

Asymptotic properties of maximum likelihood estimators for Heston models based on continuous time observations

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Abstract

We study asymptotic properties of maximum likelihood estimators for Heston models based on continuous time observations of the log-price process. We distinguish three cases: subcritical (also called ergodic), critical and supercritical. In the subcritical case, asymptotic normality is proved for all the parameters, while in the critical and supercritical cases, non-standard asymptotic behavior is described.

1 Introduction

Affine processes and especially the Heston model have been frequently applied in financial mathematics since they can be well-fitted to financial time series, and also due to their computational tractability. They are characterized by their characteristic function which is exponentially affine in the state variable. A precise mathematical formulation and a complete characterization of regular affine processes are due to Duffie et al. [19]. A very recent monograph of Baldeaux and Platen [4] gives a detailed survey on affine processes and their applications in financial mathematics.

Let us consider a Heston model

$$(1.1) \quad \begin{cases} dY_t = (a - bY_t) dt + \sigma_1 \sqrt{Y_t} dW_t, \\ dX_t = (\alpha - \beta Y_t) dt + \sigma_2 \sqrt{Y_t} (\varrho dW_t + \sqrt{1 - \varrho^2} dB_t), \end{cases} \quad t \geq 0,$$

where $a > 0$, $b, \alpha, \beta \in \mathbb{R}$, $\sigma_1 > 0$, $\sigma_2 > 0$, $\varrho \in (-1, 1)$ and $(W_t, B_t)_{t \geq 0}$ is a 2-dimensional standard Wiener process. In this paper we study maximum likelihood estimator (MLE) of (a, b, α, β) based on continuous time observations $(X_t)_{t \in [0, T]}$ with $T > 0$, starting the process (Y, X) from some known non-random initial value $(y_0, x_0) \in (0, \infty) \times \mathbb{R}$. We do not suppose the process $(Y_t)_{t \in [0, T]}$ being observed, since it can be determined using the observations $(X_t)_{t \in [0, T]}$, see Remark 2.5. We do not estimate the parameters σ_1 , σ_2 and ϱ , since these parameters could—in principle, at least—be determined (rather than estimated) using the observations $(X_t)_{t \in [0, T]}$, see Remark 2.6. Further,

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it will turn out that for the calculation of the MLE of (a, b, α, β) , one does not need to know the values of the parameters $\sigma_1 > 0$, $\sigma_2 > 0$, and $\varrho \in (-1, 1)$, see (3.4). Note also that $(Y_t, X_t)_{t \geq 0}$ is a 2-dimensional affine diffusion process with state space $[0, \infty) \times \mathbb{R}$, see Proposition 2.1. In the language of financial mathematics, provided that $\beta = \sigma_2^2/2$, one can interpret

$$S_t := \exp \left\{ X_t - \alpha + \frac{\sigma_2^2}{2} t \right\}$$

as the asset price, $X_t - \alpha + \frac{\sigma_2^2}{2} t$ as the log-price (log-spot) and $\sigma_2 \sqrt{Y_t}$ as the volatility of the asset price at time $t \geq 0$. Indeed, using (1.1), by an application of Itô's formula, if $\beta = \sigma_2^2/2$, then we have

$$dS_t = (\alpha + \sigma_2^2/2) S_t dt + \sigma_2 \sqrt{Y_t} S_t (\varrho dW_t + \sqrt{1 - \varrho^2} dB_t), \quad t \geq 0,$$

which is Equation (19) in Heston [22]. The squared volatility process $(\sigma_2^2 Y_t)_{t \geq 0}$ is a continuous time continuous state branching process with immigration, also called Cox–Ingersoll–Ross (CIR) process, first studied by Feller [21].

Parameter estimation for continuous time models has a long history, see, e.g., the monographs of Liptser and Shiryaev [33, Chapter 17], Kutoyants [29] and Bishwal [13]. For estimating continuous time models used in finance, Phillips and Yu [36] gave an overview of maximum likelihood and Gaussian methods. Since the exact likelihood can be constructed only in special cases (e.g., geometric Brownian motion, Ornstein–Uhlenbeck process, CIR process and inverse square-root process), much attention has been devoted to the development of methods designed to approximate the likelihood.

Aït-Sahalia [1] provides closed-form expansions for the log-likelihood function of multivariate diffusions based on discrete time observations. He proved that, under some conditions, the approximate maximum likelihood exists almost surely, and the difference of the approximate and the true maximum likelihood converges in probability to 0 as the time interval separating observations tends to 0. The above mentioned closed-form expansions for the Heston model can be found in Aït-Sahalia and Kimmel [2, Appendix A.1]. We note that in Sørensen [38] one can find a brief and concise summary of the approach of Aït-Sahalia. In fact, Sørensen [38] gives a survey of estimation techniques for stationary and ergodic (one-dimensional) diffusion processes observed at discrete time points. Besides the above mentioned approach of Aït-Sahalia, she recalls estimating functions with special emphasis on martingale estimating functions and so-called simple estimating functions, together with Bayesian analysis of discretely observed diffusion processes.

Azencott and Gadhyan [3] considered another parametrization of the Heston model (1.1), and they investigated only the subcritical (also called ergodic) case, i.e., when $b > 0$ (see Definition 2.3). They developed an algorithm to estimate the parameters of the Heston model based on discrete time observations for the asset price and the volatility. They supposed that $\sigma_2 = 1$ and $\beta = 1/2$, and estimated the parameters σ_1 and ϱ as well. They assumed the time interval separating two consecutive observations also to be unknown and used MLE based on Euler and Milstein discretization schemes. They showed that parameter estimates derived from the Euler scheme using constrained optimization of the approximate MLE are strongly consistent. Note that we obtain results also on the asymptotic behavior of the MLE, and not only in the subcritical case.

Hurn et al. [23] developed a quasi-maximum likelihood procedure for estimating the parameters of multi-dimensional diffusions based on discrete time observations by replacing the original transition

density by a multivariate Gaussian density with first and second moments approximating the true moments of the unknown density. For affine drift and diffusion functions, these moments are exactly those of the true transitional density. As an example, the Heston stochastic volatility model has been analyzed in the subcritical case. However, they did not investigate consistency or asymptotic behavior of their estimators.

Recently, Varughese [41] has studied parameter estimation for time inhomogeneous multi-dimensional diffusion processes given by SDEs based on discrete time observations. The likelihood of a diffusion process in question sampled at discrete time points has been estimated by a so-called saddlepoint approximation. In general, the saddlepoint approximation is an algebraic expression based on a random variable's cumulant generation function. In cases where the first few moments of a random variable are known but the corresponding probability density is difficult to obtain, the saddlepoint approximation to the density can be calculated. The parameter estimates are taken to be the values that maximize this approximate likelihood, which may be estimated by a Markov Chain Monte Carlo (MCMC) procedure. However, the asymptotic properties of the estimators have not been studied. As an example, the saddlepoint MCMC is used to fit a subcritical Heston model to the S&P 500 and the VIX indices over the period December 2009–November 2010.

In case of the one-dimensional CIR process Y , the parameter estimation of a and b goes back to Overbeck and Rydén [34] (conditional least squares estimator (LSE)), Overbeck [35] (MLE), and see also Bishwal [13, Example 7.6] and the very recent papers of Ben Alaya and Kebaier [10], [11] (MLE). We also note that Li and Ma [31] started to investigate the asymptotic behaviour of the (weighted) conditional LSE of the drift parameters for a CIR model driven by a stable noise (they call it a stable CIR model) from some discretely observed low frequency data set.

To the best knowledge of the authors the parameter estimation problem for multi-dimensional affine processes has not been tackled so far. Since affine processes are frequently used in financial mathematics, the question of parameter estimation for them needs to be well-investigated. In Barczy et al. [5] we started the discussion with a simple non-trivial 2-dimensional affine diffusion process given by the SDE

$$(1.2) \quad \begin{cases} dY_t = (a - bY_t) dt + \sqrt{Y_t} dW_t, \\ dX_t = (m - \theta X_t) dt + \sqrt{Y_t} dB_t, \end{cases} \quad t \geq 0,$$

where $a > 0$, $b, m, \theta \in \mathbb{R}$, $(W_t, B_t)_{t \geq 0}$ is a 2-dimensional standard Wiener process. Chen and Joslin [14] have found several applications of the model (1.2) in financial mathematics, see their equations (25) and (26). In the special critical case $b = 0$, $\theta = 0$ we described the asymptotic behavior of the LSE of (m, θ) based on discrete time observations X_0, X_1, \dots, X_n as $n \rightarrow \infty$. The description of the asymptotic behavior of the LSE of (m, θ) in the other critical cases $b = 0$, $\theta > 0$ or $b > 0$, $\theta = 0$ remained opened. In Barczy et al. [7] we dealt with the same model (1.2) but in the so-called subcritical (ergodic) case: $b > 0$, $\theta > 0$, and we considered the MLE of (a, b, m, θ) and the LSE of (m, θ) based on continuous time observations. To carry out the analysis in the subcritical case, we needed to examine the question of existence of a unique stationary distribution and ergodicity for the model given by (1.2). We solved this problem in a companion paper Barczy et al. [6].

Next, we summarize our results comparing with those of Overbeck [35] and Ben Alaya and Kebaier [10], [11], and give an overview of the structure of the paper. Section 2 is devoted to some preliminaries. We recall that the SDE (1.1) has a pathwise unique strong solution and show that it is a regular

affine process, see Proposition 2.1. We describe the asymptotic behaviour of the first moment of $(Y_t, X_t)_{t \geq 0}$, and, based on it, we introduce a classification of Heston processes given by the SDE (1.1), see Proposition 2.2 and Definition 2.3. Namely, we call $(Y_t, X_t)_{t \geq 0}$ subcritical, critical or supercritical if $b > 0$, $b = 0$, or $b < 0$, respectively. We recall a result about existence of a unique stationary distribution and ergodicity for the process $(Y_t)_{t \geq 0}$ given by the first equation in (1.1) in the subcritical case, see Theorem 2.4. From Section 3 we will consider the Heston model (1.1) with a non-random initial value. In Section 3 we study the existence and uniqueness of the MLE of (a, b, α, β) by giving an explicit formula for this MLE as well. It turned out that the MLE of (a, b) based on the observations $(Y_t)_{t \in [0, T]}$ for the CIR process Y is the same as the MLE of (a, b) based on the observations $(X_t)_{t \in [0, T]}$ for the Heston process (Y, X) given by the SDE (1.1), see formula (3.4) and Overbeck [35, formula (2.2)] or Ben Alaya and Kebaier [11, Section 3.1].

In Section 4 we investigate consistency of MLE. For subcritical Heston models we prove that the MLE of (a, b, α, β) is strongly consistent whenever $a \in (\frac{\sigma^2}{2}, \infty)$ (which is an extension of strong consistency of the MLE of (a, b) proved by Overbeck [35, Theorem 2 (ii)], see Remark 4.5), and weakly consistent whenever $a = \frac{\sigma^2}{2}$ (which is an extension of weak consistency of the MLE of (a, b) following from part 1 of Theorem 7 in Ben Alaya and Kebaier [11], see Remark 4.5), see Theorem 4.1. For critical Heston models with $a \in (\frac{\sigma^2}{2}, \infty)$, we obtain weak consistency of the MLE of (a, b, α, β) (as a consequence of Theorem 6.2), which is an extension of weak consistency of the MLE of (a, b) following from Theorem 6 in Ben Alaya and Kebaier [11], see Remark 4.6. For supercritical Heston models $a \in [\frac{\sigma^2}{2}, \infty)$, we get strong consistency of the MLE of b , see Theorem 4.4, and weak consistency of the MLE of β , see Theorem 7.1, and it turns out that the MLE of a and α is not even weakly consistent, see Corollary 7.3. This is an extension of Overbeck [35, Theorem 2, parts (i) and (v)], see Remark 4.7.

Sections 5, 6 and 7 are devoted to study asymptotic behaviour of the MLE of (a, b, α, β) for subcritical, critical and supercritical Heston models, respectively. In Section 5 we show that the MLE of (a, b, α, β) is asymptotically normal in the subcritical case with $a \in (\frac{\sigma^2}{2}, \infty)$, which is a generalization of the asymptotic normality of the MLE of (a, b) proved by Ben Alaya and Kebaier [11, Theorem 5], see Remark 5.2. We also show asymptotic normality with random scaling for the MLE of (a, b, α, β) generalizing the asymptotic normality with random scaling for the MLE of (a, b) due to Overbeck [35, Theorem 3 (iii)], see Remark 5.2. In Section 6 we describe the asymptotic behaviour of the MLE in the critical case with $a \in (\frac{\sigma^2}{2}, \infty)$ generalizing the second part of Theorem 6 in Ben Alaya and Kebaier [11], see Remark 6.3. It turns out that the MLE of a and α is asymptotically normal, but we have a different limit behaviour for the MLE of b and β , see Theorem 6.2. In Theorem 6.4 we incorporate random scaling for the MLE of (a, b, α, β) in case of critical Heston models generalizing part (ii) of Theorem 3 in Overbeck [35], see Remark 6.5. In Section 7 for supercritical Heston models with $a \in [\frac{\sigma^2}{2}, \infty)$, we prove that the MLE of a and α has a weak limit without any scaling (consequently, not weakly consistent, see Corollary 7.3), and the appropriately normalized MLE of b and β has a mixed normal limit distribution, which is a generalization of the second part of Theorem 3 (i) of Overbeck [35], see Remark 7.2. We also show asymptotic normality with random scaling for the MLE of (b, β) generalizing the asymptotic normality with random scaling for the MLE of b due to Overbeck [35, first part of Theorem 3 (i)], see Remark 7.2. In the Appendix we recall some limit theorems for continuous local martingales for studying asymptotic behaviour of the MLE of (a, b, α, β) .

In the proofs, mainly for the critical and supercritical cases, we extensively used the following results of Ben Alaya and Kebaier [10, Propositions 3 and 4], [11, Theorems 4 and 6]: for $b > 0$ and $a = \frac{\sigma_1^2}{2}$, weak convergence of $\frac{1}{T^2} \int_0^T \frac{ds}{Y_s}$ as $T \rightarrow \infty$; for $b = 0$ and $a > \frac{\sigma_1^2}{2}$, the explicit form of the moment generating function of the quadruplet $(\log Y_T, Y_T, \int_0^T Y_s ds, \int_0^T \frac{ds}{Y_s})$, $T > 0$; for $b < 0$ and $a \geq \frac{\sigma_1^2}{2}$, a representation of the weak limit of $(e^{bT} Y_T, \int_0^T \frac{ds}{Y_s})$ as $T \rightarrow \infty$. However, our results are not simple consequences of those of Ben Alaya and Kebaier, we will have to find appropriate decompositions of the derived MLEs and then to investigate the joint weak convergence of the components via continuity theorem.

In Barczy et al. [9], we study conditional least squares estimation for the drift parameters (a, b, α, β) of the Heston model (1.1) starting from some known non-random initial value $(y_0, x_0) \in [0, \infty) \times \mathbb{R}$ based on discrete time observations $(Y_i, X_i)_{i \in \{1, \dots, n\}}$, and in the subcritical case we describe its asymptotic properties.

Finally, note that Benke and Pap [12] study local asymptotic properties of likelihood ratios of the Heston model (1.1) under the assumption $a \in (\frac{\sigma_1^2}{2}, \infty)$. Local asymptotic normality has been proved in the subcritical case and for the submodel when $b = 0$ and $\beta \in \mathbb{R}$ are known in the critical case. Moreover, local asymptotic mixed normality has been shown for the submodel when $a \in (\frac{\sigma_1^2}{2}, \infty)$ and $\alpha \in \mathbb{R}$ are known in the supercritical case. As a consequence, there exist asymptotic minimax bounds for arbitrary estimators in these models, the MLE (for the appropriate submodels in the critical and supercritical cases) attains this bound for bounded loss function, and the MLE is asymptotically efficient in Hájek's convolution theorem sense, see Benke and Pap [12].

2 Preliminaries

Let \mathbb{N} , \mathbb{Z}_+ , \mathbb{R} , \mathbb{R}_+ , \mathbb{R}_{++} , \mathbb{R}_- and \mathbb{R}_{--} denote the sets of positive integers, non-negative integers, real numbers, non-negative real numbers, positive real numbers, non-positive real numbers and negative real numbers, respectively. For $x, y \in \mathbb{R}$, we will use the notations $x \wedge y := \min(x, y)$ and $x \vee y := \max(x, y)$. By $\|x\|$ and $\|A\|$, we denote the Euclidean norm of a vector $x \in \mathbb{R}^d$ and the induced matrix norm of a matrix $A \in \mathbb{R}^{d \times d}$, respectively. By $\mathbf{I}_d \in \mathbb{R}^{d \times d}$, we denote the d -dimensional unit matrix.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. By $C_c^2(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R})$ and $C_c^\infty(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R})$, we denote the set of twice continuously differentiable real-valued functions on $\mathbb{R}_+ \times \mathbb{R}$ with compact support, and the set of infinitely differentiable real-valued functions on $\mathbb{R}_+ \times \mathbb{R}$ with compact support, respectively.

The next proposition is about the existence and uniqueness of a strong solution of the SDE (1.1) stating also that (Y, X) is a regular affine process. Note that these statements for the first equation of (1.1) are well known.

2.1 Proposition. *Let (η_0, ζ_0) be a random vector independent of $(W_t, B_t)_{t \in \mathbb{R}_+}$ satisfying $\mathbb{P}(\eta_0 \in \mathbb{R}_+) = 1$. Then for all $a \in \mathbb{R}_{++}$, $b, \alpha, \beta \in \mathbb{R}$, $\sigma_1, \sigma_2 \in \mathbb{R}_{++}$, $\varrho \in (-1, 1)$, there is a (pathwise) unique strong solution $(Y_t, X_t)_{t \in \mathbb{R}_+}$ of the SDE (1.1) such that $\mathbb{P}((Y_0, X_0) = (\eta_0, \zeta_0)) = 1$ and*

$\mathbb{P}(Y_t \in \mathbb{R}_+ \text{ for all } t \in \mathbb{R}_+) = 1$. Further, for all $s, t \in \mathbb{R}_+$ with $s \leq t$,

$$(2.1) \quad \begin{cases} Y_t = e^{-b(t-s)} \left(Y_s + a \int_s^t e^{-b(s-u)} du + \sigma_1 \int_s^t e^{-b(s-u)} \sqrt{Y_u} dW_u \right), \\ X_t = X_s + \int_s^t (\alpha - \beta Y_u) du + \sigma_2 \int_s^t \sqrt{Y_u} (\varrho dW_u + \sqrt{1 - \varrho^2} dB_u). \end{cases}$$

Moreover, $(Y_t, X_t)_{t \in \mathbb{R}_+}$ is a regular affine process with infinitesimal generator

$$(2.2) \quad \begin{aligned} (\mathcal{A}f)(y, x) &= (a - by)f'_1(y, x) + (\alpha - \beta y)f'_2(y, x) \\ &\quad + \frac{1}{2}y(\sigma_1^2 f''_{1,1}(y, x) + 2\varrho\sigma_1\sigma_2 f''_{1,2}(y, x) + \sigma_2^2 f''_{2,2}(y, x)), \end{aligned}$$

where $(y, x) \in \mathbb{R}_+ \times \mathbb{R}$, $f \in C_c^2(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R})$, and f'_i and $f''_{i,j}$, $i, j \in \{1, 2\}$, denote the first and second order partial derivatives of f with respect to its i -th, and i -th and j -th variables, respectively.

Proof. By a theorem due to Yamada and Watanabe (see, e.g., Karatzas and Shreve [27, Proposition 5.2.13]), the strong uniqueness holds for the first equation in (1.1). By Ikeda and Watanabe [24, Example 8.2, page 221], there is a (pathwise) unique non-negative strong solution $(Y_t)_{t \in \mathbb{R}_+}$ of the first equation in (1.1) with any initial value η_0 such that $\mathbb{P}(\eta_0 \in \mathbb{R}_+) = 1$. Clearly, the second equation in (2.1) gives the (pathwise) unique strong solution $(X_t)_{t \in \mathbb{R}_+}$ of the second equation in (1.1). Next, by an application of the Itô's formula for the process $(Y_t)_{t \in \mathbb{R}_+}$, we obtain

$$d(e^{bt}Y_t) = be^{bt}Y_t dt + e^{bt}dY_t = be^{bt}Y_t dt + e^{bt}((a - bY_t) dt + \sigma_1 \sqrt{Y_t} dW_t) = ae^{bt} dt + \sigma_1 e^{bt} \sqrt{Y_t} dW_t$$

for all $t \in \mathbb{R}_+$, which implies the first equation in (2.1).

Now we turn to check that $(Y_t, X_t)_{t \in \mathbb{R}_+}$ is an affine process with the given infinitesimal generator. We may and do suppose that the initial value is deterministic, say, $(Y_0, X_0) = (y_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}$, since the infinitesimal generator of a time homogeneous Markov process does not depend on the initial value of the Markov process. By Itô's formula, for all $f \in C_c^2(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R})$ we have

$$\begin{aligned} f(Y_t, X_t) &= f(y_0, x_0) + \sigma_1 \int_0^t f'_1(Y_s, X_s) \sqrt{Y_s} dW_s + \sigma_2 \int_0^t f'_2(Y_s, X_s) \sqrt{Y_s} (\varrho dW_s + \sqrt{1 - \varrho^2} dB_s) \\ &\quad + \int_0^t f'_1(Y_s, X_s)(a - bY_s) ds + \int_0^t f'_2(Y_s, X_s)(\alpha - \beta Y_s) ds \\ &\quad + \frac{1}{2} \left(\sigma_1^2 \int_0^t f''_{1,1}(Y_s, X_s) Y_s ds + 2\varrho\sigma_1\sigma_2 \int_0^t f''_{1,2}(Y_s, X_s) Y_s ds + \sigma_2^2 \int_0^t f''_{2,2}(Y_s, X_s) Y_s ds \right) \\ &= f(y_0, x_0) + \int_0^t (\mathcal{A}f)(Y_s, X_s) ds + M_t(f), \quad t \in \mathbb{R}_+, \end{aligned}$$

where

$$M_t(f) := \sigma_1 \int_0^t f'_1(Y_s, X_s) \sqrt{Y_s} dW_s + \sigma_2 \int_0^t f'_2(Y_s, X_s) \sqrt{Y_s} (\varrho dW_s + \sqrt{1 - \varrho^2} dB_s), \quad t \in \mathbb{R}_+,$$

and $\mathcal{A}f$ is given by (2.2). It is enough to show that $(M_t(f))_{t \in \mathbb{R}_+}$ is a local martingale with respect to the augmented filtration corresponding to $(W_t, B_t)_{t \in \mathbb{R}_+}$ and (η_0, ζ_0) , constructed as in Karatzas

and Shreve [27, Section 5.2]. However, it turns out that it is a square integrable martingale with respect to this filtration, since

$$\begin{aligned} \int_0^t \mathbb{E}((f'_1(Y_s, X_s))^2 Y_s) ds &\leq C_1 \int_0^t \mathbb{E}(Y_s) ds < \infty, & t \in \mathbb{R}_+, \\ \int_0^t \mathbb{E}((f'_2(Y_s, X_s))^2 Y_s) ds &\leq C_2 \int_0^t \mathbb{E}(Y_s) ds < \infty, & t \in \mathbb{R}_+, \end{aligned}$$

with some constants $C_1, C_2 \in \mathbb{R}_{++}$, where the finiteness of the integrals follows by

$$(2.3) \quad \mathbb{E}(Y_s) = e^{-bs} y_0 + a \int_0^s e^{-bu} du, \quad s \in \mathbb{R}_+,$$

see, e.g., Cox et al. [15, Equation (19)] or Jeanblanc et al. [26, Theorem 6.3.3.1].

Finally, we check that the transition semigroup $(P_t)_{t \in \mathbb{R}_+}$ with state space $\mathbb{R}_+ \times \mathbb{R}$ corresponding to $(Y_t, X_t)_{t \in \mathbb{R}_+}$ is a regular affine semigroup having infinitesimal generator given by (2.2). With the notations of Dawson and Li [16],

$$\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \frac{1}{2} \mathbf{S}, \begin{bmatrix} a \\ \alpha \end{bmatrix}, \begin{bmatrix} -b & 0 \\ -\beta & 0 \end{bmatrix}, 0, 0 \right)$$

is a set of admissible parameters corresponding to the affine process $(Y_t, X_t)_{t \in \mathbb{R}_+}$, where

$$(2.4) \quad \mathbf{S} := \begin{bmatrix} \sigma_1^2 & \rho \sigma_1 \sigma_2 \\ \rho \sigma_1 \sigma_2 & \sigma_2^2 \end{bmatrix}.$$

Hence Theorem 2.7 in Duffie et al. [19] (see also Theorem 6.1 in Dawson and Li [16]) yields that for this set of admissible parameters, there exists a regular affine semigroup $(Q_t)_{t \in \mathbb{R}_+}$ with infinitesimal generator given by (2.2). By Theorem 2.7 in Duffie et al. [19], $C_c^\infty(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R})$ is a core of the infinitesimal generator corresponding to the affine semigroup $(Q_t)_{t \in \mathbb{R}_+}$. Since we have checked that the infinitesimal generators corresponding to the transition semigroups $(P_t)_{t \in \mathbb{R}_+}$ and $(Q_t)_{t \in \mathbb{R}_+}$ (defined on the Banach space of bounded real-valued functions on $\mathbb{R}_+ \times \mathbb{R}$) coincide on $C_c^\infty(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R})$, by the definition of a core, we get they coincide on the Banach space of bounded real-valued functions on $\mathbb{R}_+ \times \mathbb{R}$. This yields that $(Y_t, X_t)_{t \in \mathbb{R}_+}$ is a regular affine process with infinitesimal generator (2.2). We also note that we could have used Lemma 10.2 in Duffie et al. [19] for concluding that $(Y_t, X_t)_{t \in \mathbb{R}_+}$ is a regular affine process with infinitesimal generator (2.2), since we have checked that $(M_t(f))_{t \in \mathbb{R}_+}$ is a martingale with respect to the filtration $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$ for any $f \in C_c^2(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R})$. \square

Next we present a result about the first moment of $(Y_t, X_t)_{t \in \mathbb{R}_+}$. We note that Hurn et al. [23, Equation (23)] derived the same formula for the expectation of (Y_t, X_t) , $t \in \mathbb{R}_+$, by a different method. Note also that the formula for $\mathbb{E}(Y_t)$, $t \in \mathbb{R}_+$, is well known.

