

A statistical method to estimate low-energy hadronic cross sections

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Abstract. In this article we propose a model based on the Statistical Bootstrap approach to estimate the cross sections of different hadronic reactions up to a few GeV in c.m.s energy. The method is based on the idea, when two particles collide a so called fireball is formed, which after a short time period decays statistically into a specific final state. To calculate the probabilities we use a phase space description extended with quark combinatorial factors and the possibility of more than one fireball formation. In a few simple cases the probability of a specific final state can be calculated analytically, where we show that the model is able to reproduce the ratios of the considered cross sections. We also show that the model is able to describe proton - antiproton annihilation at rest. In the latter case we used a numerical method to calculate the more complicated final state probabilities. Additionally, we examined the formation of strange and charmed mesons as well, where we used existing data to fit the relevant model parameters.

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1 Introduction

Transport models are very important tools to understand the dynamics of heavy ion collisions, plan the detectors, or even understand the experimental results. Very essential ingredients of these models are the elementary hadronic cross sections. Some of these are experimentally well known in a broad energy range, however, cross sections involving short lifetime particles cannot be determined experimentally. Below 1 – 2 GeV effective field theories can be used [1,2], however, the many resonances – which are not part of the model – restrict the energy range of validity of these methods. An interesting attempt was made by J. Van de Wiele and S. Ong [3] to use Regge propagators in higher energies to extend the low energy effective field theories.

Statistical methods are useful tools to determine particle multiplicities in high energy collisions. The first attempt can be associated with Fermi (1950) who proposed a simple phase space method to describe the cross sections of binary collisions [4]. Fermi estimated the probabilities of one and multiple pion creation in nucleon - nucleon collisions based solely on statistical considerations. Although, he treated the momentum and angular momentum conservation only approximately, he could describe some of the cross sections correctly.

Latter R. Hagedorn developed the statistical bootstrap method [5,6], which was able to explain hadron multi-

plicities. The basic assumption of Hagedorn's theory is that asymptotically ($m \rightarrow \infty$, where m is the invariant mass) no particles are elementary, but instead every particle is made up of all resonances and particles, that is at very high energies “fireballs (hot hadrons) are made of fireballs, are made of fireballs ...”. This condition leads to the so-called bootstrap equation, which can be solved to get asymptotically the density of states ($\rho(m)$) inside a fireball [7,8,9]. Once $\rho(m)$ is known, the partition function is readily given by a Laplace transformation, $\mathcal{Z}(T) = \int dm \rho(m) e^{-m/T}$. From the partition function the average number of particles can be calculated, therefore the particle multiplicities in high energy collisions can be obtained.

For sufficiently high energies statistical methods [10, 11, 12, 13, 14, 15, 16] show good agreement with existing data, however, at lower energies where interesting non - perturbative effects can be important, and the particle multiplicities are low, the number of results are somewhat less (for an exception see [17], in which the authors could reach a fairly good agreement with experimental data in the 2.1 to 2.6 GeV energy range by using a one fireball model). It is worth to note that this behavior can be caused by the fact that at high energies (few hundred GeV) the quark masses become negligible and certain quantities like the quark creation probabilities and quark pair numbers – see later – become the same for each flavor. Consequently,

at such high energies the effects of new ingredients introduced here could become negligible or cancel out from the calculated cross section ratios.

In our case, we would like to calculate hadronic cross section of such reactions that have a relatively high energy (2 – 10 GeV) and can have more than two particles in the final state. It is worth to note that the calculated elementary, experimentally unknown cross sections can be used to improve our existing BUU (Boltzmann-Uehling-Uhlenbeck) code for simulating Heavy-ion collisions [18]. Thus, we tried to extend a statistical model to reach our goals. Our method incorporate that during the collision a compound system, a so-called fireball, is formed and after a short time, through possible production of subsequent fireballs, this system decays into a specific final state. In our approach the probability of the resulting final state can be calculated from the corresponding phase space, the quark content of the final state, the density of states $\rho(m)$, and from the properties of the fireballs.

The paper is organized as follows. In Section 2 we introduce the basic mathematical formulation of the model and in Section 3 we show that it can be used to describe medium energy reactions. For that reason, a Monte Carlo algorithm was developed to describe final states with more than 3 particles, in case of which multiple fireball decays could be important. The method is used to describe proton-antiproton annihilation at rest, where a very good agreement with the existing data is found. Exotic meson – containing strange and/or charm quarks – formation is also examined, in case of which the model parameters are fitted to the existing experimental data. Finally, we conclude in Section 4.

2 Basic formulation

The bootstrap idea is very useful in describing particle multiplicities, however, if the probability of a specific reaction is to be calculated, the basic method should be extended. For more on the basics of the statistical method see for instance [19]. As it was already mentioned, our starting point is the fireball formation and its subsequent decay statistically into different final states. It is worth to summarize here the most important ingredients of our approach, which amounts in the multi-fireball decay scheme and the quark combinatorial factors. Some version of multi-fireball creation was already used *e.g.* in [20,21], however, our decay scheme is different, which will be discussed in detail below. On the other hand, the quark combinatorial factors, introduced in the last part of this section – as far as we are aware – has not been used together with the bootstrap idea.

Our basic assumption is that the generalized cross section ($\sigma^{n \rightarrow k}(M)$) of the $n \rightarrow k$ reaction – containing n incoming and k outgoing particles – can be factorized into two terms, one which describes the dynamics of the collision, and the other one with mixed dynamical and statistical part, which describes the probabilities of each final

state channel,

$$\sigma^{n \rightarrow k}(M) = \left(\int \prod_{i=1}^n d^3 p_i R(M, p_1, \dots, p_n) \right) \times \left(\int \prod_{j=1}^k d^3 q_j w(M, q_1, \dots, q_k) \right), \quad (1)$$

where $M = \sqrt{s}$ is the CM energy of the colliding particles, while p_i and q_i stand for the incoming and outgoing momenta, respectively. As a further simplification we assume that the dynamical term can be substituted by the integrated total cross section of the considered reaction. Moreover, for the mixed term it is assumed that the energy-momentum conservation can be factored out, that is

$$\int \prod_{i=1}^n d^3 p_i R(M, p_1, \dots, p_n) \approx \sigma_{\text{Tot}}^{(n)}, \quad (2)$$

$$\int \prod_{j=1}^k d^3 q_j w(M, q_1, \dots, q_k) \approx \tilde{w}(M) \frac{\Phi_k}{V^{k-1}} \equiv W(M), \quad (3)$$

where

$$\Phi_k(M, m_1, \dots, m_k) = V^{k-1} \times \int \prod_{i=1}^k d^3 q_i \delta \left(\sum_{j=1}^k E_j - M \right) \delta \left(\sum_{l=1}^k \mathbf{q}_l \right) \quad (4)$$

is the k particle phase space, $\sigma_{\text{Tot}}^{(n)}$ is the generalized total cross section of n incoming particles and V is the interaction volume, which is set to $V = 4\pi/(3m_\pi^3)$. It is worth to note that if we want to give *e.g.* the angular dependence of the outgoing particles we assume every possibility may happen with a probability weighted by the corresponding phase space. This assumption seems to be a very crude one, but it is the main idea behind other statistical methods in the literature (see *e.g.* [22]) and we will show in Sec. 3.1 that despite its simplicity it can give reasonable results. The model can be extended to determine not only total, but also generalized differential cross sections, however, this extension is not necessary to describe the basics of our model. In the current general form of Eqs. (1)-(4) the model could be used to describe many body collisions, which we plan to do in a subsequent work, however, we now only concentrate on two body collisions. Consequently, from now on Eq. (2) has two incoming momenta p_1 , and p_2 , while Eqs. (3)-(4) hold. Based on Eqs. (1)-(4) we calculate the considered partial cross section according to

$$\sigma(M) = W(M) \cdot \sigma_{\text{Tot}}(M), \quad (5)$$

where $\sigma(M) (\equiv \sigma^{2 \rightarrow k}(M))$ and $\sigma_{\text{Tot}} (\equiv \sigma_{\text{Tot}}^{(2)})$ are the usual partial and total cross sections respectively. The later is usually taken from experimental data while $W(M)$, which

