

Microinertia and internal variables

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Abstract

The origin of microinertia of micromorphic theories is investigated from the point of view of non-equilibrium thermodynamics. In the framework of dual internal variables microinertia stems from a thermodynamic equation of state related to the internal variable with the properties of mechanical momentum.

1 Introduction

The prediction of a thermomechanical behavior of solids with microstructure cannot be completely correct unless the influence of the microstructure is taken into account. In spite of the possible small scale characterizing the microstructure, its influence may be not necessary small if corresponding gradients are comparable with those at the macroscale. It is well understood that microstructural effects can be described only by means of a generalization of classical thermomechanical theory. Such a generalization is usually achieved at the expense of the extension of the state space by additional independent variables like the microrotation in Cosserats media, the microdeformation in micromorphic media, or the heat flux in extended thermodynamics.

Making the distinction between the medium and its microstructural components, one implicitly admits that they can respond to dynamic loads differently. This is reflected in the notion of microinertia appeared in the description of micromorphic media [1, 2]. Mindlin [2] described the deformation of what he called "cell" explicitly, while Eringen and Suhubi [1] introduced an averaging procedure over the microelement. Both approaches results in similar balances of micromomentum in the linear case. As noted by Eringen and Suhubi [1], the corresponding microinertia tensor "resembles" the physical inertia. This means that the coefficient of the second time derivative of the microdeformation is consistent with the constitutive relations.

Mindlin as well as Eringen and Suhubi considered their microdeformation as an additional degree of freedom. Therefore, it should be governed (controlled) by boundary conditions pointed out in the general form by both approaches. However, it is difficult to imagine boundary conditions that excite the microstructure only, without affecting the medium itself. This difficulty brings the internal degrees of freedom together with internal variables of state, which even cannot be controlled by boundary conditions.

The classical treatment of internal variables suggests that their evolution equations may contain only the first-order time derivatives, and, therefore, have no relations to inertia [3–6]. This is definitely true for order parameters considered as internal variables, and for pure dissipative effects like heat conduction. However, the generalization of the internal variables theory by the dual internal variables concept allows hyperbolic evolution equations as well [7]. The question is how the coefficient of the second order time derivative of the microdeformation corresponds to microinertia.

In order to answer this question, we will show how the formalism of dual internal variables works. Starting with simplest possible case, we demonstrate the difference between the standard single internal variable theory and the dual internal variables concept. Then we compare the generalization of linear elasticity by dual internal variables with the Mindlin microelasticity model [2]. For simplicity, all the considerations are made in one-dimensional case.

2 Single internal variable: one-dimensional example

While for mechanical motion of a material point it is sufficient to prescribe its coordinates and velocities, the theory of internal variables is essentially thermodynamical. This means that we deal with a thermodynamical system and suppose that all thermodynamic quantities like temperature, energy, entropy, etc. are defined.

2.1 Entropic representation

Here we assume that the entropy density S is specified as a function of the internal energy density E and an internal variable φ and its space derivative φ_x :

$$S = \bar{S}(E, \varphi, \varphi_x). \quad (1)$$

Constitutive assumption (1) allows us to introduce the corresponding conjugate quantities as follows:

$$\frac{1}{\theta} := \frac{\partial \bar{S}}{\partial E}, \quad \tau := \frac{\partial \bar{S}}{\partial \varphi}, \quad \eta := \frac{\partial \bar{S}}{\partial \varphi_x}, \quad (2)$$

where θ is the temperature. With the usual thermodynamic notation with differentials the consequent Gibbs relation can be written as

$$dE = \theta dS - \tau d\varphi - \eta d\varphi_x. \quad (3)$$

Therefore, Eq. (1) is the equation of state of a "gradient" (or weakly non-local) internal variable theory.

The conservation of the internal energy can be represented as

$$E_t + Q_x = 0, \quad (4)$$

where Q is the heat flux, index t denotes time derivative. For the calculation of the entropy production a generalized entropy current density (i.e. entropy flux) is introduced in the form

$$J = \frac{Q - \eta\varphi}{\theta}. \quad (5)$$

Here the deviation of the entropy current density from its classical form $J_{class} = Q/\theta$ is represented by a particular additional term. This form is characteristic for second-order weakly nonlocal theories and can be justified either by a rigorous exploitation of the entropy inequality [8, 9], or by the classical separation of divergences [10, 6]. In the next section we will demonstrate the second method.