2.2 Proposition. *Let $(Y_t, X_t)_{t \in \mathbb{R}_+}$ be the unique strong solution of the SDE (1.1) satisfying $\mathbb{P}(Y_0 \in \mathbb{R}_+) = 1$ and $\mathbb{E}(Y_0) < \infty$, $\mathbb{E}(|X_0|) < \infty$. Then*

$$\begin{bmatrix} \mathbb{E}(Y_t) \\ \mathbb{E}(X_t) \end{bmatrix} = \begin{bmatrix} e^{-bt} & 0 \\ -\beta \int_0^t e^{-bu} du & 1 \end{bmatrix} \begin{bmatrix} \mathbb{E}(Y_0) \\ \mathbb{E}(X_0) \end{bmatrix} + \begin{bmatrix} \int_0^t e^{-bu} du & 0 \\ -\beta \int_0^t (\int_0^u e^{-bv} dv) du & t \end{bmatrix} \begin{bmatrix} a \\ \alpha \end{bmatrix}, \quad t \in \mathbb{R}_+.$$

Consequently, if $b \in \mathbb{R}_{++}$, then

$$\lim_{t \rightarrow \infty} \mathbb{E}(Y_t) = \frac{a}{b}, \quad \lim_{t \rightarrow \infty} t^{-1} \mathbb{E}(X_t) = \alpha - \frac{\beta a}{b},$$

if $b = 0$, then

$$\lim_{t \rightarrow \infty} t^{-1} \mathbb{E}(Y_t) = a, \quad \lim_{t \rightarrow \infty} t^{-2} \mathbb{E}(X_t) = -\frac{1}{2} \beta a,$$

if $b \in \mathbb{R}_{--}$, then

$$\lim_{t \rightarrow \infty} e^{bt} \mathbb{E}(Y_t) = \mathbb{E}(Y_0) - \frac{a}{b}, \quad \lim_{t \rightarrow \infty} e^{bt} \mathbb{E}(X_t) = \frac{\beta}{b} \mathbb{E}(Y_0) - \frac{\beta a}{b^2}.$$

Proof. It is sufficient to prove the statement in the case when $(Y_0, X_0) = (y_0, x_0)$ with an arbitrary $(y_0, x_0) \in \mathbb{R}_{++} \times \mathbb{R}$, since then the statement of the proposition follows by the law of total expectation.

The formula for $\mathbb{E}(Y_t)$, $t \in \mathbb{R}_+$, can be found, e.g., in Cox et al. [15, Equation (19)] or Jeanblanc et al. [26, Theorem 6.3.3.1]. Next we observe that

$$(2.5) \quad \left(\int_0^t \sqrt{Y_u} d(\varrho W_u + \sqrt{1 - \varrho^2} B_u) \right)_{t \in \mathbb{R}_+}$$

is a square integrable martingale, since

$$\mathbb{E} \left[\left(\int_0^t \sqrt{Y_u} d(\varrho W_u + \sqrt{1 - \varrho^2} B_u) \right)^2 \right] = \int_0^t \mathbb{E}(Y_u) du < \infty,$$

where the finiteness of the integral follows from (2.3).

Taking expectations of both sides of the second equation in (2.1) and using the martingale property of the process in (2.5), we have

$$\begin{aligned} \mathbb{E}(X_t) &= x_0 + \int_0^t (\alpha - \beta \mathbb{E}(Y_u)) du \\ &= x_0 + \alpha t - \beta \int_0^t \left(e^{-bu} y_0 + a \int_0^u e^{-bv} dv \right) du \\ &= x_0 - \beta y_0 \int_0^t e^{-bu} du + \alpha t - \beta a \int_0^t \left(\int_0^u e^{-bv} dv \right) du \end{aligned}$$

for all $t \in \mathbb{R}_+$.

Further, if $b \in \mathbb{R}_{++}$, then

$$\begin{aligned} \lim_{t \rightarrow \infty} \mathbb{E}(Y_t) &= \lim_{t \rightarrow \infty} \left(e^{-bt} y_0 - \frac{a}{b} (e^{-bt} - 1) \right) = \frac{a}{b}, \\ \lim_{t \rightarrow \infty} t^{-1} \mathbb{E}(X_t) &= \lim_{t \rightarrow \infty} \left(\frac{x_0}{t} + \frac{\beta}{b} y_0 \frac{e^{-bt} - 1}{t} + \alpha + \frac{\beta a}{bt} \left(\frac{e^{-bt} - 1}{-b} - t \right) \right) = \alpha - \frac{\beta a}{b}. \end{aligned}$$

If $b = 0$, then

$$\begin{aligned} \lim_{t \rightarrow \infty} t^{-1} \mathbb{E}(Y_t) &= \lim_{t \rightarrow \infty} t^{-1} (y_0 + at) = a, \\ \lim_{t \rightarrow \infty} t^{-2} \mathbb{E}(X_t) &= \lim_{t \rightarrow \infty} \left(\frac{x_0}{t^2} - \frac{\beta y_0}{t} + \frac{\alpha}{t} - \frac{\beta a}{2} \right) = -\frac{\beta a}{2}. \end{aligned}$$

If $b \in \mathbb{R}_{--}$, then

$$\lim_{t \rightarrow \infty} e^{bt} \mathbb{E}(Y_t) = \lim_{t \rightarrow \infty} \left(y_0 + \frac{a}{b}(e^{bt} - 1) \right) = y_0 - \frac{a}{b},$$

$$\begin{aligned} \lim_{t \rightarrow \infty} e^{bt} \mathbb{E}(X_t) &= x_0 \lim_{t \rightarrow \infty} e^{bt} + \frac{\beta}{b} y_0 \lim_{t \rightarrow \infty} (1 - e^{bt}) + \alpha \lim_{t \rightarrow \infty} t e^{bt} + \frac{\beta a}{b} \lim_{t \rightarrow \infty} \left(\frac{1 - e^{bt}}{-b} - t e^{bt} \right) \\ &= \frac{\beta}{b} y_0 - \frac{\beta a}{b^2}. \end{aligned}$$

□

Based on the asymptotic behavior of the expectations $(\mathbb{E}(Y_t), \mathbb{E}(X_t))$ as $t \rightarrow \infty$, we introduce a classification of Heston processes given by the SDE (1.1).

2.3 Definition. Let $(Y_t, X_t)_{t \in \mathbb{R}_+}$ be the unique strong solution of the SDE (1.1) satisfying $\mathbb{P}(Y_0 \in \mathbb{R}_+) = 1$. We call $(Y_t, X_t)_{t \in \mathbb{R}_+}$ *subcritical, critical or supercritical* if $b \in \mathbb{R}_{++}$, $b = 0$ or $b \in \mathbb{R}_{--}$, respectively.

In the sequel $\xrightarrow{\mathbb{P}}$, $\xrightarrow{\mathcal{D}}$ and $\xrightarrow{\text{a.s.}}$ will denote convergence in probability, in distribution and almost surely, respectively.

The following result states the existence of a unique stationary distribution and the ergodicity for the process $(Y_t)_{t \in \mathbb{R}_+}$ given by the first equation in (1.1) in the subcritical case, see, e.g., Feller [21], Cox et al. [15, Equation (20)], Li and Ma [31, Theorem 2.6] or Theorem 3.1 with $\alpha = 2$ and Theorem 4.1 in Barczy et al. [6].

2.4 Theorem. Let $a, b, \sigma_1 \in \mathbb{R}_{++}$. Let $(Y_t)_{t \in \mathbb{R}_+}$ be the unique strong solution of the first equation of the SDE (1.1) satisfying $\mathbb{P}(Y_0 \in \mathbb{R}_+) = 1$.

(i) Then $Y_t \xrightarrow{\mathcal{D}} Y_\infty$ as $t \rightarrow \infty$, and the distribution of Y_∞ is given by

$$(2.6) \quad \mathbb{E}(e^{-\lambda Y_\infty}) = \left(1 + \frac{\sigma_1^2}{2b} \lambda \right)^{-2a/\sigma_1^2}, \quad \lambda \in \mathbb{R}_+,$$

i.e., Y_∞ has Gamma distribution with parameters $2a/\sigma_1^2$ and $2b/\sigma_1^2$, hence

$$\mathbb{E}(Y_\infty^\kappa) = \frac{\Gamma\left(\frac{2a}{\sigma_1^2} + \kappa\right)}{\left(\frac{2b}{\sigma_1^2}\right)^\kappa \Gamma\left(\frac{2a}{\sigma_1^2}\right)}, \quad \kappa \in \left(-\frac{2a}{\sigma_1^2}, \infty\right).$$

Epecially, $\mathbb{E}(Y_\infty) = \frac{a}{b}$. Further, if $a \in \left(\frac{\sigma_1^2}{2}, \infty\right)$, then $\mathbb{E}\left(\frac{1}{Y_\infty}\right) = \frac{2b}{2a - \sigma_1^2}$.

(ii) Supposing that the random initial value Y_0 has the same distribution as Y_∞ , the process $(Y_t)_{t \in \mathbb{R}_+}$ is strictly stationary.

(iii) For all Borel measurable functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that $\mathbb{E}(|f(Y_\infty)|) < \infty$, we have

$$(2.7) \quad \frac{1}{T} \int_0^T f(Y_s) ds \xrightarrow{\text{a.s.}} \mathbb{E}(f(Y_\infty)) \quad \text{as } T \rightarrow \infty.$$

In the next remark we explain why we suppose only that the process X is observed.

2.5 Remark. If $a \in \mathbb{R}_{++}$, $b, \alpha, \beta \in \mathbb{R}$, $\sigma_1, \sigma_2 \in \mathbb{R}_{++}$, $\varrho \in (-1, 1)$, and $(Y_0, X_0) = (y_0, x_0) \in \mathbb{R}_{++} \times \mathbb{R}$, then, by the SDE (1.1),

$$\langle X \rangle_t = \sigma_2^2 \int_0^t Y_s ds, \quad t \in \mathbb{R}_+.$$

By Theorems I.4.47 a) and I.4.52 in Jacod and Shiryaev [25],

$$\sum_{i=1}^{\lfloor nt \rfloor} (X_{\frac{i}{n}} - X_{\frac{i-1}{n}})^2 \xrightarrow{\mathbb{P}} \langle X \rangle_t \quad \text{as } n \rightarrow \infty, \quad t \in \mathbb{R}_+.$$

This convergence holds almost surely along a suitable subsequence, the members of this sequence are measurable functions of $(X_s)_{s \in [0, t]}$, hence, using Theorems 4.2.2 and 4.2.8 in Dudley [18], we obtain that $\langle X \rangle_t = \sigma_2^2 \int_0^t Y_s ds$ is a measurable function of $(X_s)_{s \in [0, t]}$. Moreover,

$$(2.8) \quad \frac{\langle X \rangle_{t+h} - \langle X \rangle_t}{h} = \frac{\sigma_2^2}{h} \int_t^{t+h} Y_s ds \xrightarrow{\text{a.s.}} \sigma_2^2 Y_t \quad \text{as } h \rightarrow 0, \quad t \in \mathbb{R}_+,$$

since Y has almost surely continuous sample paths. In particular,

$$\frac{\langle X \rangle_h}{hy_0} = \frac{\sigma_2^2}{hy_0} \int_0^h Y_s ds \xrightarrow{\text{a.s.}} \sigma_2^2 \frac{Y_0}{y_0} = \sigma_2^2 \quad \text{as } h \rightarrow 0,$$

hence, for any fixed $T > 0$, σ_2^2 is a measurable function of $(X_s)_{s \in [0, T]}$, i.e., it can be determined from a sample $(X_s)_{s \in [0, T]}$ (provided that (Y, X) starts from some known non-random initial value $(y_0, x_0) \in (0, \infty) \times \mathbb{R}$). However, we also point out that this measurable function remains abstract. Consequently, by (2.8), for all $t \in [0, T]$, Y_t is a measurable function of $(X_s)_{s \in [0, T]}$, i.e., it can be determined from a sample $(X_s)_{s \in [0, T]}$ (provided that (Y, X) starts from some known non-random initial value $(y_0, x_0) \in (0, \infty) \times \mathbb{R}$). Finally, we note that the sample size T is fixed above, and it is enough to know any short sample $(X_s)_{s \in [0, T]}$ to carry out the above calculations. \square

Next we give statistics for the parameters σ_1 , σ_2 and ϱ using continuous time observations $(X_t)_{t \in [0, T]}$ with some $T > 0$ (provided that (Y, X) starts from some known non-random initial value $(y_0, x_0) \in (0, \infty) \times \mathbb{R}$). Due to this result we do not consider the estimation of these parameters, they are supposed to be known.

2.6 Remark. If $a \in \mathbb{R}_{++}$, $b, \alpha, \beta \in \mathbb{R}$, $\sigma_1, \sigma_2 \in \mathbb{R}_{++}$, $\varrho \in (-1, 1)$, and $(Y_0, X_0) = (y_0, x_0) \in \mathbb{R}_{++} \times \mathbb{R}$, then for all $T > 0$,

$$\mathbf{S} = \frac{1}{\int_0^T Y_s ds} \begin{bmatrix} \langle Y \rangle_T & \langle Y, X \rangle_T \\ \langle Y, X \rangle_T & \langle X \rangle_T \end{bmatrix} =: \widehat{\mathbf{S}}_T \quad \text{almost surely,}$$

where $(\langle Y, X \rangle_t)_{t \in \mathbb{R}_+}$ denotes the quadratic cross-variation process of Y and X , since, by the SDE (1.1),

$$\langle Y \rangle_T = \sigma_1^2 \int_0^T Y_s ds, \quad \langle X \rangle_T = \sigma_2^2 \int_0^T Y_s ds, \quad \langle Y, X \rangle_T = \varrho \sigma_1 \sigma_2 \int_0^T Y_s ds.$$

Here $\widehat{\mathbf{S}}_T$ is a statistic, i.e., there exists a measurable function $\Xi : C([0, T], \mathbb{R}) \rightarrow \mathbb{R}^{2 \times 2}$ such that $\widehat{\mathbf{S}}_T = \Xi((X_s)_{s \in [0, T]})$, where $C([0, T], \mathbb{R})$ denotes the space of continuous real-valued functions defined on $[0, T]$, since

$$(2.9) \quad \frac{1}{\frac{1}{n} \sum_{i=1}^{[nT]} Y_{\frac{i-1}{n}}} \sum_{i=1}^{[nT]} \begin{bmatrix} Y_{\frac{i}{n}} - Y_{\frac{i-1}{n}} \\ X_{\frac{i}{n}} - X_{\frac{i-1}{n}} \end{bmatrix} \begin{bmatrix} Y_{\frac{i}{n}} - Y_{\frac{i-1}{n}} \\ X_{\frac{i}{n}} - X_{\frac{i-1}{n}} \end{bmatrix}^\top \xrightarrow{\mathbb{P}} \widehat{\mathbf{S}}_T \quad \text{as } n \rightarrow \infty,$$

where $[x]$ denotes the integer part of a real number $x \in \mathbb{R}$, the convergence in (2.9) holds almost surely along a suitable subsequence, by Remark 2.5, the members of the sequence in (2.9) are measurable functions of $(X_s)_{s \in [0, T]}$, and one can use Theorems 4.2.2 and 4.2.8 in Dudley [18]. Next we prove (2.9). By Theorems I.4.47 a) and I.4.52 in Jacod and Shiryaev [25],

$$\begin{aligned} \sum_{i=1}^{[nT]} (Y_{\frac{i}{n}} - Y_{\frac{i-1}{n}})^2 &\xrightarrow{\mathbb{P}} \langle Y \rangle_T, & \sum_{i=1}^{[nT]} (X_{\frac{i}{n}} - X_{\frac{i-1}{n}})^2 &\xrightarrow{\mathbb{P}} \langle X \rangle_T, \\ \sum_{i=1}^{[nT]} (Y_{\frac{i}{n}} - Y_{\frac{i-1}{n}})(X_{\frac{i}{n}} - X_{\frac{i-1}{n}}) &\xrightarrow{\mathbb{P}} \langle Y, X \rangle_T \end{aligned}$$

as $n \rightarrow \infty$. Consequently,

$$\sum_{i=1}^{[nT]} \begin{bmatrix} Y_{\frac{i}{n}} - Y_{\frac{i-1}{n}} \\ X_{\frac{i}{n}} - X_{\frac{i-1}{n}} \end{bmatrix} \begin{bmatrix} Y_{\frac{i}{n}} - Y_{\frac{i-1}{n}} \\ X_{\frac{i}{n}} - X_{\frac{i-1}{n}} \end{bmatrix}^\top \xrightarrow{\mathbb{P}} \left(\int_0^T Y_s ds \right) \widehat{\mathbf{S}}_T$$

as $n \rightarrow \infty$, see, e.g., van der Vaart [40, Theorem 2.7, part (vi)]. Moreover,

$$\frac{1}{n} \sum_{i=1}^{[nT]} Y_{\frac{i-1}{n}} \xrightarrow{\text{a.s.}} \int_0^T Y_s ds \quad \text{as } n \rightarrow \infty$$

since Y has almost surely continuous sample paths. Here $\mathbb{P}(\int_0^T Y_s ds \in \mathbb{R}_{++}) = 1$. Indeed, if $\omega \in \Omega$ is such that $[0, T] \ni s \mapsto Y_s(\omega)$ is continuous and $Y_t(\omega) \in \mathbb{R}_+$ for all $t \in \mathbb{R}_+$, then we have $\int_0^T Y_s(\omega) ds = 0$ if and only if $Y_s(\omega) = 0$ for all $s \in [0, T]$. Using the method of the proof of Theorem 3.1 in Barczy et. al [5], we get $\mathbb{P}(\int_0^T Y_s = 0) = 0$, as desired. Hence (2.9) follows by properties of convergence in probability. \square

3 Existence and uniqueness of MLE

From this section, we will consider the Heston model (1.1) with a known non-random initial value $(y_0, x_0) \in \mathbb{R}_{++} \times \mathbb{R}$, and we equip $(\Omega, \mathcal{F}, \mathbb{P})$ with the augmented filtration $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$ corresponding to $(W_t, B_t)_{t \in \mathbb{R}_+}$, constructed as in Karatzas and Shreve [27, Section 5.2]. Note that $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$ satisfies the usual conditions, i.e., the filtration $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$ is right-continuous and \mathcal{F}_0 contains all the \mathbb{P} -null sets in \mathcal{F} .

Let $\mathbb{P}_{(Y, X)}$ denote the probability measure induced by $(Y_t, X_t)_{t \in \mathbb{R}_+}$ on the measurable space $(C(\mathbb{R}_+, \mathbb{R}_+ \times \mathbb{R}), \mathcal{B}(C(\mathbb{R}_+, \mathbb{R}_+ \times \mathbb{R})))$ endowed with the natural filtration $(\mathcal{G}_t)_{t \in \mathbb{R}_+}$, given by $\mathcal{G}_t := \varphi_t^{-1}(\mathcal{B}(C(\mathbb{R}_+, \mathbb{R}_+ \times \mathbb{R})))$, $t \in \mathbb{R}_+$, where $\varphi_t : C(\mathbb{R}_+, \mathbb{R}_+ \times \mathbb{R}) \rightarrow C(\mathbb{R}_+, \mathbb{R}_+ \times \mathbb{R})$ is the mapping

$\varphi_t(f)(s) := f(t \wedge s)$, $s, t \in \mathbb{R}_+$, $f \in C(\mathbb{R}_+, \mathbb{R}_+ \times \mathbb{R})$. Here $C(\mathbb{R}_+, \mathbb{R}_+ \times \mathbb{R})$ denotes the set of $\mathbb{R}_+ \times \mathbb{R}$ -valued continuous functions defined on \mathbb{R}_+ , and $\mathcal{B}(C(\mathbb{R}_+, \mathbb{R}_+ \times \mathbb{R}))$ is the Borel σ -algebra on it. Further, for all $T \in \mathbb{R}_{++}$, let $\mathbb{P}_{(Y,X),T} := \mathbb{P}_{(Y,X)}|_{\mathcal{G}_T}$ be the restriction of $\mathbb{P}_{(Y,X)}$ to \mathcal{G}_T .

3.1 Lemma. *Let $a \in [\frac{\sigma_1^2}{2}, \infty)$, $b, \alpha, \beta \in \mathbb{R}$, $\sigma_1, \sigma_2 \in \mathbb{R}_{++}$, and $\varrho \in (-1, 1)$. Let $(Y_t, X_t)_{t \in \mathbb{R}_+}$ and $(\tilde{Y}_t, \tilde{X}_t)_{t \in \mathbb{R}_+}$ be the unique strong solutions of the SDE (1.1) with initial values $(y_0, x_0) \in \mathbb{R}_{++} \times \mathbb{R}$, $(\tilde{y}_0, \tilde{x}_0) \in \mathbb{R}_{++} \times \mathbb{R}$ such that $(y_0, x_0) = (\tilde{y}_0, \tilde{x}_0)$, corresponding to the parameters $(a, b, \alpha, \beta, \sigma_1, \sigma_2, \varrho)$ and $(\sigma_1^2, 0, 0, 0, \sigma_1, \sigma_2, \varrho)$, respectively. Then for all $T \in \mathbb{R}_{++}$, the measures $\mathbb{P}_{(Y,X),T}$ and $\mathbb{P}_{(\tilde{Y},\tilde{X}),T}$ are absolutely continuous with respect to each other, and the Radon–Nikodym derivative of $\mathbb{P}_{(Y,X),T}$ with respect to $\mathbb{P}_{(\tilde{Y},\tilde{X}),T}$ (the so called likelihood ratio) takes the form*

$$L_T^{(Y,X),(\tilde{Y},\tilde{X})}((Y_s, X_s)_{s \in [0,T]}) = \exp \left\{ \int_0^T \frac{1}{Y_s} \begin{bmatrix} a - bY_s - \sigma_1^2 \\ \alpha - \beta Y_s \end{bmatrix}^\top \mathbf{S}^{-1} \begin{bmatrix} dY_s \\ dX_s \end{bmatrix} - \frac{1}{2} \int_0^T \frac{1}{Y_s} \begin{bmatrix} a - bY_s - \sigma_1^2 \\ \alpha - \beta Y_s \end{bmatrix}^\top \mathbf{S}^{-1} \begin{bmatrix} a - bY_s + \sigma_1^2 \\ \alpha - \beta Y_s \end{bmatrix} ds \right\},$$

where \mathbf{S} is defined in (2.4).

Proof. First note that the SDE (1.1) can be written in the matrix form

$$(3.1) \quad \begin{bmatrix} dY_t \\ dX_t \end{bmatrix} = \left(\begin{bmatrix} -b & 0 \\ -\beta & 0 \end{bmatrix} \begin{bmatrix} Y_t \\ X_t \end{bmatrix} + \begin{bmatrix} a \\ \alpha \end{bmatrix} \right) dt + \sqrt{Y_t} \begin{bmatrix} \sigma_1 & 0 \\ \sigma_2 \varrho & \sigma_2 \sqrt{1 - \varrho^2} \end{bmatrix} \begin{bmatrix} dW_t \\ dB_t \end{bmatrix}, \quad t \in \mathbb{R}_+.$$

Note also that under the condition $a \in [\frac{\sigma_1^2}{2}, \infty)$, we have $\mathbb{P}(Y_t \in \mathbb{R}_{++} \text{ for all } t \in \mathbb{R}_+) = 1$, see, e.g., page 442 in Revuz and Yor [37].

We intend to use formula (7.139) in Section 7.6.4 of Liptser and Shiryaev [32]. We have to check their condition (7.137) which takes the form

$$(3.2) \quad \mathbb{P} \left(\int_0^T \frac{1}{Y_s} \begin{bmatrix} a - bY_s \\ \alpha - \beta Y_s \end{bmatrix}^\top \mathbf{S}^{-1} \begin{bmatrix} a - bY_s \\ \alpha - \beta Y_s \end{bmatrix} + \frac{1}{Y_s} \begin{bmatrix} \sigma_1^2 \\ 0 \end{bmatrix}^\top \mathbf{S}^{-1} \begin{bmatrix} \sigma_1^2 \\ 0 \end{bmatrix} ds < \infty \right) = 1, \quad \forall T \in \mathbb{R}_+.$$

Here note that the matrix \mathbf{S} is invertible, since $\sigma_1, \sigma_2 \in \mathbb{R}_{++}$ and $\varrho \in (-1, 1)$. Since Y has continuous sample paths almost surely, condition (3.2) holds if

$$(3.3) \quad \mathbb{P} \left(\int_0^T \frac{1}{Y_s} ds < \infty \right) = 1 \quad \text{for all } T \in \mathbb{R}_+.$$

Since Y has continuous sample paths almost surely and $\mathbb{P}(Y_t \in \mathbb{R}_{++}, \forall t \in \mathbb{R}_+) = 1$, we have $\mathbb{P}(\inf_{t \in [0,T]} Y_t \in \mathbb{R}_{++}) = 1$ for all $T \in \mathbb{R}_+$, which yields (3.3). Note that under the condition $a \in [\frac{\sigma_1^2}{2}, \infty)$, Theorems 1 and 3 in Ben Alaya and Kebaier [10] also imply (3.3). Applying formula (7.139) in Section 7.6.4 of Liptser and Shiryaev [32] we obtain the statement.