is the total formation probability of the desired end state, is calculated from our model. More precisely, instead of calculating $W(M)$'s directly, we always use some reference channel, with which we only calculate the ratios of $W(M)$'s. Let us demonstrate this in the following example. Consider the reaction $p\pi^- \rightarrow n\pi^+\pi^-$. To calculate its cross section, we use the reference channel $p\pi^- \rightarrow p\pi^-$. It is assumed that this elastic cross section is known from somewhere else. Assuming our factorization scheme works, and we know the cross section of the reference channel, we can calculate the desired cross section as follows

$$\begin{aligned} \sigma_{p\pi^- \rightarrow n\pi^+\pi^-} &\equiv \frac{\sigma_{p\pi^- \rightarrow n\pi^+\pi^-}}{\sigma_{p\pi^- \rightarrow p\pi^-}} \sigma_{p\pi^- \rightarrow p\pi^-} \\ &= \frac{W_{n\pi^+\pi^-}}{W_{p\pi^-}} \frac{\sigma_{p\pi^-}^{Tot}}{\sigma_{p\pi^-}^{Tot}} \sigma_{p\pi^- \rightarrow p\pi^-} \\ &= \frac{W_{n\pi^+\pi^-}}{W_{p\pi^-}} \sigma_{p\pi^- \rightarrow p\pi^-}, \quad (6) \end{aligned}$$

here the only unknown part is the ratio $\frac{W_{n\pi^+\pi^-}}{W_{p\pi^-}}$, which can be calculated much easier than the W 's themselves, since common factors – for instance in the case of the same initial states the rather complicated normalization factor – cancels out. The remaining part of the section will be dedicated to the explanation of how we calculate the W 's.

It is worth to note that in the literature there are various approaches like cases where authors considered only one, see *e.g.* [23], or more than one, see *e.g.* [20, 21], fireball(s) formation. In our approach we allow more than one fireball to form in a chain of consecutive decays, that is in every step – until the energy runs out – the fireball splits into two. This is a similar approach as in the model of Frautschi [7]. Generally, when k fireball is formed after the collision with M invariant mass, and from the individual fireballs n_1, n_2, \dots, n_k hadrons are produced, the total formation probability $W_k^{n_1, \dots, n_k}$ can be written as,

$$W_k^{n_1, \dots, n_k}(M) = \mathcal{N}_k(M) P_k^{\text{fb}}(M) C_Q(M) \int_{x_{1, \min}}^{x_{1, \max}} \dots \int_{x_{k, \min}}^{x_{k, \max}} \prod_{i=1}^k dx_i P_{n_1}^{H,1}(x_1) P_{n_2}^{H,2}(x_2) \dots P_{n_k}^{H,k}(x_k) \delta\left(\sum_{i=1}^k x_i - M\right), \quad (7)$$

where $\mathcal{N}_k(M)$ is a normalization factor¹, P_k^{fb} stands for the formation probability of k fireballs, $C_Q(M)$ is the quark-combinatorial factor, x_i 's are the invariant masses of the individual fireballs, $P_{n_i}^{H,i}(x_i)$'s are the hadronization probabilities of the i th fireball that is producing n_i hadrons, while $x_{i, \min}$ and $x_{i, \max}$ are the lower and upper kinematical limits of the decay of the i th fireball, which are imposed by the choice of the final states. P_k^{fb} , $P_{n_i}^{H,i}$, and C_Q will be discussed below in detail. We assume for the hadronization that only two and three body decays may take place following the reasoning of Frautschi [7]. He calculated that the probability of a fireball decay into n hadrons – in contrast to Hagedorn's picture [5, 6] – is energy independent and the dominant ones are the $n = 2, 3$ body decays with $P_2^d = 0.69$ and $P_3^d = 0.24$ probabilities. It is worth to note that we assume – based on [7] – that the minimal number of hadrons coming out of a fireball is 2. According to that the n_i 's in Eq. (7) can take only the values 2 or 3, while the hadronization probabilities $P_2^{H,i}(x_i)$ and $P_3^{H,i}(x_i)$ can be written as

$$P_2^{H,i}(x_i) = \prod_{l=1}^2 (2s_l + 1) P_2^d \frac{\Phi_2(x_i, m_1, m_2)}{\rho(x_i) (2\pi)^3 N_l!}, \quad (8)$$

$$P_3^{H,i}(x_i) = \prod_{l=1}^3 (2s_l + 1) P_3^d \frac{\Phi_3(x_i, m_1, m_2, m_3)}{\rho(x_i) (2\pi)^6 N_l!}, \quad (9)$$

¹ It should be noted that the normalization factor depends on the initial state.

where s_l and m_l are the spin and mass of the l th outgoing physical particle, respectively, N_l is the number of identical particles in the final state, while $\rho(x_i)$ stands for the density of states with invariant mass x_i . The explicit form of $\rho(M)$ is taken from [24] and it reads

$$\rho(M) = \frac{a\sqrt{M}}{(M_0 + M)^{3.5}} e^{\frac{M}{T_0}}, \quad (10)$$

where a , M_0 , and T_0 are free parameters. In our calculations, however, only M_0 and T_0 are relevant, since a always cancels out from the ratio of the W 's. Their values are given by $M_0 = 500$ MeV and $T_0 = 130 - 170$ MeV [24]. For the latter, we calculated the

$$\frac{\sigma_{p\bar{p} \rightarrow n\bar{n}}}{\sigma_{p\bar{p} \rightarrow \pi^+\pi^-}}, \frac{\sigma_{p\bar{p} \rightarrow p\bar{p}\pi^0}}{\sigma_{p\bar{p} \rightarrow \pi^+\pi^-}}, \frac{\sigma_{p\bar{p} \rightarrow \pi^+\pi^-}}{\sigma_{p\bar{p} \rightarrow \Lambda\bar{\Lambda}}}, \frac{\sigma_{p\bar{p} \rightarrow \pi^+\pi^-}}{\sigma_{p\bar{p} \rightarrow K^+K^-}}, \text{ and } \frac{\sigma_{pp \rightarrow pp\rho^0}}{\sigma_{pp \rightarrow n\Delta^{++}}}$$

ratios and varied T_0 in order to get a good agreement with the experimentally known values, which lead finally to $T_0 = 160$ MeV.

Let's turn to the fireball formation probability P_k^{fb} . Either we can consider these probabilities as energy dependent free parameters of the model, which we can fit from measured data, or we can calculate them based on a purely statistical approach. Thus, we calculated the probabilities of one two and three fireballs formation, since four and more fireballs formation would give significant

contribution only at larger energies.² It is important to note here that, however, while in the case of analytic calculations we restricted our calculations only to maximum three fireballs, we also used a Monte-Carlo simulation, where this restriction was abolished. According to the simulations, we can justify that for the currently considered processes and energy range the inclusion of four or more fireballs have a negligible impact on the results. Here we assumed a chain like decay scheme – in one step one fireball can split into two smaller. The method is shown for the two fireballs formation case in Fig. 1, where two different subcases are possible. In the figure the circle marked with M is the initial fireball, which tries to decay into more fireballs. Here M denotes the invariant mass, which is the available energy from the collision. Next we assume that this fireball can decay into two more fireballs but one of them is permanent – denoted by a filled square in Fig. 1 –, which means it cannot decay any further. However, the other fireball can decay into another two fireballs, and so on. Those fireballs that can not decay any further stay and subsequently hadronize giving a final state with the previously described probabilities. Each permanent fireball will give some particles to the final state, so the net final state will be the sum of those particles coming from the different permanent fireballs. Back to our example case, we suppose that the permanent fireball is produced with invariant mass $(r_1 M)$ (left panel of Fig. 1), where $(r_1 \in U[0, 1])$ is a random number with uniform distribution. This fireball will hadronize, consequently it must have a minimum invariant mass that can give a real final state, which will be denoted by m_c . Practically, since minimum two particles will be produced from a fireball, we can assume that $m_c = 2m_\pi$. The maximum is also limited, because the remaining invariant mass also need to produce at least two particles. The other fireball with invariant mass $((1-r_1)M)$ in principle tries to decay further, but in this case it can not, because at the end exactly two permanent fireballs are needed. Thus, we have to make sure that this fireball is unable decay any further, but it also has to produce at least two particles, which constrain both r_1 and r_2 . The unfilled squares in Fig. 1 means that those fireballs cannot be created. A fireball can not decay any further if either one of its subsequent fireballs fails to have at least m_c invariant mass (practically can not produce two pions). With this in mind the constraints for the first case are the following

$$r_1 M > m_c, \quad (11)$$

$$(1 - r_1)M > m_c, \quad (12)$$

$$(1 - r_1)(1 - r_2)M < m_c, \text{ or } (1 - r_1)r_2 M < m_c. \quad (13)$$

The same reasoning is valid for the right panel in Fig. 1 where the only difference is that in the first step the fireballs with invariant masses $r_1 M$ and $(1 - r_1)M$ should be swapped. The probability for this decay scheme can be calculated easily if we define the following five events

$$(A_0 - A_5),$$

$$A_0 : (r_1 > \frac{m_c}{M}) \wedge (r_1 < 1 - \frac{m_c}{M}), \quad (14)$$

$$A_1 : (1 - r_1)(1 - r_2) < \frac{m_c}{M}, A_2 : (1 - r_1)r_2 < \frac{m_c}{M}, \quad (15)$$

$$A_3 : (1 - r_2)r_1 < \frac{m_c}{M}, A_4 : r_1 r_2 < \frac{m_c}{M}. \quad (16)$$

