should be insensitive for these branch cuts. As a result, quasi-momenta may be permuted up to multiples of 2π on the cuts

$$q_\ell(\lambda + i\varepsilon) - q_{\ell+1}(\lambda - i\varepsilon) = 2\pi n_{\ell,r}, \quad x \in \mathcal{C}_{\ell,r}, \quad \ell = 1, \dots, [N/2] - 1.$$

Let us analyze the analytic structure of the quasi-momenta.

2.4.1 Asymptotics at $\lambda \rightarrow +\infty$

The large λ asymptotics of the transport matrix $T(\lambda)$ can be read off from the definition of the connection (10)

$$T(\lambda) \sim \overleftarrow{\mathcal{P} \exp} \left\{ \frac{1}{\lambda} \int_0^\pi d\sigma J_\tau \right\} \sim \mathbb{I} + \frac{1}{\lambda} \int_0^\pi d\sigma J_\tau \equiv \mathbb{I} + \frac{\bar{Q}}{\lambda} + \dots \quad (41)$$

where \bar{Q} would be the conserved charge of the bulk theory. This implies, for the boundary monodromy matrix:

$$\Omega(\lambda) \sim \mathbb{I} - \frac{1}{\lambda} (U \bar{Q} U + \bar{Q}) + \dots = \mathbb{I} - \frac{2Q}{\lambda} + \dots \quad ; \quad Q \in \mathfrak{h} .$$

Clearly, only the charges corresponding to the survived symmetry appear in the asymptotic behaviour of the monodromy matrix. For its eigenvalues, this implies that

$$q_\ell(\lambda) \sim \frac{1}{\lambda} q_\ell^{[\infty]} + \mathcal{O}(\lambda^{-2}),$$

where $q_i^{[\infty]}$ are the eigenvalues of the conserved charges preserved by the boundary.

2.4.2 Reflection symmetry

The boundary nature of the monodromy matrix, together with $U^2 = \mathbb{I}$, implies the following reflection property

$$\Omega(\lambda) = U\Omega^{-1}(-\lambda)U.$$

Let us see how this relation translates to the quasi-momenta. Assuming that $\Omega(\lambda)$ is diagonalized as $\Omega(\lambda) = A(\lambda)\Omega_{\text{diag}}(\lambda)A(\lambda)^{-1}$ and that the quasi-momenta are all different for generic λ , we have

$$\Omega_{\text{diag}}(\lambda) = P\Omega_{\text{diag}}(-\lambda)^{-1}P^{-1} \quad ; \quad P = A(\lambda)^{-1}UA(-\lambda)$$

Since P connects a generic diagonal matrix to another generic diagonal matrix, it has to be a permutation, actually the same permutation for any λ . To identify the permutation, we analyze the relation at $\lambda \rightarrow \pm\infty$. In this limit

$$\Omega_{\text{diag}}(\lambda) \simeq \mathbb{I} + \frac{2Q}{\lambda} = P \left(\mathbb{I} - \frac{2Q}{-\lambda} \right) P \simeq P (\Omega_{\text{diag}}(-\lambda))^{-1} P \quad (42)$$

For a generically charged state it implies that $P = 1$, which leads to the following reflection property of the quasi-momenta

$$q_\ell(-\lambda) = -q_\ell(\lambda). \quad (43)$$

2.4.3 Singularities around $\lambda = \pm 1$

Let us first recall that

$$a_\sigma(\lambda) = \frac{1}{1-\lambda^2}J_\sigma + \frac{\lambda}{1-\lambda^2}J_\tau = \frac{J_+}{1-\lambda} - \frac{J_-}{1+\lambda} \quad (44)$$

Thus the pole singularities around $\lambda = \pm 1$ are governed by the light-cone components of the currents J_\pm . Let us denote the matrix which diagonalizes these currents by $h_\pm(\sigma)$:

$$J_\pm^{\text{diag}}(\sigma) = h_\pm(\sigma)^{-1}J_\pm(\sigma)h_\pm(\sigma). \quad (45)$$

For the monodromy matrix we have at leading order around $\lambda = \pm 1$

$$\begin{aligned} \Omega(\lambda)|_{\lambda=\pm 1} &= U h_\mp(0) \exp\left(\frac{1}{\lambda \mp 1} \int_0^\pi d\sigma J_\mp^{\text{diag}}(\sigma)\right) h_\mp(\pi)^{-1} U \times \\ & h_\pm(\pi) \exp\left(\frac{1}{\lambda \mp 1} \int_0^\pi d\sigma J_\pm^{\text{diag}}(\sigma)\right) h_\pm(0)^{-1}. \end{aligned} \quad (46)$$

Recall from (38) that at the boundaries the light-cone currents are related by the automorphism, α , as $J_-(\sigma_i) = U J_+(\sigma_i) U$, where $\sigma_i = 0, \pi$. This means that, at the boundary, the matrices which diagonalize J_- and J_+ are related, and

$$J_\mp^{\text{diag}}(\sigma_i) = h_\mp(\sigma_i)^{-1} U h_\pm(\sigma_i) J_\pm^{\text{diag}}(\sigma_i) h_\pm(\sigma_i)^{-1} U h_\mp(\sigma_i). \quad (47)$$

Actually the currents $J = m dm = -2(\mathbf{dn})\mathbf{n}^t + 2\mathbf{n}dn^t$ are rank 2 matrices and their diagonal form is

$$J_\mp^{\text{diag}}(\sigma) = \text{diag}(i j_\mp(\sigma), -i j_\mp(\sigma), 0, \dots, 0),$$

thus $h_{\mp}(\sigma_i)^{-1} U h_{\pm}(\sigma_i)$ is either the identity or the permutation matrix on the relevant 2 dimensional space. Here we assumed that $j_{\mp}(\sigma)$ is never vanishing, thus the 2 dimensional nonzero subspace is the same for any σ . Let us introduce

$$\int_0^\pi d\sigma j_{\mp}(\sigma) = \kappa_{\mp},$$

such that for the quasi-momenta we have

$$\begin{aligned} (q_1, -q_1, \dots)|_{\lambda \sim 1} &= \frac{\kappa}{\lambda - 1} (1, -1, 0, \dots, 0) + \dots \\ (q_1, -q_1, \dots)|_{\lambda \sim -1} &= \frac{\pm \kappa}{\lambda + 1} (1, -1, 0, \dots, 0) + \dots, \end{aligned}$$

where $\kappa = \kappa_+ \pm \kappa_-$ depending whether the permutation is the identity or not. However, consistency with the parity properties favors the $\kappa = \kappa_+ + \kappa_-$ choice and the plus sign in the behaviour of q_1 at $\lambda \sim -1$, i.e. at the boundary $J_{\mp}^{\text{diag}}(\sigma_i) = J_{\pm}^{\text{diag}}(\sigma_i)$.

2.4.4 Inversion symmetry

To derive the behaviour of Ω under the inversion transformation we recall that the Lax connection transforms under inversion as

$$a(\lambda^{-1}) = m(d + a(\lambda))m.$$

This implies for the transport matrix that

$$T(\lambda^{-1}) = m(\pi)T(\lambda)m(0),$$

where $m(0)$ and $m(\pi)$ are the boundary values of the $O(N)$ element. Since in general these values are different for the two different boundaries, this is not a similarity transformation. Nevertheless, using it together with $[U, m|_{\text{bdry}}] = 0$ in the definition of Ω one finds that

$$\Omega(\lambda^{-1}) = m(0)^{-1}\Omega(\lambda)m(0).$$

We can now argue similarly to the case of the reflection symmetry: the eigenvalues of the monodromy at λ^{-1} , $\Omega_{\text{diag}}(\lambda^{-1})$, and of the original monodromy matrix $\Omega_{\text{diag}}(\lambda)$, are related by a λ -independent parity transformation:

$$\Omega_{\text{diag}}(\lambda) = P\Omega_{\text{diag}}(\lambda^{-1})P^{-1} \quad ; \quad P = A(\lambda)^{-1}m(0)A(\lambda^{-1}). \quad (48)$$

Analysing the $\lambda \rightarrow \lambda^{-1}$ transformation for the singularities around $\lambda = \pm 1$ we can conclude that

$$q_1(1/\lambda) = -q_1(\lambda).$$

Restricting the relation (48) for the non-singular part of the monodromy matrix and evaluating at $\lambda = 1$ implies that the quasi-momenta q_j for $j \neq 1$ will not suffer any permutations under inversion:

$$q_j(1/\lambda) = q_j(\lambda). \quad (49)$$

Let us note one difference from the bulk theory. When there are no boundaries, inversion symmetry allows $q_1^{\text{bulk}}(1/\lambda) = 2\pi n - q_1^{\text{bulk}}(\lambda)$. But when we have boundaries, the parity condition enforces this mode number to be zero.