With the help of modified entropy flux (5) the entropy production can be calculated as follows:

$$\begin{aligned} S_t + J_x &= \frac{1}{\theta}E_t + \frac{\tau}{\theta}\varphi_t + \frac{\eta}{\theta}\varphi_{tx} + \left(\frac{Q - \eta\varphi}{\theta}\right)_x \\ &= Q \left(\frac{1}{\theta}\right)_x + \varphi_t \left(\frac{\tau}{\theta} - \left[\frac{\eta}{\theta}\right]_x\right) \\ &= (Q - \eta\varphi_t) \left(\frac{1}{\theta}\right)_x + \frac{\varphi_t}{\theta} (\tau - \eta_x) \geq 0. \end{aligned} \quad (6)$$

A constitutive relation for heat flux and an evolution equation for internal variable are needed to be specified for any considered process. The given form of the entropy production supposes two possible representations of the thermodynamic fluxes and forces as one can see inspecting the last two lines of Eq. (6). In the first form the thermal part is the classical, while in the second form the thermal part is modified. However, the linear constitutive equations are identical, as it is shown in the Appendix.

2.2 Energetic representation

Free energy is the Legendre transformation of the internal energy by the entropy $W = E - S\theta$. It is convenient for practical use, because temperature is the introduced variable of state instead of entropy or internal energy:

$$W = \overline{W}(\theta, \varphi, \varphi_x). \quad (7)$$

Partial derivatives of W define

$$S := -\frac{\partial \overline{W}}{\partial \theta}, \quad \tau := -\frac{\partial \overline{W}}{\partial \varphi}, \quad \eta := -\frac{\partial \overline{W}}{\partial \varphi_x}, \quad (8)$$

which results in the Pfaffian form

$$dW = -Sd\theta - \tau d\varphi - \eta d\varphi_x. \quad (9)$$

The energy conservation equation is

$$(S\theta)_t + Q_x = h^{int}, \quad h^{int} := -W_t, \quad (10)$$

where the right-hand side of Eq. (10)₁ is formally an internal heat source [6].

The energy conservation equation should be accompanied by the second law of thermodynamics, here written as

$$S_t + (Q/\theta + K)_x \geq 0, \quad (11)$$

where in the entropy flux is additively separated onto the classical part and the extra entropy flux K . Multiplying the second law (11) by θ

$$\theta S_t + \theta(Q/\theta + K)_x \geq 0, \quad (12)$$

and taking into account Eq. (10), we obtain

$$-(W_t + S\theta_t) + (\theta K)_x - (Q/\theta + K)\theta_x \geq 0. \quad (13)$$

The last equation can be represented in the form

$$S\theta_t + (Q/\theta + K)\theta_x \leq h^{int} + (\theta K)_x. \quad (14)$$

The internal heat source h^{int} is calculated following the constitutive assumption (7)

$$h^{int} = -W_t = -\frac{\partial W}{\partial \theta}\theta_t - \frac{\partial W}{\partial \varphi}\varphi_t - \frac{\partial W}{\partial \varphi_x}\varphi_{xt} = S\theta_t + \tau\varphi_t + \eta\varphi_{xt} = h^{th} + h^{intr}. \quad (15)$$

Accounting for Eq. (15), dissipation inequality (14) can be rewritten as

$$\Phi = \tau\varphi_t + \eta\varphi_{xt} - (Q/\theta + K)\theta_x + (\theta K)_x \geq 0. \quad (16)$$

Here Φ is the entropy production multiplied by the temperature. To rearrange the dissipation inequality, we add and subtract the same term $\eta_x\varphi_t$

$$\Phi = \tau\varphi_t + \eta\dot{\varphi}_x - \eta_x\varphi_t + \eta_x\varphi_t - (Q/\theta + K)\theta_x + (\theta K)_x \geq 0, \quad (17)$$

that leads to

$$\Phi = (\tau - \eta_x)\varphi_t - (Q/\theta + K)\theta_x + (\eta\varphi_t + \theta K)_x \geq 0. \quad (18)$$