With these events the probability can be expressed as,

$$\begin{aligned} P_2^{fb} &= \frac{1}{2} \{P(A_1 \vee A_2 | A_0) + P(A_3 \vee A_4 | A_0)\} \\ &= \frac{1}{2} \{P(A_1 | A_0) + P(A_2 | A_0) - P(A_1 \wedge A_2 | A_0) \\ &\quad + P(A_3 | A_0) + P(A_4 | A_0) - P(A_3 \wedge A_4 | A_0)\}, \quad (17) \end{aligned}$$

where the factor 1/2 reflects the two existing subcases for the two fireballs formation case as it is shown in Fig. 1. We carried out the calculations – based on geometric probabilities – for the one, two and three fireballs formation, however, the probability has a closed form only in the first two cases, which are

$$P_1^{fb}(M) = \begin{cases} \frac{2m_c}{M}, & 0 < \frac{m_c}{M} < \frac{1}{2} \\ 1, & \frac{1}{2} < \frac{m_c}{M} < 1 \end{cases}, \quad (18)$$

$$P_2^{fb}(M) = \begin{cases} \frac{2m_c}{M} \left[\ln \left(\frac{M}{2m_c} - \frac{1}{2} \right) + \frac{1}{2} \right], & 0 < \frac{m_c}{M} < \frac{1}{3} \\ 1 - \frac{2m_c}{M}, & \frac{1}{3} < \frac{m_c}{M} < \frac{1}{2} \end{cases}, \quad (19)$$

while in the third case it can be found in Appendix B due to its lengthy formula. Important consequence are the ratios of the fireball formation probabilities P_k^{fb} , since they clearly show that if the invariant mass – or collision energy – is increased, it is more likely to get more and more fireballs. Thus more particles can be produced. The ratio of P_2^{fb}/P_1^{fb} can be seen in Fig. 2, where the result of a Monte-Carlo simulation is also shown.

The last ingredient in Eq. (7) is the the quark-combinatorial factor $C_Q(M)$, which is a dynamical – that is energy dependent – parameter of the model and is necessary to describe baryonic final states. A similar approach can be found in [25], where the abundances of baryons containing strange quarks was described by quark combinatorial probabilities. The basic assumption is that the probability of the creation of a specific hadron is proportional to the number of ways that the hadron can be formed from a given number of u, d, b, s, c, t quark-antiquark pairs. Thus, it is necessary to know as a function of energy/invariant mass the number of quark-antiquark pairs and the distribution of the various quark species once the average number of pairs is known. The number of quark-antiquark pairs is calculated in [12] based on phase space considerations, which reads

$$N(M) = \frac{1 + \sqrt{1 + M^2/T_0^2}}{2}, \quad (20)$$

where T_0 is the interaction temperature, which is found to be $T_0 = 160$ MeV, the Hagedorn temperature. It should be noted that only u, d, s, c quark-antiquark pairs are considered in our investigation. It is assumed that the light

² In some cases, like $p - \bar{p}$ annihilation, where multiple particles creation (≥ 10) is possible even at lower energies it can happen that four fireballs formation becomes significant.

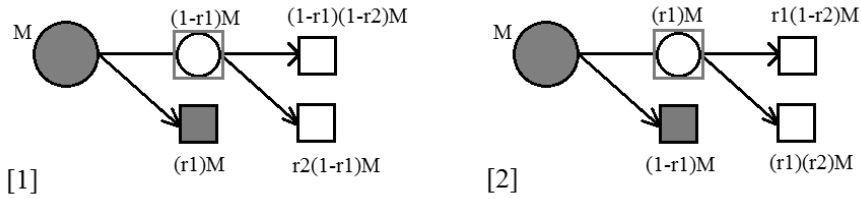


Fig. 1. The decay scheme for the two fireballs formation case. Left and right figure shows the two possible subcases.

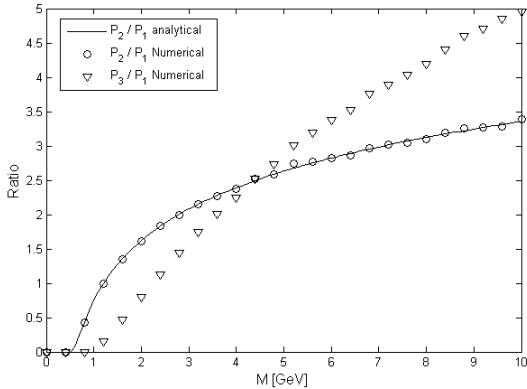


Fig. 2. Ratio of the fireball formation probabilities $P_2^{\text{fb}}/P_1^{\text{fb}}$ and $P_3^{\text{fb}}/P_1^{\text{fb}}$ as function of the invariant mass. Analytic and Monte-Carlo simulation results are shown by solid line and circles/triangles, respectively.

up and down quark-antiquark pairs are formed with the same $P_u = P_d$ probability due to their almost equal mass, but heavier pairs such as the strange and charm have suppression in their creation probabilities $P_s, P_c < P_u$. These probabilities are important if there are strange or charm particles in the final state such as K or J/Ψ . The P_s and P_c quark-antiquark pair probabilities are fitted to experimental data, which is discussed in Sec. 3.2. It is worth to note that in the four quark approximation the pair formation probabilities should satisfy the $P_u + P_d + P_s + P_c = 1$ relation. Once the pair probabilities are known we also need a probability mass function $F(n_u, n_d, n_s, n_c; N(M))$, which gives the probability that from the $N(M)$ quark-antiquark pairs we have exactly n_u, n_d, n_s, n_c number of $u\bar{u}, d\bar{d}, s\bar{s}, c\bar{c}$ pairs, respectively. We used the multinomial distribution that has the following probability mass function,

$$F(n_u, n_d, n_s, n_c; N(M)) = \frac{\left(\frac{N(M)!}{n_u!n_d!n_s!n_c!}\right) P_u^{n_u} P_d^{n_d} P_s^{n_s} P_c^{n_c}}{P_{\text{tot}}(N)}, \quad (21)$$

where $P_{\text{tot}}(N)$ is the normalization factor, which sums the nominator over all the possible $\{n_u, n_d, n_s, n_c\}$ combinations that satisfy the $n_u + n_d + n_s + n_c = N$ constraint. It is very important to note that we assumed that such n_u, n_d, n_s, n_c values will be realized that maximize $F(n_u, n_d, n_s, n_c; N(M))$.

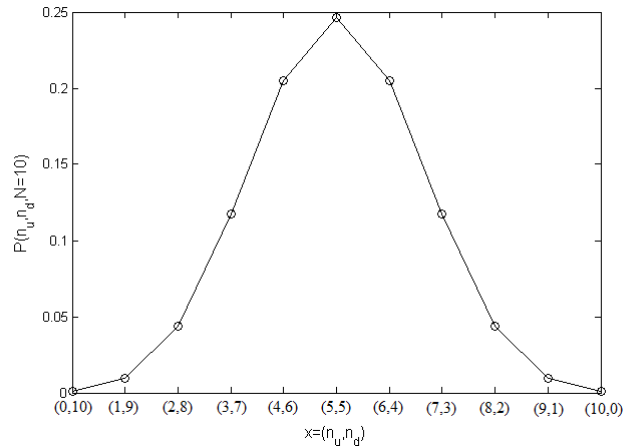


Fig. 3. The distribution function in the two quark case for $N = 10$ from Monte-Carlo simulation. On the x -axis $\{n_u, n_d\} = \{n_u, N - n_u\}$ is shown.