2.5 Explicit example in the $O(4)$ model

In the following, we construct explicitly the spectral curve of a solution having $SO(2) \times SO(2)$ symmetry. At the language of the \mathbf{n} variables it corresponds to two Dirichlet and two Neumann boundary conditions

$$\partial_\sigma n_1| = 0 \quad ; \quad \partial_\sigma n_2| = 0 \quad ; \quad n_3| = 0 \quad ; \quad n_4| = 0.$$

The U matrix in the Lax formulation is

$$U = \text{diag}(1, 1, -1, -1).$$

A solution satisfying this boundary condition is

$$n_1 = \cos n\sigma \cos \omega\tau \quad ; \quad n_2 = \cos n\sigma \sin \omega\tau \quad ; \quad n_3 = \sin n\sigma \cos \omega\tau \quad ; \quad n_4 = \sin n\sigma \sin \omega\tau.$$

One can check that both the bulk equations of motion and the boundary conditions are satisfied, provided $n \in \mathbb{Z}$. The solution corresponds to a circular rotating string in S^3 , where the end points of the string rotate on the same $S^1 \subset S^3$. The Lax connection, $a_\sigma(\lambda)$, is a complicated function of σ , making it very difficult to calculate the path-order exponential. To avoid this problem we switch to the coset formulation. One possible evolution on the group manifold can be described as

$$g(\tau, \sigma) = e^{-n\sigma(J_{13}+J_{24})} e^{-\omega\tau(J_{12}+J_{34})} \quad ; \quad (J_{ik})_{lm} = \delta_{il}\delta_{km} - \delta_{im}\delta_{kl}$$

Note that we can recover the constrained fields of the circular string via the relations (15). A gauge equivalent evolution can be obtained by changing the sign of J_{34} . What is nice about this choice is that $[J_{13} + J_{24}, J_{12} + J_{34}] = 0$, and the components of the Maurer-Cartan one-form are constants:

$$\omega_\sigma = g^{-1}\partial_\sigma g = -n(J_{13} + J_{24}) \quad ; \quad \omega_\tau = g^{-1}\partial_\tau g = -\omega(J_{12} + J_{34}).$$

The projected currents are

$$A_\sigma = -nJ_{24} \quad ; \quad A_\tau = -\omega J_{34} \quad ; \quad K_\sigma = -nJ_{13} \quad ; \quad K_\tau = -\omega J_{12},$$

which satisfy the equation of motion (14). The σ component of the Lax connection is

$$L_\sigma(\lambda) = -nJ_{24} - n\frac{\lambda^2 + 1}{\lambda^2 - 1}J_{13} - \omega\frac{2\lambda}{\lambda^2 - 1}J_{12}.$$

Since

$$-UL_\sigma(-\lambda)U = L_\sigma(\lambda) \quad \longrightarrow \quad UT(-\lambda)^{-1}U = T(\lambda)$$

we can simply diagonalize $L_\sigma(\lambda)$ and exponentiate it to get the eigenvalues of the monodromy matrix. We find

$$q_{1,2} = \frac{2\pi}{1-\lambda^2} i \sqrt{n^2(\lambda^4 + 1) + 2\omega^2\lambda^2 \pm 2A} \quad ; \quad A = \lambda^2 \sqrt{(\omega^2 + n^2\lambda^2)(n^2/\lambda^2 + \omega^2)}.$$

It is instructive to write the quasi-momenta as

$$q_1 = \frac{2\pi i\lambda}{1-\lambda^2} \left(\sqrt{\omega^2 + n^2\lambda^2} + \sqrt{\frac{n^2}{\lambda^2} + \omega^2} \right) \quad ; \quad q_2 = \frac{2\pi i\lambda}{1-\lambda^2} \left(\sqrt{\omega^2 + n^2\lambda^2} - \sqrt{\frac{n^2}{\lambda^2} + \omega^2} \right).$$

This solution is analogous to the open string solution of the $Y = 0$ brane [15]. The point-like string solution does not depend on σ and corresponds to $n = 0$. Its quasi-momenta are

$$q_1 = \frac{4\pi i\lambda\omega}{1-\lambda^2} \quad ; \quad q_2 = 0$$

One can easily check that the spectral curve has the right asymptotics and residues around $\lambda = \pm 1$, and satisfies the inversion and reflection properties. Let us also note that the point-like string solution satisfies the all Neumann boundary condition, too.

3 Quantum integrability

The quantum integrability of the $O(N)$ non-linear σ -models can be shown by following the argumentations of Polyakov [22], Goldschmidt and Witten (GW) [23]. The idea is to analyze the classical conservation laws (5) and their possible quantum corrections. Since products of operators are not well-defined at the quantum level they have to be regularized, leading to the appearance of new terms which can make the classical symmetry anomalous. To decide whether a higher spin symmetry is maintained or not, one has to classify the possible anomaly terms. In the case of T_{++}^2 , the global symmetry, the dimensionality of the fields and the Lorentz transformation property fix the anomaly of the form

$$\partial_- T_{++}^2 = c_1 \partial_- (\partial_+^2 \mathbf{n}^t \partial_+^2 \mathbf{n}) + c_2 \partial_+ (\partial_- \mathbf{n}^t \partial_+ \mathbf{n} \partial_+ \mathbf{n}^t \partial_+ \mathbf{n}) + c_3 \partial_+ (\partial_+^3 \mathbf{n}^t \partial_- \mathbf{n}) .$$

Grouping the terms a conservation law can be established at the quantum level – of the form (6)

$$\partial_- T^{(4)} = \partial_+ \Theta^{(2)} \tag{50}$$

with

$$T = T_{++}^2 - c_1 (\partial_+^2 \mathbf{n}^t \partial_+^2 \mathbf{n}) \quad ; \quad \Theta^{(2)} = c_2 (\partial_- \mathbf{n}^t \partial_+ \mathbf{n} \partial_+ \mathbf{n}^t \partial_+ \mathbf{n}) + c_3 (\partial_+^3 \mathbf{n}^t \partial_- \mathbf{n}) ,$$

which implies factorized scattering. We can obtain a similar equation by exchanging $\partial_+ \leftrightarrow \partial_-$.

Integrability in the presence of boundaries, similarly to the classical case, requires the fulfillment of the equation

$$\left[T^{(s+1)} - \bar{T}^{(s+1)} + \bar{\Theta}^{(s-1)} - \Theta^{(s-1)} \right] | = \frac{d\Sigma}{dt} .$$

Since at the quantum level the bulk-boundary OPEs can get quantum corrections, classical integrable boundary conditions can be anomalous. In particular, for the $O(N)$ symmetric Neumann boundary condition, $\partial_1 \mathbf{n} = 0$, symmetry and dimensionality allow for an anomaly of the form

$$T_{++} - T_{--} | = c \partial_\tau \mathbf{n}^t \partial_\tau \mathbf{n} ,$$

which would even spoil the existence of a conserved energy. Since we have no control of such terms, nor have we a systematic quantization approach which can decide in these questions, we classify the integrable boundary conditions based on the existence of reflection factors, which satisfy the boundary bootstrap equations.

3.1 Reflection factors

As we explained, the GW argument implies that the $O(N)$ non-linear σ -model is integrable at the quantum level. There are N particles with the same mass, m , and they transform in the vector representation of $O(N)$. The index structure of the scattering matrix, \mathcal{S} , compatible with this symmetry, has the form

$$S_{ij}^{kj}(\theta) = \sigma_1(\theta) \delta_{ij} \delta^{kl} + \sigma_2(\theta) \delta_i^k \delta_j^l + \sigma_3(\theta) \delta_i^l \delta_j^k .$$

Factorized scattering implies the Yang-Baxter equation (YBE) which, together with unitarity and crossing symmetry, restricts the amplitudes to

$$\sigma_1(\theta) = -\frac{i\lambda}{i\pi - \theta} \sigma_2(\theta) \quad ; \quad \sigma_3(\theta) = -\frac{i\lambda}{\theta} \sigma_2(\theta) \quad ; \quad \lambda = \frac{2\pi}{N-2} ,$$

where

$$\sigma_2(\theta) = \frac{\Gamma(\frac{1}{2} + \frac{\lambda}{2\pi} + \frac{i\theta}{2\pi}) \Gamma(1 + \frac{i\theta}{2\pi}) \Gamma(\frac{1}{2} - \frac{i\theta}{2\pi}) \Gamma(\frac{\lambda}{2\pi} - \frac{i\theta}{2\pi})}{\Gamma(\frac{1}{2} + \frac{\lambda}{2\pi} - \frac{i\theta}{2\pi}) \Gamma(-\frac{i\theta}{2\pi}) \Gamma(\frac{1}{2} + \frac{i\theta}{2\pi}) \Gamma(1 + \frac{\lambda}{2\pi} + \frac{i\theta}{2\pi})}$$

In the presence of an integrable boundary particles reflect from the boundary by a reflection matrix $R_i^j(\theta)$, which satisfies the boundary analogue of the Yang-Baxter equation

$$S_{ji}^{nm}(\theta_1 - \theta_2) R_n^p(\theta_1) S_{mp}^{ql}(\theta_1 + \theta_2) R_q^k(\theta_2) = R_i^p(\theta_2) S_{jp}^{nm}(\theta_1 + \theta_2) R_n^q(\theta_1) S_{mq}^{kl}(\theta_1 - \theta_2).$$

The scalar factor is fixed from unitarity and boundary crossing unitarity [5]

$$R_i^k(\theta) R_k^j(-\theta) = \delta_i^j \quad ; \quad R_i^j\left(\frac{i\pi}{2} - \theta\right) = S_{kl}^{ij}(2\theta) R_k^l\left(\frac{i\pi}{2} + \theta\right).$$

The solutions of these equations were classified and they fall into the following classes:

1. **Diagonal** [8]. The diagonal reflection factors have the form

$$\mathcal{R}(\theta) = \text{diag}\left(\underbrace{R_1(\theta), \dots, R_1(\theta)}_k, \underbrace{R_2(\theta), \dots, R_2(\theta)}_{N-k}\right)$$

where from the boundary Yang-Baxter equation it follows that

$$\frac{R_2(\theta)}{R_1(\theta)} = \frac{c - \theta}{c + \theta} \quad ; \quad c = -i \frac{\pi N - 2k}{2N - 2}.$$