Therefore we can define the "extra" entropy flux by the elimination of the divergence term in Eq. (18)

$$K = -\theta^{-1}\eta\varphi_t. \quad (19)$$

This method, the separation of divergences, has its roots in the classical irreversible thermodynamics [10] and formulated explicitly in case of internal variables by Maugin [11]. Then the dissipation inequality reduces to

$$\Phi = (\tau - \eta_x)\varphi_t - (Q + \eta\varphi_t)(\log(\theta))_x \geq 0. \quad (20)$$

It is remarkable that in the isothermal case ($\theta_x = 0$) the dissipation is determined by the internal variable only. It is easy to check that inequalities (20) and (6) are identical.

2.3 Evolution equation for a single internal variable

In the *isothermal* case the dissipation inequality (20) is

$$\Phi = (\tau - \eta_x)\varphi_t \geq 0, \quad (21)$$

The linear solution is

$$\varphi_t = k(\tau - \eta_x), \quad k \geq 0, \quad (22)$$

since dissipation inequality (21) is satisfied automatically in this case

$$\Phi = k\varphi_t^2 \geq 0, \quad \text{if } k \geq 0. \quad (23)$$

It is easy to see that the dissipation is the product of the thermodynamic flux φ_t and the thermodynamic force $(\tau - \eta_x)$. The proportionality between the thermodynamic flux and the conjugated force is the standard choice to satisfy the dissipation inequality.

To see how the obtained evolution equation looks like, we specialize free energy dependence (1) in the isothermal case to a quadratic one

$$\bar{W} = \frac{1}{2}B\varphi^2 + \frac{1}{2}C\varphi_x^2, \quad (24)$$

where B and C are material parameters. It follows from equations of state (2) that

$$\tau := -\frac{\partial \bar{W}}{\partial \varphi} = -B\varphi, \quad \eta := -\frac{\partial \bar{W}}{\partial \varphi_x} = -C\varphi_x, \quad (25)$$

and evolution equation (22) is an equation of reaction-diffusion type

$$\varphi_t = k(C\varphi_{xx} - B\varphi), \quad k \geq 0. \quad (26)$$

The given standard formalism of the single internal variable of state is sufficient for many cases [12–19]. In the case of a more general free energy, Eq. (26) is the Ginzburg-Landau equation [20].

2.4 Dual internal variables in one dimension

Now we would like to extend the same technique to the case of two internal variables. However, none of them is presupposed to be definitely an internal variable of state or an internal degree of freedom. Let us consider them as internal variables “in general”. Here we treat the isothermal case, therefore we extend the free energy focused train of thought of the previous section. The corresponding entropic treatment is straightforward and equivalent.

Let us suppose that the free energy density depends on the internal variables φ, ψ and their space derivatives

$$W = \bar{W}(\theta, \varphi, \varphi_x, \psi, \psi_x). \quad (27)$$

The equations of state in the case of two internal variables read

$$S = -\frac{\partial \bar{W}}{\partial \theta}, \quad \tau := -\frac{\partial \bar{W}}{\partial \varphi}, \quad \eta := -\frac{\partial \bar{W}}{\partial \varphi_x}, \quad \xi := -\frac{\partial \bar{W}}{\partial \psi}, \quad \zeta := -\frac{\partial \bar{W}}{\partial \psi_x}. \quad (28)$$

The Gibbs relation follows accordingly

$$dW = -Sd\theta - \tau d\varphi - \eta d\varphi_x - \xi d\psi - \zeta d\psi_x. \quad (29)$$

Then we consider a non-zero extra entropy flux following the case of a single internal variable and set

$$K = -\theta^{-1}\eta\varphi_t - \theta^{-1}\zeta\zeta_t. \quad (30)$$

It can be checked that the intrinsic heat source is determined in the considered case as follows

$$\tilde{h}^{intr} := (\tau - \eta_x)\varphi_t + (\xi - \zeta_x)\psi_t. \quad (31)$$