Suppose for a moment that we have only u and d quarks, then since $n_u + n_d = N$ and $P_u + P_d = 1$, $F(n_u, n_d = N - n_u; N) = F(n_u; N)$ and $P_u = P_d = 0.5$. One particular case, $N = 10$, can be seen in Fig. 3. It can be seen that the distribution is symmetric and have a unique maximal value, which is the case with more quarks as well. Consequently, by using the values given by the maximum of (21) the energy dependent quark-combinatorial factors can be easily written for the various final states. For instance, in case of $M(\text{fireball}) \rightarrow p(uud) + n(udd)$ one has,

$$C_{Q,pn}(M) = P_{(u,d,s,c)}^{\text{max}} 3^2 n_u (n_u - 1) n_d \times (n_u - 2)(n_d - 1)(n_d - 2), \quad (22)$$

where $P_{(u,d,s,c)}^{\text{max}} \equiv \max(F(n_u, n_d, n_s, n_c; N(M)))$ is the maximal value of the probability that n_u, n_d, n_s, n_c number of quark-antiquark pairs were created, and the 3^2 factor is for the number of colorless combinations of the two baryons. Connected to Eq. (22) it should be noted that currently we do not care about the fate of the created antiquarks or quarks, which are not used for any hadrons. That is for instance in Eq. (22) we disregard three \bar{u} 's and three \bar{d} 's. We plan to address this issue in a subsequent work.

To get $C_Q(M)$ the final step is to calculate a normalization factor according to the multiplicity of the decay

(two or three body):

$$C_Q(M) = C_{Q,pn} \mathcal{N}_{C,2}(M) \quad (23)$$

In the normalization factor we have to sum over all the possible quark combinations that is allowed by the conservation laws of the fireball. In the cases of two and three body decays the normalizations are

$$\mathcal{N}_{C,2}(M) = \left[\sum_{\langle ij \rangle \in S} C_{Q,(ij)} \right]^{-1}, \quad (24)$$

$$\mathcal{N}_{C,3}(M) = \left[\sum_{\langle ijk \rangle \in S} C_{Q,(ijk)} \right]^{-1}, \quad (25)$$

where S represents the set of all possible hadron combinations that has the same quantum numbers as the fireball. With this normalization $C_Q(M)$ can be considered as a quark combinatorial probability. In the two fireball case we will have two combinatorial factors, hence the final combinatorial probability will be $C_Q = C_{Q_1} C_{Q_2}$. For other final states C_Q can be calculated similarly. In our model $P_{(u,d,s,c)}^{max}$ and the number of quark-antiquark pairs n_i are also energy dependent. This will be important when the pair creation probability is very small, e.g. channels with charm quarks.

Finally, let us give the explicit form of the total formation probability W in the cases of one and two fireball formation. As was already discussed, one fireball can decay into two or three hadrons, thus $n_i = 2, 3$ in $P_{n_i}^{H,i}$, while the number of outgoing hadrons can be two or three in case of one fireball, while four to six in case of two fireballs. The total probabilities are

$$W_1^{2,3}(M) = \mathcal{N}_1(M) P_1^{fb}(M) C_Q(M) P_{2,3}^H(M), \quad (26)$$

$$W_2^{2,3;2,3}(M) = \mathcal{N}_2(M) P_2^{fb}(M) C_Q(M) \times \int_{x_{min}}^{x_{max}} dx P_{2,3}^{H,1}(M-x) P_{2,3}^{H,2}(x), \quad (27)$$

where $C_Q(M)$, as was already discussed, depends on the considered final state, and $P_{1,2}^{fb}$ and $P_{2,3}^H$ are given by Eqs. (18), (19) and Eqs. (8), (9), respectively. Moreover, the integration limits x_{min} and x_{max} have to respect the kinematic limits of the final states. For instance in case of $n_1 = n_2 = 2$ – four hadron final state – with fireball₁ $\rightarrow a+b$ and fireball₂ $\rightarrow c+d$ decay scheme one has $x_{min} = m_c + m_d$ and $x_{max} = M - m_a - m_b$.

For a few simple processes with low multiplicity – e.g. two or three particles – the probabilities, and thus the ratios of probabilities of different processes can be expressed analytically, however, for more complicated processes numerical methods are necessary. In the following section we show that the model is able to describe low energy cross sections and their ratios with good accuracy.

3 Results

3.1 Final states with up and down quarks

The simplest processes that can be calculated are the ones with two particles in the final state. In this subsection only the lightest u and d quarks are considered and for their formation probabilities $P_u = P_d = 0.5$ is assumed. As a first example we would like to reconstruct the following ratio

$$R_{\pi^+\pi^-}^{n\bar{n}}(M) = \left(\frac{\sigma_{p\bar{p} \rightarrow n\bar{n}}}{\sigma_{p\bar{p}}^{Tot}} \right) / \left(\frac{\sigma_{p\bar{p} \rightarrow \pi^+\pi^-}}{\sigma_{p\bar{p}}^{Tot}} \right). \quad (28)$$

Consequently, $W_{n\bar{n}}$ and $W_{\pi^+\pi^-}$ should be calculated from the model. Since the two-particle final states can only come from a one-fireball decay, Eq. (26) should be used. The quark combinatorics can be given based on Eq. (22) and on the quark content of the final state, which is $n \sim udd$, $\bar{n} \sim \bar{u}\bar{d}\bar{d}$, $\pi^+ \sim u\bar{d}$, $\pi^- \sim \bar{u}d$; and the combinatorial factors are,

$$C_{Q,n\bar{n}} = 3^2 P_{u,d}^{max} n_u^2 (n_d - 1)^2 n_u^2, \quad (29)$$

$$C_{Q,\pi^+\pi^-} = 3^2 P_{u,d}^{max} n_u^2 n_d^2, \quad (30)$$

where $n_u = n_d = N/2$, since these values give the maximum of the probability mass function $F(n_u, n_d; N(M))$, and $P_{u,d}^{max} = F(n_u = N/2, n_d = N/2; N)$. The resulting expression will be quite simple due to the common factors that cancel out and it reads

$$R_{\pi^+\pi^-}^{n\bar{n}}(M) = \frac{W_{n\bar{n}}(M)}{W_{\pi^+\pi^-}(M)} = 4 \left(\frac{1 + \sqrt{1 + M^2/T_0^2}}{4} - 1 \right)^2 \times \left(\frac{M^2 - 4m_n^2}{M^2 - 4m_\pi^2} \right)^{1/2}, \quad (31)$$

where m_n and m_π are the masses of the neutron and pion, respectively. The result together with the measured data can be seen in Fig. 4, which shows a good agreement in the given energy range, however, the error bars are quite large.

As another example we calculated the ratio of the cross sections of the reactions $(p\bar{p} \rightarrow p\bar{p}\pi^0)$ and $(p\bar{p} \rightarrow \pi^+\pi^-)$, which is slightly more complicated than the previous one, because the $p\pi^0$ two-particle final state can arise from many resonances. Thus, we have to take into account all the possible $p\bar{p} \rightarrow R\bar{p}$, or $p\bar{p} \rightarrow \bar{R}p$ final states as well, where R can be any resonance which decays into $p\pi^0$. The possible decay schemes can be seen in Fig. 5. It is worth to note that in the figure the zeroth scheme corresponds to a three particle hadronization process, while all the rest to two particle hadronization processes. In this calculation, such nucleon and delta resonances are considered – taken from PDG [27] – that has a larger branching ratio than 15% for the $p\pi^0$ channel. These are listed in Table 1 together with their masses, spins and branching ratios of the $p\pi^0$ channel $B_i^{p\pi^0}$. We considered resonances and antiresonances alike. The necessary quark combinatorial factors