Clearly, for the transformation $k \leftrightarrow N - k$ the reflection factors exchange $R_1 \leftrightarrow R_2$, thus it is enough to consider the cases $k \leq N/2$. Unitarity and boundary crossing unitarity fix the scalar factor up to a CDD factor as

$$\begin{aligned} R_1(\theta) &= -R_0(\theta) \frac{\Gamma\left(\frac{1}{4} + \frac{\lambda}{4\pi} + \frac{i\theta}{2\pi}\right) \Gamma\left(\frac{3}{4} + \frac{\lambda}{4\pi} - \frac{i\theta}{2\pi}\right) \Gamma\left(\frac{1}{4} + \frac{\lambda(N-k-1)}{4\pi} + \frac{i\theta}{2\pi}\right) \Gamma\left(\frac{3}{4} + \frac{\lambda(N-k-1)}{4\pi} - \frac{i\theta}{2\pi}\right)}{\Gamma\left(\frac{1}{4} + \frac{\lambda}{4\pi} - \frac{i\theta}{2\pi}\right) \Gamma\left(\frac{3}{4} + \frac{\lambda}{4\pi} + \frac{i\theta}{2\pi}\right) \Gamma\left(\frac{1}{4} + \frac{\lambda(N-k-1)}{4\pi} - \frac{i\theta}{2\pi}\right) \Gamma\left(\frac{3}{4} + \frac{\lambda(N-k-1)}{4\pi} + \frac{i\theta}{2\pi}\right)}, \\ R_0(\theta) &= \frac{\Gamma\left(\frac{1}{2} + \frac{\lambda}{4\pi} - \frac{i\theta}{2\pi}\right) \Gamma\left(1 + \frac{i\theta}{2\pi}\right) \Gamma\left(\frac{1}{4} - \frac{i\theta}{2\pi}\right) \Gamma\left(\frac{3}{4} + \frac{\lambda}{4\pi} + \frac{i\theta}{2\pi}\right)}{\Gamma\left(\frac{1}{2} + \frac{\lambda}{4\pi} + \frac{i\theta}{2\pi}\right) \Gamma\left(1 - \frac{i\theta}{2\pi}\right) \Gamma\left(\frac{1}{4} + \frac{i\theta}{2\pi}\right) \Gamma\left(\frac{3}{4} + \frac{\lambda}{4\pi} - \frac{i\theta}{2\pi}\right)}. \end{aligned}$$

The symmetry of this boundary condition is $SO(k) \times SO(N - k)$. The case $k = N - 1$ corresponds to the fixed boundary condition of Ghoshal [6], i.e. all boundary conditions are Dirichlet, while the case $k = 0$ corresponds to the $O(N)$ symmetric Neumann boundary condition of Ghoshal [6]. Finally, we also note that in the $\theta \rightarrow \infty$ limit the reflection factor agrees with the boundary Lax matrix U .

2. **One-block** [10]. The reflection factor is diagonal except a 2×2 block

$$\mathcal{R}(\theta) = \begin{pmatrix} A_\xi(\theta) & B_\xi(\theta) & 0 & \cdots & 0 \\ -B_\xi(\theta) & A_\xi(\theta) & 0 & \cdots & 0 \\ 0 & 0 & R_\xi(\theta) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & R_\xi(\theta) \end{pmatrix}$$

where

$$A(\theta) = \frac{1}{2} \left(\frac{c - \theta}{c + \theta} + \frac{c' - \theta}{c' + \theta} \right) R(\theta) \quad ; \quad B(\theta) = \frac{1}{2i} \left(\frac{c - \theta}{c + \theta} - \frac{c' - \theta}{c' + \theta} \right) R(\theta),$$

with the constraint

$$c = -\frac{i\pi N - 4}{2N - 2} + \xi \quad ; \quad c' = -\frac{i\pi N - 4}{2N - 2} - \xi.$$

The symmetry of this boundary condition is $SO(2) \times SO(N - 2)$. The case $\xi = 0$ reduces to the diagonal solution with $k = 2$ above. There is another diagonal limit of the reflection factor, namely by sending $\xi \rightarrow \infty$ we can recover the $O(N)$ symmetric boundary condition. Unitarity, together with boundary crossing unitarity, fixes the reflection factors to be:

$$R_\xi(\theta) = -R_0(\theta) \frac{\Gamma\left(\frac{1}{4} + \frac{\lambda+2i\xi}{4\pi} + \frac{i\theta}{2\pi}\right) \Gamma\left(\frac{3}{4} + \frac{\lambda+2i\xi}{4\pi} - \frac{i\theta}{2\pi}\right) \Gamma\left(\frac{1}{4} + \frac{\lambda-2i\xi}{4\pi} + \frac{i\theta}{2\pi}\right) \Gamma\left(\frac{3}{4} + \frac{\lambda-2i\xi}{4\pi} - \frac{i\theta}{2\pi}\right)}{\Gamma\left(\frac{1}{4} + \frac{\lambda+2i\xi}{4\pi} - \frac{i\theta}{2\pi}\right) \Gamma\left(\frac{3}{4} + \frac{\lambda+2i\xi}{4\pi} + \frac{i\theta}{2\pi}\right) \Gamma\left(\frac{1}{4} + \frac{\lambda-2i\xi}{4\pi} - \frac{i\theta}{2\pi}\right) \Gamma\left(\frac{3}{4} + \frac{\lambda-2i\xi}{4\pi} + \frac{i\theta}{2\pi}\right)}.$$

3. **All blocks the same** [10]. There is also a completely non-diagonal reflection factor for any even N of the form

$$\mathcal{R}(\theta) = \begin{pmatrix} A(\theta) & B(\theta) & 0 & 0 & \cdots \\ -B(\theta) & A(\theta) & 0 & 0 & \cdots \\ 0 & 0 & A(\theta) & B(\theta) & \cdots \\ 0 & 0 & -B(\theta) & A(\theta) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

From the boundary YBE it follows that

$$B(\theta) = \alpha \theta A(\theta).$$

The symmetry of the boundary condition is $U(N/2)$. Unitarity and boundary crossing unitarity fix the scalar factor to

$$A(\theta) = -\frac{1}{2\pi\alpha} \frac{\Gamma\left(\frac{1}{2} - \frac{1}{2\pi\alpha} + \frac{i\theta}{2\pi}\right) \Gamma\left(-\frac{1}{2\pi\alpha} - \frac{i\theta}{2\pi}\right)}{\Gamma\left(\frac{1}{2} - \frac{1}{2\pi\alpha} - \frac{i\theta}{2\pi}\right) \Gamma\left(1 - \frac{1}{2\pi\alpha} + \frac{i\theta}{2\pi}\right)} R_0(\theta).$$

4. **Exceptional** boundary conditions for the $O(4)$ model. The above classification was confirmed in [11], and additionally a new family of boundary conditions was found in the $O(4)$ model. Since $SO(4) \equiv SU(2)_l \times SU(2)_r$, the $O(4)$ model is an $SU(2)$ principal chiral model, which allows a two parameter, (ξ_l, ξ_r) family of reflection factors:

$$\mathcal{R}(\theta) = \begin{pmatrix} A_+(\theta) & B_+(\theta) & 0 & 0 \\ -B_+(\theta) & A_+(\theta) & 0 & 0 \\ 0 & 0 & A_-(\theta) & B_-(\theta) \\ 0 & 0 & -B_-(\theta) & A_-(\theta) \end{pmatrix},$$

where

$$A_{\pm}(\theta) = R(\theta) \frac{\xi_l \xi_r \pm \theta^2}{(\xi_l + \theta)(\xi_r + \theta)} \quad ; \quad B_{\pm}(\theta) = \frac{-i\theta(\xi_r \pm \xi_l)}{(\xi_l + \theta)(\xi_r + \theta)}$$

and

$$R(\theta) = \frac{\Gamma\left(\frac{1}{2} - \frac{i\xi_l}{2\pi} + \frac{i\theta}{2\pi}\right) \Gamma\left(1 - \frac{i\xi_l}{2\pi} - \frac{i\theta}{2\pi}\right) \Gamma\left(\frac{1}{2} - \frac{i\xi_r}{2\pi} + \frac{i\theta}{2\pi}\right) \Gamma\left(1 - \frac{i\xi_r}{2\pi} - \frac{i\theta}{2\pi}\right)}{\Gamma\left(\frac{1}{2} - \frac{i\xi_l}{2\pi} - \frac{i\theta}{2\pi}\right) \Gamma\left(1 - \frac{i\xi_l}{2\pi} + \frac{i\theta}{2\pi}\right) \Gamma\left(\frac{1}{2} - \frac{i\xi_r}{2\pi} - \frac{i\theta}{2\pi}\right) \Gamma\left(1 - \frac{i\xi_r}{2\pi} + \frac{i\theta}{2\pi}\right)} R_0(\theta).$$

The symmetry of this boundary condition is $U(1)_l \times U(1)_r$. By choosing $\xi_l = -\xi_r$, the solution reduces to the one-block case, while choosing $\xi_r = i/\alpha$ and taking the $\xi_l \rightarrow \infty$ limit we can recover the two block, same reflection factor.

Comparing this classification with the classical case, we can see that the ‘‘all block different’’ boundary reflection factor exists only in the $O(4)$ case. It would be interesting to understand how the general integrable boundary conditions, formulated in terms of the antisymmetric matrix M , become anomalous during the quantization procedure. From the existence of the nondiagonal reflection factors in the quantum case it is expected that they also have classical limits, which should be described by monodromy matrices and spectral curves. In order to make a connection to the classical formulations, we analyze the classical limit of the spectrum via the Bethe-Yang equations.

3.2 Bethe-Yang equations

In this section we analyze the large volume spectrum in a finite volume. The energy of an n -particle state on an interval of size L , with rapidities $\theta_1, \dots, \theta_n$, can be written as

$$E = \sum_{i=1}^n m \cosh \theta_i + O(e^{-mL}). \quad (51)$$

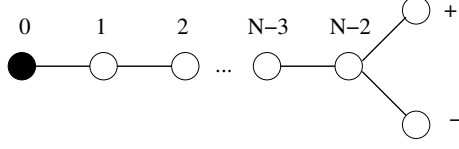


Figure 1: Extended Dynkin diagram of $O(N)$ for even N .