The latter means that the dissipation inequality in the isothermal case reduces to

$$\Phi = (\tau - \eta_x)\varphi_t + (\xi - \zeta_x)\psi_t \geq 0. \quad (32)$$

The solution of the dissipation inequality can be represented as [7]

$$\begin{pmatrix} \varphi_t \\ \psi_t \end{pmatrix} = \mathbf{L} \begin{pmatrix} \tau - \eta_x \\ \xi - \zeta_x \end{pmatrix}, \quad \text{where } \mathbf{L} = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}. \quad (33)$$

Nonnegativity of the entropy production (32) results in the positive semidefiniteness of the symmetric part of the conductivity matrix \mathbf{L} , which requires

$$L_{11} \geq 0, \quad L_{22} \geq 0, \quad L_{11}L_{22} - \frac{(L_{12} + L_{21})^2}{2} \geq 0. \quad (34)$$

To be more specific, we keep a quadratic free energy density

$$\bar{W} = \frac{1}{2}B\varphi^2 + \frac{1}{2}C\varphi_x^2 + \frac{1}{2}D\psi^2 + \frac{1}{2}F\psi_x^2. \quad (35)$$

Calculating the quantities defined in Eq. (28)

$$\tau := -\frac{\partial \bar{W}}{\partial \varphi} = -B\varphi, \quad \eta := -\frac{\partial \bar{W}}{\partial \varphi_x} = -C\varphi_x, \quad (36)$$

$$\xi := -\frac{\partial \bar{W}}{\partial \psi} = -D\psi, \quad \zeta := -\frac{\partial \bar{W}}{\partial \psi_x} = -F\psi_x, \quad (37)$$

we can represent system of Eqs. (33) in the form

$$\varphi_t = L_{11}(-B\varphi + C\varphi_{xx}) + L_{12}(-D\psi + F\psi_{xx}), \quad (38)$$

$$\psi_t = L_{21}(-B\varphi + C\varphi_{xx}) + L_{22}(-D\psi + F\psi_{xx}). \quad (39)$$

Now we will derive a single evolution equation for the primary internal variable. To this end a suitable rearrangement of the previous equations is performed:

$$\left(\frac{\partial}{\partial t} + L_{11}(B - C\frac{\partial^2}{\partial x^2}) \right) \varphi + L_{12} \left(D - F\frac{\partial^2}{\partial x^2} \right) \psi = \hat{L}_{11}\varphi + \hat{L}_{12}\psi = 0, \quad (40)$$

$$L_{21} \left(B - C\frac{\partial^2}{\partial x^2} \right) \varphi + \left(\frac{\partial}{\partial t} + L_{22}(D - F\frac{\partial^2}{\partial x^2}) \right) \psi = \hat{L}_{21}\varphi + \hat{L}_{22}\psi = 0. \quad (41)$$

Here the traditional notation of time and space derivatives is used for the clear distinction of the differential operators \widehat{L}_{11} , \widehat{L}_{12} , \widehat{L}_{21} and \widehat{L}_{22} . Then the easiest way to eliminate ψ is to multiply the first equation, (40), by \widehat{L}_{22} and the second one, (41), by \widehat{L}_{12} and subtract them:

$$\begin{aligned} \widehat{L}_{22}\widehat{L}_{11}\varphi - \widehat{L}_{12}\widehat{L}_{21}\varphi &= \varphi_{tt} - \det L(CD + BF)\varphi_{xx} + \\ &+ (BL_{11} + DL_{22})\varphi_t - (CL_{11} + FL_{22})\varphi_{txx} + \det L(BD\varphi + CF\varphi_{xxxx}) = 0, \end{aligned} \quad (42)$$

where $\det L = L_{11}L_{22} - L_{12}L_{21}$ is the determinant of L . The free energy density W is non-negative by default, which results in non-negativity of material parameters B, C, D , and F . This means that Eq. (42) is the hyperbolic wave equation with dispersion and dissipation.

Thus, extending the state space of our thermodynamic system by an additional internal variable and keeping the quadratic form for the free energy density, we arrive at the hyperbolic evolution equation for the primary internal variable. The corresponding evolution equation for the secondary internal variable can be derived similarly. The natural question is the following: can we associate the abstract internal variables with the description of any physical process?