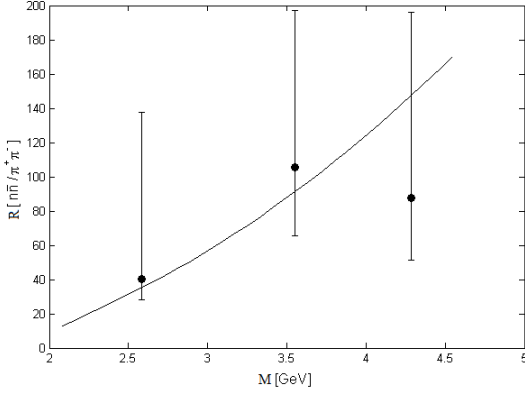


Fig. 4. The ratio $R_{\pi^+\pi^-}^{n\bar{n}}$ – see Eq. (31) – as a function of M (solid line) together with the measured data (filled circles) from [26]

Table 1. Considered nuclear and delta resonances their masses m_{R_i} , spins s_{R_i} and branching ratios $B_i^{p\pi^0}$. We excluded resonances with $B_i^{p\pi^0} < 0.15$. The same values apply to the corresponding antiresonances.

i	R_i	m_{R_i} [GeV]	s_{R_i}	$B_i^{p\pi^0}$
1	N_{1440}	1.430	1/2	0.22
2	N_{1520}	1.515	3/2	0.20
3	N_{1535}	1.535	1/2	0.15
4	N_{1650}	1.655	1/2	0.23
5	N_{1680}	1.685	5/2	0.23
6	Δ_{1232}	1.232	3/2	0.66
7	Δ_{1620}	1.630	1/2	0.17
8	Δ_{1910}	1.890	1/2	0.15
9	Δ_{1950}	1.930	7/2	0.27

for this ratio are

$$C_{Q,p\bar{p}\pi^0} = 3^3 P_{u,d}^{max} n_u^2 (n_u - 1)^2 n_d^2 \times \frac{(n_u - 2)^2 + (n_d - 1)^2}{2}, \quad (32)$$

$$C_{Q,\bar{p}R} = C_{Q,p\bar{R}} = 3^2 P_{u,d}^{max} n_u^2 (n_u - 1)^2 n_d^2, \quad (33)$$

while $C_{Q,\pi^+\pi^-}$ is already given in Eq. (30). After some manipulation we find,

$$R_{\pi^+\pi^-}^{p\bar{p}\pi^0}(M) = \frac{W_{p\bar{p}\pi^0}}{W_{\pi^+\pi^-}} = \frac{(n_u - 1)^2}{\Phi_2(M, m_\pi, m_\pi)} \times \left[\frac{6P_3^d}{P_2^d (2\pi)^3} \frac{\mathcal{N}_{\mathcal{C},3}}{\mathcal{N}_{\mathcal{C},2}} [(n_u - 2)^2 + (n_d - 1)^2] \times \Phi_3(M, m_p, m_p, m_{\pi^0}) + 4 \sum_{i=1}^9 B_i^{p\pi^0} (2s_{R_i} + 1) \times \Phi_2(M, m_p, m_{R_i}) \right], \quad (34)$$

where m_p and m_{R_i} are the masses of the proton and the resonances, respectively, while the explicit form of the phase space integrals $\Phi_{2,3}$ can be found in Appendix A. Moreover, the factor of 4 in front of the last term comes from two source: a factor of 2 from the summation over the antiresonances, and another factor of 2 from the proton spin degeneration. The ratio as a function of energy/invariant mass and its comparison with experimental data can be seen in Fig. 6, where a remarkably good match was found.

3.2 Final states with strange and charm quarks

For final states that also contain heavier quarks – in our case strange or charm – the assumption of equal quark creation probabilities is not a good approximation [28]. In this case one can fit the creation probability parameters P_s and P_c to existing data. Besides, it is also possible to estimate these parameters from some theory (see e.g. the Hawking-Unruh hadronization model [29]). In order to determine the strange quark probability P_s , we calculated the ratio $\frac{\sigma_{p\bar{p} \rightarrow \pi^+\pi^-}}{\sigma_{p\bar{p} \rightarrow K^+K^-}}$ in the energy range $E \in \{1.876, 2.602\}$ GeV, which has a similar expression as Eq. (31) and it reads

$$R_{K^+K^-}^{\pi^+\pi^-}(M) = \frac{n_d^2}{n_s^2} \left(\frac{M^2 - 4m_\pi^2}{M^2 - 4m_K^2} \right)^{1/2}, \quad (35)$$

where m_K is the kaon mass. As it can be seen, $R_{K^+K^-}^{\pi^+\pi^-}(M)$ strongly depends on the n_d/n_s ratio, which was adjusted through the change of P_u and P_s to get the best agreement with the measured data. We used the following iterative procedure, first we set some initial values for P_u and P_s then calculated the maximum of the probability mass function F in Eq. (21), which gave the values of n_u and n_s . With these values we calculated the ratio in Eq. (35) and compared with the experimental data, then changed the P_u and P_s and recalculated the ratio $R_{K^+K^-}^{\pi^+\pi^-}(M)$. We repeated this process until we got the smallest deviation from the experimental data. It turns out that we get the well-known strange quark suppression [10, 30, 31]. In Fig. 7 the original equal probability case $P_u = P_d = P_s = 1/3$ can be seen together with the fitted strange suppression case, where we get for the probabilities, $P_u = P_d = 0.38$, and $P_s = 0.24$. In the figure the experimental data from [26] marked with crosses and error bars is also shown. It can be seen that when the equal probability assumption is used (dotted line) the calculated ratio underestimates by an approximate factor of three the measured one, which means that the strange channel is overestimated by the model. When the strange quark creation probability was decreased, however, a better agreement could be achieved (solid line). In conclusion we can say that inclusion of an extra suppression parameter is not necessary in our approach, but in exchange we have to tune the P_u, P_d, P_s probability parameters. The previous example of fitting P_s was merely pedagogical due to the large errors and the few data points. Other recent

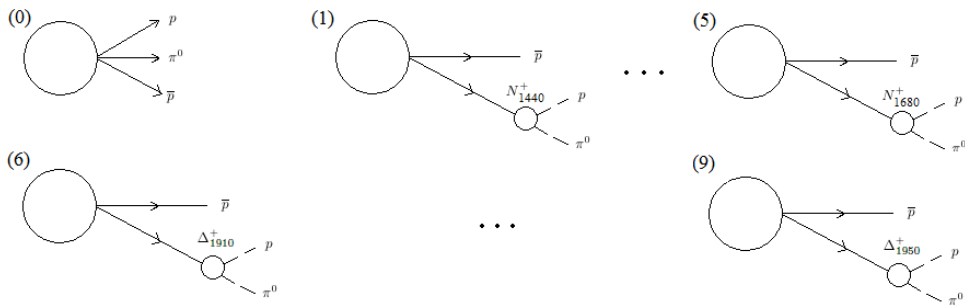


Fig. 5. Possible one-fireball decay schemes for the $p\bar{p} \rightarrow p\bar{p}\pi^0$ process. Similar set of diagrams can be shown for the antiresonances, where the protons and antiprotons should also be exchanged.

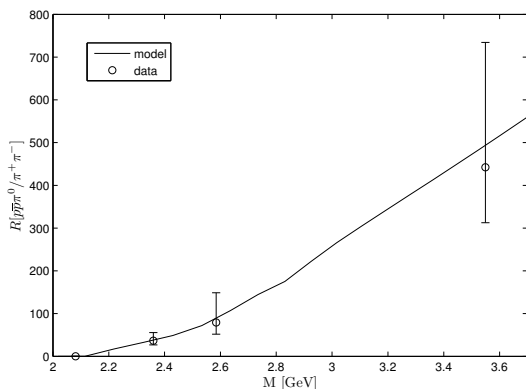


Fig. 6. The ratio $R_{\pi^+\pi^-}^{p\bar{p}\pi^0}$ – see Eq. (34) – as a function of M (solid line) together with the measured data (circles) from [26]

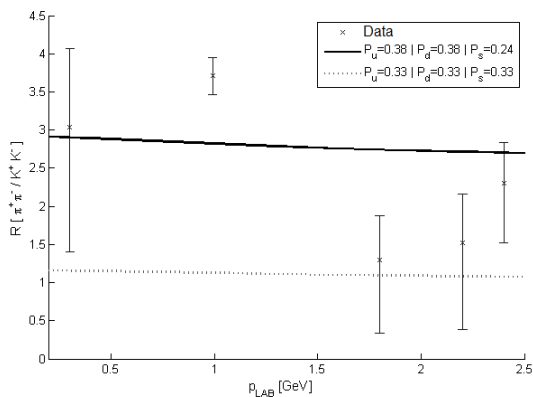


Fig. 7. Calculated values and experimental data of the ratio $R_{K^+K^-}^{\pi^+\pi^-}(M)$. Dotted line correspond to the equal probability case $P_u = P_d = P_s$, while the solid line to the strange suppression scenario.

fits – not discussed here – using channels involving Λ and $\bar{\Lambda}$ particles shows us that P_s could have a little smaller, while P_u, P_d a little higher values, however, it is necessary to include more channels to make a reliable fit. Another important point to note is that for a wider energy range P_s should be energy dependent. This is due to the natu-

ral assumption that at very large energy $P_u = P_d = P_s$ should hold. For the energy dependant suppression factor please see [32].