Volume dependence comes through momentum quantisation, which formulates the periodicity of the wave function. The requirement is called the Bethe-Yang equation, which is based on the infinite volume scattering and reflection matrices as

$$e^{im2L \sinh \theta_i} \prod_{j=i+1}^n \mathcal{S}(\theta_i - \theta_j) \mathcal{R}(\theta_i) \prod_{j=n:j \neq i}^1 \mathcal{S}(\theta_j + \theta_i) \mathcal{R}(-\theta_i) \prod_{j=1}^{i-1} \mathcal{S}(\theta_i - \theta_j) = 1, \quad (52)$$

where we assumed that the left and the right boundaries are the same. These matrix equations can be solved by diagonalizing the double row transfer matrix

$$T(\theta_0 | \{\theta_i\}) = \text{tr}_0 \left(\prod_{j=1}^n \mathcal{S}(\theta_0 - \theta_j) \mathcal{R}(\theta_0) \prod_{j=n}^1 \mathcal{S}(\theta_j + \theta_0) \mathcal{R}^c(-\theta_0) \right), \quad (53)$$

where the charge conjugation of the reflection factor ensures the equivalence to eq. (52) (see [24] for details). The diagonalization can be done either via the analytic [11] or the algebraic [25] Bethe Ansatz (BA). Results are available only for even N so we restrict ourselves to those cases. In the analytic BA, the regularity of the transfer matrix at the positions of the Bethe roots provides the BA equations. They can be most compactly described using roots and functions related to the extended Dynkin diagram for $O(N)$. The original Dynkin diagram of $D_{N/2}$ is extended by a black dot, with label 0, which represents the massive particle, see Figure 3.2. Additionally there are magnonic particles with labels, $i = 1, \dots, N/2 - 2, +, -$. For each index we associate two Q -functions:

$$Q_i(\theta) = \prod_{k=1}^{n_i} (\theta - u_k^{(i)})(\theta + u_k^{(i)}), \quad (54)$$

$$\tilde{Q}_i(\theta) = \prod_{\substack{k=1 \\ \theta \neq u_k^{(i)}}}^{n_i} (\theta - u_k^{(i)})(\theta + u_k^{(i)}). \quad (55)$$

The one with tilde leaves out the root, if it is evaluated at the root position. All the magnonic BA equations can be formulated very compactly as:

$$\frac{Q_{i-1}^-(u_k^{(i)})}{Q_{i-1}^+(u_k^{(i)})} \frac{\tilde{Q}_i^{++}(u_k^{(i)})}{\tilde{Q}_i^{--}(u_k^{(i)})} \frac{Q_{i+1}^-(u_k^{(i)})}{Q_{i+1}^+(u_k^{(i)})} = r_i^0(u_k^{(i)}) r_i^\pi(u_k^{(i)}), \quad 0 < i < \frac{N}{2} - 2, \quad (56)$$

$$\frac{Q_{i-1}^-(u_k^{(i)})}{Q_{i-1}^+(u_k^{(i)})} \frac{\tilde{Q}_i^{++}(u_k^{(i)})}{\tilde{Q}_i^{--}(u_k^{(i)})} \frac{Q_{i+1}^-(u_k^{(i)})}{Q_{i+1}^+(u_k^{(i)})} = r_i^0(u_k^{(i)}) r_i^\pi(u_k^{(i)}), \quad i = \frac{N}{2} - 2, \quad (57)$$

$$\frac{Q_{N/2-2}^-(u_k^{(-)})}{Q_{N/2-2}^+(u_k^{(-)})} \frac{\tilde{Q}_{-}^{++}(u_k^{(-)})}{\tilde{Q}_{-}^{--}(u_k^{(-)})} = r_{-}^0(u_k^{(-)}) r_{-}^\pi(u_k^{(-)}), \quad (58)$$

$$\frac{Q_{N/2-2}^-(u_k^{(+)})}{Q_{N/2-2}^+(u_k^{(+)})} \frac{\tilde{Q}_{+}^{++}(u_k^{(+)})}{\tilde{Q}_{+}^{--}(u_k^{(+)})} = r_{+}^0(u_k^{(+)}) r_{+}^\pi(u_k^{(+)}), \quad (59)$$

where $f^\pm(\theta) = f(\theta \pm \frac{i\pi}{N-2})$ and the BA equations for the massive particles reads as:

$$e^{i2p_k L} \prod_{\substack{j=1 \\ j \neq k}}^{n_0} S(u_k^{(0)} - u_j^{(0)}) \frac{Q_1^-(u_k^{(0)})}{Q_1^+(u_k^{(0)})} = r_0^0(u_k^{(i)}) r_0^\pi(u_k^{(i)}).$$

Above, we introduced the function

$$S(\theta) = \frac{\Gamma(-\frac{i\theta}{2\pi})}{\Gamma(1 - \frac{i\theta}{2\pi})} \sigma_2(\theta).$$

The dependence on the boundary conditions sits in the various reflection phases, of which the non-trivial ones ($\neq 1$) are the following:

1. $\text{SO}(k) \times \text{SO}(N - k)$ case with even k ($k \leq N/2$):

$$r_0(\theta) = R_1(\theta), \tag{60}$$

$$r_i(\theta) = \frac{i\frac{\pi}{2} \frac{N-k}{N-2} - \theta}{i\frac{\pi}{2} \frac{N-k}{N-2} + \theta} \delta_{i,k/2}. \tag{61}$$

2. $\text{SO}(2) \times \text{SO}(N - 2)$ case:

$$r_0(\theta) = A(\theta) + iB(\theta), \tag{62}$$

$$r_i(\theta) = \frac{i\frac{\pi}{2} - \xi - \theta}{i\frac{\pi}{2} - \xi + \theta} \delta_{i,1}. \tag{63}$$

3. $\text{U}(N/2)$ case:

$$r_0(\theta) = A(\theta) + iB(\theta), \tag{64}$$

$$r_i(\theta) = \frac{i\frac{\pi}{2} + \frac{i}{\alpha} - \theta}{i\frac{\pi}{2} + \frac{i}{\alpha} + \theta} \delta_{i,+}. \tag{65}$$

4. $\text{U}(1)_l \times \text{U}(1)_r$ case:

$$r_0(\theta) = R(\theta), \tag{66}$$

$$r_-(\theta) = \frac{i\frac{\pi}{2} + \xi_l - \theta}{i\frac{\pi}{2} + \xi_l + \theta} \delta_{i,-}, \tag{67}$$

$$r_+(\theta) = \frac{i\frac{\pi}{2} + \xi_r - \theta}{i\frac{\pi}{2} + \xi_r + \theta} \delta_{i,+}. \tag{68}$$

3.3 Spectral curve as the limit of the BY equations

In this section we investigate the (quasi-) classical limit of the boundary Bethe-Yang equations in the $O(4)$ model, for two boundary conditions: the free boundary conditions (all four Neumann) and the the mixed diagonal reflection with two Neumann and two Dirichlet ones.

The quantum $O(4)$ σ -model is asymptotically free with a dynamically generated mass $m = \Lambda e^{-\frac{\sqrt{\Lambda}}{2}}$ (Λ being the cutoff and λ is the 't Hooft coupling evaluated at Λ). Similarly to the periodic case [26], we can only compare this model to a quantum field theory defined by a Lagrangian in the classical limit $\lambda \rightarrow \infty$ ($m \rightarrow 0$), in which case it exhibits the classical conformal symmetry of the latter. In this limit, the dimensionless parameter $\mu = mL = \Lambda L e^{-\frac{\sqrt{\Lambda}}{2}}$ appearing in the BA equations tends to zero, $\mu \rightarrow 0$. The quasi-momenta of the corresponding classical spectral curve then have cuts, originated from the condensation of Bethe roots, see [27] for details. Indeed, if

we take both the number of particles n_0 and the number of roots $2n_+ \sim 2n_- \sim 2n_0$ to infinity ($n_0 \rightarrow \infty$), while keeping the quantization number fixed, it implies that all roots $\pm\theta_\alpha$, $\pm u_j$ and $\pm v_j$ -s become large together with their differences. In an appropriately rescaled variable, relevant for the classical limit, they start to condense on cuts. If one introduces the densities of these condensed roots, then the Bethe ansatz equations provide integral equations restricting these densities. Finally, the solutions for the resolvent of the densities can be mapped to the quasi-momenta of the classical spectral curve. Let us see, how this can be achieved in the simplest case.