We call the attention that conditions (4.110) and (4.111) are also required for Section 7.6.4 in Liptser and Shiryaev [32], but the Lipschitz condition (4.110) in Liptser and Shiryaev [32] does not hold for the SDE (1.1). However, we can use formula (7.139) in Liptser and Shiryaev [32], since

they use their conditions (4.110) and (4.111) only in order to ensure that the SDE they consider in Section 7.6.4 has a unique strong solution (see, the proof of Theorem 7.19 in Liptser and Shiryaev [32]). By Proposition 2.1, under the conditions of the present lemma, there is a (pathwise) unique strong solution of the SDE (1.1). \square

By Lemma 3.1, under its conditions the log-likelihood function satisfies

$$\begin{aligned}
& (1 - \varrho^2) \log L_T^{(Y,X),(\tilde{Y},\tilde{X})}((Y_s, X_s)_{s \in [0,T]}) \\
&= \int_0^T \frac{1}{Y_s} \left[\left(\frac{a - bY_s - \sigma_1^2}{\sigma_1^2} - \frac{\varrho(\alpha - \beta Y_s)}{\sigma_1 \sigma_2} \right) dY_s + \left(-\frac{\varrho(a - bY_s - \sigma_1^2)}{\sigma_1 \sigma_2} + \frac{\alpha - \beta Y_s}{\sigma_2^2} \right) dX_s \right] \\
&\quad - \frac{1}{2} \int_0^T \frac{1}{Y_s} \left[\frac{(a - bY_s)^2 - \sigma_1^4}{\sigma_1^2} - \frac{2\varrho(a - bY_s)(\alpha - \beta Y_s)}{\sigma_1 \sigma_2} + \frac{(\alpha - \beta Y_s)^2}{\sigma_2^2} \right] ds \\
&= a \int_0^T \left(\frac{dY_s}{\sigma_1^2 Y_s} - \frac{\varrho dX_s}{\sigma_1 \sigma_2 Y_s} \right) + b \int_0^T \left(-\frac{dY_s}{\sigma_1^2} + \frac{\varrho dX_s}{\sigma_1 \sigma_2} \right) \\
&\quad + \alpha \int_0^T \left(-\frac{\varrho dY_s}{\sigma_1 \sigma_2 Y_s} + \frac{dX_s}{\sigma_2^2 Y_s} \right) + \beta \int_0^T \left(\frac{\varrho dY_s}{\sigma_1 \sigma_2} - \frac{dX_s}{\sigma_2^2} \right) \\
&\quad - \frac{1}{2} a^2 \int_0^T \frac{ds}{\sigma_1^2 Y_s} + ab \int_0^T \frac{ds}{\sigma_1^2} - \frac{1}{2} b^2 \int_0^T \frac{Y_s ds}{\sigma_1^2} - \frac{1}{2} \alpha^2 \int_0^T \frac{ds}{\sigma_2^2 Y_s} + \alpha \beta \int_0^T \frac{ds}{\sigma_2^2} - \frac{1}{2} \beta^2 \int_0^T \frac{Y_s ds}{\sigma_2^2} \\
&\quad + a\alpha \int_0^T \frac{\varrho ds}{\sigma_1 \sigma_2 Y_s} - (b\alpha + a\beta) \int_0^T \frac{\varrho ds}{\sigma_1 \sigma_2} + b\beta \int_0^T \frac{\varrho Y_s ds}{\sigma_1 \sigma_2} - \int_0^T \frac{dY_s}{Y_s} + \int_0^T \frac{\varrho \sigma_1 dX_s}{\sigma_2 Y_s} + \frac{1}{2} \int_0^T \frac{\sigma_1^2 ds}{Y_s} \\
&= \boldsymbol{\theta}^\top \mathbf{d}_T - \frac{1}{2} \boldsymbol{\theta}^\top \mathbf{A}_T \boldsymbol{\theta} - \int_0^T \frac{dY_s}{Y_s} + \int_0^T \frac{\varrho \sigma_1 dX_s}{\sigma_2 Y_s} + \frac{1}{2} \int_0^T \frac{\sigma_1^2 ds}{Y_s},
\end{aligned}$$

where

$$\boldsymbol{\theta} := \begin{bmatrix} a \\ b \\ \alpha \\ \beta \end{bmatrix}, \quad \mathbf{d}_T := \mathbf{d}_T^{(\sigma_1, \sigma_2, \varrho)}((Y_s, X_s)_{s \in [0,T]}) := \begin{bmatrix} \int_0^T \left(\frac{dY_s}{\sigma_1^2 Y_s} - \frac{\varrho dX_s}{\sigma_1 \sigma_2 Y_s} \right) \\ \int_0^T \left(-\frac{dY_s}{\sigma_1^2} + \frac{\varrho dX_s}{\sigma_1 \sigma_2} \right) \\ \int_0^T \left(-\frac{\varrho dY_s}{\sigma_1 \sigma_2 Y_s} + \frac{dX_s}{\sigma_2^2 Y_s} \right) \\ \int_0^T \left(\frac{\varrho dY_s}{\sigma_1 \sigma_2} - \frac{dX_s}{\sigma_2^2} \right) \end{bmatrix},$$

$$\mathbf{A}_T := \mathbf{A}_T^{(\sigma_1, \sigma_2, \varrho)}((Y_s, X_s)_{s \in [0,T]}) := \begin{bmatrix} \int_0^T \frac{ds}{\sigma_1^2 Y_s} & -\int_0^T \frac{ds}{\sigma_1^2} & -\int_0^T \frac{\varrho ds}{\sigma_1 \sigma_2 Y_s} & \int_0^T \frac{\varrho ds}{\sigma_1 \sigma_2} \\ -\int_0^T \frac{ds}{\sigma_1^2} & \int_0^T \frac{Y_s ds}{\sigma_1^2} & \int_0^T \frac{\varrho ds}{\sigma_1 \sigma_2} & -\int_0^T \frac{\varrho Y_s ds}{\sigma_1 \sigma_2} \\ -\int_0^T \frac{\varrho ds}{\sigma_1 \sigma_2 Y_s} & \int_0^T \frac{\varrho ds}{\sigma_1 \sigma_2} & \int_0^T \frac{ds}{\sigma_2^2 Y_s} & -\int_0^T \frac{ds}{\sigma_2^2} \\ \int_0^T \frac{\varrho ds}{\sigma_1 \sigma_2} & -\int_0^T \frac{\varrho Y_s ds}{\sigma_1 \sigma_2} & -\int_0^T \frac{ds}{\sigma_2^2} & \int_0^T \frac{Y_s ds}{\sigma_2^2} \end{bmatrix}.$$

If we fix $\sigma_1, \sigma_2 \in \mathbb{R}_{++}$, $\varrho \in (-1, 1)$, the initial value $(y_0, x_0) \in \mathbb{R}_{++} \times \mathbb{R}$, and $T \in \mathbb{R}_{++}$, then the probability measures $\mathbb{P}_{(Y,X),T}$ induced by $(Y_t, X_t)_{t \in \mathbb{R}_+}$ corresponding to the parameters $(a, b, \alpha, \beta, \sigma_1, \sigma_2, \varrho)$, where $a \in [\frac{\sigma_1^2}{2}, \infty)$, $b, \alpha, \beta \in \mathbb{R}$, are absolutely continuous with respect to each other. Hence it does not matter which measure is taken as a reference measure for defining the MLE (we have chosen the measure corresponding to the parameters $(\sigma_1^2, 0, 0, 0, \sigma_1, \sigma_2, \varrho)$). For more details, see, e.g., Liptser and Shiryaev [32, page 35].

The random symmetric matrix \mathbf{A}_T can be written as a Kronecker product of a deterministic symmetric matrix and a random symmetric matrix, namely,

$$\mathbf{A}_T = \begin{bmatrix} \frac{1}{\sigma_1^2} & -\frac{\varrho}{\sigma_1\sigma_2} \\ -\frac{\varrho}{\sigma_1\sigma_2} & \frac{1}{\sigma_2^2} \end{bmatrix} \otimes \begin{bmatrix} \int_0^T \frac{ds}{Y_s} & -\int_0^T 1 ds \\ -\int_0^T 1 ds & \int_0^T Y_s ds \end{bmatrix}.$$

The first matrix is strictly positive definite. The second matrix is strictly positive definite if and only if $\int_0^T Y_s ds \int_0^T \frac{ds}{Y_s} > T^2$. The eigenvalues of \mathbf{A}_T coincides with the products of the eigenvalues of the two matrices in question (taking into account their multiplicities), hence the matrix \mathbf{A}_T is strictly positive definite if and only if $\int_0^T Y_s ds \int_0^T \frac{ds}{Y_s} > T^2$, and in this case the inverse \mathbf{A}_T^{-1} has the form (applying the identity $(\mathbf{A} \otimes \mathbf{B})^{-1} = \mathbf{A}^{-1} \otimes \mathbf{B}^{-1}$)

$$\mathbf{A}_T^{-1} = \begin{bmatrix} \frac{1}{\sigma_1^2} & -\frac{\varrho}{\sigma_1\sigma_2} \\ -\frac{\varrho}{\sigma_1\sigma_2} & \frac{1}{\sigma_2^2} \end{bmatrix}^{-1} \otimes \begin{bmatrix} \int_0^T \frac{ds}{Y_s} & -T \\ -T & \int_0^T Y_s ds \end{bmatrix}^{-1} = \frac{\mathbf{S} \otimes \begin{bmatrix} \int_0^T Y_s ds & T \\ T & \int_0^T \frac{ds}{Y_s} \end{bmatrix}}{(1 - \varrho^2) \left(\int_0^T Y_s ds \int_0^T \frac{ds}{Y_s} - T^2 \right)}.$$

Hence we have

$$\begin{aligned} & 2(1 - \varrho^2) \log L_T^{(Y,X),(\tilde{Y},\tilde{X})}((Y_s, X_s)_{s \in [0,T]}) \\ &= -(\boldsymbol{\theta} - \mathbf{A}_T^{-1} \mathbf{d}_T)^\top \mathbf{A}_T (\boldsymbol{\theta} - \mathbf{A}_T^{-1} \mathbf{d}_T) + \mathbf{d}_T^\top \mathbf{A}_T^{-1} \mathbf{d}_T - 2 \int_0^T \frac{dY_s}{Y_s} + 2 \int_0^T \frac{\varrho \sigma_1 dX_s}{\sigma_2 Y_s} + \int_0^T \frac{\sigma_1^2 ds}{Y_s}, \end{aligned}$$

provided that $\int_0^T Y_s ds \int_0^T \frac{ds}{Y_s} > T^2$. Recall that $\sigma_1, \sigma_2 \in \mathbb{R}_{++}$ and $\varrho \in (-1, 1)$ are supposed to be known. Then maximizing $(1 - \varrho^2) \log L_T^{(Y,X),(\tilde{Y},\tilde{X})}((Y_s, X_s)_{s \in [0,T]})$ in $(a, b, \alpha, \beta) \in \mathbb{R}^4$ gives the MLE of (a, b, α, β) based on the observations $(X_t)_{t \in [0,T]}$ having the form

$$\hat{\boldsymbol{\theta}}_T = \begin{bmatrix} \hat{a}_T \\ \hat{b}_T \\ \hat{\alpha}_T \\ \hat{\beta}_T \end{bmatrix} = \mathbf{A}_T^{-1} \mathbf{d}_T,$$

provided that $\int_0^T Y_s ds \int_0^T \frac{ds}{Y_s} > T^2$. The random vector \mathbf{d}_T can be expressed as

$$\mathbf{d}_T = \begin{bmatrix} \frac{1}{\sigma_1^2} \\ -\frac{\varrho}{\sigma_1\sigma_2} \end{bmatrix} \otimes \begin{bmatrix} \int_0^T \frac{dY_s}{Y_s} \\ -\int_0^T dY_s \end{bmatrix} + \begin{bmatrix} -\frac{\varrho}{\sigma_1\sigma_2} \\ \frac{1}{\sigma_2^2} \end{bmatrix} \otimes \begin{bmatrix} \int_0^T \frac{dX_s}{Y_s} \\ -\int_0^T dX_s \end{bmatrix}.$$

Applying the identity $(\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D}) = (\mathbf{AC}) \otimes (\mathbf{BD})$, we can calculate

$$\begin{aligned}
& \left(\mathbf{S} \otimes \begin{bmatrix} \int_0^T Y_s ds & T \\ T & \int_0^T \frac{ds}{Y_s} \end{bmatrix} \right) \mathbf{d}_T \\
&= \left(\mathbf{S} \begin{bmatrix} \frac{1}{\sigma_1^2} \\ -\frac{\rho}{\sigma_1 \sigma_2} \end{bmatrix} \right) \otimes \left(\begin{bmatrix} \int_0^T Y_s ds & T \\ T & \int_0^T \frac{ds}{Y_s} \end{bmatrix} \begin{bmatrix} \int_0^T \frac{dY_s}{Y_s} \\ -\int_0^T dY_s \end{bmatrix} \right) \\
&+ \left(\mathbf{S} \begin{bmatrix} -\frac{\rho}{\sigma_1 \sigma_2} \\ \frac{1}{\sigma_2^2} \end{bmatrix} \right) \otimes \left(\begin{bmatrix} \int_0^T Y_s ds & T \\ T & \int_0^T \frac{ds}{Y_s} \end{bmatrix} \begin{bmatrix} \int_0^T \frac{dX_s}{Y_s} \\ -\int_0^T dX_s \end{bmatrix} \right) \\
&= \begin{bmatrix} 1 - \rho^2 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} \int_0^T Y_s ds \int_0^T \frac{dY_s}{Y_s} - T(Y_T - y_0) \\ T \int_0^T \frac{dY_s}{Y_s} - (Y_T - y_0) \int_0^T \frac{ds}{Y_s} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 - \rho^2 \end{bmatrix} \otimes \begin{bmatrix} \int_0^T Y_s ds \int_0^T \frac{dX_s}{Y_s} - T(X_T - x_0) \\ T \int_0^T \frac{dX_s}{Y_s} - (X_T - x_0) \int_0^T \frac{ds}{Y_s} \end{bmatrix}.
\end{aligned}$$

Consequently, we obtain

$$(3.4) \quad \begin{bmatrix} \widehat{a}_T \\ \widehat{b}_T \\ \widehat{\alpha}_T \\ \widehat{\beta}_T \end{bmatrix} = \frac{1}{\int_0^T Y_s ds \int_0^T \frac{ds}{Y_s} - T^2} \begin{bmatrix} \int_0^T Y_s ds \int_0^T \frac{dY_s}{Y_s} - T(Y_T - y_0) \\ T \int_0^T \frac{dY_s}{Y_s} - (Y_T - y_0) \int_0^T \frac{ds}{Y_s} \\ \int_0^T Y_s ds \int_0^T \frac{dX_s}{Y_s} - T(X_T - x_0) \\ T \int_0^T \frac{dX_s}{Y_s} - (X_T - x_0) \int_0^T \frac{ds}{Y_s} \end{bmatrix},$$

provided that $\int_0^T Y_s ds \int_0^T \frac{ds}{Y_s} > T^2$. In fact, it turned out that for the calculation of the MLE of (a, b, α, β) , one does not need to know the values of the parameters $\sigma_1, \sigma_2 \in \mathbb{R}_{++}$ and $\rho \in (-1, 1)$. Note that the MLE of (a, b) based on the observations $(X_t)_{t \in [0, T]}$ for the Heston model (Y, X) is the same as the MLE of (a, b) based on the observations $(Y_t)_{t \in [0, T]}$ for the CIR process Y , see, e.g., Overbeck [35, formula (2.2)] or Ben Alaya and Kebaier [11, Section 3.1].

In the next remark we point out that the MLE (3.4) of (a, b, α, β) can be approximated using discrete time observations for X , which can be reassuring for practical applications, where data in continuous record is not available.

3.2 Remark. For the stochastic integrals $\int_0^T \frac{dX_s}{Y_s}$ and $\int_0^T \frac{dY_s}{Y_s}$ in (3.4), we have

$$(3.5) \quad \sum_{i=1}^{\lfloor nT \rfloor} \frac{X_{\frac{i}{n}} - X_{\frac{i-1}{n}}}{Y_{\frac{i-1}{n}}} \xrightarrow{\mathbb{P}} \int_0^T \frac{dX_s}{Y_s} \quad \text{and} \quad \sum_{i=1}^{\lfloor nT \rfloor} \frac{Y_{\frac{i}{n}} - Y_{\frac{i-1}{n}}}{Y_{\frac{i-1}{n}}} \xrightarrow{\mathbb{P}} \int_0^T \frac{dY_s}{Y_s} \quad \text{as } n \rightarrow \infty,$$

following from Proposition I.4.44 in Jacod and Shiryaev [25] with the Riemann sequence of deterministic subdivisions $(\frac{i}{n} \wedge T)_{i \in \mathbb{N}}$, $n \in \mathbb{N}$. Thus, there exist measurable functions $\Phi, \Psi : C([0, T], \mathbb{R}) \rightarrow \mathbb{R}$ such that $\int_0^T \frac{dX_s}{Y_s} = \Phi((X_s)_{s \in [0, T]})$ and $\int_0^T \frac{dY_s}{Y_s} = \Psi((X_s)_{s \in [0, T]})$, since the convergences in (3.5) hold almost surely along suitable subsequences, by Remark 2.5, the members of both sequences in (3.5) are measurable functions of $(X_s)_{s \in [0, T]}$, and one can use Theorems 4.2.2 and 4.2.8 in Dudley [18]. Moreover, since Y has continuous sample paths almost surely,

$$\frac{1}{n} \sum_{i=1}^{\lfloor nT \rfloor} Y_{\frac{i-1}{n}} \xrightarrow{\text{a.s.}} \int_0^T Y_s ds \quad \text{as } n \rightarrow \infty, \quad \text{and} \quad \frac{1}{n} \sum_{i=1}^{\lfloor nT \rfloor} \frac{1}{Y_{\frac{i-1}{n}}} \xrightarrow{\text{a.s.}} \int_0^T \frac{ds}{Y_s} \quad \text{as } n \rightarrow \infty,$$

hence the right hand side of (3.4) is a measurable function of $(X_s)_{s \in [0, T]}$, i.e., it is a statistic. Further, one can define a sequence $(\widehat{\boldsymbol{\theta}}_{T, n})_{n \in \mathbb{N}}$ of estimators of $\boldsymbol{\theta} = (a, b, \alpha, \beta)^\top$ based only on the discrete time observations $(Y_{\frac{i}{n}}, X_{\frac{i}{n}})_{i \in \{1, \dots, [nT]\}}$ such that $\widehat{\boldsymbol{\theta}}_{T, n} \xrightarrow{\mathbb{P}} \widehat{\boldsymbol{\theta}}_T$ as $n \rightarrow \infty$. This is also called infill asymptotics. This phenomenon is similar to the approximate MLE, used by Ait-Sahalia [1], as discussed in the Introduction. \square

Using the SDE (1.1) one can check that

$$(3.6) \quad \begin{bmatrix} \widehat{a}_T - a \\ \widehat{b}_T - b \\ \widehat{\alpha}_T - \alpha \\ \widehat{\beta}_T - \beta \end{bmatrix} = \frac{1}{\int_0^T Y_s ds \int_0^T \frac{ds}{Y_s} - T^2} \begin{bmatrix} \int_0^T Y_s ds \int_0^T \frac{dY_s}{Y_s} - T(Y_T - y_0) - a \int_0^T Y_s ds \int_0^T \frac{ds}{Y_s} + aT^2 \\ T \int_0^T \frac{dY_s}{Y_s} - (Y_T - y_0) \int_0^T \frac{ds}{Y_s} - b \int_0^T Y_s ds \int_0^T \frac{ds}{Y_s} + bT^2 \\ \int_0^T Y_s ds \int_0^T \frac{dX_s}{Y_s} - T(X_T - x_0) - \alpha \int_0^T Y_s ds \int_0^T \frac{ds}{Y_s} + \alpha T^2 \\ T \int_0^T \frac{dX_s}{Y_s} - (X_T - x_0) \int_0^T \frac{ds}{Y_s} - \beta \int_0^T Y_s ds \int_0^T \frac{ds}{Y_s} + \beta T^2 \end{bmatrix}$$

$$= \frac{1}{\int_0^T Y_s ds \int_0^T \frac{ds}{Y_s} - T^2} \begin{bmatrix} \sigma_1 \int_0^T Y_s ds \int_0^T \frac{dW_s}{\sqrt{Y_s}} - \sigma_1 T \int_0^T \sqrt{Y_s} dW_s \\ \sigma_1 T \int_0^T \frac{dW_s}{\sqrt{Y_s}} - \sigma_1 \int_0^T \frac{ds}{Y_s} \int_0^T \sqrt{Y_s} dW_s \\ \sigma_2 \int_0^T Y_s ds \int_0^T \frac{d\widetilde{W}_s}{\sqrt{Y_s}} - \sigma_2 T \int_0^T \sqrt{Y_s} d\widetilde{W}_s \\ \sigma_2 T \int_0^T \frac{d\widetilde{W}_s}{\sqrt{Y_s}} - \sigma_2 \int_0^T \frac{ds}{Y_s} \int_0^T \sqrt{Y_s} d\widetilde{W}_s \end{bmatrix},$$

provided that $\int_0^T Y_s ds \int_0^T \frac{ds}{Y_s} > T^2$, where the process

$$\widetilde{W}_s := \varrho W_s + \sqrt{1 - \varrho^2} B_s, \quad s \in \mathbb{R}_+,$$

is a standard Wiener process.

The next lemma is about the existence of $(\widehat{a}_T, \widehat{b}_T, \widehat{\alpha}_T, \widehat{\beta}_T)$.

3.3 Lemma. *If $a \in [\frac{\sigma_1^2}{2}, \infty)$, $b \in \mathbb{R}$, $\sigma_1 \in \mathbb{R}_{++}$, and $Y_0 = y_0 \in \mathbb{R}_{++}$, then*

$$(3.7) \quad \mathbb{P} \left(\int_0^T Y_s ds \int_0^T \frac{1}{Y_s} ds > T^2 \right) = 1 \quad \text{for all } T \in \mathbb{R}_{++},$$

and hence, supposing also that $\alpha, \beta \in \mathbb{R}$, $\sigma_2 \in \mathbb{R}_{++}$, $\varrho \in (-1, 1)$, and $X_0 = x_0 \in \mathbb{R}$, there exists a unique MLE $(\widehat{a}_T, \widehat{b}_T, \widehat{\alpha}_T, \widehat{\beta}_T)$ for all $T \in \mathbb{R}_{++}$.

Proof. First note that $\mathbb{P}(Y_t \in \mathbb{R}_{++} \text{ for all } t \in \mathbb{R}_+) = 1$ as it was detailed in the proof of Lemma 3.1. We have $\mathbb{P}(\int_0^T Y_s ds < \infty) = 1$ for all $T \in \mathbb{R}_+$, since Y has continuous trajectories almost surely, and further, $\mathbb{P}(\int_0^T \frac{1}{Y_s} ds < \infty) = 1$ by (3.3). For each $T \in \mathbb{R}_{++}$, put

$$A_T := \{\omega \in \Omega : t \mapsto Y_t(\omega) \text{ is continuous and positive on } [0, T]\}.$$

Then $A_T \in \mathcal{F}$, $\mathbb{P}(A_T) = 1$, and for all $\omega \in A_T$, by the Cauchy-Schwarz's inequality, we have

$$\int_0^T Y_s(\omega) ds \int_0^T \frac{1}{Y_s(\omega)} ds \in [T^2, \infty),$$

and $\int_0^T Y_s(\omega) ds \int_0^T \frac{1}{Y_s(\omega)} ds = T^2$ if and only if $K_T(\omega) Y_s(\omega) = \frac{L_T(\omega)}{Y_s(\omega)}$ for almost every $s \in [0, T]$ with some $K_T(\omega), L_T(\omega) \in \mathbb{R}_+$ satisfying $K_T(\omega)^2 + L_T(\omega)^2 \in \mathbb{R}_{++}$. Clearly, $K_T(\omega) = 0$ would

imply $L_T(\omega) = 0$, thus $K_T(\omega) \neq 0$ and $Y_s(\omega) = \left(\frac{L_T(\omega)}{K_T(\omega)}\right)^{1/2}$ for almost every $s \in [0, T]$. Hence $Y_s(\omega) = y_0$ for all $s \in [0, T]$ if $\omega \in A_T$ and $\int_0^T Y_s(\omega) ds \int_0^T \frac{1}{Y_s(\omega)} ds = T^2$. Since the quadratic variation of a deterministic process is the identically zero process, the quadratic variation process $(\langle Y \rangle_t)_{t \in [0, T]}$ of $(Y_t)_{t \in [0, T]}$ should be identically zero on the event

$$A_T \cap \left\{ \omega \in \Omega : \int_0^T Y_s(\omega) ds \int_0^T \frac{1}{Y_s(\omega)} ds = T^2 \right\}.$$

Since $\langle Y \rangle_t = \sigma_1^2 \int_0^t Y_s ds$, $t \in \mathbb{R}_+$, we have $\int_0^t Y_s(\omega) ds = 0$ for all $t \in [0, T]$ on the event

$$A_T \cap \left\{ \omega \in \Omega : \int_0^T Y_s(\omega) ds \int_0^T \frac{1}{Y_s(\omega)} ds = T^2 \right\}.$$

However, $\left\{ \omega \in \Omega : \int_0^T Y_s(\omega) ds = 0 \right\} \cap A_T = \emptyset$, since $t \mapsto Y_t(\omega)$ is continuous and positive on $[0, T]$ for all $\omega \in A_T$. Consequently, since $\mathbb{P}(A_T) = 1$, we have $\mathbb{P}\left(\int_0^T Y_s ds \int_0^T \frac{1}{Y_s} ds = T^2\right) = 0$. \square

4 Consistency of MLE

First we consider the case of subcritical Heston models, i.e., when $b \in \mathbb{R}_{++}$.

4.1 Theorem. *If $b \in \mathbb{R}_{++}$, $\alpha, \beta \in \mathbb{R}$, $\sigma_1, \sigma_2 \in \mathbb{R}_{++}$, $\varrho \in (-1, 1)$, and $(Y_0, X_0) = (y_0, x_0) \in \mathbb{R}_{++} \times \mathbb{R}$, then the MLE of (a, b, α, β) is strongly consistent, i.e., $(\hat{a}_T, \hat{b}_T, \hat{\alpha}_T, \hat{\beta}_T) \xrightarrow{\text{a.s.}} (a, b, \alpha, \beta)$ as $T \rightarrow \infty$, whenever $a \in \left(\frac{\sigma_1^2}{2}, \infty\right)$, and it is weakly consistent, i.e., $(\hat{a}_T, \hat{b}_T, \hat{\alpha}_T, \hat{\beta}_T) \xrightarrow{\mathbb{P}} (a, b, \alpha, \beta)$ as $T \rightarrow \infty$, whenever $a = \frac{\sigma_1^2}{2}$.*

Proof. In both cases we have to show coordinate-wise convergences. Indeed, for the almost sure convergence, one can use that the intersection of four events with probability one is an event with probability one, and for the convergence in probability one can apply, e.g., van der Vaart [40, Theorem 2.7, part (vi)].