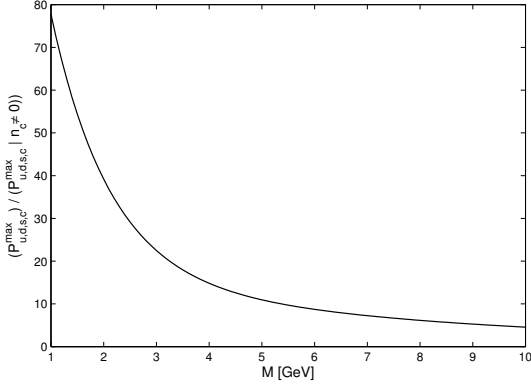
This technique can be applied to determine the charm quark creation probability P_c . Subsequently, P_c can be used to estimate cross sections with final states containing charm quarks, like for instance to the reaction $p + \bar{p} \rightarrow \pi^0 + J/\Psi$. It should be noted, however, that the existing measured data are still very limited, thus only a crude estimation can be made.

To determine P_c we considered the inclusive reaction ($p + p \rightarrow p + p + J/\Psi + X$) in the low energy regime and used a parametrization of the $\sigma_{J/\Psi}^{pp}$ cross section – based on experimental data – taken from [33]. Since calculation of the inclusive reaction probability is rather involved, we used the data near the ($p + p + J/\Psi$) threshold instead, where no other particles are created. For the fit we calculated the ratio $R_{ppJ/\Psi}^{pp\pi^0} = W_{pp\pi^0}/W_{ppJ/\Psi}$ and adjusted the result – by changing the quark pair creation probabilities P_u, P_d, P_s , and P_c – to the measured ratio $\sigma_{pp\pi^0}/\sigma_{ppJ/\Psi}$ near the threshold, *i.e.* $\sqrt{s} \equiv M = 5$ GeV. Since the charm quark mass 1.28 GeV is much larger than the u, d, s quark masses 2 MeV, 5 MeV, and 95 MeV, respectively, it is expected that its pair creation probability P_c will be much lower. If we simply used in C_Q , as we did previously, the maximum value $P_{u,d,s,c}^{\max}$ of the probability mass function F , then most probably we would get zero c quarks, which was checked explicitly. Obviously that maximum for $P_{u,d,s,c}$ can not be used to calculate the $W_{ppJ/\Psi}$ probability. Thus for a reaction, where the final state involves charm quark(s) we use the conditional maximum $P_{u,d,s,c}^{\max}|_{n_c \neq 0}$ instead, while for other reactions we retain the original $P_{u,d,s,c}^{\max}$. The calculated ratio reads as,

$$R_{ppJ/\Psi}^{pp\pi^0}(M) = \frac{P_{u,d,s,c}^{\max}}{P_{u,d,s,c}^{\max}|_{n_c \neq 0}} \frac{1}{n_c^2} \left[((n_u - 4)n_u + (n_d - 2)n_d) \frac{\Phi_3(M, m_p, m_p, m_\pi)}{2} + \frac{P_2^d (2\pi)^3 \mathcal{N}_{C,2}}{3P_3^d \mathcal{N}_{C,3}} \right] \left/ \sum_{i=1}^3 (B_i^{p\pi^0} (2s_{R_i} + 1) \Phi_2(M, m_p, m_{R_i})) \right/ \sum_{i=1}^3 (B_i^J)$$

Table 2. Considered J/Ψ resonances their masses $m_{R_i^J}$, spins $s_{R_i^J}$ and branching ratios B_i^J

i	R_i	$m_{R_i^J}$ [GeV]	$s_{R_i^J}$	B_i^J
1	Ξ_{c1}	3.511	1	0.30
2	Ξ_{c2}	3.556	2	0.30
3	Ψ'	3.686	1	0.56

**Fig. 8.** M dependence of the ratio r_P of the probabilities appearing in Eq. (36) (see text for details).

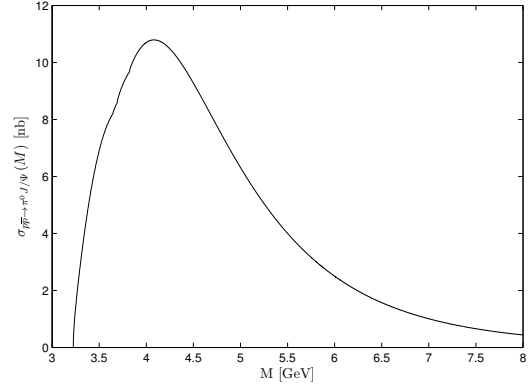
$$\times (2s_{R_i^J} + 1)\Phi_3(M, m_p, m_p, m_{R_i^J}), \quad (36)$$

where m_{R_i} , $B_i^{p\pi^0}$, and s_{R_i} are the masses, branching ratios and spins of the nucleon and delta resonances already given in Table 1, while $m_{R_i^J}$, B_i^J and $s_{R_i^J}$ are the masses, branching ratios – of J/Ψ resonances decaying into J/Ψ – and spins of the J/Ψ resonances and are listed in Table 2. The branching ratios were taken from [34]. Based on [33, 34] the experimental value for the ratio at threshold is $R_{ppJ/\Psi}^{pp\pi^0}(M = 5\text{GeV}) \approx 8.73 \cdot 10^7$. After fitting our calculated ratio – similarly as in the case of P_s – to the experimental value, it was found for the quark-antiquark creation probabilities

$$P_u = P_d = 0.38, P_s = 0.2386, P_c = 0.0014 \quad (37)$$

It turns out that in this energy range the $n_c = 1$ condition gives the local maximum $P_{u,d,s,c}^{max}|_{n_c \neq 0}$, thus in the quark combinatorial factor the $n_c = 1$ value was used. With these values we calculated the invariant mass/energy dependence of the probability ratio $r_P \equiv P_{u,d,s,c}^{max}/P_{u,d,s,c}^{max}|_{n_c \neq 0}$ appearing in Eq. (36), which can be seen in Fig. 8. The ratio of the global and the local maximums tends to unity (not shown explicitly), which means that at higher energies – where the ratio is close to 1 – only the global maximums of the probability mass distribution function is needed as the number of $c\bar{c}$ pairs will always be larger than zero.

As an application we calculated the cross section of the reaction $p + \bar{p} \rightarrow \pi^0 + J/\Psi$, which is an important ingredient to describe antiproton induced J/Ψ production – an

**Fig. 9.** M dependence of the cross section $\sigma_{p\bar{p} \rightarrow \pi^0 J/\Psi}$

important part of the PANDA/FAIR research plan. For this a reference channel was needed, where the cross section in the desired energy range is known. Consequently, we choose the $p + \bar{p} \rightarrow n + \bar{n}$ reaction, for which the cross section is well known in a relatively wide energy range. The desired cross section can be expressed as,

$$\sigma_{p\bar{p} \rightarrow \pi^0 J/\Psi} = \frac{\sigma_{p\bar{p} \rightarrow n\bar{n}}}{R_{\pi^0 J/\Psi}^{n\bar{n}}}, \quad (38)$$

where ($R_{\pi^0 J/\Psi}^{n\bar{n}} = W_{n\bar{n}}/W_{\pi^0 J/\Psi}$) can be analytically calculated from our model resulting in a similar expression as in Eq. (36). By using the previously fitted values for the P_u, P_d, P_s, P_c probabilities given in Eq. (37), the resulting cross section can be seen in Fig. 9.