The Bethe-Yang equations for the roots $\theta_\beta = u_\beta^{(0)}$ take the following form:

$$e^{2ip_\beta L} \prod_{\alpha:\alpha\neq\beta}^{n_0} S_0^2(\theta_\beta + \theta_\alpha) S_0^2(\theta_\beta - \theta_\alpha) r_i^2(\theta_\beta) \frac{Q_+^+(\theta_\beta)}{Q_+^-(\theta_\beta)} \frac{Q_-^+(\theta_\beta)}{Q_-^-(\theta_\beta)} = 1, \quad i = f, m \quad \beta = 1, \dots, n_0 \quad (69)$$

where

$$S_0(\theta) = i \frac{\Gamma(\frac{1}{2} - \frac{i\theta}{2\pi})}{\Gamma(\frac{1}{2} + \frac{i\theta}{2\pi})} \frac{\Gamma(\frac{i\theta}{2\pi})}{\Gamma(-\frac{i\theta}{2\pi})}$$

and

$$r_f(\theta) = \frac{\Gamma(\frac{3}{4} - \frac{i\theta}{2\pi})}{\Gamma(\frac{3}{4} + \frac{i\theta}{2\pi})} \frac{\Gamma(1 + \frac{i\theta}{2\pi})^2}{\Gamma(1 - \frac{i\theta}{2\pi})^2} \frac{\Gamma(\frac{1}{4} - \frac{i\theta}{2\pi})}{\Gamma(\frac{1}{4} + \frac{i\theta}{2\pi})} \quad ; \quad r_m(\theta) = \frac{\Gamma(\frac{3}{4} - \frac{i\theta}{2\pi})}{\Gamma(\frac{3}{4} + \frac{i\theta}{2\pi})} \frac{\Gamma(\frac{1}{2} + \frac{i\theta}{2\pi})^2}{\Gamma(\frac{1}{2} - \frac{i\theta}{2\pi})^2} \frac{\Gamma(\frac{1}{4} - \frac{i\theta}{2\pi})}{\Gamma(\frac{1}{4} + \frac{i\theta}{2\pi})}.$$

The accompanying Bethe equations for the roots $u_j^{(+)} \equiv u_j$ are

$$1 = \frac{Q_0^-(u_j) \tilde{Q}_+^{++}(u_j)}{Q_0^+(u_j) \tilde{Q}_+^{--}(u_j)}, \quad j = 1, \dots, n^+ \quad (70)$$

for the free case, while for the mixed case they change to

$$\left(\frac{u_j - \frac{i\pi}{2}}{u_j + \frac{i\pi}{2}} \right)^2 = \frac{Q_0^-(u_j) \tilde{Q}_+^{++}(u_j)}{Q_0^+(u_j) \tilde{Q}_+^{--}(u_j)}, \quad j = 1, \dots, n^+. \quad (71)$$

(The equations for the $u_j^{(-)}$ roots, denoted here by v_j , are obtained from eqs. (70), (71) by the $(u_j, n^+) \rightarrow (v_j, n^-)$ substitutions). It is straightforward to show, using the transformation properties of the various functions appearing in eq. (69), that if θ_β is a solution, then so is $-\theta_\beta$. Note that substituting $u_j \rightarrow -u_j$ (but keeping u_i for $i \neq j$ the same) changes eqs. (70), (71) to their inverses, thus the roots are also doubled: to every root u_j solving eqs. (69)-(70) (or eqs. (69)-(71)) there is another one $-u_j$.

Apart from the $r_i^2(\theta_\beta)$ factors in eq. (69) and the l.h.s. in eq. (71) the system consisting of (69)-(70) (or (69)-(71)) is identical to the system of BA equations in the periodic case [26] but for $2n_0$ particles with rapidities coming in pairs $(\theta_\beta, -\theta_\beta)$, accompanied by $2n_+$ “left” roots $(u_j, -u_j)$ ($2n_+$ “right” roots $(v_j, -v_j)$) coming also in pairs. Thus the effect of the integrability preserving boundaries is twofold: on the one hand they double the particles and the left/right roots, while on the other hand they introduce the $r_i^2(\theta_\beta)$ factors and the l.h.s. in eq. (71). However, in the light of the observations in the previous paragraph, even in the presence of these modifications the solutions of the boundary BA equations above are n_0 pairs of $(\theta_\beta, -\theta_\beta)$, accompanied by $2n_+$ ($2n_-$) pairs of roots $(u_j, -u_j)$ ($(v_j, -v_j)$).

Let us now consider the limit classical limit when $\mu \rightarrow 0$. We also let both the number of particles n_0 , and the number of roots $2n_+ \sim 2n_- \sim 2n_0$ go to infinity enforcing that all $\pm\theta_\alpha$, $\pm u_j$ and $\pm v_j$ -s become large allowing us to take the logarithms of the Bethe equations and use the “Coulomb approximations” (large θ or u):

$$-i \log S^2(\theta) \rightarrow -\frac{\pi}{\theta}, \quad -i \log r_1^2(\theta) \rightarrow -\frac{3\pi}{2\theta}, \quad -i \log r_2^2(\theta) \rightarrow \frac{\pi}{2\theta}, \quad \log \frac{u+iB}{u-iB} \rightarrow \frac{2iB}{u}.$$

We consider here the classical limit of eq.(69) in the absence of any roots. Then, after taking the log of both sides and using the previous approximations, we find:

$$\frac{2\mu}{\pi} \sinh(\theta_\beta) - \sum_{\alpha \neq \beta} \left(\frac{1}{\theta_\beta + \theta_\alpha} + \frac{1}{\theta_\beta - \theta_\alpha} \right) + \frac{-\frac{3}{2\theta_\beta}}{\frac{1}{2\theta_\beta}} = 2m_\beta, \quad m_\beta \in \mathbb{Z} \quad \beta = 1, \dots, n_0.$$

where the upper line applies for the free and the lower line for the mixed diagonal boundary conditions.

These equations describe a system of (1D) particles put into the combination of constant external forces ($2m_\beta$) and a (confining) potential $V(\theta) = \frac{2\mu}{\pi} \cosh(\theta)$, which interact by the Coulomb repulsion not only with each other but also with their “mirror images” at $-\theta_\beta$. The interaction of the particles with the boundaries is described by the last terms. To emphasize our interpretation we rewrite

$$\frac{-\frac{3}{2\theta_\beta}}{\frac{1}{2\theta_\beta}} = -\frac{1}{2\theta_\beta} + \frac{-\frac{1}{\theta_\beta}}{\frac{1}{\theta_\beta}}$$

since the $1/(2\theta_\beta)$ term reproduces the Coulomb repulsion between the β -th particle and its mirror image (and thus can be absorbed into the first term of the sum), and the interactions with the boundaries differs only by a sign between the free and the mixed diagonal cases. This interaction with the mirror images and with the boundaries are the two new phenomena which are absent in the periodic case.

It is convenient to handle the asymptotically large nature of θ_β by an appropriate rescaling $\theta_\beta = M\xi_\beta$, where we choose (following [26])

$$M = -\frac{\log \mu}{2} \sim n_0.$$

Then, according to [26], in the $n_0 \rightarrow \infty$ limit the potential $V_M(\xi) = \frac{2\mu}{\pi} \cosh(M\xi)$ becomes an infinitely deep (square) potential well confining ξ to the interval $(-2, 2)$, and in this interval the equations become

$$\frac{1}{M} \sum_{\alpha}^L \frac{1}{\xi_\beta + \xi_\alpha} + \frac{1}{M} \sum_{\alpha \neq \beta}^L \frac{1}{\xi_\beta - \xi_\alpha} + \frac{\frac{1}{M\xi_\beta}}{-\frac{1}{M\xi_\beta}} = -2m_\beta. \quad (72)$$

Note that in the limit $n_0 \rightarrow \infty$, n_0/M finite, the last terms (originated in the interaction with the boundaries) become sub-leading. Apart from the boundary contributions, this system of equations is identical to the one considered in the periodic case in [26], but for $2n_0$ particles located at $\xi_\beta, -\xi_\beta$; indeed changing $\xi_\beta \rightarrow -\xi_\beta$ in (72) changes $m_\beta \rightarrow -m_\beta$, thus providing the (identical) other half of the system. In [26] the analogous system was investigated by putting all mode numbers m_β equal to a common m , implying that in the continuous limit there is only one ξ cut. Looking at eq. (72) as a system for $2n_0$ particles coming in pairs, shows that we cannot choose a common mode number unless $m = 0$.

To implement the confining flat potential we introduce boundaries with charges q at $\xi = \pm 2$, which we eventually will take to zero. In the presence of the boundary charges q , the equilibrium condition for the system of charges and mirror charges becomes

$$\sum_{\alpha \neq \beta}^{n_0} \left(\frac{1}{\xi_\beta - \xi_\alpha} + \frac{1}{\xi_\beta + \xi_\alpha} \right) + \frac{\frac{3}{2\xi_\beta}}{-\frac{1}{2\xi_\beta}} = \frac{2qM\xi_\beta}{4 - \xi_\beta^2} \quad \beta = 1, \dots, n_0.$$

Now defining

$$Q(z) = z \prod_i^{n_0} (z - \xi_i)(z + \xi_i) = z \prod_i^{n_0} (z^2 - \xi_i^2), \quad P(z) = \frac{1}{z} \prod_i^{n_0} (z^2 - \xi_i^2),$$

one readily proves, that

$$\begin{aligned}\frac{Q''(\xi_\beta)}{Q'(\xi_\beta)} &= 2 \left(\sum_{\alpha \neq \beta}^{n_0} \left(\frac{1}{\xi_\beta - \xi_\alpha} + \frac{1}{\xi_\beta + \xi_\alpha} \right) + \frac{3}{2\xi_\beta} \right); \\ \frac{P''(\xi_\beta)}{P'(\xi_\beta)} &= 2 \left(\sum_{\alpha \neq \beta}^{n_0} \left(\frac{1}{\xi_\beta - \xi_\alpha} + \frac{1}{\xi_\beta + \xi_\alpha} \right) - \frac{1}{2\xi_\beta} \right).\end{aligned}$$

In the definition of P, Q the product terms represent the $2n_0$ particles at $\pm\xi_\beta$, and the prefactors are introduced to account for the boundary contributions. Though they look similar, they are rather different: $Q(z)$ is a polynomial of order $2n_0 + 1$ having zeros at $z = 0, \pm\xi_i$, while $P(z)$ is an analytical function with zeros at $\pm\xi_i$, z^{2n_0-1} asymptotic behaviour at $z \rightarrow \infty$, and a single pole at $z = 0$.