2.5 Example I: Linear elasticity

To answer the question posed above, we simplify the situation as much as possible. We note first that dissipation will be absent if the diagonal components of the conductivity matrix are zero ($L_{11} = L_{22} = 0$). Due to inequalities (34) this requires antisymmetric conductivity matrix, that is, Casimir type reciprocal relations $L_{12} = -L_{21}$. Next, dispersion will be eliminated if we choose the material parameters $B = 0$ and $F = 0$.

It follows immediately that the evolution equations for the dual internal variables (38) and (39) can be rewritten as

$$\varphi_t = -L_{12}D\psi, \quad (43)$$

$$\psi_t = -L_{12}C(\varphi_x)_x. \quad (44)$$

Without loss of generality, we can choose the value $L_{12} = -1$. We should emphasize that the two evolution equations express the duality between internal variables: one internal variable is driven by another one and vice versa.

Now we are ready to relate the introduced internal variables to well-known physical quantities. First, we note that Eqs. (43), (44) can be represented in the form

$$\varphi_{xt} = D\psi_x, \quad (45)$$

$$\psi_t = C(\varphi_x)_x. \quad (46)$$

Secondly, denoting φ_x as ε , and ψ as ρv we arrive at the system of equations

$$\varepsilon_t = D(\rho v)_x, \quad (47)$$

$$(\rho v)_t = C\varepsilon_x, \quad (48)$$

which coincides with one-dimensional equations for linear elasticity (with the standard notation for derivatives), and ρ is constant:

$$\frac{\partial \varepsilon}{\partial t} = \frac{\partial v}{\partial x'}, \quad (49)$$

$$\rho \frac{\partial v}{\partial t} = \frac{\partial \sigma}{\partial x'}, \quad (50)$$

after the following choice of coefficients: $D = 1/\rho$ and $C = \rho c^2$, where c is the elastic wave velocity and $\sigma = \rho c^2 \varepsilon$. This means that the internal variable φ can be interpreted as the displacement and the internal variable ψ as the momentum. These variables are controlled by boundary conditions and, therefore, they are true mechanical degrees of freedom, while appeared as internal degrees of freedom first. No doubts arise about inertia in this case.

We have considered a thermodynamic system (body) under assumption that there may be certain internal phenomena which we tried to describe by internal variables. We have derived possible evolution equations for these internal variables from the first and second laws of thermodynamics. It has been demonstrated that evolution equations for internal variables may be hyperbolic for a quadratic free energy density. Moreover, in the non-dissipative and non-dispersive case these evolution equations can be identified with equations of linear elasticity.

It follows that both internal variables of state and internal degrees of freedom can be considered as particular cases in the framework of unified formalism. This formalism can help us to consider generalized continuum models by means of internal variables. In our next example, we look at the Mindlin micromorphic elasticity theory.

3 Example II: One-dimensional microelasticity

Following [2], in our one-dimensional treatment we introduce the macrostrain ε , the microdeformation ψ , the relative deformation $\gamma = \varepsilon - \psi$, and the microdeformation gradient ψ_x . The corresponding quadratic free energy density [2] is reduced to

$$W(\varepsilon, \gamma, \psi_x) = \frac{1}{2} \rho c^2 \varepsilon^2 + g \varepsilon \gamma + \frac{b}{2} \gamma^2 + \frac{a}{2} \psi_x^2. \quad (51)$$

Equations of state determine the corresponding stresses

$$\tau = \frac{\partial W}{\partial \varepsilon} = \rho c^2 \varepsilon + g \gamma, \quad \sigma = \frac{\partial W}{\partial \gamma} = g \varepsilon + b \gamma, \quad \mu = \frac{\partial W}{\partial \psi_x} = a \psi_x, \quad (52)$$

and equations of motion have the form [2] (without body forces)

$$\rho u_{tt} = \tau_x + \sigma_x, \quad (53)$$

$$\frac{1}{3} \rho' d^2 \psi_{tt} = \mu_x + \sigma, \quad (54)$$

where u is the displacement, and $\rho' d^2/3$ plays the role of microinertia.