3.3 Proton - antiproton annihilation at rest

An important application, and also a good validity check of the model if we calculate a few, more complicated final states for $p\bar{p}$ annihilation at rest – *i.e.* at $\sqrt{s} = 2m_p$ – and compare with existing measured data; probabilities of the most important channels and the end state pion distribution can be found in [35, 36]. Because of the higher multiplicities – two to six –, there will be more than one fireball. Owing to the complexity of the problem the calculations were carried out with the help of a Monte-Carlo simulation. The results are shown in Fig. 10, where the histograms show the calculated, while the crosses with errorbars the measured values. The plot shows a very good agreement, which is promising considering future applications. The final state pion distribution was also checked, where again a good match was found to the experimentally known normal distribution. Fig. 11 shows the result, where the following fit could be used,

$$P(N_\pi) \approx \frac{1}{\sqrt{2\pi}D} \exp\left(-\frac{(N_\pi - \langle N_\pi \rangle)^2}{2D^2}\right), \quad (39)$$

where $\langle N_\pi \rangle \approx 5$ is the average pion multiplicity, and $D = 0.97$ is the standard deviation, which values are taken from [37]. The main consequence is that the many fire-

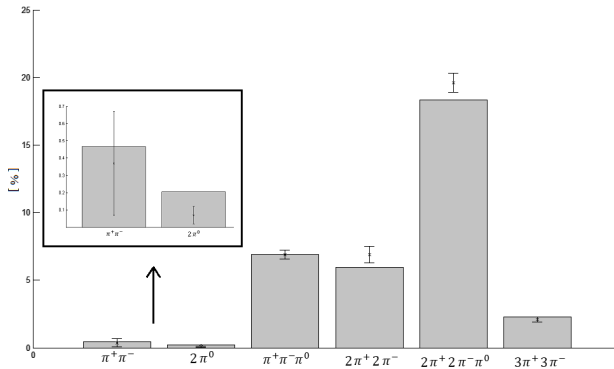


Fig. 10. Probabilities of the multiple pion final states of the $p\bar{p}$ collision at rest

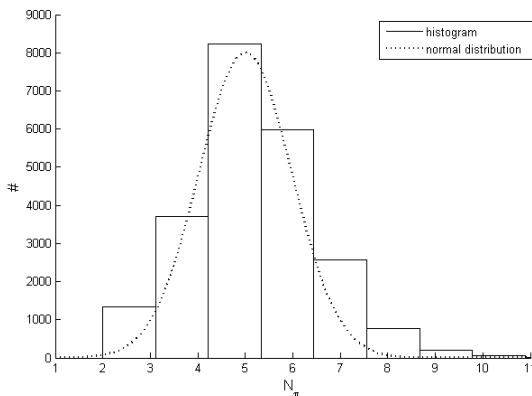


Fig. 11. Pion multiplicities in $p\bar{p}$ annihilation at rest

ball model seems to be a good phenomenological approximation for processes with more than three particle final states, and can be used to estimate various yet experimentally unknown cross sections.

4 Conclusions

In this article we proposed a statistical model based on earlier works [5, 6, 7] and extended by extra dynamical factors that can be used to describe medium energy hadronic processes. The original Bootstrap idea of Hagedorn and Frautschi was extended with quark combinatorics and a specific many-fireball decay scheme. The method can be used to calculate ratios of different processes, and consequently let us – by using a known reference channel – calculate yet unknown cross sections. The most important parameters of the model are the interaction temperature T_0 and the quark creation probabilities P_u, P_d, P_s, P_c , which were determined by fitting to existing experimental data. Another parameter, which appears in ratios with different number of fireballs is the interaction volume $V (= 4\pi/(3m_\pi^3))$. To test our method we calculated some known cross section ratios, which gave very good match to the measured data. In case of proton-antiproton annihilation, where analytical calculations were not feasible

we used a Monte-Carlo simulations, which also showed a good agreement to the existing data. This somewhat confirms our idea of the many-fireball decay scheme. In the future, we intend to use this method in a BUU transport code to include unknown elementary cross sections, especially concentrating on the charmonium formation processes. The method could also be important to describe possible many-body collisions in strongly interacting dense matter.

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This work is devoted to the memory of Walter Greiner.

A Explicit form of the phase space integrals

$$\begin{aligned} \Phi_2(M, m_1, m_2) &= V \int d^3 q_1 d^3 q_2 \delta(M - E_1 - E_2) \\ &\times \delta^{(3)}(q_1 + q_2) = \frac{V\pi}{2M^4} \left(M^4 - (m_1^2 - m_2^2)^2 \right) \\ &\times \sqrt{\lambda(M^2, m_1^2, m_2^2)}, \end{aligned} \quad (40)$$

where $\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz$ is the Källén function, and V is the interaction volume which is set to $V = \frac{4\pi}{3m_\pi^3}$

$$\begin{aligned} \Phi_3(M, m_1, m_2, m_3) &= V^2 \int d^3 q_1 d^3 q_2 d^3 q_3 \\ &\times \delta(M - E_1 - E_2 - E_3) \delta^{(3)}(q_1 + q_2 + q_3) \\ &= V^2 \left\{ \frac{M^5}{120} - \frac{M^3}{12} \sum_{i=1}^3 m_i^2 + \frac{M^2}{6} \sum_{i=1}^3 m_i^3 \right. \\ &+ \frac{M}{4} \left[\sum_{i=1}^3 \sum_{j=2}^3 m_i^2 m_j^2 - \frac{1}{2} \sum_{i=1}^3 m_i^4 \right] + \frac{1}{30} \sum_{i=1}^3 m_i^5 \\ &\left. - \frac{1}{6} \sum_{i=1}^3 \sum_{j=1}^3 m_i^3 m_j^2 \right\} \end{aligned} \quad (41)$$

B Formation probability of three fireballs

To calculate the three fireball case there are four different topologies we have to consider, which can be seen in Fig. 12. For the first two decay schemes we can define a

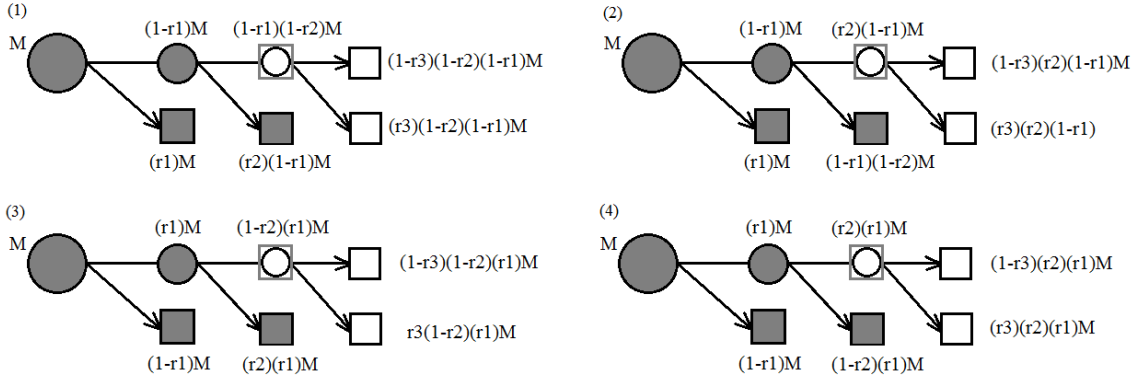


Fig. 12. The decay scheme for the three fireballs formation. The four possible subcases are labeled from (1) to (4).

common event A as

$$A : \left(r_1 > \frac{m_c}{M} \right) \wedge \left(r_1 < 1 - \frac{m_c}{M} \right) \wedge \left(r_2 > \frac{m_c}{M(1-r_1)} \right) \wedge \left(r_2 < 1 - \frac{m_c}{M(1-r_1)} \right) \quad (42)$$

This event describes the first part of the decay schemes, which is the same in the first two topologies. For the first decay scheme have to define the following set of events.