Let us analyze the free boundary condition with $m = 0$. As a result of the equilibrium conditions, the condition

$$(4 - \xi_\beta^2)Q''(\xi_\beta) - 4qM\xi_\beta Q'(\xi_\beta) = 0$$

holds (and similarly for $\xi_\beta \rightarrow -\xi_\beta$). Since the polynomial $r(z) = (4 - z^2)Q''(z) - 4qMzQ'(z)$ has a zero at $z = 0$, and is also of order $2n_0 + 1$, it must be proportional to $Q(z)$. Matching the coefficients of the highest powers of z in them, we get

$$(4 - z^2)Q''(z) - 4qMzQ'(z) + (2n_0 + 1)(2n_0 + 4qM)Q(z) = 0.$$

With $z = 2y$ this is the defining equation of the Jacobi polynomials $P_n^{(\alpha, \beta)}(y)$ with $\alpha = \beta = 2qM - 1$, and $n = 2n_0 + 1$. Therefore

$$Q(z) = P_{2n_0+1}^{(2qM-1, 2qM-1)}\left(\frac{z}{2}\right),$$

and the ξ_i are twice the positive roots of this polynomial.

Realizing that the “free boundary” resolvent $G(z)$ is related to $Q(z)$

$$G(z) = \frac{1}{M} \left(\frac{1}{z} + \sum_i^L \left(\frac{1}{z - \xi_i} + \frac{1}{z + \xi_i} \right) \right) = \frac{1}{M} \frac{Q'(z)}{Q(z)}, \quad (73)$$

one can derive an equation for it. Indeed using this relation and the equilibrium conditions, we obtain

$$\frac{1}{M}G' = -G^2 + \frac{4qz}{4 - z^2}G - \frac{(2n_0 + 1)(2n_0 + 4qM)}{M^2(4 - z^2)}.$$

In the continuum limit when $M \rightarrow \infty, n_0 \rightarrow \infty, n_0/M \sim \mathcal{O}(1)$, one can drop the l.h.s. and obtain an algebraic equation for $G(z)$. Particularly interesting is the solution for $q \rightarrow 0$:

$$G(z) = \pm \frac{2n_0}{M} \frac{1}{z\sqrt{1 - \frac{4}{z^2}}}.$$

On the one hand it shows an inverse square root type singularity at $z = \pm 2$, on the other hand it differs only in the $n_0 \rightarrow 2n_0$ substitution from the analogous periodic expression [26].

Now we return to the investigation of the original problem eq. (72) for the free boundary case. We choose a common mode number m for the positive solutions ξ_β (likewise $-m$ for the negative ones $-\xi_\beta$). Therefore in the continuum limit, the resolvent has two cuts, one running between $(0, 2)$, the other between $(-2, 0)$, i.e

$$G(z) = \frac{1}{2\pi} \int_0^2 dw \frac{\rho_+(w)}{z - w} + \frac{1}{2\pi} \int_{-2}^0 dw \frac{\rho_-(w)}{z - w}.$$

$G(z)$, introduced explicitly in eq. (73), satisfies $G(-z) = -G(z)$; imposing this symmetry on this expression relates the two ρ -s to each other

$$\rho_-(-w) = \rho_+(w) \equiv \rho(w),$$

thus

$$G(z) = \frac{1}{2\pi} \int_0^2 dw \rho(w) \left(\frac{1}{z-w} + \frac{1}{z+w} \right).$$

In terms $G(z)$ one can write the continuum limit of eq.(72) as

$$\mathcal{G}(z) = G(z + i\epsilon) + G(z - i\epsilon) = -2m, \quad x \in (0, 2). \quad (74)$$

We solve this by making an Ansatz for the $\rho(w)$ density motivated by the finding of [26]

$$\rho(w) = \frac{B}{\sqrt{4-w^2}},$$

where B is a constant. Computing the principal value (PV) integral one finds that $\mathcal{G}(z) = 0$, which becomes a constant – as required by eq.(74) – implying that $m = 0$. B is determined from the $z \rightarrow \infty$ asymptotics of G : $G(z) \rightarrow \frac{2n_0+1}{M} \frac{1}{z} \sim \frac{2n_0}{M} \frac{1}{z}$, leading to $B = \frac{4n_0}{M}$. With these values one finds (for any z)

$$G(z) = \pm \frac{2n_0}{M} \frac{1}{z\sqrt{1-\frac{4}{z^2}}},$$

which, after applying the Zhukowski map, $z = \lambda + \frac{1}{\lambda}$, becomes

$$G(\lambda) = \pm \frac{2n_0}{M} \frac{\lambda}{\lambda^2 - 1}.$$

We can see that $G(\lambda)$ is related to the (single) quasi-momentum of the $O(4)$ σ -model with integrable (free) boundaries, which corresponds to the uncharged, constant solution $n = 0$.

The situation with the mixed diagonal boundary, i.e. the one described by the second version of eq. (72) with the $-1/\xi_\beta$ boundary contribution, is a bit different from the free case. First, we found numerical solutions for $m = 0$ always with two imaginary roots at $\xi_0 = \pm i\alpha$. In the $M \rightarrow \infty$ and $n_0 \rightarrow \infty$ limit, however, $\alpha \rightarrow 0$ and we recover the free case. Defining the resolvent as

$$\tilde{G}(z) = \frac{1}{M} \left(-\frac{1}{z} + \sum_i^{n_0} \left(\frac{1}{z-\xi_i} + \frac{1}{z+\xi_i} \right) \right),$$

one can repeat the previous consideration leading to an identical form for $\tilde{G}(z)$ in the continuum limit. The quasi-momenta is related to the $n = 0$ case in the classical analysis. To recover the quasi-momenta of the one-cut solution, one has to introduce magnonic roots and let them condense on an imaginary cut.

4 Conclusions

In this paper we analyzed the integrable boundary conditions of the $O(N)$ non-linear σ -models at various levels. Classically, the models are conformal and conformality of the boundary condition implies the existence of a special set of infinitely many conserved charges. Indeed, in the conformal boundary conditions, the boundary limit of the difference of the light-cone components of the energy-momentum tensor vanishes. In particular, this implies that the difference of any integer power of the same light-cone components of the energy-momentum tensor will also vanish, leading to an infinite family of conserved charges. This conformality requirement can be guaranteed by connecting the time derivative of the fundamental field to its space derivative at the boundary

by an anti-symmetric matrix, M [10]. By a similarity transformation, this matrix can be brought into a 2 by 2 block diagonal form, with different matrix entries. Taking various limits of the matrix, one can recover boundary conditions with Dirichlet and Neumann directions. In the coset formulation one has to ensure that the time and space components of the conserved currents are orthogonal for the trace at the boundary. This can be achieved if one is obtained from the other by a commutation with another matrix. However, this description is equivalent to the previous one – formulated on the fundamental fields – only if the constraint is added to the boundary Lagrangian. Thus the boundary condition in [10] has to be modified with a non-linear term (25). Conformality of the boundary condition ensures the existence of infinitely many conserved charges. Whether these charges are in involution or whether they provide enough conserved charges for the theory to be integrable is not investigated in the paper.

In order to classify classical solutions and to have a relation to the quantum theory we introduced the boundary Lax formulation of the problem. Integrable boundary conditions are classified by $O(N)$ group valued matrices, U , located at the boundaries. They are the classical analogues of the reflection matrices and have to satisfy an evolution equation (32). Unfortunately we could find only constant matrix solutions of these equations, which are related to the mixture of Neumann and vanishing Dirichlet boundary conditions. We characterized the analytical structure and symmetry properties of the spectral curve of these boundary conditions.

The quantization of the model is a highly non-trivial task. In the bulk case the anomaly terms of the higher spin equations of motion were classified and shown to have the same structure as the original one, leading to quantum conservation laws. Unfortunately, anomalous terms can appear at the boundary too, which can spoil the integrability of all the boundary conditions. As we have no control over the boundary anomalous terms, instead, assuming integrability, we classified the quantum integrable boundary conditions by the solutions of the boundary Yang-Baxter equations. We list all solutions found so far: these contain diagonal reflection matrices with two different entries. They correspond to Dirichlet and vanishing Neumann boundary conditions; there are also boundary conditions with a single 2 by 2 block and otherwise diagonal; in the $O(2N)$ case there is a boundary condition in which all 2 by 2 blocks are the same. Bethe Ansatz equations in spin-chains have been formulated in all cases, except when we have odd Dirichlet (or Neumann) directions [11]. We used them to formulate the Bethe-Yang equations which determine the asymptotically large volume spectrum of the models on the interval with identical boundary conditions on the two sides. Since the classifications at the quantum and classical levels do not match, we analyzed the classical limit of certain solutions in the $O(4)$ models with all Neumann (free) and two Dirichlet and two Neumann (mixed) boundary conditions. We found that they correspond to the classical solution whose spectral curve we previously calculated explicitly.

There are many open questions. There is obviously a mismatch between the boundary conditions found at the various levels. This could be related to the fact that we analyzed in many cases conformal boundary conditions. It would be very interesting to see whether the corresponding special set of conserved charges do commute or, if we can find higher spin Casimir charges as well. It would be also challenging to find time- and even field-dependent boundary U matrices for any boundary conditions, which can be described by an anti-symmetric matrix, M . The quantum theory suggests that they may exist only for the cases of one single block or all blocks the same. To get some insight one can calculate the classical limit of all the Bethe-Yang equations including also magnonic roots and other groups. In doing so, the derivation of the Bethe Ansatz equations in the missing cases, i.e. when we have odd number of Neumann or Dirichlet directions, are crucial.

It would be also of interest to figure out a quantisation of the boundary system in which the anomalous terms can be directly calculated, and the existence of quantum conservation laws can be explicitly checked.