Now we compare the Mindlin microelasticity theory with the dual internal variables approach. To avoid the confusion with the original notation of Mindlin, we use other symbols for internal variables. We suppose that the free energy density depends on the gradient of the displacement, u_x , the internal variables α, β and their space derivatives

$$W = \overline{W}(u_x, \alpha, \alpha_x, \beta, \beta_x). \quad (55)$$

Then the equations of state follow

$$\sigma' := \frac{\partial \overline{W}}{\partial u_x}, \quad \tau' := -\frac{\partial \overline{W}}{\partial \alpha}, \quad \eta := -\frac{\partial \overline{W}}{\partial \alpha_x}, \quad \xi := -\frac{\partial \overline{W}}{\partial \beta}, \quad \zeta := -\frac{\partial \overline{W}}{\partial \beta_x}. \quad (56)$$

As in [2], we use a quadratic function for the free energy density

$$\overline{W} = \frac{1}{2}\rho c^2 u_x^2 + A\alpha u_x + \frac{1}{2}B\alpha^2 + \frac{1}{2}C\alpha_x^2 + \frac{1}{2}D\beta^2. \quad (57)$$

Considering the non-dissipative case, we have for the evolution equation for internal variables (cf. Eqs. (38), (39) with $L_{11} = L_{22} = 0$)

$$\alpha_t = L_{12}(\xi - \zeta_x) = -L_{12}D\beta, \quad (58)$$

$$\beta_t = -L_{12}(\tau' - \eta_x) = -L_{12}(C\alpha_{xx} - Au_x - B\alpha). \quad (59)$$

As a result, we arrive at the hyperbolic evolution equation equation for the primary internal variable

$$I\alpha_{tt} = C\alpha_{xx} - Au_x - B\alpha, \quad (60)$$

where $I = 1/(L_{12}^2 D)$ is the measure of microinertia.

The balance of linear momentum

$$\rho v_t = \sigma'_x, \quad (61)$$

results in the form

$$\rho u_{tt} = \rho c^2 u_{xx} + A\alpha_x. \quad (62)$$

Let us rewrite the governing equations of the Mindlin microelasticity (53) and (54) using the notation u_x instead of ε and α instead of the microdeformation ψ . We have, respectively,

$$\rho u_{tt} = \tau_x + \sigma_x = \rho c^2 u_{xx} + (b + 2g)u_{xx} - (g + b)\alpha_x, \quad (63)$$

$$\frac{1}{3}\rho' d^2 \alpha_{tt} = a\alpha_{xx} + (g + b)u_x - b\alpha, \quad (64)$$

It is easy to see that systems of equations (60), (62) and (63), (64) coincide if the following values of parameters are chosen: $A = -(b + g)$, $B = b$, $C = a$, $I = 1/3\rho' d^2$, $b + 2g = 0$. This means that the dual internal variable theory can recover the generalized continuum theory of Mindlin [2] and the corresponding microinertia measure has the same justification as in the Mindlin microelasticity.

4 Summary and discussion

In this short note we have analyzed the relation of mechanical and thermodynamical approaches of extending the state space of continuum theories. Our analysis focused on the origin of inertial terms in the evolution equations. In the first part of the paper we have presented the thermodynamic background of single and dual internal variable theories both in the entropic and in the energetic representation [21].

The recent methodology of dual internal variables unifies these treatments and we have shown that both classical linear elasticity and Mindlin type microelasticity are well suited in this general background without a direct reference to mechanical principles or notions. Our treatment was restricted to one spatial dimension and without a mention of ispace-time related conceptual questions. The 3D tensorial representation of the application of the dual internal variables approach is given in [22, 23].

What is the origin of inertia in the light of our approach? The two ingredients were the thermodynamic equation of state and the antisymmetric part of the constitutively constructed dynamics. The more surprising part is the first: in our treatment inertia and microinertia stem from an equation of state. The coefficients C and D in the quadratic free energy function of linear elasticity, (Eq. (35)), are connected with the density, which is the measure of inertia. In case of microinertia the coefficient D is enough, the gradient term of the second variable was not necessary. In our dual internal variable approach one of the variables plays the role of momentum when compared to non-dissipative mechanics, on the other hand it is a state variable from a thermodynamic point of view.