By Lemma 3.3, there exists a unique MLE $(\hat{a}_T, \hat{b}_T, \hat{\alpha}_T, \hat{\beta}_T)$ of (a, b, α, β) for all $T \in \mathbb{R}_{++}$, which has the form given in (3.4). By (3.6), we have

$$(4.1) \quad \hat{a}_T - \alpha = \frac{\sigma_2 \cdot \frac{\int_0^T \frac{d\tilde{W}_s}{\sqrt{Y_s}}}{\int_0^T \frac{ds}{Y_s}} - \frac{\sigma_2}{\frac{1}{T} \int_0^T \frac{ds}{Y_s}} \cdot \frac{\int_0^T \sqrt{Y_s} d\tilde{W}_s}{\int_0^T Y_s ds}}{1 - \frac{1}{\frac{1}{T} \int_0^T Y_s ds \cdot \frac{1}{T} \int_0^T \frac{ds}{Y_s}}}$$

provided that $\int_0^T Y_s ds \int_0^T \frac{ds}{Y_s} > T^2$ (implying $\int_0^T Y_s ds \int_0^T \frac{ds}{Y_s} \in \mathbb{R}_{++}$) which holds a.s.

First we consider the case of $a \in \left(\frac{\sigma_1^2}{2}, \infty\right)$. The strong consistency of the MLE of (a, b) has been proved by Overbeck [35, Theorem 2, part (ii)]. By part (i) of Theorem 2.4, $\mathbb{E}(Y_\infty) = \frac{a}{b}$ and $\mathbb{E}\left(\frac{1}{Y_\infty}\right) = \frac{2b}{2a - \sigma_1^2}$, and hence, part (iii) of Theorem 2.4 implies

$$(4.2) \quad \frac{1}{T} \int_0^T Y_s ds \xrightarrow{\text{a.s.}} \mathbb{E}(Y_\infty) \quad \text{and} \quad \frac{1}{T} \int_0^T \frac{ds}{Y_s} \xrightarrow{\text{a.s.}} \mathbb{E}\left(\frac{1}{Y_\infty}\right) \quad \text{as } T \rightarrow \infty.$$

Further, since $\mathbb{E}(Y_\infty), \mathbb{E}\left(\frac{1}{Y_\infty}\right) \in \mathbb{R}_{++}$, (4.2) yields

$$\int_0^T Y_s ds \xrightarrow{\text{a.s.}} \infty \quad \text{and} \quad \int_0^T \frac{ds}{Y_s} \xrightarrow{\text{a.s.}} \infty \quad \text{as } T \rightarrow \infty.$$

Applying a strong law of large numbers for continuous local martingales (see, e.g., Theorem A.1), we obtain

$$\hat{\alpha}_T - \alpha \xrightarrow{\text{a.s.}} \frac{\sigma_2 \cdot 0 - \frac{\sigma_2}{2b} \cdot 0}{1 - \frac{1}{\frac{a}{b} \cdot \frac{2b}{2a - \sigma_1^2}}} = 0 \quad \text{as } T \rightarrow \infty,$$

where we also used that the denominator above is not zero due to $\sigma_1 \in \mathbb{R}_{++}$.

Next we consider the case of $a = \frac{\sigma^2}{2}$. Weak consistency of the MLE of (a, b) follows from part 1 of Theorem 7 in Ben Alaya and Kebaier [11]. We have again $\mathbb{E}(Y_\infty) = \frac{a}{b} \in \mathbb{R}_{++}$, implying

$$(4.3) \quad \frac{1}{T} \int_0^T Y_s ds \xrightarrow{\text{a.s.}} \mathbb{E}(Y_\infty) \quad \text{and} \quad \int_0^T Y_s ds \xrightarrow{\text{a.s.}} \infty \quad \text{as } T \rightarrow \infty.$$

Due to Ben Alaya and Kebaier [10, Proposition 4], we have

$$(4.4) \quad \frac{1}{T^2} \int_0^T \frac{ds}{Y_s} \xrightarrow{\mathcal{D}} \tau \quad \text{as } T \rightarrow \infty,$$

where $\tau := \inf\{t \in \mathbb{R}_{++} : \mathcal{W}_t = \frac{b}{\sigma_1}\}$ with a standard Wiener process $(\mathcal{W}_t)_{t \in \mathbb{R}_+}$. Since $\mathbb{P}(\tau \in \mathbb{R}_{++}) = 1$, we conclude

$$\frac{1}{T} \int_0^T \frac{ds}{Y_s} = \frac{1}{T} \frac{1}{\frac{1}{T^2} \int_0^T \frac{ds}{Y_s}} \xrightarrow{\mathcal{D}} 0 \cdot \frac{1}{\tau} = 0 \quad \text{as } T \rightarrow \infty,$$

and hence,

$$(4.5) \quad \frac{1}{\frac{1}{T} \int_0^T \frac{ds}{Y_s}} \xrightarrow{\mathbb{P}} 0 \quad \text{as } T \rightarrow \infty,$$

implying also

$$\frac{1}{\int_0^T \frac{ds}{Y_s}} = \frac{1}{T} \frac{1}{\frac{1}{T} \int_0^T \frac{ds}{Y_s}} \xrightarrow{\mathbb{P}} 0 \quad \text{as } T \rightarrow \infty.$$

Since the function $\mathbb{R}_{++} \ni T \mapsto \frac{1}{\int_0^T \frac{ds}{Y_s}}$ is monotone decreasing, we obtain

$$\frac{1}{\int_0^T \frac{ds}{Y_s}} \xrightarrow{\text{a.s.}} 0 \quad \text{and} \quad \int_0^T \frac{ds}{Y_s} \xrightarrow{\text{a.s.}} \infty \quad \text{as } T \rightarrow \infty.$$

Using (4.1) and a strong law of large numbers for continuous local martingales (see, e.g., Theorem A.1), we obtain

$$\hat{\alpha}_T - \alpha \xrightarrow{\mathbb{P}} \frac{\sigma_2 \cdot 0 - 0 \cdot 0}{1 - \frac{b}{a} \cdot 0} = 0 \quad \text{as } T \rightarrow \infty.$$

Here we have convergence only in probability because of (4.5).

By (3.6), we have

$$(4.6) \quad \widehat{\beta}_T - \beta = \frac{\frac{\sigma_2}{\frac{1}{T} \int_0^T Y_s ds} \cdot \frac{\int_0^T \frac{d\widetilde{W}_s}{\sqrt{Y_s}}}{\int_0^T \frac{ds}{Y_s}} - \sigma_2 \cdot \frac{\int_0^T \sqrt{Y_s} d\widetilde{W}_s}{\int_0^T Y_s ds}}{1 - \frac{1}{\frac{1}{T} \int_0^T Y_s ds \cdot \frac{1}{T} \int_0^T \frac{ds}{Y_s}}}$$

provided that $\int_0^T Y_s ds \int_0^T \frac{ds}{Y_s} > T^2$ (implying $\int_0^T Y_s ds \int_0^T \frac{ds}{Y_s} \in \mathbb{R}_{++}$) which holds a.s.

First we consider the case of $a \in \left(\frac{\sigma_1^2}{2}, \infty\right)$. Applying again a strong law of large numbers for continuous local martingales (see, e.g., Theorem A.1), we obtain

$$\widehat{\beta}_T - \beta \xrightarrow{\text{a.s.}} \frac{\frac{\sigma_2}{\mathbb{E}(Y_\infty)} \cdot 0 - \sigma_2 \cdot 0}{1 - \frac{1}{\mathbb{E}(Y_\infty) \mathbb{E}\left(\frac{1}{Y_\infty}\right)}} = 0 \quad \text{as } T \rightarrow \infty,$$

where we also used that the denominator above is not zero due to $\sigma_1 \in \mathbb{R}_{++}$.

Next we consider the case of $a = \frac{\sigma_1^2}{2}$. Using (4.5) and (4.6), we obtain

$$\widehat{\beta}_T - \beta \xrightarrow{\mathbb{P}} \frac{\frac{\sigma_2}{\mathbb{E}(Y_\infty)} \cdot 0 - \sigma_2 \cdot 0}{1 - \frac{1}{\mathbb{E}(Y_\infty)} \cdot 0} = 0 \quad \text{as } T \rightarrow \infty. \quad \square$$

In order to handle supercritical Heston models, i.e., when $b \in \mathbb{R}_{--}$, we need the following integral version of the Toeplitz Lemma, due to Dietz and Kutoyants [17].

4.2 Lemma. *Let $\{\varphi_T : T \in \mathbb{R}_+\}$ be a family of probability measures on \mathbb{R}_+ such that $\varphi_T([0, T]) = 1$ for all $T \in \mathbb{R}_+$, and $\lim_{T \rightarrow \infty} \varphi_T([0, K]) = 0$ for all $K \in \mathbb{R}_{++}$. Then for every bounded and measurable function $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ for which the limit $f(\infty) := \lim_{t \rightarrow \infty} f(t)$ exists, we have*

$$\lim_{T \rightarrow \infty} \int_0^\infty f(t) \varphi_T(dt) = f(\infty).$$

As a special case, we have the following integral version of the Kronecker Lemma, see K uchler and S orensen [30, Lemma B.3.2].

4.3 Lemma. *Let $a : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a measurable function. Put $b(T) := \int_0^T a(t) dt$, $T \in \mathbb{R}_+$. Suppose that $\lim_{T \rightarrow \infty} b(T) = \infty$. Then for every bounded and measurable function $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ for which the limit $f(\infty) := \lim_{t \rightarrow \infty} f(t)$ exists, we have*

$$\lim_{T \rightarrow \infty} \frac{1}{b(T)} \int_0^T a(t) f(t) dt = f(\infty).$$

The next theorem states strong consistency of the MLE of b in the supercritical case. Overbeck [35, Theorem 2, part (i)] contains this result for CIR processes with a slightly incomplete proof.

4.4 Theorem. *If $a \in \left[\frac{\sigma_1^2}{2}, \infty\right)$, $b \in \mathbb{R}_{--}$, $\alpha, \beta \in \mathbb{R}$, $\sigma_1, \sigma_2 \in \mathbb{R}_{++}$, $\varrho \in (-1, 1)$, and $(Y_0, X_0) = (y_0, x_0) \in \mathbb{R}_{++} \times \mathbb{R}$, then the MLE of b is strongly consistent, i.e., $\widehat{b}_T \xrightarrow{\text{a.s.}} b$ as $T \rightarrow \infty$.*

Proof. By Lemma 3.3, there exists a unique MLE \widehat{b}_T of b for all $T \in \mathbb{R}_{++}$ which has the form given in (3.4). First we check that

$$\mathbb{E}(Y_t | \mathcal{F}_s^Y) = \mathbb{E}(Y_t | Y_s) = e^{-b(t-s)}Y_s + a \int_s^t e^{-b(t-u)} du$$

for all $s, t \in \mathbb{R}_+$ with $0 \leq s \leq t$, where \mathcal{F}_s^Y denotes the σ -algebra $\sigma(\{Y_u, u \in [0, s]\})$. The first equality follows from the Markov property of the process $(Y_t)_{t \in \mathbb{R}_+}$. The second equality is a consequence of the time-homogeneity of the Markov process Y and

$$\mathbb{E}(Y_t | (Y_0, X_0) = (y_0, x_0)) = e^{-bt}y_0 + a \int_0^t e^{-b(t-u)} du, \quad t \in \mathbb{R}_+,$$

valid for all $(y_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}$, following from Proposition 2.2. Thus

$$\mathbb{E}(e^{bt}Y_t | \mathcal{F}_s^Y) = e^{bs}Y_s + a \int_s^t e^{bu} du \geq e^{bs}Y_s$$

for all $s, t \in \mathbb{R}_+$ with $0 \leq s \leq t$, consequently, the process $(e^{bt}Y_t)_{t \in \mathbb{R}_+}$ is a non-negative submartingale with respect to the filtration $(\mathcal{F}_t^Y)_{t \in \mathbb{R}_+}$. Moreover,

$$\mathbb{E}(e^{bt}Y_t) = y_0 + a \int_0^t e^{bu} du \leq y_0 + a \int_0^\infty e^{bu} du = y_0 - \frac{a}{b} < \infty, \quad t \in \mathbb{R}_+,$$

hence, by the submartingale convergence theorem, there exists a non-negative random variable V such that

$$(4.7) \quad e^{bt}Y_t \xrightarrow{\text{a.s.}} V \quad \text{as } t \rightarrow \infty.$$

Note that the distribution of V coincides with the distribution of $\widetilde{\mathcal{Y}}_{-1/b}$, where $(\widetilde{\mathcal{Y}}_t)_{t \in \mathbb{R}_+}$ is a CIR process given by the SDE

$$d\widetilde{\mathcal{Y}}_t = a dt + \sigma_1 \sqrt{\widetilde{\mathcal{Y}}_t} d\mathcal{W}_t, \quad t \in \mathbb{R}_+,$$

with initial value $\widetilde{\mathcal{Y}}_0 = y_0$, where $(\mathcal{W}_t)_{t \in \mathbb{R}_+}$ is a standard Wiener process, see Ben Alaya and Kebaier [10, Proposition 3]. Consequently, $\mathbb{P}(V \in \mathbb{R}_{++}) = 1$ due to $\mathbb{P}(\widetilde{\mathcal{Y}}_t \in \mathbb{R}_{++}, \forall t \in \mathbb{R}_+) = 1$. If $\omega \in \Omega$ such that $\mathbb{R}_+ \ni t \mapsto Y_t(\omega)$ is continuous and $e^{bt}Y_t(\omega) \rightarrow V(\omega)$ as $t \rightarrow \infty$, then, by the integral Kronecker Lemma 4.3 with $f(t) = e^{bt}Y_t(\omega)$ and $a(t) = e^{-bt}$, $t \in \mathbb{R}_+$, we have

$$\frac{1}{\int_0^t e^{-bu} du} \int_0^t e^{-bu} (e^{bu} Y_u(\omega)) du \rightarrow V(\omega) \quad \text{as } t \rightarrow \infty.$$

Here $\int_0^t e^{-bu} du = \frac{e^{-bt}-1}{-b}$, $t \in \mathbb{R}_+$, thus we conclude

$$(4.8) \quad e^{bt} \int_0^t Y_u du \xrightarrow{\text{a.s.}} -\frac{V}{b} \quad \text{as } t \rightarrow \infty.$$

Further,

$$(4.9) \quad \int_0^t \frac{du}{Y_u} \xrightarrow{\text{a.s.}} \int_0^\infty \frac{du}{Y_u} \quad \text{as } t \rightarrow \infty,$$

where $\int_0^\infty \frac{du}{Y_u} \stackrel{\mathcal{D}}{=} \int_0^{-1/b} \tilde{\mathcal{Y}}_u du$, see Ben-Alaya and Kebaier [10, Proposition 4]. Consequently, $\mathbb{P}(\int_0^\infty \frac{du}{Y_u} \in \mathbb{R}_{++}) = 1$ due to $\mathbb{P}(\tilde{\mathcal{Y}}_t \in \mathbb{R}_{++}, \forall t \in \mathbb{R}_+) = 1$.

Since $\mathbb{P}(Y_t \in \mathbb{R}_{++} \text{ for all } t \in \mathbb{R}_+) = 1$, one can apply Itô's rule to the function $f(x) = \log x$, $x \in \mathbb{R}_{++}$, for which $f'(x) = 1/x$, $f''(x) = -1/x^2$, $x \in \mathbb{R}_{++}$, and we obtain

$$(4.10) \quad \log Y_t = \log y_0 + \int_0^t \frac{dY_s}{Y_s} - \frac{\sigma_1^2}{2} \int_0^t \frac{ds}{Y_s}, \quad t \in \mathbb{R}_+,$$

for all $b \in \mathbb{R}$, see von Weizsäcker and Winkler [43, Theorem 8.1.1].

Using (3.4) and (4.10), we have

$$\hat{b}_T = \frac{\frac{T \int_0^T \frac{dY_s}{Y_s}}{\int_0^T Y_s ds \int_0^T \frac{ds}{Y_s}} - \frac{Y_T - y_0}{\int_0^T Y_s ds}}{1 - \frac{1}{T^2} \int_0^T Y_s ds \int_0^T \frac{ds}{Y_s}} = \frac{\frac{T(\log Y_T - \log y_0)}{\int_0^T Y_s ds \int_0^T \frac{ds}{Y_s}} - \frac{Y_T - y_0 - \frac{\sigma_1^2}{2} T}{\int_0^T Y_s ds}}{1 - \frac{1}{T^2} \int_0^T Y_s ds \int_0^T \frac{ds}{Y_s}}$$

provided that $\int_0^T Y_s ds \int_0^T \frac{ds}{Y_s} > T^2$ (implying $\int_0^T Y_s ds \int_0^T \frac{ds}{Y_s} \in \mathbb{R}_{++}$) which holds a.s. Applying (4.7), (4.8) and (4.9), we conclude

$$\hat{b}_T = \frac{\frac{T e^{bT} \log(e^{bT} Y_T) - b T^2 e^{bT} - T e^{bT} \log y_0}{(e^{bT} \int_0^T Y_s ds) \int_0^T \frac{ds}{Y_s}} - \frac{e^{bT} Y_T - e^{bT} y_0 - \frac{\sigma_1^2}{2} T e^{bT}}{e^{bT} \int_0^T Y_s ds}}{1 - T^2 e^{bT} \frac{1}{(e^{bT} \int_0^T Y_s ds) \int_0^T \frac{ds}{Y_s}}} \xrightarrow{\text{a.s.}} \frac{\frac{0 \cdot \log V - 0}{-\frac{V}{b} \int_0^\infty \frac{ds}{Y_s}} - \frac{V - 0}{-\frac{V}{b}}}{1 - 0 \cdot \frac{1}{-\frac{V}{b} \int_0^\infty \frac{ds}{Y_s}}} = b$$

as $T \rightarrow \infty$. □

4.5 Remark. For subcritical (i.e., $b \in \mathbb{R}_{++}$) CIR models with $a \in (\frac{\sigma_1^2}{2}, \infty)$, Overbeck [35, Theorem 2, part (ii)] proved strong consistency of the MLE of (a, b) . For subcritical (i.e., $b \in \mathbb{R}_{++}$) CIR models with $a = \frac{\sigma_1^2}{2}$, weak consistency of the MLE of (a, b) follows from part 1 of Theorem 7 in Ben Alaya and Kebaier [11]. □

4.6 Remark. For critical (i.e., $b = 0$) CIR models with $a \in [\frac{\sigma_1^2}{2}, \infty)$, weak consistency of the MLE of (a, b) follows from Theorem 2 (iii) in Overbeck [35] or Theorem 6 in Ben Alaya and Kebaier [11]. For critical Heston models with $a \in (\frac{\sigma_1^2}{2}, \infty)$, weak consistency of the MLE of (a, b, α, β) is a consequence of Theorem 6.2. □

4.7 Remark. For supercritical (i.e., $b \in \mathbb{R}_{--}$) CIR models with $a \in [\frac{\sigma_1^2}{2}, \infty)$, Overbeck [35, Theorem 2, parts (i) and (v)] proved that the MLE of b is strongly consistent, however, there is no strongly consistent estimator of a . See also Ben Alaya and Kebaier [11, Theorem 7, part 2]. For supercritical Heston models with $a \in [\frac{\sigma_1^2}{2}, \infty)$, it will turn out that the MLE of a and α is not even weakly consistent, but the MLE of β is weakly consistent, see Theorem 7.1. □

5 Asymptotic behaviour of MLE: subcritical case

We consider subcritical Heston models, i.e., when $b \in \mathbb{R}_{++}$.

5.1 Theorem. If $a \in \left(\frac{\sigma_1^2}{2}, \infty\right)$, $b \in \mathbb{R}_{++}$, $\alpha, \beta \in \mathbb{R}$, $\sigma_1, \sigma_2 \in \mathbb{R}_{++}$, $\varrho \in (-1, 1)$, and $(Y_0, X_0) = (y_0, x_0) \in \mathbb{R}_{++} \times \mathbb{R}$, then the MLE of (a, b, α, β) is asymptotically normal, i.e.,

$$(5.1) \quad \sqrt{T} \begin{bmatrix} \widehat{a}_T - a \\ \widehat{b}_T - b \\ \widehat{\alpha}_T - \alpha \\ \widehat{\beta}_T - \beta \end{bmatrix} \xrightarrow{\mathcal{D}} \mathcal{N}_4 \left(\mathbf{0}, \mathbf{S} \otimes \begin{bmatrix} \frac{2b}{2a - \sigma_1^2} & -1 \\ -1 & \frac{a}{b} \end{bmatrix}^{-1} \right) \quad \text{as } T \rightarrow \infty,$$

where \mathbf{S} is defined in (2.4).

With a random scaling, we have

$$(5.2) \quad \frac{1}{\left(\int_0^T \frac{ds}{Y_s}\right)^{1/2}} \left(\mathbf{I}_2 \otimes \begin{bmatrix} \int_0^T \frac{ds}{Y_s} & -T \\ 0 & \left(\int_0^T Y_s ds \int_0^T \frac{ds}{Y_s} - T^2\right)^{1/2} \end{bmatrix} \right) \begin{bmatrix} \widehat{a}_T - a \\ \widehat{b}_T - b \\ \widehat{\alpha}_T - \alpha \\ \widehat{\beta}_T - \beta \end{bmatrix} \xrightarrow{\mathcal{D}} \mathcal{N}_4(\mathbf{0}, \mathbf{S} \otimes \mathbf{I}_2)$$

as $T \rightarrow \infty$.

Proof. By Lemma 3.3, there exists a unique MLE $(\widehat{a}_T, \widehat{b}_T, \widehat{\alpha}_T, \widehat{\beta}_T)$ of (a, b, α, β) for all $T \in \mathbb{R}_{++}$, which has the form given in (3.4). By (3.6), we have

$$\begin{aligned} \sqrt{T}(\widehat{a}_T - a) &= \frac{\frac{1}{T} \int_0^T Y_s ds \cdot \frac{\sigma_1}{\sqrt{T}} \int_0^T \frac{dW_s}{\sqrt{Y_s}} - \frac{\sigma_1}{\sqrt{T}} \int_0^T \sqrt{Y_s} dW_s}{\frac{1}{T} \int_0^T Y_s ds \cdot \frac{1}{T} \int_0^T \frac{ds}{Y_s} - 1}, \\ \sqrt{T}(\widehat{b}_T - b) &= \frac{\frac{\sigma_1}{\sqrt{T}} \int_0^T \frac{dW_s}{\sqrt{Y_s}} - \frac{1}{T} \int_0^T \frac{ds}{Y_s} \cdot \frac{\sigma_1}{\sqrt{T}} \int_0^T \sqrt{Y_s} dW_s}{\frac{1}{T} \int_0^T Y_s ds \cdot \frac{1}{T} \int_0^T \frac{ds}{Y_s} - 1}, \\ \sqrt{T}(\widehat{\alpha}_T - \alpha) &= \frac{\frac{1}{T} \int_0^T Y_s ds \cdot \frac{\sigma_2}{\sqrt{T}} \int_0^T \frac{d\widetilde{W}_s}{\sqrt{Y_s}} - \frac{\sigma_2}{\sqrt{T}} \int_0^T \sqrt{Y_s} d\widetilde{W}_s}{\frac{1}{T} \int_0^T Y_s ds \cdot \frac{1}{T} \int_0^T \frac{ds}{Y_s} - 1}, \\ \sqrt{T}(\widehat{\beta}_T - \beta) &= \frac{\frac{\sigma_2}{\sqrt{T}} \int_0^T \frac{d\widetilde{W}_s}{\sqrt{Y_s}} - \frac{1}{T} \int_0^T \frac{ds}{Y_s} \cdot \frac{\sigma_2}{\sqrt{T}} \int_0^T \sqrt{Y_s} d\widetilde{W}_s}{\frac{1}{T} \int_0^T Y_s ds \cdot \frac{1}{T} \int_0^T \frac{ds}{Y_s} - 1} \end{aligned}$$

provided that $\int_0^T Y_s ds \int_0^T \frac{1}{Y_s} ds > T^2$ which holds a.s. Consequently,

$$\begin{aligned} \sqrt{T} \begin{bmatrix} \widehat{a}_T - a \\ \widehat{b}_T - b \\ \widehat{\alpha}_T - \alpha \\ \widehat{\beta}_T - \beta \end{bmatrix} &= \frac{1}{\frac{1}{T} \int_0^T Y_s ds \cdot \frac{1}{T} \int_0^T \frac{ds}{Y_s} - 1} \left(\mathbf{I}_2 \otimes \begin{bmatrix} \frac{1}{T} \int_0^T Y_s ds & 1 \\ 1 & \frac{1}{T} \int_0^T \frac{ds}{Y_s} \end{bmatrix} \right) \frac{1}{\sqrt{T}} \mathbf{M}_T \\ &= \left(\mathbf{I}_2 \otimes \begin{bmatrix} \frac{1}{T} \int_0^T \frac{ds}{Y_s} & -1 \\ -1 & \frac{1}{T} \int_0^T Y_s ds \end{bmatrix}^{-1} \right) \frac{1}{\sqrt{T}} \mathbf{M}_T, \end{aligned}$$

where

$$\mathbf{M}_t := \begin{bmatrix} \sigma_1 \int_0^t \frac{dW_s}{\sqrt{Y_s}} \\ -\sigma_1 \int_0^t \sqrt{Y_s} dW_s \\ \sigma_2 \int_0^t \frac{d\widetilde{W}_s}{\sqrt{Y_s}} \\ -\sigma_2 \int_0^t \sqrt{Y_s} d\widetilde{W}_s \end{bmatrix}, \quad t \in \mathbb{R}_+.$$