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A Symmetries of the boundary conditions

In this appendix we investigate the residual symmetries of the model (17), for the various boundary conditions in the language of the unconstrained variable. First we recall that as a consequence of the (bulk) conservation of the currents, $\partial^\alpha J_\alpha^A = 0$ ($A = ij, iN$), the charges satisfy

$$\partial_\tau Q^A = \int_0^\pi d\sigma \partial^\tau J_\tau^A = - \int_0^\pi d\sigma \partial^\sigma J_\sigma^A = J_\sigma^A(\pi) - J_\sigma^A(0). \quad A = ij, iN.$$

Assuming again the lack of interplay between the two boundaries (“locality”), we conclude that those components of the bulk charges stay conserved in the presence of boundaries for which $J_\sigma^A| = 0$.

Now consider the case when l ($0 \leq l \leq N - 1$) of the ξ^i -s satisfy Neumann ($\partial_\sigma \xi^i = 0$), while the rest are Dirichlet ($\partial_\tau \xi^i = 0$, $i = l + 1, \dots, N - 1$) b.c.-s. To ease the notation we write the $\vec{\xi}$ field as $\vec{\xi} = (\vec{r}, \vec{s})$ with boundary values $\partial_\sigma \vec{r} = 0$ and $\partial_\tau \vec{s} = 0$, where $\vec{r} = (r^1, \dots, r^l)$ and $\vec{s} = (s^1, \dots, s^{N-1-l})$. (Note that in this notation $\xi^2 = \vec{r}^2 + \vec{s}^2$ and on the boundary $\vec{\xi} \cdot \partial_\sigma \vec{\xi} = \vec{s} \cdot \partial_\sigma \vec{s}$, $\vec{\xi} \cdot \partial_\tau \vec{\xi} = \vec{r} \cdot \partial_\tau \vec{r}$). This notation is useful since the currents J_α^{ij} can be split into three sets

$$J_\alpha^{ij} = \frac{4}{(1 + \xi^2)^2} (r^i \partial_\alpha r^j - r^j \partial_\alpha r^i), \quad i, j = 1, \dots, l; \quad J_\alpha^{l+i, l+j} = \frac{4}{(1 + \xi^2)^2} (s^i \partial_\alpha s^j - s^j \partial_\alpha s^i)$$

where $i, j = 1, \dots, N - l - 1$, and

$$J_\alpha^{i, l+j} = \frac{4}{(1 + \xi^2)^2} (r^i \partial_\alpha s^j - s^j \partial_\alpha r^i), \quad i = 1, \dots, l, \quad j = 1, \dots, N - l - 1,$$

while those of J_α^{iN} split into two sets

$$J_\alpha^{iN} = - \frac{2}{(1 + \xi^2)^2} (2r^i \vec{\xi} \cdot \partial_\alpha \vec{\xi} + (1 - \xi^2) \partial_\alpha r^i), \quad i = 1, \dots, l;$$

$$J_\alpha^{l+j, N} = - \frac{2}{(1 + \xi^2)^2} (2s^j \vec{\xi} \cdot \partial_\alpha \vec{\xi} + (1 - \xi^2) \partial_\alpha s^j). \quad j = 1, \dots, N - l - 1.$$

It is obvious that the boundary values of $J_\sigma^{ij}| = 0$ ($ij = 1, \dots, l$) as a result of the Neumann b.c. on \vec{r} , and consequently the $SO(l)$ symmetry generated by these currents survives. It is slightly more complicated to see the symmetry coming from the fields with Dirichlet b.c., however one can verify that the combinations

$$\tilde{J}_\sigma^{l+i, l+j} = J_\sigma^{l+i, l+j} + \frac{2s^i}{1 - \xi^2} |J_\sigma^{jN} - \frac{2s^j}{1 - \xi^2} |J_\sigma^{iN}$$

(where $|$ stand for the boundary values of the expressions in question) do vanish on the boundary, and the $SO(N - 1 - l)$ symmetry generated by them also survives. Thus the complete symmetry compatible with the boundaries is $SO(l) \times SO(N - 1 - l)$ when l of the ξ^i fields satisfy Neumann while the remaining $N - 1 - l$ of them generic Dirichlet b.c.-s.

If l takes its maximal value $l = N - 1$, then all $\vec{\xi}$ fields satisfy Neumann b.c., and looking at (8) we see that also all components of n^I do the same. From the expressions of the currents above, it follows that all of the currents vanish at the boundary, and the full $SO(N)$ symmetry of the bulk theory is preserved by this b.c.. If, on the other hand, l vanishes ($l = 0$), then all the $\vec{\xi}$ fields satisfy Dirichlet b.c., and from (8) it follows that all components of n^I do the same. From the

previous argument leading to $\tilde{J}_\sigma^{l+i,l+j}$ above, it follows that the symmetry of this “all Dirichlet” b.c. is $SO(N-1)$, i.e. the bulk $SO(N)$ symmetry is broken.

In the following we derive the boundary conditions for the constrained variables. Without loss of generality we assume that $\vec{s}| = (\alpha, 0, \dots, 0)$. This, in particular, implies the vanishing Dirichlet boundary condition for n_i :

$$n_i| = \frac{2s_i}{1+r^2+\alpha^2}| = 0 \quad ; \quad i = l+2, \dots, N-1$$

Let us introduce primed variables by the combinations

$$\begin{aligned} n'_{l+1} &= \frac{1}{\sqrt{1+\alpha^2}} (n_{l+1} - \alpha n_N); \\ n'_N &= \frac{1}{\sqrt{1+\alpha^2}} (\alpha n_{l+1} + n_N), \end{aligned}$$

as well as $n'_i = n_i$ for $i \neq l+1, N$. One can easily see that n'_{l+1} satisfies Dirichlet boundary condition

$$n'_{l+1}| = \frac{\alpha}{\sqrt{1+\alpha^2}}.$$

Actually the primed coordinates have length 1 and can be obtained from n_i by an orthogonal transformation. One can check that from these new coordinates, $N-l-1$ satisfy a Dirichlet boundary condition, while $l+1$ satisfy the generalized Neumann boundary condition (24). This makes the two formulations completely equivalent. Choosing, in particular, $\forall s^j| = 0$ (i.e. $\vec{s}| = 0$, “vanishing” Dirichlet) makes both $\partial_\sigma n^i| = 0$ and $\partial_\tau n^{l+j}| = 0$ in addition to guaranteeing $\partial_\sigma n^N| = 0$. Looking at the previous expressions for the various currents reveals that for this vanishing Dirichlet b.c. not only $J_\sigma^{l+i,l+j}$ vanish on the boundary but also J_σ^{iN} (while $J_\sigma^{l+j,N}| \neq 0$). The – now conserved – J_α^{iN} combine with J_α^{ij} to generate an $SO(l+1)$ symmetry. Thus in this case the total symmetry compatible with the boundaries is $SO(l+1) \times SO(N-l-1)$.

Now that we determined the symmetries compatible with the consistent boundary conditions and the (matrix) form of J_σ and J_τ on the boundary, we can search for a constant U matrix which satisfies (37). When the remaining symmetry is $SO(l+1) \times SO(N-l-1)$, the construction of such a U is known from the mathematical literature [21], since $SO(N)/SO(l+1) \times SO(N-l-1)$ is a symmetric space. However, when the residual symmetry is $SO(l) \times SO(N-l-1)$, no such U matrix exists, since $SO(N)/SO(l) \times SO(N-l-1)$ is NOT a symmetric space for $1 \leq l < N-1$. This would be an example when a model is conformal but not Lax integrable.

The b.c.-s (21) for the ξ fields can be translated into b.c.-s for the currents J_σ^{ij} and J_σ^{iN} : we take eq. (3) and (4) and replace $\partial_\sigma \xi^k|$ in them by the r.h.s. of eq. (21); then, after some algebraic manipulations, we try to identify J_τ^{ij} and J_τ^{iN} in what is obtained. Since we previously derived the b.c.

$$J_\sigma^{IJ} = \frac{1}{2} \left([M, J_\tau] + [mMm, J_\tau] \right)^{IJ}, \quad m^{IJ} = \delta^{IJ} - 2n^I n^J, \quad (75)$$

we need only to check whether this condition is a consequence of (21). A direct computation gives

$$\begin{aligned} \left([M, J_\tau] + [mMm, J_\tau] \right)^{im} &= \frac{8}{(1+\xi^2)^2} \left(\xi^i M_{ml} \dot{\xi}^l - \xi_m M_{il} \dot{\xi}^l \right. \\ &\quad \left. - \frac{2\vec{\xi} \cdot \dot{\vec{\xi}}}{1+\xi^2} \left(\xi^i (M_{ml} \xi^l + M_{mN}) - \xi^m (M_{il} \xi^l + M_{iN}) \right) \right), \end{aligned}$$

$$\begin{aligned} \left([M, J_\tau] + [mMm, J_\tau] \right)^{iN} &= \frac{4}{(1+\xi^2)^2} \left(2\xi^i M_{Ni} \dot{\xi}^l - (1-\xi^2) M_{il} \dot{\xi}^l \right. \\ &\quad \left. + \frac{2\vec{\xi} \cdot \dot{\vec{\xi}}}{1+\xi^2} \left((1-\xi^2) (M_{il} \xi^l + M_{iN}) + 2\xi^i M_{Ni} \xi^l \right) \right), \end{aligned}$$

and after a not very informative computation one obtains that our currents do indeed satisfy (75).

Next we translate these b.c.-s for boundary conditions for the n^I fields. Starting from eq. (21) one finds

$$\partial_\sigma n^K = \left(\sum_I M_{IK} \dot{n}^I - n^K \sum_{L,I} n^L M_{IL} \dot{n}^I \right).$$

This b.c. is highly non-linear and depends also on the boundary values of n^I , not only on its derivatives. The first term on the r.h.s. gives the naïve (and inconsistent) b.c. derived by Moriconi [10] and analyzed by He and Zhao [12], and the second non-linear term guarantees that this b.c. is consistent, i.e. the r.h.s vanishes also when it is multiplied by n^K and summed over K . For $N = 3$, when the antisymmetric M_{IJ} can be described in terms of a three vector $\vec{q} = (q^1, q^2, q^3)$ as $M_{IJ} = \epsilon_{IJK} q^K$, this b.c. is identical to the one derived by Corrigan and Sheng [7].