This kind of close connection of mechanics and thermodynamics is not too surprising from the broader perspective of relativistic theories. Therein momentum cannot be separated from energy in a covariant treatment, therefore an energy dependent entropy depends naturally on the momentum as well. This is the root of the famous problem of the proper Lorentz transformation of the temperature. The first formulation of Planck and Einstein considered energy and momentum as variables of the entropy [24–26]. Then Blanuša and Ott seemingly have eliminated this dependence by fixing the thermometer to the body [27, 28]. However, in a proper covariant treatment these approaches are only two sides of the same coin, where the entropy depends on energy-momentum [29, 30].

In our nonrelativistic treatment the integration of mechanics and thermodynamics looks like more involved than usually taken. From the point of view of the presented approach it is remarkable, that Hamiltonian dynamics is a particular case characterized by zero dissipation. Moreover, the possibility of a pure variational description can be a consequence of ideal dynamics without dissipation [31].

In summary, we have shown that inertial terms are natural in a thermodynamic theory with dual variables and the conditions of their appearance is well understandable in terms of mechanical notions. However, a true space-time related frame independent treatment is required to a final clarification of this issue.

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5 Appendix: Representations of entropy production

In some cases entropy production $(6)_2$ appears in the following form

$$\Sigma = J_1 X_1 + J_2 (X_2 - AX_1) \geq 0, \quad (65)$$

where J_1 and J_2 are constitutive quantities, and they are to be determined as functions of X_1 and X_2 , A is a given function of X_1 and X_2 . One may introduce the following choice of thermodynamic fluxes and forces:

Fluxes	J_1	J_2
Forces	X_1	$X_2 - AX_1$

Table 1. First choice of thermodynamic fluxes and forces

The linear solution of inequality (65) results in:

$$J_1 = l_1 X_1 + l_{12} (X_2 - AX_1), \quad (66)$$

$$J_2 = l_{21} X_1 + l_2 (X_2 - AX_1). \quad (67)$$

Here the inequality requires that the symmetric part of the coefficient matrix is positive definite, therefore l_1, l_2, l_{12} and l_{21} coefficients fulfill the following inequalities:

$$l_1 > 0, \quad l_2 > 0, \quad \text{and} \quad l_1 l_2 - \frac{(l_{12} + l_{21})^2}{4} \geq 0. \quad (68)$$

However, the entropy production may be written in a distinct but equivalent form

$$\Sigma = (J_1 - AJ_2)X_1 + J_2 X_2 \geq 0. \quad (69)$$

This grouping of the terms suggest a different choice of the thermodynamic forces and fluxes:

Fluxes	$J_1 - AJ_2$	J_2
Forces	X_1	X_2

Table 1. Second choice of thermodynamic fluxes and forces

The consequent linear solution introduces the constitutive relations as follows:

$$J_1 - AJ_2 = k_1 X_1 + k_{12} X_2, \quad (70)$$

$$J_2 = k_{21} X_1 + k_2 X_2, \quad (71)$$

where the coefficient matrix must fulfill the following inequalities:

$$k_1 > 0, \quad k_2 > 0, \quad \text{and} \quad k_1 k_2 - \frac{(k_{12} + k_{21})^2}{4} \geq 0. \quad (72)$$

The material coefficients l_1, l_2, l_{12} and l_{21} are not independent of k_1, k_2, k_{12} and k_{21} and their relations can be expressed easily:

$$\begin{pmatrix} l_1 & l_{12} \\ l_{21} & l_2 \end{pmatrix} = \begin{pmatrix} k_1 + A(k_{12} + k_{21}) + A^2k_2 & k_{12} + Ak_2 \\ k_{21} + Ak_2 & k_2 \end{pmatrix} \quad (73)$$

The inverted form is also explicit

$$\begin{pmatrix} k_1 & k_{12} \\ k_{21} & k_2 \end{pmatrix} = \begin{pmatrix} l_1 - A(l_{12} + l_{21}) + A^2l_2 & l_{12} - Al_2 \\ l_{21} - Al_2 & l_2 \end{pmatrix} \quad (74)$$

As one can check, inequalities (68) and (72) are equivalent and therefore the constitutive equations (66)-(67) are equivalent to (70)-(71). The seemingly different thermodynamic fluxes and forces lead to equivalent constitutive relations.

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