Next, we show that

$$(5.3) \quad \frac{1}{\sqrt{T}} \mathbf{M}_T \xrightarrow{\mathcal{D}} \boldsymbol{\eta} \mathbf{Z} \quad \text{as } T \rightarrow \infty,$$

where \mathbf{Z} is a 4-dimensional standard normally distributed random vector and $\boldsymbol{\eta} \in \mathbb{R}^{4 \times 4}$ such that

$$\boldsymbol{\eta} \boldsymbol{\eta}^\top = \mathbf{S} \otimes \begin{bmatrix} \mathbb{E}\left(\frac{1}{Y_\infty}\right) & -1 \\ -1 & \mathbb{E}(Y_\infty) \end{bmatrix}.$$

Here the two symmetric matrices on the right hand side are positive definite, since

$$\sigma_1^2 \sigma_2^2 (1 - \varrho^2) \in \mathbb{R}_{++} \quad \text{and} \quad \mathbb{E}\left(\frac{1}{Y_\infty}\right) \mathbb{E}(Y_\infty) - 1 = \frac{\sigma_1^2}{2a - \sigma_1^2} \in \mathbb{R}_{++},$$

so $\boldsymbol{\eta}$ can be chosen, for instance, as the uniquely defined symmetric positive definite square root of the Kronecker product of the two matrices in question. The process $(\mathbf{M}_t)_{t \in \mathbb{R}_+}$ is a 4-dimensional continuous local martingale with quadratic variation process

$$\langle \mathbf{M} \rangle_t = \mathbf{S} \otimes \begin{bmatrix} \int_0^t \frac{1}{Y_s} ds & -t \\ -t & \int_0^t Y_s ds \end{bmatrix}, \quad t \in \mathbb{R}_+.$$

By Theorem 2.4, we have

$$\mathbf{Q}(t) \langle \mathbf{M} \rangle_t \mathbf{Q}(t)^\top \xrightarrow{\text{a.s.}} \mathbf{S} \otimes \begin{bmatrix} \mathbb{E}\left(\frac{1}{Y_\infty}\right) & -1 \\ -1 & \mathbb{E}(Y_\infty) \end{bmatrix} \quad \text{as } t \rightarrow \infty$$

with $\mathbf{Q}(t) := t^{-1/2} \mathbf{I}_4$, $t \in \mathbb{R}_{++}$. Hence, Theorem A.2 yields (5.3). Then Slutsky's lemma yields

$$\sqrt{T} \begin{bmatrix} \widehat{a}_T - a \\ \widehat{b}_T - b \\ \widehat{\alpha}_T - \alpha \\ \widehat{\beta}_T - \beta \end{bmatrix} \xrightarrow{\mathcal{D}} \left(\mathbf{I}_2 \otimes \begin{bmatrix} \mathbb{E}\left(\frac{1}{Y_\infty}\right) & -1 \\ -1 & \mathbb{E}(Y_\infty) \end{bmatrix}^{-1} \right) \boldsymbol{\eta} \mathbf{Z} \stackrel{\mathcal{D}}{=} \mathcal{N}_4(\mathbf{0}, \boldsymbol{\Sigma}_1) \quad \text{as } T \rightarrow \infty,$$

where (applying the identities $(\mathbf{A} \otimes \mathbf{B})^\top = \mathbf{A}^\top \otimes \mathbf{B}^\top$ and $(\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D}) = (\mathbf{AC}) \otimes (\mathbf{BD})$)

$$\begin{aligned} \boldsymbol{\Sigma}_1 &:= \left(\mathbf{I}_2 \otimes \begin{bmatrix} \mathbb{E}\left(\frac{1}{Y_\infty}\right) & -1 \\ -1 & \mathbb{E}(Y_\infty) \end{bmatrix}^{-1} \right) \left(\mathbf{S} \otimes \begin{bmatrix} \mathbb{E}\left(\frac{1}{Y_\infty}\right) & -1 \\ -1 & \mathbb{E}(Y_\infty) \end{bmatrix} \right) \left(\mathbf{I}_2 \otimes \begin{bmatrix} \mathbb{E}\left(\frac{1}{Y_\infty}\right) & -1 \\ -1 & \mathbb{E}(Y_\infty) \end{bmatrix}^{-1} \right)^\top \\ &= (\mathbf{I}_2 \mathbf{S} \mathbf{I}_2) \otimes \left(\begin{bmatrix} \mathbb{E}\left(\frac{1}{Y_\infty}\right) & -1 \\ -1 & \mathbb{E}(Y_\infty) \end{bmatrix}^{-1} \begin{bmatrix} \mathbb{E}\left(\frac{1}{Y_\infty}\right) & -1 \\ -1 & \mathbb{E}(Y_\infty) \end{bmatrix} \left(\begin{bmatrix} \mathbb{E}\left(\frac{1}{Y_\infty}\right) & -1 \\ -1 & \mathbb{E}(Y_\infty) \end{bmatrix}^{-1} \right)^\top \right) \end{aligned}$$

$$= \mathbf{S} \otimes \begin{bmatrix} \mathbb{E}\left(\frac{1}{Y_\infty}\right) & -1 \\ -1 & \mathbb{E}(Y_\infty) \end{bmatrix}^{-1},$$

which yields (5.1) recalling $\mathbb{E}(Y_\infty) = \frac{a}{b}$ and $\mathbb{E}\left(\frac{1}{Y_\infty}\right) = \frac{2b}{2a - \sigma_1^2}$.

Slutsky's lemma and (5.1) yield

$$\begin{aligned} & \frac{1}{\left(\int_0^T \frac{ds}{Y_s}\right)^{1/2}} \left(\mathbf{I}_2 \otimes \begin{bmatrix} \int_0^T \frac{ds}{Y_s} & -T \\ 0 & \left(\int_0^T Y_s ds \int_0^T \frac{ds}{Y_s} - T^2\right)^{1/2} \end{bmatrix} \right) \begin{bmatrix} \widehat{a}_T - a \\ \widehat{b}_T - b \\ \widehat{\alpha}_T - \alpha \\ \widehat{\beta}_T - \beta \end{bmatrix} \\ &= \frac{1}{\left(\frac{1}{T} \int_0^T \frac{ds}{Y_s}\right)^{1/2}} \left(\mathbf{I}_2 \otimes \begin{bmatrix} \frac{1}{T} \int_0^T \frac{ds}{Y_s} & -1 \\ 0 & \left(\frac{1}{T} \int_0^T Y_s ds \cdot \frac{1}{T} \int_0^T \frac{ds}{Y_s} - 1\right)^{1/2} \end{bmatrix} \right) \sqrt{T} \begin{bmatrix} \widehat{a}_T - a \\ \widehat{b}_T - b \\ \widehat{\alpha}_T - \alpha \\ \widehat{\beta}_T - \beta \end{bmatrix} \\ &\xrightarrow{\mathcal{D}} \frac{1}{\left(\mathbb{E}\left(\frac{1}{Y_\infty}\right)\right)^{1/2}} \left(\mathbf{I}_2 \otimes \begin{bmatrix} \mathbb{E}\left(\frac{1}{Y_\infty}\right) & -1 \\ 0 & \left(\mathbb{E}(Y_\infty) \mathbb{E}\left(\frac{1}{Y_\infty}\right) - 1\right)^{1/2} \end{bmatrix} \right) \mathcal{N}_4 \left(\mathbf{0}, \mathbf{S} \otimes \begin{bmatrix} \mathbb{E}\left(\frac{1}{Y_\infty}\right) & -1 \\ -1 & \mathbb{E}(Y_\infty) \end{bmatrix}^{-1} \right) \\ &\stackrel{\mathcal{D}}{=} \mathcal{N}_4(\mathbf{0}, \boldsymbol{\Sigma}_2) \quad \text{as } T \rightarrow \infty, \end{aligned}$$

where (applying the identities $(\mathbf{A} \otimes \mathbf{B})^\top = \mathbf{A}^\top \otimes \mathbf{B}^\top$ and $(\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D}) = (\mathbf{AC}) \otimes (\mathbf{BD})$)

$$\begin{aligned} \boldsymbol{\Sigma}_2 &:= \frac{1}{\mathbb{E}\left(\frac{1}{Y_\infty}\right)} \left(\mathbf{I}_2 \otimes \begin{bmatrix} \mathbb{E}\left(\frac{1}{Y_\infty}\right) & -1 \\ 0 & \left(\mathbb{E}(Y_\infty) \mathbb{E}\left(\frac{1}{Y_\infty}\right) - 1\right)^{1/2} \end{bmatrix} \right) \left(\mathbf{S} \otimes \begin{bmatrix} \mathbb{E}\left(\frac{1}{Y_\infty}\right) & -1 \\ -1 & \mathbb{E}(Y_\infty) \end{bmatrix}^{-1} \right) \\ &\quad \times \left(\mathbf{I}_2 \otimes \begin{bmatrix} \mathbb{E}\left(\frac{1}{Y_\infty}\right) & -1 \\ 0 & \left(\mathbb{E}(Y_\infty) \mathbb{E}\left(\frac{1}{Y_\infty}\right) - 1\right)^{1/2} \end{bmatrix} \right)^\top \\ &= \frac{1}{\mathbb{E}\left(\frac{1}{Y_\infty}\right)} (\mathbf{I}_2 \mathbf{S} \mathbf{I}_2) \\ &\quad \otimes \left(\begin{bmatrix} \mathbb{E}\left(\frac{1}{Y_\infty}\right) & -1 \\ 0 & \left(\mathbb{E}(Y_\infty) \mathbb{E}\left(\frac{1}{Y_\infty}\right) - 1\right)^{1/2} \end{bmatrix} \begin{bmatrix} \mathbb{E}\left(\frac{1}{Y_\infty}\right) & -1 \\ -1 & \mathbb{E}(Y_\infty) \end{bmatrix}^{-1} \begin{bmatrix} \mathbb{E}\left(\frac{1}{Y_\infty}\right) & 0 \\ -1 & \left(\mathbb{E}(Y_\infty) \mathbb{E}\left(\frac{1}{Y_\infty}\right) - 1\right)^{1/2} \end{bmatrix} \right) \\ &= \mathbf{S} \otimes \mathbf{I}_2, \end{aligned}$$

since

$$\begin{bmatrix} \mathbb{E}\left(\frac{1}{Y_\infty}\right) & -1 \\ -1 & \mathbb{E}(Y_\infty) \end{bmatrix} = \frac{1}{\mathbb{E}\left(\frac{1}{Y_\infty}\right)} \begin{bmatrix} \mathbb{E}\left(\frac{1}{Y_\infty}\right) & 0 \\ -1 & \left(\mathbb{E}(Y_\infty) \mathbb{E}\left(\frac{1}{Y_\infty}\right) - 1\right)^{1/2} \end{bmatrix} \begin{bmatrix} \mathbb{E}\left(\frac{1}{Y_\infty}\right) & -1 \\ 0 & \left(\mathbb{E}(Y_\infty) \mathbb{E}\left(\frac{1}{Y_\infty}\right) - 1\right)^{1/2} \end{bmatrix}.$$

Thus we obtain (5.2). \square

5.2 Remark. For subcritical (i.e., $b \in \mathbb{R}_{++}$) CIR models, for the MLE of (a, b) , Ben Alaya and Kebaier [11, Theorems 5 and 7] proved asymptotic normality whenever $a \in (\frac{\sigma_1^2}{2}, \infty)$, and derived a limit theorem with a non-normal limit distribution whenever $a = \frac{\sigma_1^2}{2}$. For subcritical (i.e., $b \in \mathbb{R}_{++}$) CIR models, for the MLE of (a, b) , with random scaling, Overbeck [35, Theorem 3 (iii)] showed asymptotic normality. \square

6 Asymptotic behaviour of MLE: critical case

We consider critical Heston models, i.e., when $b = 0$. First we present an auxiliary lemma.

6.1 Lemma. *The mapping $C(\mathbb{R}_+, \mathbb{R}) \ni f \mapsto (\int_0^t f(u) du)_{t \in \mathbb{R}_+} \in C(\mathbb{R}_+, \mathbb{R})$ is continuous, hence measurable, where $C(\mathbb{R}_+, \mathbb{R})$ denotes the set of real-valued continuous functions defined on \mathbb{R}_+ .*

Proof. The space $C(\mathbb{R}_+, \mathbb{R})$ is topologized by the locally uniform metric

$$\delta_{\text{lu}}(f, g) := \sum_{N=1}^{\infty} 2^{-N} \min \left\{ 1, \sup_{t \in [0, N]} |f(t) - g(t)| \right\}, \quad f, g \in C(\mathbb{R}_+, \mathbb{R}),$$

see, e.g., Jacod and Shiryaev [25, Chapter VI, Section 1]. Let $f \in C(\mathbb{R}_+, \mathbb{R})$ and $f_n \in C(\mathbb{R}_+, \mathbb{R})$, $n \in \mathbb{N}$, such that $\delta_{\text{lu}}(f, f_n) \rightarrow 0$ as $n \rightarrow \infty$. Put $F(t) := \int_0^t f(s) ds$, $t \in \mathbb{R}_+$, and $F_n(t) := \int_0^t f_n(s) ds$, $t \in \mathbb{R}_+$, $n \in \mathbb{N}$. Then $\sup_{t \in [0, N]} |F(t) - F_n(t)| \leq N \sup_{t \in [0, N]} |f(t) - f_n(t)|$ for all $N \in \mathbb{N}$, hence for each $K \in \mathbb{N}$, we have

$$\begin{aligned} \delta_{\text{lu}}(F, F_n) &= \sum_{N=K+1}^{\infty} 2^{-N} \min \left\{ 1, \sup_{t \in [0, N]} |F(t) - F_n(t)| \right\} + \sum_{N=1}^K 2^{-N} \min \left\{ 1, \sup_{t \in [0, N]} |F(t) - F_n(t)| \right\} \\ &\leq \sum_{N=K+1}^{\infty} 2^{-N} + \sum_{N=1}^K 2^{-N} \min \left\{ 1, N \sup_{t \in [0, N]} |f(t) - f_n(t)| \right\} \\ &\leq 2^{-K} + \sum_{N=1}^K N 2^{-N} \min \left\{ 1, \sup_{t \in [0, N]} |f(t) - f_n(t)| \right\} \\ &\leq 2^{-K} + \delta_{\text{lu}}(f, f_n) \sum_{N=1}^K N = 2^{-K} + \frac{K(K+1)}{2} \delta_{\text{lu}}(f, f_n) \rightarrow 2^{-K} \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Consequently,

$$\limsup_{n \rightarrow \infty} \delta_{\text{lu}}(F, F_n) \leq 2^{-K} \quad \text{for all } K \in \mathbb{N},$$

thus we obtain the statement.

We present another short proof. Applying Problem 3.11.26 in Ethier and Kurtz [20] and Proposition VI.1.17 in Jacod and Shiryaev [25], the mapping $C(\mathbb{R}_+, \mathbb{R}) \ni f \mapsto (\int_0^t f(u) du)_{t \in \mathbb{R}_+} \in C(\mathbb{R}_+, \mathbb{R})$ is continuous, hence measurable. \square

The next result can be considered as a generalization of part 2 of Theorem 6 in Ben Alaya and Kebaier [11] for critical Heston models.

6.2 Theorem. If $a \in (\frac{\sigma_1^2}{2}, \infty)$, $b = 0$, $\alpha, \beta \in \mathbb{R}$, $\sigma_1, \sigma_2 \in \mathbb{R}_{++}$, $\varrho \in (-1, 1)$ and $(Y_0, X_0) = (y_0, x_0) \in \mathbb{R}_{++} \times \mathbb{R}$, then

$$(6.1) \quad \begin{bmatrix} \sqrt{\log T}(\hat{a}_T - a) \\ \sqrt{\log T}(\hat{\alpha}_T - \alpha) \\ T\hat{b}_T \\ T(\hat{\beta}_T - \beta) \end{bmatrix} \xrightarrow{\mathcal{D}} \begin{bmatrix} \left(a - \frac{\sigma_1^2}{2}\right)^{1/2} \mathbf{S}^{1/2} \mathbf{Z}_2 \\ \frac{a - \mathcal{Y}_1}{\int_0^1 \mathcal{Y}_s ds} \\ \frac{\alpha - \mathcal{X}_1}{\int_0^1 \mathcal{Y}_s ds} \end{bmatrix} \quad \text{as } T \rightarrow \infty,$$

where $(\mathcal{Y}_t, \mathcal{X}_t)_{t \in \mathbb{R}_+}$ is the unique strong solution of the SDE

$$(6.2) \quad \begin{cases} d\mathcal{Y}_t = a dt + \sigma_1 \sqrt{\mathcal{Y}_t} dW_t, \\ d\mathcal{X}_t = \alpha dt + \sigma_2 \sqrt{\mathcal{Y}_t} (\varrho dW_t + \sqrt{1 - \varrho^2} dB_t), \end{cases} \quad t \in \mathbb{R}_+,$$

with initial value $(\mathcal{Y}_0, \mathcal{X}_0) = (0, 0)$, where $(W_t, B_t)_{t \in \mathbb{R}_+}$ is a 2-dimensional standard Wiener process, \mathbf{Z}_2 is a 2-dimensional standard normally distributed random vector independent of $(\mathcal{Y}_1, \int_0^1 \mathcal{Y}_t dt, \mathcal{X}_1)$, \mathbf{S} is defined in (2.4), and $\mathbf{S}^{1/2}$ denotes its uniquely determined symmetric, positive definite square root.

Proof. By Lemma 3.3, there exists a unique MLE $(\hat{a}_T, \hat{b}_T, \hat{\alpha}_T, \hat{\beta}_T)$ of (a, b, α, β) for all $T \in \mathbb{R}_{++}$, which has the form given in (3.4). By (3.6), we have

$$\sqrt{\log T}(\hat{a}_T - a) = \frac{\frac{1}{\left(\frac{1}{\log T} \int_0^T \frac{ds}{Y_s}\right)^{1/2}} \frac{\sigma_1 \int_0^T \frac{dW_s}{\sqrt{Y_s}}}{\left(\int_0^T \frac{ds}{Y_s}\right)^{1/2}} - \frac{\frac{1}{\sqrt{\log T}}}{\log T} \frac{T\sigma_1 \int_0^T \sqrt{Y_s} dW_s}{\int_0^T Y_s ds}}{1 - \frac{1}{\frac{1}{T^2} \int_0^T Y_s ds} \frac{1}{\int_0^T \frac{ds}{Y_s}}}},$$

$$\sqrt{\log T}(\hat{\alpha}_T - \alpha) = \frac{\frac{1}{\left(\frac{1}{\log T} \int_0^T \frac{ds}{Y_s}\right)^{1/2}} \frac{\sigma_2 \int_0^T \frac{d\tilde{W}_s}{\sqrt{Y_s}}}{\left(\int_0^T \frac{ds}{Y_s}\right)^{1/2}} - \frac{\frac{1}{\sqrt{\log T}}}{\log T} \frac{T\sigma_2 \int_0^T \sqrt{Y_s} d\tilde{W}_s}{\int_0^T Y_s ds}}{1 - \frac{1}{\frac{1}{T^2} \int_0^T Y_s ds} \frac{1}{\int_0^T \frac{ds}{Y_s}}}},$$

$$T\hat{b}_T = \frac{\frac{1}{\frac{1}{T^2} \int_0^T Y_s ds} \frac{1}{\left(\int_0^T \frac{ds}{Y_s}\right)^{1/2}} \frac{\sigma_1 \int_0^T \frac{dW_s}{\sqrt{Y_s}}}{\left(\int_0^T \frac{ds}{Y_s}\right)^{1/2}} - \frac{T\sigma_1 \int_0^T \sqrt{Y_s} dW_s}{\int_0^T Y_s ds}}{1 - \frac{1}{\frac{1}{T^2} \int_0^T Y_s ds} \frac{1}{\int_0^T \frac{ds}{Y_s}}}},$$

and

$$T(\hat{\beta}_T - \beta) = \frac{\frac{1}{\frac{1}{T^2} \int_0^T Y_s ds} \frac{1}{\left(\int_0^T \frac{ds}{Y_s}\right)^{1/2}} \frac{\sigma_2 \int_0^T \frac{d\tilde{W}_s}{\sqrt{Y_s}}}{\left(\int_0^T \frac{ds}{Y_s}\right)^{1/2}} - \frac{T\sigma_2 \int_0^T \sqrt{Y_s} d\tilde{W}_s}{\int_0^T Y_s ds}}{1 - \frac{1}{\frac{1}{T^2} \int_0^T Y_s ds} \frac{1}{\int_0^T \frac{ds}{Y_s}}}},$$

provided that $\int_0^T Y_s ds \int_0^T \frac{1}{Y_s} ds > T^2$ which holds a.s. It is known that

$$(6.3) \quad \frac{1}{\log T} \int_0^T \frac{ds}{Y_s} \xrightarrow{\mathbb{P}} \left(a - \frac{\sigma_1^2}{2}\right)^{-1} \quad \text{as } T \rightarrow \infty,$$

see, e.g., Overbeck [35, Lemma 5] or Ben Alaya and Kebaier [10, Proposition 2]. Consequently,

$$(6.4) \quad \frac{1}{\int_0^T \frac{ds}{Y_s}} \xrightarrow{\text{a.s.}} 0 \quad \text{and} \quad \int_0^T \frac{ds}{Y_s} \xrightarrow{\text{a.s.}} \infty \quad \text{as } T \rightarrow \infty,$$

where we used that $(\int_0^t \frac{ds}{Y_s})_{t \in \mathbb{R}_+}$ is monotone increasing and convergence in probability implies the existence of a subsequence which converges almost surely. Note that

$$(6.5) \quad \frac{T\sigma_1 \int_0^T \sqrt{Y_s} dW_s}{\int_0^T Y_s ds} = \frac{\frac{1}{T}(Y_T - y_0) - a}{\frac{1}{T^2} \int_0^T Y_s ds}, \quad T \in \mathbb{R}_{++},$$

$$(6.6) \quad \frac{T\sigma_2 \int_0^T \sqrt{Y_s} d\widetilde{W}_s}{\int_0^T Y_s ds} = \frac{\sigma_2 \varrho}{\sigma_1} \frac{T\sigma_1 \int_0^T \sqrt{Y_s} dW_s}{\int_0^T Y_s ds} + \frac{\sigma_2 \sqrt{1 - \varrho^2}}{(\frac{1}{T^2} \int_0^T Y_s ds)^{1/2}} \frac{\int_0^T \sqrt{Y_s} dB_s}{(\int_0^T Y_s ds)^{1/2}}, \quad T \in \mathbb{R}_{++}.$$

Consequently, (6.1) will follow from

$$(6.7) \quad \left(\frac{\sigma_1 \int_0^T \frac{dW_s}{\sqrt{Y_s}}}{(\int_0^T \frac{ds}{Y_s})^{1/2}}, \frac{\sigma_2 \int_0^T \frac{d\widetilde{W}_s}{\sqrt{Y_s}}}{(\int_0^T \frac{ds}{Y_s})^{1/2}}, \frac{\int_0^T \sqrt{Y_s} dB_s}{(\int_0^T Y_s ds)^{1/2}}, \frac{1}{T} Y_T, \frac{1}{T^2} \int_0^T Y_s ds \right) \xrightarrow{\mathcal{D}} \left(\mathbf{S}^{1/2} \mathbf{Z}_2, Z_3, \mathcal{Y}_1, \int_0^1 \mathcal{Y}_s ds \right)$$

as $T \rightarrow \infty$, where Z_3 is a standard normally distributed random variable independent of $(\mathbf{Z}_2, \mathcal{Y}_1, \int_0^1 \mathcal{Y}_s ds)$, from (6.3), (6.4), (6.5), (6.6), Slutsky's lemma, continuous mapping theorem, and $\mathbb{P}(\int_0^1 \mathcal{Y}_s ds \in \mathbb{R}_{++}) = 1$ (which has been shown in the proof of Theorem 3.1 in Barczy et al. [5]). Indeed,

$$\begin{bmatrix} \sqrt{\log T}(\widehat{a}_T - a) \\ \sqrt{\log T}(\widehat{\alpha}_T - \alpha) \\ T\widehat{b}_T \\ T(\widehat{\beta}_T - \beta) \end{bmatrix} \xrightarrow{\mathcal{D}} \frac{1}{1 - \frac{1}{\int_0^1 \mathcal{Y}_s ds} \cdot 0} \begin{bmatrix} \frac{1}{(a - \frac{\sigma_1^2}{2})^{-1/2}} (\mathbf{S}^{1/2} \mathbf{Z}_2)_1 - \frac{0}{(a - \frac{\sigma_1^2}{2})^{-1}} \frac{\mathcal{Y}_1 - a}{\int_0^1 \mathcal{Y}_s ds} \\ \frac{1}{(a - \frac{\sigma_1^2}{2})^{-1/2}} (\mathbf{S}^{1/2} \mathbf{Z}_2)_2 - \frac{0}{(a - \frac{\sigma_1^2}{2})^{-1}} \frac{\mathcal{X}_1 - \alpha}{\int_0^1 \mathcal{Y}_s ds} \\ \frac{1}{\int_0^1 \mathcal{Y}_s ds} \cdot 0 \cdot (\mathbf{S}^{1/2} \mathbf{Z}_2)_1 - \frac{\mathcal{Y}_1 - a}{\int_0^1 \mathcal{Y}_s ds} \\ \frac{1}{\int_0^1 \mathcal{Y}_s ds} \cdot 0 \cdot (\mathbf{S}^{1/2} \mathbf{Z}_2)_2 - \frac{\mathcal{X}_1 - \alpha}{\int_0^1 \mathcal{Y}_s ds} \end{bmatrix}$$

as $T \rightarrow \infty$, where $\mathbf{S}^{1/2} \mathbf{Z}_2 =: ((\mathbf{S}^{1/2} \mathbf{Z}_2)_1, (\mathbf{S}^{1/2} \mathbf{Z}_2)_2)^\top$, since

$$(6.8) \quad \left(\mathbf{Z}_2, \mathcal{Y}_1, \int_0^1 \mathcal{Y}_s ds, \frac{\sigma_2 \varrho}{\sigma_1} \frac{\mathcal{Y}_1 - a}{\int_0^1 \mathcal{Y}_s ds} + \frac{\sigma_2 \sqrt{1 - \varrho^2}}{(\int_0^1 \mathcal{Y}_s ds)^{1/2}} Z_3 \right) \stackrel{\mathcal{D}}{=} \left(\mathbf{Z}_2, \mathcal{Y}_1, \int_0^1 \mathcal{Y}_s ds, \frac{\mathcal{X}_1 - \alpha}{\int_0^1 \mathcal{Y}_s ds} \right).$$