To see what residual symmetries are compatible with eqs. (21) and (75) we determine which parts of the bulk $SO(N)$ transformations leave the boundary Lagrangian $\sum_{IJ} n^I M_{IJ} \dot{n}^J$ invariant. Since the infinitesimal $SO(N)$ transformations can be written as

$$n^K \rightarrow n^K + \epsilon^{AB} \Sigma_{AB}^{KL} n^L, \quad \Sigma_{AB}^{KL} = \delta_A^K \delta_B^L - \delta_A^L \delta_B^K, \quad \epsilon^{AB} = -\epsilon^{BA},$$

one readily obtains that the transformations commuting with M ,

$$[M, \epsilon^{AB} \Sigma_{AB}] = 0,$$

are the ones that leave the boundary Lagrangian invariant. Since, by an appropriate orthogonal transformation, any antisymmetric $N \times N$ matrix can be brought to the block diagonal form

$$M_{IJ} = \begin{pmatrix} 0 & m_1 & & & & \\ -m_1 & 0 & & & & \\ & & 0 & m_2 & & \\ & & -m_2 & 0 & & \\ & & & & & \\ & & & & & 0 \end{pmatrix}$$

(all m_i -s, $i = 1, \dots, \lfloor \frac{N}{2} \rfloor$, are real) it is enough to analyze the symmetries of the model when we use these matrices in the boundary Lagrangian. If we assume that the first k m_i -s are all different $m_i \neq m_j$ $i, j = 1, \dots, k$ and non zero, while the rest vanishes, then the subgroup of $SO(N)$ commuting with this M is $\underbrace{SO(2) \times \dots \times SO(2)}_k \times SO(N - 2k)$. If, on the other hand, $N = 2L$ is

even, and all m_i -s are equal and non-vanishing, then the subgroup of $SO(2L)$ commuting with this M is $U(L)$ [21].

B Comments on boundary conditions with not-maximal radius

In this appendix we consider the boundary condition when the fields are restricted to a sphere S^{n-k-1} of radius $\cos \phi$. In this case the constraints at the boundary are the following:

$$\begin{aligned} \tilde{\mathbf{n}}^t \tilde{\mathbf{n}} &= \cos(\phi)^2, \\ \hat{\mathbf{n}}^t \hat{\mathbf{n}} &= \sin(\phi)^2, \\ \partial_0 \hat{\mathbf{n}} &= 0. \end{aligned} \tag{76}$$

If we want to use stereographic projection to obtain the unconstrained variables in the bulk we have to introduce some extra constraints at the boundary, as (76) is not a flat hypersurface in

\mathbb{R}^{n-1} . We can use the following coordinates:

$$\begin{aligned}\hat{\mathbf{n}} &= (0, \dots, 0, \sin(\phi)), \\ n_i &= \frac{2\xi_i}{1+\xi^2}, \quad n_n = \frac{1-\xi^2}{1+\xi^2}, \quad \xi^2 = \xi_i\xi_i.\end{aligned}$$

At the boundary, however, the ξ_i -s live only on an $(n-k-1)$ -dimensional sphere because

$$\xi^2 = \frac{1 - \sin(\phi)}{1 + \sin(\phi)}.$$

We can see that the ξ_i -s are subject to a constraint at the boundary. Therefore, in the language of ξ_i -s and \mathbf{n} -s this boundary condition cannot be written in any simple homogeneous Neumann and Dirichlet form.

In the coset language there is a subalgebra \mathfrak{h} (algebra of the little group) where $\mathfrak{h}\mathbf{n}_0 = 0$ and another subalgebra \mathfrak{g}_1 (which generates the S^{n-k-1} , $G_1 = O(n-k)$) where $\hat{\mathbf{n}} = \exp(\mathfrak{g}_1)\mathbf{n}_0$. In this case the reference vector is $\mathbf{n}_0 = (\cos(\phi), 0, \dots, 0, \sin(\phi))$.

We can decompose \mathfrak{g}_1 to $\mathfrak{h}_1 \oplus \mathfrak{f}_1$, where $\mathfrak{h}_1 \subset \mathfrak{h}$, and we can convince ourselves that \mathfrak{f}_1 is not a subset of \mathfrak{f} . Therefore the algebra of the currents cannot be decomposed into a $\mathbb{Z}_2 \times \mathbb{Z}_2$ graded algebra, which means that the boundary conditions cannot be written as a commutator and an anticommutator.

References

- [1] Alexander B. Zamolodchikov and Alexei B. Zamolodchikov. Factorized s Matrices in Two-Dimensions as the Exact Solutions of Certain Relativistic Quantum Field Models. *Annals Phys.*, 120:253–291, 1979.
- [2] P. Hasenfratz and F. Niedermayer. The Exact mass gap of the O(N) sigma model for arbitrary N is ≥ 3 in d = 2. *Phys. Lett.*, B245:529–532, 1990.
- [3] Niklas Beisert et al. Review of AdS/CFT Integrability: An Overview. *Lett. Math. Phys.*, 99:3–32, 2012.
- [4] Konstantinos Zoubos. Review of AdS/CFT Integrability, Chapter IV.2: Deformations, Orbifolds and Open Boundaries. *Lett. Math. Phys.*, 99:375–400, 2012.
- [5] Subir Ghoshal and Alexander B. Zamolodchikov. Boundary S matrix and boundary state in two-dimensional integrable quantum field theory. *Int.J.Mod.Phys.*, A9:3841–3886, 1994.
- [6] Subir Ghoshal. Boundary S matrix of the O(N) symmetric nonlinear sigma model. *Phys. Lett.*, B334:363–368, 1994.
- [7] Edward Corrigan and Z-M. Sheng. Classical integrability of the O(N) nonlinear sigma model on a half line. *Int. J. Mod. Phys.*, A12:2825–2834, 1997.
- [8] M. Moriconi and A. De Martino. Quantum integrability of certain boundary conditions. *Phys. Lett.*, B447:292–297, 1999.
- [9] Marco Moriconi. Integrable boundary conditions for the O(n) nonlinear sigma model. In *NATO Advanced Research Workshop on Statistical Field Theories Villa Olmo, Como, Italy, June 18-23, 2001*, 2001.
- [10] M. Moriconi. Integrable boundary conditions and reflection matrices for the O(N) nonlinear sigma model. *Nucl. Phys.*, B619:396–414, 2001.

- [11] D. Arnaudon, J. Avan, N. Crampe, A. Doikou, L. Frappat, and E. Ragoucy. Classification of reflection matrices related to (super) Yangians and application to open spin chain models. *Nucl. Phys.*, B668:469–505, 2003.
- [12] Wen-li He and Liu Zhao. Classical $O(N)$ nonlinear sigma model on the half line: A Study on consistent Hamiltonian description. *Phys. Lett.*, B570:251–259, 2003.
- [13] Nelia Mann and Samuel E. Vazquez. Classical Open String Integrability. *JHEP*, 04:065, 2007.
- [14] Amit Dekel and Yaron Oz. Integrability of Green-Schwarz Sigma Models with Boundaries. *JHEP*, 08:004, 2011.
- [15] Zoltan Bajnok, Minkyoo Kim, and Laszlo Palla. Spectral curve for open strings attached to the $Y=0$ brane. *JHEP*, 04:035, 2014.
- [16] N. J. MacKay and B. J. Short. Boundary scattering, symmetric spaces and the principal chiral model on the half line. *Commun. Math. Phys.*, 233:313–354, 2003. [Erratum: *Commun. Math. Phys.* 245,425(2004)].
- [17] N. J. MacKay and C. A. S. Young. Classically integrable boundary conditions for symmetric space sigma models. *Phys. Lett.*, B588:221–227, 2004.
- [18] J. M. Evans and A. J. Mountain. Commuting charges and symmetric spaces. *Phys. Lett. B* **483**, 290 (2000).
- [19] N. Beisert, V. A. Kazakov, and K. Sakai. Algebraic curve for the $SO(6)$ sector of AdS/CFT. *Commun. Math. Phys.*, 263:611–657, 2006.
- [20] N. Beisert, V. A. Kazakov, K. Sakai, and K. Zarembo. The Algebraic curve of classical superstrings on $AdS(5) \times S^5$. *Commun. Math. Phys.*, 263:659–710, 2006.
- [21] Sigurdur Helgason. *Differential geometry, Lie groups, and symmetric spaces*, volume 34 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2001. Corrected reprint of the 1978 original.
- [22] Alexander M. Polyakov. Hidden Symmetry of the Two-Dimensional Chiral Fields. *Phys. Lett.*, B72:224–226, 1977.
- [23] Y. Y. Goldschmidt and Edward Witten. Conservation Laws in Some Two-dimensional Models. *Phys. Lett.*, B91:392–396, 1980.
- [24] Zoltan Bajnok and Laszlo Palla. Boundary finite size corrections for multiparticle states and planar AdS/CFT. *JHEP*, 01:011, 2011.
- [25] Tamas Gombor and Laszlo Palla. Algebraic Bethe Ansatz for $O(2N)$ sigma models with integrable diagonal boundaries. *JHEP*, 02:158, 2016.
- [26] Nikolay Gromov, Vladimir Kazakov, Kazuhiro Sakai, and Pedro Vieira. Strings as multiparticle states of quantum sigma-models. *Nucl. Phys.*, B764:15–61, 2007.
- [27] V. A. Kazakov, A. Marshakov, J. A. Minahan, and K. Zarembo. Classical/quantum integrability in AdS/CFT. *JHEP*, 05:024, 2004.