$$A1 : r_3 < \frac{m_c}{(1-r_1)(1-r_2)M}, \quad (43)$$

$$A2 : r_3 > 1 - \frac{m_c}{(1-r_1)(1-r_2)M}. \quad (44)$$

For the second decay scheme the corresponding events are

$$B1 : r_3 < \frac{m_c}{(1-r_1)r_2M}, \quad (45)$$

$$B2 : r_3 > 1 - \frac{m_c}{(1-r_1)r_2M}. \quad (46)$$

The third and the fourth topologies also have a common event, because they differ only at the end of the chain,

$$C : \left(r_1 > \frac{m_c}{M} \right) \wedge \left(r_1 < 1 - \frac{m_c}{M} \right) \wedge \left(r_2 > \frac{m_c}{Mr_1} \right) \wedge \left(r_2 < 1 - \frac{m_c}{Mr_1} \right). \quad (47)$$

For the third decay scheme the followings events were defined,

$$C1 : r_3 < \frac{m_c}{r_1(1-r_2)M}, \quad (48)$$

$$C2 : r_3 > 1 - \frac{m_c}{r_1(1-r_2)M}. \quad (49)$$

The last decay scheme also needs two events to be defined,

$$D1 : r_3 < \frac{m_c}{r_1r_2M}, \quad (50)$$

$$D2 : r_3 > 1 - \frac{m_c}{r_1r_2M}. \quad (51)$$

From the previously defined events we can express the total probability of three-fireballs formation as,

$$P_3(M) = \frac{1}{4} \left[P(A1|A) + P(A2|A) - P(A1 \wedge A2|A) + P(B1|A) + P(B2|A) - P(B1 \wedge B2|A) + P(C1|C) + P(C2|C) - P(C1 \wedge C2|C) + P(D1|C) + P(D2|C) - P(D1 \wedge D2|C) \right], \quad (52)$$

where the $1/4$ factor reflects the four existing subcases. As it can be seen we have to calculate conditional probabilities for each decay scheme and then sum up all of the possible events. These probabilities can be expressed in closed form, however for a few cases numerical integration is required. The different probabilities are the following

$$P(B1|A) = \left[\int_{\frac{m_c}{M}}^{1-2\frac{m_c}{M}} \int_{\frac{m_c}{M(1-r_1)}}^{1-\frac{m_c}{M(1-r_1)}} dr_1 dr_2 \times \left(\frac{m_c}{r_2(1-r_1)M} \right) \right]_{0 < \frac{m_c}{M} < \frac{1}{3}}, \quad (53)$$

$$P(B2|A) = \left[1 - \frac{3m_c}{M} + \frac{2m_c}{M} \ln \left(\frac{M}{m_c} - 2 \right) - \int_{\frac{m_c}{M}}^{1-2\frac{m_c}{M}} \int_{\frac{m_c}{M(1-r_1)}}^{1-\frac{m_c}{M(1-r_1)}} dr_1 dr_2 \left(\frac{m_c}{M(1-r_1)r_2} \right) \right]_{0 < \frac{m_c}{M} < \frac{1}{3}}, \quad (54)$$

$$P(B1 \wedge B2|A) = \left[\int_{1-3\frac{m_c}{M}}^{1-\frac{m_c}{M}} \int_{\frac{m_c}{M(1-r_1)}}^{1-\frac{m_c}{M(1-r_1)}} dr_1 dr_2 \left(\frac{2m_c}{Mr_2(1-r_1)} - 1 \right) \right]_{0 < \frac{m_c}{M} < \frac{1}{4}} + \left[\int_{\frac{m_c}{M}}^{1-3\frac{m_c}{M}} \int_{\frac{m_c}{M(1-r_1)}}^{2\frac{m_c}{M(1-r_1)}} dr_1 dr_2 \left(\frac{2m_c}{Mr_2(1-r_1)} - 1 \right) \right]_{0 < \frac{m_c}{M} < \frac{1}{4}}$$

$$+ \left[\int_{\frac{m_c}{M}}^{1-\frac{m_c}{M}} \int_{\frac{m_c}{M(1-r_1)}}^{1-\frac{m_c}{M(1-r_1)}} dr_1 dr_2 \left(\frac{2m_c}{Mr_2(1-r_1)} - 1 \right) \right]_{\frac{1}{4} < \frac{m_c}{M} < \frac{1}{2}} \quad (55) \quad + \left[\int_{\frac{m_c}{M}}^{1-\frac{m_c}{M}} \int_{\frac{m_c}{Mr_1}}^{1-\frac{m_c}{Mr_1}} dr_1 dr_2 \left(\frac{2m_c}{Mr_1 r_2} - 1 \right) \right]_{\frac{1}{4} < \frac{m_c}{M} < \frac{1}{2}} \quad (61)$$

$$P(C1|C) = \left[\int_{\frac{m_c}{M}}^{1-\frac{m_c}{M}} \int_{\frac{m_c}{Mr_1}}^{1-\frac{m_c}{Mr_1}} dr_1 dr_2 \times \left(\frac{m_c}{Mr_1(1-r_2)} \right) \right]_{0 < \frac{m_c}{M} < \frac{1}{3}} \quad (56)$$

$$P(C2|C) = \left[1 - \frac{3m_c}{M} - \frac{2m_c}{M} \ln \left(\frac{M}{2m_c} - \frac{1}{2} \right) - \int_{\frac{m_c}{M}}^{1-\frac{m_c}{M}} \int_{\frac{m_c}{Mr_1}}^{1-\frac{m_c}{Mr_1}} dr_1 dr_2 \left(1 - \frac{m_c}{Mr_1(1-r_2)} \right) \right]_{0 < \frac{m_c}{M} < \frac{1}{3}} \quad (57)$$

$$P(C1 \wedge C2|C) = \left[\int_{\frac{m_c}{M}}^{3\frac{m_c}{M}} \int_{\frac{m_c}{Mr_1}}^{1-\frac{m_c}{Mr_1}} dr_1 dr_2 \left(\frac{2m_c}{Mr_1(1-r_2)} - 1 \right) \right]_{0 < \frac{m_c}{M} < \frac{1}{4}} + \left[\int_{\frac{m_c}{M}}^{1-\frac{m_c}{M}} \int_{\frac{m_c}{Mr_1}}^{1-\frac{m_c}{Mr_1}} dr_1 dr_2 \left(\frac{2m_c}{Mr_1(1-r_2)} - 1 \right) \right]_{0 < \frac{m_c}{M} < \frac{1}{4}} + \left[\int_{\frac{m_c}{M}}^{1-\frac{m_c}{M}} \int_{\frac{m_c}{Mr_1}}^{1-\frac{m_c}{Mr_1}} dr_1 dr_2 \left(\frac{2m_c}{Mr_1(1-r_2)} - 1 \right) \right]_{\frac{1}{4} < \frac{m_c}{M} < \frac{1}{2}} \quad (58)$$

$$P(D1|C) = \left[\int_{\frac{m_c}{M}}^{1-\frac{m_c}{M}} \int_{\frac{m_c}{Mr_1}}^{1-\frac{m_c}{Mr_1}} dr_1 dr_2 \left(\frac{m_c}{Mr_1 r_2} \right) \right]_{0 < \frac{m_c}{M} < \frac{1}{3}} \quad (59)$$

$$P(D2|C) = \left[1 - \frac{3m_c}{M} - \frac{2m_c}{M} \ln \left(\frac{M}{2m_c} - \frac{1}{2} \right) - \int_{\frac{m_c}{M}}^{1-\frac{m_c}{M}} \int_{\frac{m_c}{Mr_1}}^{1-\frac{m_c}{Mr_1}} dr_1 dr_2 \left(1 - \frac{m_c}{Mr_1 r_2} \right) \right]_{0 < \frac{m_c}{M} < \frac{1}{3}} \quad (60)$$

$$P(D1 \wedge D2|C) = \left[\int_{\frac{m_c}{M}}^{1-\frac{m_c}{M}} \int_{\frac{m_c}{Mr_1}}^{2\frac{m_c}{Mr_1}} dr_1 dr_2 \left(\frac{2m_c}{Mr_1 r_2} - 1 \right) \right]_{0 < \frac{m_c}{M} < \frac{1}{4}} + \left[\int_{\frac{m_c}{M}}^{3\frac{m_c}{M}} \int_{\frac{m_c}{Mr_1}}^{1-\frac{m_c}{Mr_1}} dr_1 dr_2 \left(\frac{2m_c}{Mr_1 r_2} - 1 \right) \right]_{0 < \frac{m_c}{M} < \frac{1}{4}}$$

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