The statement (6.8) is equivalent to

$$(6.9) \quad \left(\mathcal{Y}_1, \int_0^1 \mathcal{Y}_s ds, \frac{\sigma_2 \varrho}{\sigma_1} \frac{\mathcal{Y}_1 - a}{\int_0^1 \mathcal{Y}_s ds} + \frac{\sigma_2 \sqrt{1 - \varrho^2}}{(\int_0^1 \mathcal{Y}_s ds)^{1/2}} Z_3 \right) \stackrel{\mathcal{D}}{=} \left(\mathcal{Y}_1, \int_0^1 \mathcal{Y}_s ds, \frac{\mathcal{X}_1 - \alpha}{\int_0^1 \mathcal{Y}_s ds} \right),$$

since \mathbf{Z}_2 is independent of $(Z_3, \mathcal{Y}_1, \int_0^1 \mathcal{Y}_s ds)$ and of $(\mathcal{Y}_1, \int_0^1 \mathcal{Y}_s ds, \mathcal{X}_1)$. The equality of the distributions in (6.9) follows from the equality of their characteristic functions. Namely, for all $(q_1, q_2, r) \in \mathbb{R}^3$ and $T \in \mathbb{R}_{++}$,

$$\mathbb{E} \left(\exp \left\{ i q_1 \mathcal{Y}_1 + i q_2 \int_0^1 \mathcal{Y}_s ds + i r \left(\frac{\sigma_2 \varrho}{\sigma_1} \frac{\mathcal{Y}_1 - a}{\int_0^1 \mathcal{Y}_s ds} + \frac{\sigma_2 \sqrt{1 - \varrho^2}}{(\int_0^1 \mathcal{Y}_s ds)^{1/2}} Z_3 \right) \right\} \middle| \mathcal{Y}_1, \int_0^1 \mathcal{Y}_s ds \right)$$

$$\begin{aligned}
&= \exp \left\{ iq_1 \mathcal{Y}_1 + iq_2 \int_0^1 \mathcal{Y}_s ds + ir \frac{\sigma_2 \varrho}{\sigma_1} \frac{\mathcal{Y}_1 - a}{\int_0^1 \mathcal{Y}_s ds} \right\} \mathbb{E} \left(\exp \left\{ ir \frac{\sigma_2 \sqrt{1 - \varrho^2}}{(\int_0^1 \mathcal{Y}_s ds)^{1/2}} Z_3 \right\} \middle| \mathcal{Y}_1, \int_0^1 \mathcal{Y}_s ds \right) \\
&= \exp \left\{ iq_1 \mathcal{Y}_1 + iq_2 \int_0^1 \mathcal{Y}_s ds + ir \frac{\sigma_2 \varrho}{\sigma_1} \frac{\mathcal{Y}_1 - a}{\int_0^1 \mathcal{Y}_s ds} \right\} \exp \left\{ -\frac{1}{2} r^2 \frac{\sigma_2^2 (1 - \varrho^2)}{\int_0^1 \mathcal{Y}_s ds} \right\},
\end{aligned}$$

thus

$$\begin{aligned}
&\mathbb{E} \left(\exp \left\{ iq_1 \mathcal{Y}_1 + iq_2 \int_0^1 \mathcal{Y}_s ds + ir \left(\frac{\sigma_2 \varrho}{\sigma_1} \frac{\mathcal{Y}_1 - a}{\int_0^1 \mathcal{Y}_s ds} + \frac{\sigma_2 \sqrt{1 - \varrho^2}}{(\int_0^1 \mathcal{Y}_s ds)^{1/2}} Z_3 \right) \right\} \right) \\
&= \mathbb{E} \left(\exp \left\{ iq_1 \mathcal{Y}_1 + iq_2 \int_0^1 \mathcal{Y}_s ds + ir \frac{\sigma_2 \varrho}{\sigma_1} \frac{\mathcal{Y}_1 - a}{\int_0^1 \mathcal{Y}_s ds} - \frac{1}{2} r^2 \frac{\sigma_2^2 (1 - \varrho^2)}{\int_0^1 \mathcal{Y}_s ds} \right\} \right).
\end{aligned}$$

Further, by (6.2),

$$\mathcal{X}_1 - \alpha = \sigma_2 \int_0^1 \sqrt{\mathcal{Y}_s} (\varrho d\mathcal{W}_s + \sqrt{1 - \varrho^2} d\mathcal{B}_s) = \frac{\sigma_2 \varrho}{\sigma_1} (\mathcal{Y}_1 - a) + \sigma_2 \sqrt{1 - \varrho^2} \int_0^1 \sqrt{\mathcal{Y}_s} d\mathcal{B}_s,$$

hence for all $(q_1, q_2, r) \in \mathbb{R}^3$ and $T \in \mathbb{R}_{++}$, we have

$$\begin{aligned}
&\mathbb{E} \left(\exp \left\{ iq_1 \mathcal{Y}_1 + iq_2 \int_0^1 \mathcal{Y}_s ds + ir \frac{\mathcal{X}_1 - \alpha}{\int_0^1 \mathcal{Y}_s ds} \right\} \middle| \mathcal{Y}_s, s \in [0, 1] \right) \\
&= \mathbb{E} \left(\exp \left\{ iq_1 \mathcal{Y}_1 + iq_2 \int_0^1 \mathcal{Y}_s ds + ir \left(\frac{\sigma_2 \varrho}{\sigma_1} \frac{\mathcal{Y}_1 - a}{\int_0^1 \mathcal{Y}_s ds} + \frac{\sigma_2 \sqrt{1 - \varrho^2}}{\int_0^1 \mathcal{Y}_s ds} \int_0^1 \sqrt{\mathcal{Y}_s} d\mathcal{B}_s \right) \right\} \middle| \mathcal{Y}_s, s \in [0, 1] \right) \\
&= \exp \left\{ iq_1 \mathcal{Y}_1 + iq_2 \int_0^1 \mathcal{Y}_s ds + ir \frac{\sigma_2 \varrho}{\sigma_1} \frac{\mathcal{Y}_1 - a}{\int_0^1 \mathcal{Y}_s ds} \right\} \\
&\quad \times \mathbb{E} \left(\exp \left\{ ir \frac{\sigma_2 \sqrt{1 - \varrho^2}}{\int_0^1 \mathcal{Y}_s ds} \int_0^1 \sqrt{\mathcal{Y}_s} d\mathcal{B}_s \right\} \middle| \mathcal{Y}_s, s \in [0, 1] \right) \\
&= \exp \left\{ iq_1 \mathcal{Y}_1 + iq_2 \int_0^1 \mathcal{Y}_s ds + ir \frac{\sigma_2 \varrho}{\sigma_1} \frac{\mathcal{Y}_1 - a}{\int_0^1 \mathcal{Y}_s ds} \right\} \exp \left\{ -\frac{1}{2} r^2 \frac{\sigma_2^2 (1 - \varrho^2)}{\int_0^1 \mathcal{Y}_s ds} \right\},
\end{aligned}$$

where the last equality follows from the independence of $(\mathcal{Y}_t)_{t \in \mathbb{R}_+}$ and $(\mathcal{B}_t)_{t \in \mathbb{R}_+}$ yielding that the conditional distribution of $\int_0^1 \sqrt{\mathcal{Y}_s} d\mathcal{B}_s$ given $(\mathcal{Y}_s)_{s \in [0, 1]}$ is normal. Thus

$$\begin{aligned}
&\mathbb{E} \left(\exp \left\{ iq_1 \mathcal{Y}_1 + iq_2 \int_0^1 \mathcal{Y}_s ds + ir \frac{\mathcal{X}_1 - \alpha}{\int_0^1 \mathcal{Y}_s ds} \right\} \right) \\
&= \mathbb{E} \left(\exp \left\{ iq_1 \mathcal{Y}_1 + iq_2 \int_0^1 \mathcal{Y}_s ds + ir \frac{\sigma_2 \varrho}{\sigma_1} \frac{\mathcal{Y}_1 - a}{\int_0^1 \mathcal{Y}_s ds} - \frac{1}{2} r^2 \frac{\sigma_2^2 (1 - \varrho^2)}{\int_0^1 \mathcal{Y}_s ds} \right\} \right),
\end{aligned}$$

and hence we obtain (6.9).

Now we turn to prove (6.7). Using that

$$(6.10) \quad \sigma_2 \int_0^T \frac{d\widetilde{W}_s}{\sqrt{\mathcal{Y}_s}} = \sigma_2 \varrho \int_0^T \frac{dW_s}{\sqrt{\mathcal{Y}_s}} + \sigma_2 \sqrt{1 - \varrho^2} \int_0^T \frac{dB_s}{\sqrt{\mathcal{Y}_s}}, \quad T \in \mathbb{R}_{++},$$

and

$$(6.11) \quad \begin{bmatrix} \sigma_1 & 0 \\ \sigma_2\rho & \sigma_2\sqrt{1-\rho^2} \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 \\ \sigma_2\rho & \sigma_2\sqrt{1-\rho^2} \end{bmatrix}^\top = \mathbf{S},$$

by continuous mapping theorem, to prove (6.7), it is sufficient to verify

$$(6.12) \quad \left(\frac{\int_0^T \frac{dW_s}{\sqrt{Y_s}}}{\left(\int_0^T \frac{ds}{Y_s}\right)^{1/2}}, \frac{\int_0^T \frac{dB_s}{\sqrt{Y_s}}}{\left(\int_0^T \frac{ds}{Y_s}\right)^{1/2}}, \frac{\int_0^T \sqrt{Y_s} dB_s}{\left(\int_0^T Y_s ds\right)^{1/2}}, \frac{1}{T}Y_T, \frac{1}{T^2} \int_0^T Y_s ds \right) \xrightarrow{\mathcal{D}} \left(\mathbf{Z}_2, Z_3, \mathcal{Y}_1, \int_0^1 \mathcal{Y}_s ds \right)$$

as $T \rightarrow \infty$. First we prove

$$(6.13) \quad \left(\frac{1}{T}Y_T, \frac{1}{T^2} \int_0^T Y_s ds \right) \xrightarrow{\mathcal{D}} \left(\mathcal{Y}_1, \int_0^1 \mathcal{Y}_s ds \right) \quad \text{as } T \rightarrow \infty.$$

By part (ii) of Remark 2.7 in Barczy et al. [5], we have

$$\left(\frac{1}{T}\mathcal{Y}_{Tt} \right)_{t \in \mathbb{R}_+} \stackrel{\mathcal{D}}{=} (\mathcal{Y}_t)_{t \in \mathbb{R}_+} \quad \text{for all } T \in \mathbb{R}_{++}.$$

Indeed, by Proposition 2.1, $(\mathcal{Y}_t)_{t \in \mathbb{R}_+}$ is a regular affine process, and the so-called admissible set of parameters corresponding to $(\mathcal{Y}_t)_{t \in \mathbb{R}_+}$ takes the form $(0, \frac{1}{2}\sigma_1^2, a, 0, 0, 0)$, and then part (ii) of Remark 2.7 in Barczy et al. [5] can be applied. Hence, by Lemma 6.1, we obtain

$$\left(\mathcal{Y}_1, \int_0^1 \mathcal{Y}_s ds \right) \stackrel{\mathcal{D}}{=} \left(\frac{1}{T}\mathcal{Y}_T, \frac{1}{T^2} \int_0^T \mathcal{Y}_s ds \right) \quad \text{for all } T \in \mathbb{R}_{++}.$$

Then, by Slutsky's lemma, in order to prove (6.13), it suffices to show convergences

$$(6.14) \quad \frac{1}{T}(Y_T - \mathcal{Y}_T) \xrightarrow{\mathbb{P}} 0, \quad \frac{1}{T^2} \int_0^T (Y_s - \mathcal{Y}_s) ds \xrightarrow{\mathbb{P}} 0, \quad \text{as } T \rightarrow \infty.$$

By (3.21) in Barczy et al. [5], we have

$$(6.15) \quad \mathbb{E}(|Y_t - \mathcal{Y}_t|) \leq y_0, \quad t \in \mathbb{R}_+,$$

hence

$$\begin{aligned} \mathbb{E} \left(\left| \frac{1}{T}(Y_T - \mathcal{Y}_T) \right| \right) &\leq \frac{1}{T}y_0 \rightarrow 0, \\ \mathbb{E} \left(\left| \frac{1}{T^2} \int_0^T (Y_s - \mathcal{Y}_s) ds \right| \right) &\leq \frac{1}{T^2} \int_0^T \mathbb{E}(|Y_s - \mathcal{Y}_s|) ds \leq \frac{1}{T}y_0 \rightarrow 0, \end{aligned}$$

as $T \rightarrow \infty$ implying (6.14). Thus we conclude (6.13).

We will prove (6.12) using continuity theorem. Applying (4.10), one can write

$$(6.16) \quad \sigma_1 \int_0^T \frac{dW_s}{\sqrt{Y_s}} = \log Y_T - \log y_0 + \left(\frac{\sigma_1^2}{2} - a \right) \int_0^T \frac{ds}{Y_s}, \quad T \in \mathbb{R}_{++},$$

hence $\int_0^T \frac{dW_s}{\sqrt{Y_s}}$ is measurable with respect to the σ -algebra $\sigma(Y_s, s \in [0, T])$. For all $(u_1, u_2, u_3, v_1, v_2) \in \mathbb{R}^5$ and $T \in \mathbb{R}_{++}$, we have

$$\begin{aligned}
& \mathbb{E} \left(\exp \left\{ iu_1 \frac{\int_0^T \frac{dW_s}{\sqrt{Y_s}}}{\left(\int_0^T \frac{ds}{Y_s}\right)^{1/2}} + iu_2 \frac{\int_0^T \frac{dB_s}{\sqrt{Y_s}}}{\left(\int_0^T \frac{ds}{Y_s}\right)^{1/2}} + iu_3 \frac{\int_0^T \sqrt{Y_s} dB_s}{\left(\int_0^T Y_s ds\right)^{1/2}} \right. \right. \\
& \quad \left. \left. + iv_1 \frac{1}{T} Y_T + iv_2 \frac{1}{T^2} \int_0^T Y_s ds \right\} \middle| Y_s, s \in [0, T] \right) \\
&= \exp \left\{ iu_1 \frac{\int_0^T \frac{dW_s}{\sqrt{Y_s}}}{\left(\int_0^T \frac{ds}{Y_s}\right)^{1/2}} + iv_1 \frac{1}{T} Y_T + iv_2 \frac{1}{T^2} \int_0^T Y_s ds \right\} \\
& \quad \times \mathbb{E} \left(\exp \left\{ i \int_0^T \left(\frac{u_2}{\left(\int_0^T \frac{dt}{Y_t}\right)^{1/2}} \frac{1}{\sqrt{Y_s}} + \frac{u_3}{\left(\int_0^T Y_t dt\right)^{1/2}} \sqrt{Y_s} \right) dB_s \right\} \middle| Y_s, s \in [0, T] \right) \\
&= \exp \left\{ iu_1 \frac{\int_0^T \frac{dW_s}{\sqrt{Y_s}}}{\left(\int_0^T \frac{ds}{Y_s}\right)^{1/2}} + iv_1 \frac{1}{T} Y_T + iv_2 \frac{1}{T^2} \int_0^T Y_s ds \right\} \\
& \quad \times \exp \left\{ -\frac{1}{2} \int_0^T \left(\frac{u_2^2}{\int_0^T \frac{dt}{Y_t}} \frac{1}{Y_s} + \frac{u_3^2}{\int_0^T Y_t dt} Y_s + \frac{2u_2 u_3}{\left(\int_0^T \frac{dt}{Y_t} \int_0^T Y_t dt\right)^{1/2}} \right) ds \right\} \\
&= \exp \left\{ iu_1 \frac{\int_0^T \frac{dW_s}{\sqrt{Y_s}}}{\left(\int_0^T \frac{ds}{Y_s}\right)^{1/2}} + iv_1 \frac{1}{T} Y_T + iv_2 \frac{1}{T^2} \int_0^T Y_s ds \right\} \exp \left\{ -\frac{1}{2} (u_2^2 + u_3^2) - \frac{T u_2 u_3}{\left(\int_0^T \frac{dt}{Y_t} \int_0^T Y_t dt\right)^{1/2}} \right\},
\end{aligned}$$

where we used the independence of Y and B . Consequently, the joint characteristic function of the random vector on the left hand side of (6.12) takes the form

$$\begin{aligned}
& \mathbb{E} \left(\exp \left\{ iu_1 \frac{\int_0^T \frac{dW_s}{\sqrt{Y_s}}}{\left(\int_0^T \frac{ds}{Y_s}\right)^{1/2}} + iu_2 \frac{\int_0^T \frac{dB_s}{\sqrt{Y_s}}}{\left(\int_0^T \frac{ds}{Y_s}\right)^{1/2}} + iu_3 \frac{\int_0^T \sqrt{Y_s} dB_s}{\left(\int_0^T Y_s ds\right)^{1/2}} + iv_1 \frac{1}{T} Y_T + iv_2 \frac{1}{T^2} \int_0^T Y_s ds \right\} \right) \\
&= e^{-(u_2^2 + u_3^2)/2} \mathbb{E} \left(\exp \left\{ \xi_T(u_1, v_1, v_2) - \frac{T u_2 u_3}{\left(\int_0^T \frac{dt}{Y_t} \int_0^T Y_t dt\right)^{1/2}} \right\} \right),
\end{aligned}$$

where

$$\xi_T(u_1, v_1, v_2) := iu_1 \frac{\int_0^T \frac{dW_s}{\sqrt{Y_s}}}{\left(\int_0^T \frac{ds}{Y_s}\right)^{1/2}} + iv_1 \frac{1}{T} Y_T + iv_2 \frac{1}{T^2} \int_0^T Y_s ds.$$

Ben Alaya and Kebaier [11, proof of Theorem 6] proved

$$(6.17) \quad \left(\frac{\log Y_T - \log y_0 + \left(\frac{\sigma_1^2}{2} - a\right) \int_0^T \frac{ds}{Y_s}}{\sqrt{\log T}}, \frac{Y_T}{T}, \frac{1}{T^2} \int_0^T Y_s ds \right) \xrightarrow{\mathcal{D}} \left(\frac{\sigma_1}{\sqrt{a - \frac{\sigma_1^2}{2}}} Z_1, \mathcal{Y}_1, \int_0^1 \mathcal{Y}_s ds \right)$$

as $T \rightarrow \infty$, where Z_1 is a 1-dimensional standard normally distributed random variable independent of $(\mathcal{Y}_1, \int_0^1 \mathcal{Y}_t dt)$. Using (6.16) we have

$$\frac{\int_0^T \frac{dW_s}{\sqrt{Y_s}}}{\left(\int_0^T \frac{ds}{Y_s}\right)^{1/2}} = \frac{\frac{1}{\sqrt{\log T}} \frac{1}{\sigma_1} \left(\log Y_T - \log y_0 + \left(\frac{\sigma_1^2}{2} - a\right) \int_0^T \frac{ds}{Y_s} \right)}{\left(\frac{1}{\log T} \int_0^T \frac{ds}{Y_s} \right)^{1/2}}, \quad T \in \mathbb{R}_{++},$$

and, by (6.3) and (6.17), we conclude

$$(6.18) \quad \left(\frac{\int_0^T \frac{dW_s}{\sqrt{Y_s}}}{\left(\int_0^T \frac{ds}{Y_s}\right)^{1/2}}, \frac{Y_T}{T}, \frac{1}{T^2} \int_0^T Y_s ds \right) \xrightarrow{\mathcal{D}} \left(Z_1, \mathcal{Y}_1, \int_0^1 \mathcal{Y}_s ds \right) \quad \text{as } T \rightarrow \infty,$$

thus we derived joint convergence of three coordinates of the left hand side of (6.12). Hence

$$(6.19) \quad \mathbb{E}(\exp\{\xi_T(u_1, v_1, v_2)\}) \rightarrow \mathbb{E}\left(\exp\left\{iu_1 Z_1 + iv_1 \mathcal{Y}_1 + iv_2 \int_0^1 \mathcal{Y}_s ds\right\}\right) \quad \text{as } T \rightarrow \infty$$

for all $(u_1, v_1, v_2) \in \mathbb{R}^3$. Using $|\exp\{\xi_T(u_1, v_1, v_2)\}| = 1$, we have

$$\begin{aligned} & \left| \mathbb{E}\left(\exp\left\{\xi_T(u_1, v_1, v_2) - \frac{Tu_2 u_3}{\left(\int_0^T \frac{dt}{Y_t} \int_0^T Y_t dt\right)^{1/2}}\right\}\right) - \mathbb{E}(\exp\{\xi_T(u_1, v_1, v_2)\}) \right| \\ & \leq \mathbb{E}\left(\left|\exp\{\xi_T(u_1, v_1, v_2)\}\right| \left|\exp\left\{-\frac{Tu_2 u_3}{\left(\int_0^T \frac{dt}{Y_t} \int_0^T Y_t dt\right)^{1/2}}\right\} - 1\right|\right) \\ & = \mathbb{E}\left(\left|\exp\left\{-\frac{Tu_2 u_3}{\left(\int_0^T \frac{dt}{Y_t} \int_0^T Y_t dt\right)^{1/2}}\right\} - 1\right|\right) \rightarrow 0 \quad \text{as } T \rightarrow \infty, \end{aligned}$$

by the moment convergence theorem (see, e.g., Stroock [39, Lemma 2.2.1]). Indeed, by (6.4), (6.18), continuous mapping theorem and Slutsky's lemma,

$$\left|\exp\left\{-\frac{Tu_2 u_3}{\left(\int_0^T \frac{dt}{Y_t} \int_0^T Y_t dt\right)^{1/2}}\right\} - 1\right| = \left|\exp\left\{-\frac{u_2 u_3}{\left(\int_0^T \frac{dt}{Y_t} \cdot \frac{1}{T^2} \int_0^T Y_t dt\right)^{1/2}}\right\} - 1\right| \xrightarrow{\mathbb{P}} 0 \quad \text{as } T \rightarrow \infty,$$

and the family

$$\left\{ \left|\exp\left\{-\frac{Tu_2 u_3}{\left(\int_0^T \frac{dt}{Y_t} \int_0^T Y_t dt\right)^{1/2}}\right\} - 1\right|, T \in \mathbb{R}_{++} \right\}$$

is uniformly integrable, since, by Cauchy–Schwarz inequality,

$$\left|\exp\left\{-\frac{Tu_2 u_3}{\left(\int_0^T \frac{dt}{Y_t} \int_0^T Y_t dt\right)^{1/2}}\right\} - 1\right|^2 \leq \left(\exp\left\{\frac{T|u_2 u_3|}{\left(\int_0^T \frac{dt}{Y_t} \int_0^T Y_t dt\right)^{1/2}}\right\} + 1\right)^2 \leq (\exp\{|u_2 u_3|\} + 1)^2$$

for all $T \in \mathbb{R}_{++}$. Using (6.19), we conclude

$$\begin{aligned} & \mathbb{E}\left(\exp\left\{iu_1 \frac{\int_0^T \frac{dW_s}{\sqrt{Y_s}}}{\left(\int_0^T \frac{ds}{Y_s}\right)^{1/2}} + iu_2 \frac{\int_0^T \frac{dB_s}{\sqrt{Y_s}}}{\left(\int_0^T \frac{ds}{Y_s}\right)^{1/2}} + iu_3 \frac{\int_0^T \sqrt{Y_s} dB_s}{\left(\int_0^T Y_s ds\right)^{1/2}} + iv_1 \frac{1}{T} Y_T + iv_2 \frac{1}{T^2} \int_0^T Y_s ds\right\}\right) \\ & \rightarrow e^{-(u_2^2 + u_3^2)/2} \mathbb{E}\left(\exp\left\{iu_1 Z_1 + iv_1 \mathcal{Y}_1 + iv_2 \int_0^1 \mathcal{Y}_s ds\right\}\right) \quad \text{as } T \rightarrow \infty. \end{aligned}$$

Note that, since Z_1 is independent of $(\mathcal{Y}_1, \int_0^1 \mathcal{Y}_s ds)$, we have

$$\begin{aligned} & e^{-(u_2^2 + u_3^2)/2} \mathbb{E}\left(\exp\left\{iu_1 Z_1 + iv_1 \mathcal{Y}_1 + iv_2 \int_0^1 \mathcal{Y}_s ds\right\}\right) \\ & = \mathbb{E}(e^{iu_1 Z_1}) \mathbb{E}(e^{iu_2 Z_2}) \mathbb{E}(e^{iu_3 Z_3}) \mathbb{E}\left(\exp\left\{iv_1 \mathcal{Y}_1 + iv_2 \int_0^1 \mathcal{Y}_s ds\right\}\right), \end{aligned}$$

where (Z_2, Z_3) is a 2-dimensional standard normally distributed random vector, independent of $(Z_1, \mathcal{Y}_1, \int_0^1 \mathcal{Y}_s ds)$, hence we obtain (6.12) with $\mathbf{Z}_2 := (Z_1, Z_2)$. \square

6.3 Remark. (i) As a consequence of Theorem 6.2 we get back the description of the asymptotic behavior of the MLE of (a, b) for the CIR process $(Y_t)_{t \in \mathbb{R}_+}$ in the critical case whenever $a \in (\frac{\sigma_1^2}{2}, \infty)$ proved by Ben Alaya and Kebaier [11, Theorem 6, part 2]. We note that Ben Alaya and Kebaier [11, Theorem 6, part 1] described the asymptotic behavior of the MLE of (a, b) in the critical case for the CIR process $(Y_t)_{t \in \mathbb{R}_+}$ with $a = \frac{\sigma_1^2}{2}$ as well.

(ii) Theorem 6.2 does not cover the case $a = \frac{\sigma_1^2}{2}$, we renounce to consider it.

(iii) Ben Alaya and Kebaier's proof of part 2 of their Theorem 6 relies on an explicit form of the moment generating-Laplace transform of the quadruplet

$$\left(\log Y_t, Y_t, \int_0^t Y_s ds, \int_0^t \frac{ds}{Y_s} \right), \quad t \in \mathbb{R}_+.$$

Using this explicit form, they derived convergence (6.17), which is a corner stone of the proof of our Theorem 6.2. \square

The next theorem can be considered as a counterpart of Theorem 6.2 by incorporating random scaling.

6.4 Theorem. *If $a \in (\frac{\sigma_1^2}{2}, \infty)$, $b = 0$, $\alpha, \beta \in \mathbb{R}$, $\sigma_1, \sigma_2 \in \mathbb{R}_{++}$, $\varrho \in (-1, 1)$ and $(Y_0, X_0) = (y_0, x_0) \in \mathbb{R}_{++} \times \mathbb{R}$, then*

$$(6.20) \quad \begin{bmatrix} \left(\int_0^T \frac{ds}{Y_s} \right)^{1/2} (\widehat{a}_T - a) \\ \left(\int_0^T \frac{ds}{Y_s} \right)^{1/2} (\widehat{\alpha}_T - \alpha) \\ \left(\int_0^T Y_s ds \right)^{1/2} \widehat{b}_T \\ \left(\int_0^T Y_s ds \right)^{1/2} (\widehat{\beta}_T - \beta) \end{bmatrix} \xrightarrow{\mathcal{D}} \begin{bmatrix} \mathbf{S}^{1/2} \mathbf{Z}_2 \\ \frac{a - \mathcal{Y}_1}{\left(\int_0^1 \mathcal{Y}_s ds \right)^{1/2}} \\ \frac{\alpha - \mathcal{X}_1}{\left(\int_0^1 \mathcal{Y}_s ds \right)^{1/2}} \end{bmatrix} \quad \text{as } T \rightarrow \infty,$$

where $(\mathcal{Y}_t, \mathcal{X}_t)_{t \in \mathbb{R}_+}$ is the unique strong solution of the SDE (6.2) with initial value $(\mathcal{Y}_0, \mathcal{X}_0) = (0, 0)$, \mathbf{Z}_2 is a 2-dimensional standard normally distributed random vector independent of $(\mathcal{Y}_1, \int_0^1 \mathcal{Y}_t dt, \mathcal{X}_1)$, and \mathbf{S} is defined in (2.4).

Proof. By Lemma 3.3, there exists a unique MLE $(\widehat{a}_T, \widehat{b}_T, \widehat{\alpha}_T, \widehat{\beta}_T)$ of (a, b, α, β) for all $T \in \mathbb{R}_{++}$, which has the form given in (3.4). By (3.6), we have

$$\begin{aligned} \left(\int_0^T \frac{ds}{Y_s} \right)^{1/2} (\widehat{a}_T - a) &= \frac{\frac{\sigma_1 \int_0^T \frac{dW_s}{\sqrt{Y_s}}}{\left(\int_0^T \frac{ds}{Y_s} \right)^{1/2}} - \frac{1}{\left(\int_0^T \frac{ds}{Y_s} \right)^{1/2}} \frac{T \sigma_1 \int_0^T \sqrt{Y_s} dW_s}{\int_0^T Y_s ds}}{1 - \frac{1}{T^2} \int_0^T Y_s ds \int_0^T \frac{ds}{Y_s}}, \\ \left(\int_0^T Y_s ds \right)^{1/2} \widehat{b}_T &= \frac{\frac{1}{\left(\frac{1}{T^2} \int_0^T Y_s ds \right)^{1/2}} \frac{1}{\left(\int_0^T \frac{ds}{Y_s} \right)^{1/2}} \frac{\sigma_1 \int_0^T \frac{dW_s}{\sqrt{Y_s}}}{\left(\int_0^T \frac{ds}{Y_s} \right)^{1/2}} - \frac{\sigma_1 \int_0^T \sqrt{Y_s} dW_s}{\left(\int_0^T Y_s ds \right)^{1/2}}}{1 - \frac{1}{T^2} \int_0^T Y_s ds \int_0^T \frac{ds}{Y_s}}, \end{aligned}$$

$$\left(\int_0^T \frac{ds}{Y_s}\right)^{1/2} (\widehat{\alpha}_T - \alpha) = \frac{\frac{\sigma_2 \int_0^T \frac{d\widetilde{W}_s}{\sqrt{Y_s}}}{\left(\int_0^T \frac{ds}{Y_s}\right)^{1/2}} - \frac{1}{\left(\int_0^T \frac{ds}{Y_s}\right)^{1/2}} \frac{T\sigma_2 \int_0^T \sqrt{Y_s} d\widetilde{W}_s}{\int_0^T Y_s ds}}{1 - \frac{1}{\frac{1}{T^2} \int_0^T Y_s ds} \frac{1}{\int_0^T \frac{ds}{Y_s}}},$$

and

$$\left(\int_0^T Y_s ds\right)^{1/2} (\widehat{\beta}_T - \beta) = \frac{\frac{1}{\left(\frac{1}{T^2} \int_0^T Y_s ds\right)^{1/2}} \frac{1}{\left(\int_0^T \frac{ds}{Y_s}\right)^{1/2}} \frac{\sigma_2 \int_0^T \frac{d\widetilde{W}_s}{\sqrt{Y_s}}}{\left(\int_0^T \frac{ds}{Y_s}\right)^{1/2}} - \frac{\sigma_2 \int_0^T \sqrt{Y_s} d\widetilde{W}_s}{\left(\int_0^T Y_s ds\right)^{1/2}}}{1 - \frac{1}{\frac{1}{T^2} \int_0^T Y_s ds} \frac{1}{\int_0^T \frac{ds}{Y_s}}},$$

provided that $\int_0^T Y_s ds \int_0^T \frac{1}{Y_s} ds > T^2$ which holds a.s. We have

$$(6.21) \quad \frac{\sigma_1 \int_0^T \sqrt{Y_s} dW_s}{\left(\int_0^T Y_s ds\right)^{1/2}} = \frac{Y_T - y_0 - aT}{\left(\int_0^T Y_s ds\right)^{1/2}} = \frac{\frac{1}{T}(Y_T - y_0) - a}{\left(\frac{1}{T^2} \int_0^T Y_s ds\right)^{1/2}}, \quad T \in \mathbb{R}_{++},$$

$$(6.22) \quad \frac{\sigma_2 \int_0^T \sqrt{Y_s} d\widetilde{W}_s}{\left(\int_0^T Y_s ds\right)^{1/2}} = \frac{\sigma_2 \varrho \sigma_1 \int_0^T \sqrt{Y_s} dW_s}{\sigma_1 \left(\int_0^T Y_s ds\right)^{1/2}} + \sigma_2 \sqrt{1 - \varrho^2} \frac{\int_0^T \sqrt{Y_s} dB_s}{\left(\int_0^T Y_s ds\right)^{1/2}}, \quad T \in \mathbb{R}_{++},$$

hence (6.20) follows from (6.3), (6.4), (6.5), (6.6), (6.7), (6.8), (6.10), Slutsky's lemma, continuous mapping theorem, and $\mathbb{P}(\int_0^1 \mathcal{Y}_s ds \in \mathbb{R}_{++}) = 1$ (which has been shown in the proof of Theorem 3.1 in Barczy et al. [5]). Indeed,

$$\begin{bmatrix} \left(\int_0^T \frac{ds}{Y_s}\right)^{1/2} (\widehat{\alpha}_T - a) \\ \left(\int_0^T \frac{ds}{Y_s}\right)^{1/2} (\widehat{\alpha}_T - \alpha) \\ \left(\int_0^T Y_s ds\right)^{1/2} \widehat{b}_T \\ \left(\int_0^T Y_s ds\right)^{1/2} (\widehat{\beta}_T - \beta) \end{bmatrix} \xrightarrow{\mathcal{D}} \frac{1}{1 - \frac{1}{\int_0^1 \mathcal{Y}_s ds} \cdot 0} \begin{bmatrix} (\mathbf{S}^{1/2} \mathbf{Z}_2)_1 - 0 \cdot \frac{\mathcal{Y}_1 - a}{\int_0^1 \mathcal{Y}_s ds} \\ (\mathbf{S}^{1/2} \mathbf{Z}_2)_2 - 0 \cdot \frac{\mathcal{X}_1 - \alpha}{\int_0^1 \mathcal{Y}_s ds} \\ \frac{1}{\left(\int_0^1 \mathcal{Y}_s ds\right)^{1/2}} \cdot 0 \cdot (\mathbf{S}^{1/2} \mathbf{Z}_2)_1 - \frac{\mathcal{Y}_1 - a}{\left(\int_0^1 \mathcal{Y}_s ds\right)^{1/2}} \\ \frac{1}{\left(\int_0^1 \mathcal{Y}_s ds\right)^{1/2}} \cdot 0 \cdot (\mathbf{S}^{1/2} \mathbf{Z}_2)_2 - \frac{\mathcal{X}_1 - \alpha}{\left(\int_0^1 \mathcal{Y}_s ds\right)^{1/2}} \end{bmatrix}$$

as $T \rightarrow \infty$, where $\mathbf{S}^{1/2} \mathbf{Z}_2 = ((\mathbf{S}^{1/2} \mathbf{Z}_2)_1, (\mathbf{S}^{1/2} \mathbf{Z}_2)_2)^\top$, since

$$\left(\mathbf{Z}_2, \mathcal{Y}_1, \int_0^1 \mathcal{Y}_s ds, \frac{\sigma_2 \varrho}{\sigma_1} \frac{\mathcal{Y}_1 - a}{\left(\int_0^1 \mathcal{Y}_s ds\right)^{1/2}} + \sigma_2 \sqrt{1 - \varrho^2} \mathcal{Z}_3\right) \stackrel{\mathcal{D}}{=} \left(\mathbf{Z}_2, \mathcal{Y}_1, \int_0^1 \mathcal{Y}_s ds, \frac{\mathcal{X}_1 - \alpha}{\left(\int_0^1 \mathcal{Y}_s ds\right)^{1/2}}\right),$$

which can be shown in the same way as (6.8). \square

6.5 Remark. For a critical (i.e., $b = 0$) CIR models with $a \in (\frac{\sigma_1^2}{2}, \infty)$, using random scaling, Overbeck [35, Theorem 3, part (ii)] has already described the asymptotic behaviour of \widehat{a}_T and \widehat{b}_T separately, but he did not consider their joint asymptotic behaviour. \square

7 Asymptotic behaviour of MLE: supercritical case

We consider supercritical Heston models, i.e., when $b \in \mathbb{R}_{--}$.

7.1 Theorem. If $a \in \left[\frac{\sigma_1^2}{2}, \infty\right)$, $b \in \mathbb{R}_{--}$, $\alpha, \beta \in \mathbb{R}$, $\sigma_1, \sigma_2 \in \mathbb{R}_{++}$, $\varrho \in (-1, 1)$, and $(Y_0, X_0) = (y_0, x_0) \in \mathbb{R}_{++} \times \mathbb{R}$, then

$$(7.1) \quad \begin{bmatrix} \widehat{a}_T - a \\ \widehat{\alpha}_T - \alpha \\ e^{-bT/2}(\widehat{b}_T - b) \\ e^{-bT/2}(\widehat{\beta}_T - \beta) \end{bmatrix} \xrightarrow{\mathcal{D}} \begin{bmatrix} \widetilde{\mathcal{Y}} \\ \varrho \frac{\sigma_2}{\sigma_1} \widetilde{\mathcal{Y}} + \sigma_2 \sqrt{1 - \varrho^2} \left(\int_0^{-1/b} \widetilde{\mathcal{Y}}_u du \right)^{-1/2} \mathbf{Z}_1 \\ \left(-\frac{\widetilde{\mathcal{Y}}_{-1/b}}{b} \right)^{-1/2} \mathbf{S}^{1/2} \mathbf{Z}_2 \end{bmatrix}$$

as $T \rightarrow \infty$, where $(\widetilde{\mathcal{Y}}_t)_{t \in \mathbb{R}_+}$ is a CIR process given by the SDE

$$d\widetilde{\mathcal{Y}}_t = a dt + \sigma_1 \sqrt{\widetilde{\mathcal{Y}}_t} dW_t, \quad t \in \mathbb{R}_+,$$

with initial value $\widetilde{\mathcal{Y}}_0 = y_0$, where $(W_t)_{t \in \mathbb{R}_+}$ is a standard Wiener process,

$$\widetilde{\mathcal{Y}} := \frac{\log \widetilde{\mathcal{Y}}_{-1/b} - \log y_0}{\int_0^{-1/b} \widetilde{\mathcal{Y}}_u du} + \frac{\sigma_1^2}{2} - a,$$

\mathbf{Z}_1 is a 1-dimensional standard normally distributed random variable, \mathbf{Z}_2 is a 2-dimensional standard normally distributed random vector such that $(\widetilde{\mathcal{Y}}_{-1/b}, \int_0^{-1/b} \widetilde{\mathcal{Y}}_u du)$, \mathbf{Z}_1 and \mathbf{Z}_2 are independent, and \mathbf{S} is defined in (2.4).

With a random scaling, we have

$$(7.2) \quad \begin{bmatrix} \widehat{a}_T - a \\ \widehat{\alpha}_T - \alpha \\ \left(\int_0^T Y_s ds \right)^{1/2} (\widehat{b}_T - b) \\ \left(\int_0^T Y_s ds \right)^{1/2} (\widehat{\beta}_T - \beta) \end{bmatrix} \xrightarrow{\mathcal{D}} \begin{bmatrix} \widetilde{\mathcal{Y}} \\ \varrho \frac{\sigma_2}{\sigma_1} \widetilde{\mathcal{Y}} + \sigma_2 \sqrt{1 - \varrho^2} \left(\int_0^{-1/b} \widetilde{\mathcal{Y}}_u du \right)^{-1/2} \mathbf{Z}_1 \\ \mathbf{S}^{1/2} \mathbf{Z}_2 \end{bmatrix}$$

as $T \rightarrow \infty$.

Proof. By Lemma 3.3, there exists a unique MLE $(\widehat{a}_T, \widehat{b}_T, \widehat{\alpha}_T, \widehat{\beta}_T)$ of (a, b, α, β) for all $T \in \mathbb{R}_{++}$, which has the form given in (3.4). By (3.6) and

$$\sigma_2 \int_0^T \frac{d\widetilde{W}_s}{\sqrt{Y_s}} = \sigma_2 \varrho \int_0^T \frac{dW_s}{\sqrt{Y_s}} + \sigma_2 \sqrt{1 - \varrho^2} \int_0^T \frac{dB_s}{\sqrt{Y_s}},$$

we obtain

$$\begin{aligned} \widehat{a}_T - a &= \frac{\frac{\sigma_1 \int_0^T \frac{dW_s}{\sqrt{Y_s}}}{\int_0^T \frac{ds}{Y_s}} - \frac{T e^{bT/2}}{\int_0^T \frac{ds}{Y_s}} \frac{1}{(e^{bT} \int_0^T Y_s ds)^{1/2}} \frac{\sigma_1 \int_0^T \sqrt{Y_s} dW_s}{\left(\int_0^T Y_s ds \right)^{1/2}}}{1 - \frac{T^2 e^{bT}}{e^{bT} \int_0^T Y_s ds \int_0^T \frac{ds}{Y_s}}}, \\ \widehat{\alpha}_T - \alpha &= \frac{\frac{\sigma_2 \varrho \int_0^T \frac{dW_s}{\sqrt{Y_s}}}{\int_0^T \frac{ds}{Y_s}} + \frac{\sigma_2 \sqrt{1 - \varrho^2}}{\left(\int_0^T \frac{ds}{Y_s} \right)^{1/2}} \frac{\int_0^T \frac{dB_s}{\sqrt{Y_s}}}{\left(\int_0^T \frac{ds}{Y_s} \right)^{1/2}} - \frac{T e^{bT/2}}{\int_0^T \frac{ds}{Y_s}} \frac{1}{(e^{bT} \int_0^T Y_s ds)^{1/2}} \frac{\sigma_2 \int_0^T \sqrt{Y_s} d\widetilde{W}_s}{\left(\int_0^T Y_s ds \right)^{1/2}}}{1 - \frac{T^2 e^{bT}}{e^{bT} \int_0^T Y_s ds \int_0^T \frac{ds}{Y_s}}}, \end{aligned}$$

$$e^{-bT/2}(\widehat{b}_T - b) = \frac{\frac{T e^{bT/2}}{e^{bT} \int_0^T Y_s ds} \frac{\sigma_1 \int_0^T \frac{dW_s}{\sqrt{Y_s}}}{\int_0^T \frac{ds}{Y_s}} - \frac{1}{(e^{bT} \int_0^T Y_s ds)^{1/2}} \frac{\sigma_1 \int_0^T \sqrt{Y_s} dW_s}{(\int_0^T Y_s ds)^{1/2}}}{1 - \frac{T^2 e^{bT}}{e^{bT} \int_0^T Y_s ds \int_0^T \frac{ds}{Y_s}}},$$

and

$$e^{-bT/2}(\widehat{\beta}_T - \beta) = \frac{\frac{T e^{bT/2}}{e^{bT} \int_0^T Y_s ds} \left(\frac{\sigma_2 \varrho \int_0^T \frac{dW_s}{\sqrt{Y_s}}}{\int_0^T \frac{ds}{Y_s}} + \frac{\sigma_2 \sqrt{1-\varrho^2}}{(\int_0^T \frac{ds}{Y_s})^{1/2}} \frac{\int_0^T \frac{dB_s}{\sqrt{Y_s}}}{(\int_0^T \frac{ds}{Y_s})^{1/2}} \right) - \frac{1}{(e^{bT} \int_0^T Y_s ds)^{1/2}} \frac{\sigma_2 \int_0^T \sqrt{Y_s} d\widetilde{W}_s}{(\int_0^T Y_s ds)^{1/2}}}{1 - \frac{T^2 e^{bT}}{e^{bT} \int_0^T Y_s ds \int_0^T \frac{ds}{Y_s}}},$$

provided that $\int_0^T Y_s ds \int_0^T \frac{1}{Y_s} ds > T^2$ which holds a.s. Applying (4.10), one can write

$$\sigma_1 \int_0^T \frac{dW_s}{\sqrt{Y_s}} = \log Y_T - \log y_0 + \left(\frac{\sigma_1^2}{2} - a \right) \int_0^T \frac{ds}{Y_s} + bT, \quad T \in \mathbb{R}_{++},$$

thus, by (4.7) and (4.9),

$$(7.3) \quad \frac{\sigma_1 \int_0^T \frac{dW_s}{\sqrt{Y_s}}}{\int_0^T \frac{ds}{Y_s}} = \frac{\log(e^{bT} Y_T) - \log y_0}{\int_0^T \frac{ds}{Y_s}} + \frac{\sigma_1^2}{2} - a \xrightarrow{\text{a.s.}} \frac{\log V - \log y_0}{\int_0^\infty \frac{ds}{Y_s}} + \frac{\sigma_1^2}{2} - a$$

as $T \rightarrow \infty$. By Theorem 4 in Ben Alaya and Kebaier [11],

$$\frac{\log V - \log y_0}{\int_0^\infty \frac{ds}{Y_s}} + \frac{\sigma_1^2}{2} - a \stackrel{\mathcal{D}}{=} \frac{\log \widetilde{\mathcal{Y}}_{-1/b} - \log y_0}{\int_0^{-1/b} \widetilde{\mathcal{Y}}_u du} + \frac{\sigma_1^2}{2} - a =: \widetilde{\mathcal{Y}}.$$

Moreover, (4.8) and (4.9) yield

$$(7.4) \quad \frac{T^2 e^{bT}}{e^{bT} \int_0^T Y_s ds \int_0^T \frac{ds}{Y_s}} \xrightarrow{\text{a.s.}} \frac{0}{(-\frac{V}{b}) \int_0^\infty \frac{ds}{Y_s}} = 0 \quad \text{as } T \rightarrow \infty,$$

$$(7.5) \quad \frac{T e^{bT/2}}{e^{bT} \int_0^T Y_s ds} \xrightarrow{\text{a.s.}} \frac{0}{-\frac{V}{b}} = 0 \quad \text{as } T \rightarrow \infty.$$

Consequently, (7.1) will follow from

$$(7.6) \quad \left(\frac{\int_0^T \frac{dB_s}{\sqrt{Y_s}}}{(\int_0^T \frac{ds}{Y_s})^{1/2}}, \frac{\sigma_1 \int_0^T \sqrt{Y_s} dW_s}{(\int_0^T Y_s ds)^{1/2}}, \frac{\sigma_2 \int_0^T \sqrt{Y_s} d\widetilde{W}_s}{(\int_0^T Y_s ds)^{1/2}}, e^{bT} Y_T, e^{bT} \int_0^T Y_s ds, \int_0^T \frac{ds}{Y_s} \right) \\ \xrightarrow{\mathcal{D}} \left(Z_1, \mathbf{S}^{1/2} \mathbf{Z}_2, \widetilde{\mathcal{Y}}_{-1/b}, -\frac{\widetilde{\mathcal{Y}}_{-1/b}}{b}, \int_0^{-1/b} \widetilde{\mathcal{Y}}_u du \right) \quad \text{as } T \rightarrow \infty,$$

from (4.8), (7.3), (7.4), (7.5), Slutsky's lemma, continuous mapping theorem and $\mathbb{P}(\widetilde{\mathcal{Y}}_{-1/b} \in \mathbb{R}_{++}) = 1$, $\mathbb{P}(\int_0^{-1/b} \widetilde{\mathcal{Y}}_u du \in \mathbb{R}_{++}) = 1$ (due to $\mathbb{P}(\widetilde{\mathcal{Y}}_t \in \mathbb{R}_{++}, \forall t \in \mathbb{R}_+) = 1$). Indeed,

$$\begin{bmatrix} \widehat{a}_T - a \\ \widehat{\alpha}_T - \alpha \\ e^{-bT/2}(\widehat{b}_T - b) \\ e^{-bT/2}(\widehat{\beta}_T - \beta) \end{bmatrix} \xrightarrow{\mathcal{D}}$$

$$\xrightarrow{\mathcal{D}} \frac{1}{1 - \frac{0}{-\frac{\tilde{y}_{-1/b}}{b} \int_0^{-1/b} \tilde{y}_u du}} \left[\begin{array}{c} \tilde{\mathcal{V}} - \frac{0}{\int_0^{-1/b} \tilde{y}_u du} \frac{1}{\left(-\frac{\tilde{y}_{-1/b}}{b}\right)^{1/2}} (\mathbf{S}^{1/2} \mathbf{Z}_2)_1 \\ \varrho \frac{\sigma_2}{\sigma_1} \tilde{\mathcal{V}} + \frac{\sigma_2 \sqrt{1-\varrho^2}}{\left(\int_0^{-1/b} \tilde{y}_u du\right)^{1/2}} Z_1 - \frac{0}{\int_0^{-1/b} \tilde{y}_u du} \frac{1}{\left(-\frac{\tilde{y}_{-1/b}}{b}\right)^{1/2}} (\mathbf{S}^{1/2} \mathbf{Z}_2)_2 \\ \frac{0}{-\frac{\tilde{y}_{-1/b}}{b}} \tilde{\mathcal{V}} - \frac{1}{\left(-\frac{\tilde{y}_{-1/b}}{b}\right)^{1/2}} (\mathbf{S}^{1/2} \mathbf{Z}_2)_1 \\ \frac{0}{-\frac{\tilde{y}_{-1/b}}{b}} \left(\varrho \frac{\sigma_2}{\sigma_1} \tilde{\mathcal{V}} + \frac{\sigma_2 \sqrt{1-\varrho^2}}{\left(\int_0^{-1/b} \tilde{y}_u du\right)^{1/2}} Z_1 \right) - \frac{1}{\left(-\frac{\tilde{y}_{-1/b}}{b}\right)^{1/2}} (\mathbf{S}^{1/2} \mathbf{Z}_2)_2 \end{array} \right]$$

as $T \rightarrow \infty$, where $\mathbf{S}^{1/2} \mathbf{Z}_2 = ((\mathbf{S}^{1/2} \mathbf{Z}_2)_1, (\mathbf{S}^{1/2} \mathbf{Z}_2)_2)^\top$.

Using that

$$\sigma_2 \int_0^T \sqrt{Y_s} d\tilde{W}_s = \sigma_2 \varrho \int_0^T \sqrt{Y_s} dW_s + \sigma_2 \sqrt{1-\varrho^2} \int_0^T \sqrt{Y_s} dB_s, \quad T \in \mathbb{R}_+,$$

and (6.11), by continuous mapping theorem, to prove (7.6), it is sufficient to verify

$$(7.7) \quad \left(\frac{\int_0^T \frac{dB_s}{\sqrt{Y_s}}}{\left(\int_0^T \frac{ds}{Y_s}\right)^{1/2}}, \frac{\int_0^T \sqrt{Y_s} dW_s}{\left(\int_0^T Y_s ds\right)^{1/2}}, \frac{\int_0^T \sqrt{Y_s} dB_s}{\left(\int_0^T Y_s ds\right)^{1/2}}, e^{bT} Y_T, e^{bT} \int_0^T Y_s ds, \int_0^T \frac{ds}{Y_s} \right) \\ \xrightarrow{\mathcal{D}} \left(Z_1, Z_2, \tilde{\mathcal{Y}}_{-1/b}, -\frac{\tilde{\mathcal{Y}}_{-1/b}}{b}, \int_0^{-1/b} \tilde{y}_u du \right) \quad \text{as } T \rightarrow \infty,$$

Applying Theorem A.2 for the continuous local martingale $M_t := \int_0^t \sqrt{Y_s} dW_s$, $t \in \mathbb{R}_+$, with quadratic variation process $\langle M \rangle_t = \int_0^t Y_s ds$, $t \in \mathbb{R}_+$, for $Q(t) := e^{bt/2}$, $t \in \mathbb{R}_{++}$, and for $\mathbf{v} := \left(V, -\frac{V}{b}, \int_0^\infty \frac{ds}{Y_s} \right)$ (defined also on $(\Omega, \mathcal{F}, \mathbb{P})$), we obtain

$$\left(e^{bt/2} \int_0^t \sqrt{Y_s} dW_s, V, -\frac{V}{b}, \int_0^\infty \frac{ds}{Y_s} \right) \xrightarrow{\mathcal{D}} \left(\left(-\frac{V}{b} \right)^{1/2} \xi_2, V, -\frac{V}{b}, \int_0^\infty \frac{ds}{Y_s} \right)$$

as $t \rightarrow \infty$, where ξ_2 is a standard normally distributed random variable independent of V and $\int_0^\infty \frac{ds}{Y_s}$. Indeed, by (4.8), we have $e^{bt} \langle M \rangle_t = e^{bt} \int_0^t Y_s ds \xrightarrow{\text{a.s.}} -\frac{V}{b}$ as $t \rightarrow \infty$. Here, by Ben Alaya and Kebaier [11, Theorem 4],

$$\left(\left(-\frac{V}{b} \right)^{1/2} \xi_2, V, -\frac{V}{b}, \int_0^\infty \frac{ds}{Y_s} \right) \stackrel{\mathcal{D}}{=} \left(\left(-\frac{\tilde{\mathcal{Y}}_{-1/b}}{b} \right)^{1/2} Z_2, \tilde{\mathcal{Y}}_{-1/b}, -\frac{\tilde{\mathcal{Y}}_{-1/b}}{b}, \int_0^{-1/b} \tilde{y}_u du \right),$$

where Z_2 is a standard normally distributed random variable independent of $\tilde{\mathcal{Y}}_{-1/b}$ and $\int_0^{-1/b} \tilde{y}_u du$. By (4.7), (4.8), (4.9) and Lemma A.3, we obtain

$$\left(e^{bt/2} \int_0^t \sqrt{Y_s} dW_s, e^{bt} Y_t, e^{bt} \int_0^t Y_s ds, \int_0^t \frac{ds}{Y_s} \right) - \left(e^{bt/2} \int_0^t \sqrt{Y_s} dW_s, V, -\frac{V}{b}, \int_0^\infty \frac{ds}{Y_s} \right) \xrightarrow{\mathbb{P}} 0$$

as $t \rightarrow \infty$, hence

$$\left(e^{bT/2} \int_0^T \sqrt{Y_s} dW_s, e^{bT} Y_T, e^{bT} \int_0^T Y_s ds, \int_0^T \frac{ds}{Y_s} \right) \xrightarrow{\mathcal{D}}$$

$$\xrightarrow{\mathcal{D}} \left(\left(-\frac{\tilde{\mathcal{Y}}_{-1/b}}{b} \right)^{1/2} Z_2, \tilde{\mathcal{Y}}_{-1/b}, -\frac{\tilde{\mathcal{Y}}_{-1/b}}{b}, \int_0^{-1/b} \tilde{\mathcal{Y}}_u du \right) \quad \text{as } T \rightarrow \infty.$$

Applying continuous mapping theorem, since $\mathbb{P}(\tilde{\mathcal{Y}}_{-1/b} \in \mathbb{R}_{++}) = 1$, we obtain

$$(7.8) \quad \left(\frac{\int_0^T \sqrt{Y_s} dW_s}{\left(\int_0^T Y_s ds \right)^{1/2}}, e^{bT} Y_T, e^{bT} \int_0^T Y_s ds, \int_0^T \frac{ds}{Y_s} \right) \xrightarrow{\mathcal{D}} \left(Z_2, \tilde{\mathcal{Y}}_{-1/b}, -\frac{\tilde{\mathcal{Y}}_{-1/b}}{b}, \int_0^{-1/b} \tilde{\mathcal{Y}}_u du \right)$$

as $T \rightarrow \infty$, hence we derived joint convergence of four coordinates of the left hand side of (7.7).

We will prove (7.7) using continuity theorem. Applying (1.1), one can write

$$\sigma_1 \int_0^T \sqrt{Y_s} dW_s = Y_T - y_0 - \int_0^T (a - bY_s) ds, \quad T \in \mathbb{R}_{++},$$

hence $\int_0^T \sqrt{Y_s} dW_s$ is measurable with respect to the σ -algebra $\sigma(Y_s, s \in [0, T])$. For all $(u_1, u_2, u_3, v_1, v_2, v_3) \in \mathbb{R}^6$ and $T \in \mathbb{R}_{++}$, we have

$$\begin{aligned} & \mathbb{E} \left(\exp \left\{ iu_1 \frac{\int_0^T \frac{dB_s}{\sqrt{Y_s}}}{\left(\int_0^T \frac{ds}{Y_s} \right)^{1/2}} + iu_2 \frac{\int_0^T \sqrt{Y_s} dW_s}{\left(\int_0^T Y_s ds \right)^{1/2}} + iu_3 \frac{\int_0^T \sqrt{Y_s} dB_s}{\left(\int_0^T Y_s ds \right)^{1/2}} \right. \right. \\ & \quad \left. \left. + iv_1 e^{bT} Y_T + iv_2 e^{bT} \int_0^T Y_s ds + iv_3 \int_0^T \frac{ds}{Y_s} \right\} \middle| Y_s, s \in [0, T] \right) \\ &= \exp \left\{ iu_2 \frac{\int_0^T \sqrt{Y_s} dW_s}{\left(\int_0^T Y_s ds \right)^{1/2}} + iv_1 e^{bT} Y_T + iv_2 e^{bT} \int_0^T Y_s ds + iv_3 \int_0^T \frac{ds}{Y_s} \right\} \\ & \quad \times \mathbb{E} \left(\exp \left\{ i \int_0^T \left(\frac{u_1}{\left(\int_0^T \frac{dt}{Y_t} \right)^{1/2}} \frac{1}{\sqrt{Y_s}} + \frac{u_3}{\left(\int_0^T Y_t dt \right)^{1/2}} \sqrt{Y_s} \right) dB_s \right\} \middle| Y_s, s \in [0, T] \right) \\ &= \exp \left\{ iu_2 \frac{\int_0^T \sqrt{Y_s} dW_s}{\left(\int_0^T Y_s ds \right)^{1/2}} + iv_1 e^{bT} Y_T + iv_2 e^{bT} \int_0^T Y_s ds + iv_3 \int_0^T \frac{ds}{Y_s} \right\} \\ & \quad \times \exp \left\{ -\frac{1}{2} \int_0^T \left(\frac{u_1}{\left(\int_0^T \frac{dt}{Y_t} \right)^{1/2}} \cdot \frac{1}{\sqrt{Y_s}} + \frac{u_3}{\left(\int_0^T Y_t dt \right)^{1/2}} \sqrt{Y_s} \right)^2 ds \right\} \\ &= \exp \left\{ iu_2 \frac{\int_0^T \sqrt{Y_s} dW_s}{\left(\int_0^T Y_s ds \right)^{1/2}} + iv_1 e^{bT} Y_T + iv_2 e^{bT} \int_0^T Y_s ds + iv_3 \int_0^T \frac{ds}{Y_s} \right\} \\ & \quad \times \exp \left\{ -\frac{1}{2} (u_1^2 + u_3^2) - \frac{T u_1 u_3}{\left(\int_0^T Y_t dt \int_0^T \frac{dt}{Y_t} \right)^{1/2}} \right\}, \end{aligned}$$

where we used the independence of Y and B . Consequently, the characteristic function of the random vector on the left hand side of (7.7) takes the form

$$\begin{aligned} & \mathbb{E} \left(\exp \left\{ iu_1 \frac{\int_0^T \frac{dB_s}{\sqrt{Y_s}}}{\left(\int_0^T \frac{ds}{Y_s} \right)^{1/2}} + iu_2 \frac{\int_0^T \sqrt{Y_s} dW_s}{\left(\int_0^T Y_s ds \right)^{1/2}} + iu_3 \frac{\int_0^T \sqrt{Y_s} dB_s}{\left(\int_0^T Y_s ds \right)^{1/2}} \right. \right. \\ & \quad \left. \left. + iv_1 e^{bT} Y_T + iv_2 e^{bT} \int_0^T Y_s ds + iv_3 \int_0^T \frac{ds}{Y_s} \right\} \right) \\ & = e^{-(u_1^2 + u_3^2)/2} \mathbb{E} \left(\exp \left\{ \xi_T(u_2, v_1, v_2, v_3) - \frac{T u_1 u_3}{\left(\int_0^T Y_t dt \int_0^T \frac{dt}{Y_t} \right)^{1/2}} \right\} \right), \end{aligned}$$

where

$$\xi_T(u_2, v_1, v_2, v_3) := iu_2 \frac{\int_0^T \sqrt{Y_s} dW_s}{\left(\int_0^T Y_s ds \right)^{1/2}} + iv_1 e^{bT} Y_T + iv_2 e^{bT} \int_0^T Y_s ds + iv_3 \int_0^T \frac{ds}{Y_s}.$$

By (7.8), for all $(u_2, v_1, v_2, v_3) \in \mathbb{R}^4$,

$$(7.9) \quad \begin{aligned} & \mathbb{E}(\exp\{\xi_T(u_2, v_1, v_2, v_3)\}) \\ & \rightarrow \mathbb{E} \left(\exp \left\{ iu_2 Z_2 + iv_1 \tilde{\mathcal{Y}}_{-1/b} + iv_2 \left(-\frac{\tilde{\mathcal{Y}}_{-1/b}}{b} \right) + iv_3 \int_0^{-1/b} \tilde{\mathcal{Y}}_u du \right\} \right) \end{aligned}$$

as $T \rightarrow \infty$. Using $|\exp\{\xi_T(u_2, v_1, v_2, v_3)\}| = 1$, we have

$$\begin{aligned} & \left| \mathbb{E} \left(\exp \left\{ \xi_T(u_2, v_1, v_2, v_3) - \frac{T u_1 u_3}{\left(\int_0^T Y_t dt \int_0^T \frac{dt}{Y_t} \right)^{1/2}} \right\} \right) - \mathbb{E}(\exp\{\xi_T(u_2, v_1, v_2, v_3)\}) \right| \\ & \leq \mathbb{E} \left(\left| \exp\{\xi_T(u_2, v_1, v_2, v_3)\} \left| \exp \left\{ -\frac{T u_1 u_3}{\left(\int_0^T Y_t dt \int_0^T \frac{dt}{Y_t} \right)^{1/2}} \right\} - 1 \right| \right) \\ & = \mathbb{E} \left(\left| \exp \left\{ -\frac{T u_1 u_3}{\left(\int_0^T Y_t dt \int_0^T \frac{dt}{Y_t} \right)^{1/2}} \right\} - 1 \right| \right) \rightarrow 0 \quad \text{as } T \rightarrow \infty, \end{aligned}$$

by dominated convergence theorem, since, by (4.8) and (4.9),

$$\exp \left\{ -\frac{T u_1 u_3}{\left(\int_0^T Y_t dt \int_0^T \frac{dt}{Y_t} \right)^{1/2}} \right\} - 1 = \exp \left\{ -\frac{T e^{bT/2} u_1 u_3}{\left(e^{bT} \int_0^T Y_t dt \int_0^T \frac{dt}{Y_t} \right)^{1/2}} \right\} - 1 \xrightarrow{\text{a.s.}} 0 \quad \text{as } T \rightarrow \infty,$$

and, by Cauchy–Schwarz inequality,

$$\left| \exp \left\{ -\frac{T u_1 u_3}{\left(\int_0^T Y_t dt \int_0^T \frac{dt}{Y_t} \right)^{1/2}} \right\} - 1 \right| \leq \exp \left\{ \frac{T |u_1 u_3|}{\left(\int_0^T Y_t dt \int_0^T \frac{dt}{Y_t} \right)^{1/2}} \right\} + 1 \leq \exp\{|u_1 u_3|\} + 1$$

for all $T \in \mathbb{R}_{++}$. Using (7.9), we conclude

$$\begin{aligned} & \mathbb{E} \left(\exp \left\{ iu_1 \frac{\int_0^T \frac{dB_s}{\sqrt{Y_s}}}{\left(\int_0^T \frac{ds}{Y_s} \right)^{1/2}} + iu_2 \frac{\int_0^T \sqrt{Y_s} dW_s}{\left(\int_0^T Y_s ds \right)^{1/2}} + iu_3 \frac{\int_0^T \sqrt{Y_s} dB_s}{\left(\int_0^T Y_s ds \right)^{1/2}} \right. \right. \\ & \quad \left. \left. + iv_1 e^{bT} Y_T + iv_2 e^{bT} \int_0^T Y_s ds + iv_3 \int_0^T \frac{ds}{Y_s} \right\} \right) \\ & \rightarrow e^{-(u_1^2 + u_3^2)/2} \mathbb{E} \left(\exp \left\{ iu_2 Z_2 + iv_1 \tilde{\mathcal{Y}}_{-1/b} + iv_2 \left(-\frac{\tilde{\mathcal{Y}}_{-1/b}}{b} \right) + iv_3 \int_0^{-1/b} \tilde{\mathcal{Y}}_u du \right\} \right) \end{aligned}$$

as $T \rightarrow \infty$. Note that, since Z_2 is independent of $\tilde{\mathcal{Y}}_{-1/b}$ and $\int_0^{-1/b} \tilde{\mathcal{Y}}_u du$, we have

$$\begin{aligned} & e^{-(u_1^2 + u_3^2)/2} \mathbb{E} \left(\exp \left\{ iu_2 Z_2 + iv_1 \tilde{\mathcal{Y}}_{-1/b} + iv_2 \left(-\frac{\tilde{\mathcal{Y}}_{-1/b}}{b} \right) + iv_3 \int_0^{-1/b} \tilde{\mathcal{Y}}_u du \right\} \right) \\ & = \mathbb{E}(e^{iu_1 Z_1}) \mathbb{E}(e^{iu_2 Z_2}) \mathbb{E}(e^{iu_3 Z_3}) \mathbb{E} \left(\exp \left\{ iv_1 \tilde{\mathcal{Y}}_{-1/b} + iv_2 \left(-\frac{\tilde{\mathcal{Y}}_{-1/b}}{b} \right) + iv_3 \int_0^{-1/b} \tilde{\mathcal{Y}}_u du \right\} \right), \end{aligned}$$

where (Z_1, Z_3) is a 2-dimensional standard normally distributed random vector, independent of $(Z_2, \tilde{\mathcal{Y}}_{-1/b}, \int_0^{-1/b} \tilde{\mathcal{Y}}_u du)$, hence we obtain (7.7) with $\mathbf{Z}_2 := (Z_2, Z_3)$.

Finally, we prove (7.2). In a similar way, by (3.6), we have

$$\left(\int_0^T Y_s ds \right)^{1/2} (\hat{b}_T - b) = \frac{\frac{T e^{bT/2}}{(e^{bT} \int_0^T Y_s ds)^{1/2}} \frac{\sigma_1 \int_0^T \frac{dW_s}{\sqrt{Y_s}}}{\int_0^T \frac{ds}{Y_s}} - \sigma_1 \frac{\int_0^T \sqrt{Y_s} dW_s}{\left(\int_0^T Y_s ds \right)^{1/2}}}{1 - \frac{T^2 e^{bT}}{e^{bT} \int_0^T Y_s ds \int_0^T \frac{ds}{Y_s}}},$$

and

$$\left(\int_0^T Y_s ds \right)^{1/2} (\hat{\beta}_T - \beta) = \frac{\frac{T e^{bT/2}}{(e^{bT} \int_0^T Y_s ds)^{1/2}} \left(\frac{\sigma_2 \varrho \int_0^T \frac{dW_s}{\sqrt{Y_s}}}{\int_0^T \frac{ds}{Y_s}} + \frac{\sigma_2 \sqrt{1-\varrho^2}}{\left(\int_0^T \frac{ds}{Y_s} \right)^{1/2}} \frac{\int_0^T \frac{dB_s}{\sqrt{Y_s}}}{\left(\int_0^T \frac{ds}{Y_s} \right)^{1/2}} \right) - \sigma_2 \frac{\int_0^T \sqrt{Y_s} d\tilde{W}_s}{\left(\int_0^T Y_s ds \right)^{1/2}}}{1 - \frac{T^2 e^{bT}}{e^{bT} \int_0^T Y_s ds \int_0^T \frac{ds}{Y_s}}},$$

provided that $\int_0^T Y_s ds \int_0^T \frac{1}{Y_s} ds > T^2$ which holds a.s. By (4.8), we get

$$\frac{T e^{bT/2}}{\left(e^{bT} \int_0^T Y_s ds \right)^{1/2}} \xrightarrow{\text{a.s.}} \frac{0}{\left(-\frac{V}{b} \right)^{1/2}} = 0 \quad \text{as } T \rightarrow \infty,$$

hence (4.8), (7.3), (7.4), (7.5), (7.6), (7.7), Slutsky's lemma, continuous mapping theorem and $\mathbb{P}(\tilde{\mathcal{Y}}_{-1/b} \in \mathbb{R}_{++}) = 1$, $\mathbb{P}(\int_0^{-1/b} \tilde{\mathcal{Y}}_u du \in \mathbb{R}_{++}) = 1$ (due to $\mathbb{P}(\tilde{\mathcal{Y}}_t \in \mathbb{R}_{++}, \forall t \in \mathbb{R}_+) = 1$) yield the second statement. Indeed,

$$\begin{bmatrix} \hat{a}_T - a \\ \hat{\alpha}_T - \alpha \\ \left(\int_0^T Y_s ds \right)^{1/2} (\hat{b}_T - b) \\ \left(\int_0^T Y_s ds \right)^{1/2} (\hat{\beta}_T - \beta) \end{bmatrix} \xrightarrow{\mathcal{D}} \begin{bmatrix} \hat{a}_T - a \\ \hat{\alpha}_T - \alpha \\ \left(\int_0^T Y_s ds \right)^{1/2} (\hat{b}_T - b) \\ \left(\int_0^T Y_s ds \right)^{1/2} (\hat{\beta}_T - \beta) \end{bmatrix}$$

$$\xrightarrow{\mathcal{D}} \frac{1}{1 - \frac{0}{-\frac{\tilde{y}_{-1/b}}{b} \int_0^{-1/b} \tilde{\mathcal{Y}}_u du}} \begin{bmatrix} \tilde{\mathcal{V}} - \frac{0}{\int_0^{-1/b} \tilde{\mathcal{Y}}_u du} \frac{1}{\left(-\frac{\tilde{y}_{-1/b}}{b}\right)^{1/2}} (\mathbf{S}^{1/2} \mathbf{Z}_2)_1 \\ \varrho \frac{\sigma_2}{\sigma_1} \tilde{\mathcal{V}} + \frac{\sigma_2 \sqrt{1-\varrho^2}}{\left(\int_0^{-1/b} \tilde{\mathcal{Y}}_u du\right)^{1/2}} Z_1 - \frac{0}{\int_0^{-1/b} \tilde{\mathcal{Y}}_u du} \frac{1}{\left(-\frac{\tilde{y}_{-1/b}}{b}\right)^{1/2}} (\mathbf{S}^{1/2} \mathbf{Z}_2)_2 \\ \frac{0}{\left(-\frac{\tilde{y}_{-1/b}}{b}\right)^{1/2}} \tilde{\mathcal{V}} - (\mathbf{S}^{1/2} \mathbf{Z}_2)_1 \\ \frac{0}{\left(-\frac{\tilde{y}_{-1/b}}{b}\right)^{1/2}} \left[\varrho \frac{\sigma_2}{\sigma_1} \tilde{\mathcal{V}} + \frac{\sigma_2 \sqrt{1-\varrho^2}}{\left(\int_0^{-1/b} \tilde{\mathcal{Y}}_u du\right)^{1/2}} Z_1 \right] - (\mathbf{S}^{1/2} \mathbf{Z}_2)_2 \end{bmatrix}$$

as $T \rightarrow \infty$, where $\mathbf{S}^{1/2} \mathbf{Z}_2 = ((\mathbf{S}^{1/2} \mathbf{Z}_2)_1, (\mathbf{S}^{1/2} \mathbf{Z}_2)_2)^\top$. \square

7.2 Remark. Overbeck [35, Theorem 3] has already derived the asymptotic behaviour of \hat{b}_T with non-random and random scaling for supercritical CIR processes. We also note that Ben Alaya and Kebaier [10, Theorem 1, Case 3] described the asymptotic behavior of the MLE of b for supercritical CIR processes supposing that $a \in \mathbb{R}_{++}$ is known. It turns out that in this case the limit distribution is different from that we have in (7.1). \square

7.3 Corollary. *Under the conditions of Theorem 7.1, the MLEs of b and β are weakly consistent, however, the MLEs of a and α are not weakly consistent. (Recall also that earlier it turned out that the MLE of b is in fact strongly consistent, see Theorem 4.4.)*

Proof. In order to show that the MLEs of a and α are not weakly consistent, it suffices to show $\mathbb{P}(\tilde{\mathcal{V}} \neq 0) > 0$, since Z_1 is independent of the random vector $(\tilde{\mathcal{Y}}_{-1/b}, \int_0^{-1/b} \tilde{\mathcal{Y}}_u du)$, and $\mathbb{P}(\int_0^{-1/b} \tilde{\mathcal{Y}}_u du > 0) = 1$ (see the end of Remark 2.6). We have

$$\mathbb{P}(\tilde{\mathcal{V}} = 0) = \mathbb{P}\left(\log \tilde{\mathcal{Y}}_{-1/b} - \log y_0 = \left(a - \frac{\sigma_1^2}{2}\right) \int_0^{-1/b} \tilde{\mathcal{Y}}_u du\right) \leq \mathbb{P}(\tilde{\mathcal{Y}}_{-1/b} \geq y_0) < 1,$$

where $y_0 \in \mathbb{R}_{++}$. Indeed, by Ikeda and Watanabe [24, page 222],

$$\mathbb{E}(e^{-\lambda \tilde{\mathcal{Y}}_{-1/b}}) = \left(1 + \frac{\sigma_1^2}{(-2b)} \lambda\right)^{-2a/\sigma_1^2}, \quad \lambda \in \mathbb{R}_+,$$

hence $\tilde{\mathcal{Y}}_{-1/b}$ has Gamma distribution with parameters $2a/\sigma_1^2$ and $-2b/\sigma_1^2$. \square

Appendix

A Limit theorems for continuous local martingales

In what follows we recall some limit theorems for continuous local martingales. We use these limit theorems for studying the asymptotic behaviour of the MLE of (a, b, α, β) . First we recall a strong law of large numbers for continuous local martingales.

A.1 Theorem. (Liptser and Shiryaev [33, Lemma 17.4]) Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}_+}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions. Let $(M_t)_{t \in \mathbb{R}_+}$ be a square-integrable continuous local martingale with respect to the filtration $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$ such that $\mathbb{P}(M_0 = 0) = 1$. Let $(\xi_t)_{t \in \mathbb{R}_+}$ be a progressively measurable process such that

$$\mathbb{P} \left(\int_0^t \xi_u^2 d\langle M \rangle_u < \infty \right) = 1, \quad t \in \mathbb{R}_+,$$

and

$$(A.1) \quad \int_0^t \xi_u^2 d\langle M \rangle_u \xrightarrow{\text{a.s.}} \infty \quad \text{as } t \rightarrow \infty,$$

where $(\langle M \rangle_t)_{t \in \mathbb{R}_+}$ denotes the quadratic variation process of M . Then

$$(A.2) \quad \frac{\int_0^t \xi_u dM_u}{\int_0^t \xi_u^2 d\langle M \rangle_u} \xrightarrow{\text{a.s.}} 0 \quad \text{as } t \rightarrow \infty.$$

If $(M_t)_{t \in \mathbb{R}_+}$ is a standard Wiener process, the progressive measurability of $(\xi_t)_{t \in \mathbb{R}_+}$ can be relaxed to measurability and adaptedness to the filtration $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$.

The next theorem is about the asymptotic behaviour of continuous multivariate local martingales, see van Zanten [42, Theorem 4.1].

A.2 Theorem. (van Zanten [42, Theorem 4.1]) Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}_+}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions. Let $(\mathbf{M}_t)_{t \in \mathbb{R}_+}$ be a d -dimensional square-integrable continuous local martingale with respect to the filtration $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$ such that $\mathbb{P}(\mathbf{M}_0 = \mathbf{0}) = 1$. Suppose that there exists a function $\mathbf{Q} : \mathbb{R}_+ \rightarrow \mathbb{R}^{d \times d}$ such that $\mathbf{Q}(t)$ is an invertible (non-random) matrix for all $t \in \mathbb{R}_+$, $\lim_{t \rightarrow \infty} \|\mathbf{Q}(t)\| = 0$ and

$$\mathbf{Q}(t) \langle \mathbf{M} \rangle_t \mathbf{Q}(t)^\top \xrightarrow{\mathbb{P}} \boldsymbol{\eta} \boldsymbol{\eta}^\top \quad \text{as } t \rightarrow \infty,$$

where $\boldsymbol{\eta}$ is a $d \times d$ random matrix. Then, for each \mathbb{R}^k -valued random vector \mathbf{v} defined on $(\Omega, \mathcal{F}, \mathbb{P})$, we have

$$(\mathbf{Q}(t) \mathbf{M}_t, \mathbf{v}) \xrightarrow{\mathcal{D}} (\boldsymbol{\eta} \mathbf{Z}, \mathbf{v}) \quad \text{as } t \rightarrow \infty,$$

where \mathbf{Z} is a d -dimensional standard normally distributed random vector independent of $(\boldsymbol{\eta}, \mathbf{v})$.

We note that Theorem A.2 remains true if the function \mathbf{Q} is defined only on an interval $[t_0, \infty)$ with some $t_0 \in \mathbb{R}_{++}$.

To derive consequences of Theorem A.2 one can use the following lemma which is a multidimensional version of Lemma 3 due to Kátaı and Mogyoródi [28], see Barczy and Pap [8, Lemma 3].

A.3 Lemma. Let $(\mathbf{U}_t)_{t \in \mathbb{R}_+}$ be a k -dimensional stochastic process such that \mathbf{U}_t converges in distribution as $t \rightarrow \infty$. Let $(\mathbf{V}_t)_{t \in \mathbb{R}_+}$ be an ℓ -dimensional stochastic process such that $\mathbf{V}_t \xrightarrow{\mathbb{P}} \mathbf{V}$ as $t \rightarrow \infty$, where \mathbf{V} is an ℓ -dimensional random vector. If $g : \mathbb{R}^k \times \mathbb{R}^\ell \rightarrow \mathbb{R}^d$ is a continuous function, then

$$g(\mathbf{U}_t, \mathbf{V}_t) - g(\mathbf{U}_t, \mathbf{V}) \xrightarrow{\mathbb{P}} \mathbf{0} \quad \text{as } t \rightarrow \infty.$$

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