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# Asymptotic behavior of the generalized St. Petersburg sum conditioned on its maximum

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*Dedicated to the memory of Sándor Csörgő*

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In this paper, we revisit the classical results on the generalized St. Petersburg sums. We determine the limit distribution of the St. Petersburg sum conditioning on its maximum, and we analyze how the limit depends on the value of the maximum. As an application, we obtain an infinite sum representation of the distribution function of the possible semistable limits. In the representation, each term corresponds to a given maximum, in particular this result explains that the semistable behavior is caused by the typical values of the maximum.

*Keywords:* conditional limit theorem; generalized St. Petersburg distribution; merging theorem; semistable law

## 1. Introduction

Peter offers to let Paul toss a possibly biased coin repeatedly until it lands heads and pays him  $r^{k/\alpha}$  ducats if this happens on the  $k$ th toss, where  $k \in \mathbb{N} = \{1, 2, \dots\}$ ,  $p \in (0, 1)$  is the probability of heads at each throw,  $q = 1 - p$ ,  $r = q^{-1}$ , while  $\alpha > 0$  is a payoff parameter. This is the so-called generalized St. Petersburg game with parameter  $(\alpha, p)$ . The classical St. Petersburg game corresponds to  $\alpha = 1$  and  $p = 1/2$ . If  $X$  denotes Paul's winning in this St. Petersburg  $(\alpha, p)$  game, then  $\mathbb{P}\{X = r^{k/\alpha}\} = q^{k-1}p$ ,  $k \in \mathbb{N}$ . Put  $\lfloor x \rfloor$  for the lower integer part,  $\lceil x \rceil$  for the upper integer part and  $\{x\}$  for the fractional part of

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$x$ . Then the distribution function of the gain is

$$F(x) = \mathbb{P}\{X \leq x\} = \begin{cases} 0, & x < r^{1/\alpha}, \\ 1 - q^{\lfloor \alpha \log_r x \rfloor} = 1 - \frac{r^{\{\alpha \log_r x\}}}{x^\alpha}, & x \geq r^{1/\alpha}, \end{cases} \quad (1)$$

where  $\log_r$  stands for the logarithm to the base  $r$ .

In the following all the functions, constants and random variables depend on the parameters  $\alpha$  and  $p$ . For the sake of readability we suppress everywhere the upper index  $\alpha, p$ .

We see that the payoff parameter  $\alpha > 0$  is in fact a tail parameter of the distribution. In particular,  $\mathbb{E}(X^\alpha) = \infty$ , but  $\mathbb{E}(X^\beta) = p/(q^{\beta/\alpha} - q)$  is finite for  $\beta \in (0, \alpha)$ , so for  $\alpha > 2$  Paul's gain  $X$  has a finite variance, so Lévy's central limit theorem holds. As Csörgő pointed out in [5], even for  $\alpha = 2$  the St. Petersburg(2,  $p$ ) distribution is in the domain of attraction of the normal law. This can be checked by straightforward calculation, using the well-known characterization of the domain of attraction of the normal law. Hence, the case  $\alpha \geq 2$  is substantially different from the more difficult case  $\alpha < 2$ . In Section 2, when we are dealing with asymptotic behavior of the sums as  $n \rightarrow \infty$  we usually assume that  $\alpha < 2$ . We indicate the possible values of  $\alpha$  in all of the statements. Of course, the most interesting case is the classical one, when  $\alpha = 1$ , for which the mean is infinite.

### 1.1. The sum

Let  $X, X_1, X_2, \dots$  be i.i.d. St. Petersburg( $\alpha, p$ ) random variables, let  $S_n = X_1 + \dots + X_n$  denote their partial sum, and  $X_n^* = \max_{1 \leq i \leq n} X_i$  their maximum. Since the bounded oscillating function  $r^{\{\alpha \log_r x\}}$  in the numerator of the distribution function in (1) is not slowly varying at infinity, by the classical Doeblin–Gnedenko criterion (cf. [11]) the underlying St. Petersburg distribution is *not* in the domain of attraction of any stable law. That is there is no asymptotic distribution for  $(S_n - c_n)/a_n$ , in the usual sense, whatever the centering and norming constants are. This is where the main difficulty lies in analyzing the St. Petersburg games.

However, asymptotic distributions do exist along subsequences of the natural numbers. In the classical case, when  $\alpha = 1$ ,  $p = 1/2$ , Martin-Löf [17] “clarified the St. Petersburg paradox,” showing that  $S_{2^k}/2^k - k$  converges in distribution, as  $k \rightarrow \infty$ . Csörgő and Dodunekova [7] showed that there are continuum of different types of asymptotic distributions of  $S_n/n - \log_2 n$  along different subsequences of  $\mathbb{N}$ .

In order to state the necessary and sufficient condition for the existence of the limit, we introduce the positional parameter

$$\gamma_n = \frac{n}{r^{\lceil \log_r n \rceil}} \in (q, 1], \quad (2)$$

which shows the position of  $n$  between two consecutive powers of  $r$ . Put

$$\mu_n = \begin{cases} n^{1-\alpha^{-1}} \frac{p}{q^{1/\alpha} - q}, & \text{for } \alpha \neq 1, \\ \frac{p}{q} \log_r n, & \text{for } \alpha = 1. \end{cases} \quad (3)$$

In Theorem 1 in [5], Csörgő showed that the following merging theorem holds (in fact a sharp estimate for the rate is also provided):

$$\sup_{x \in \mathbb{R}} \left| \mathbb{P} \left\{ \frac{S_n}{n^{1/\alpha}} - \mu_n \leq x \right\} - G_{\gamma_n}(x) \right| \rightarrow 0 \quad \text{as } n \rightarrow \infty, \quad (4)$$

where  $G_\gamma$  is the distribution function of the infinitely divisible random variable  $W_\gamma$ ,  $\gamma \in (q, 1]$  with characteristic function

$$\mathbb{E}(e^{itW_\gamma}) = e^{y_\gamma(t)} = \exp \left( it[s_\gamma + u_\gamma] + \int_0^\infty \left( e^{itx} - 1 - \frac{itx}{1+x^2} \right) dR_\gamma(x) \right) \quad (5)$$

with

$$s_\gamma = \begin{cases} \frac{p}{q - q^{1/\alpha}} \frac{1}{\gamma^{(1-\alpha)/\alpha}}, & \alpha \neq 1, \\ \frac{p}{q} \log_r \frac{1}{\gamma}, & \alpha = 1, \end{cases}$$

$$u_\gamma = \frac{p}{q} \gamma^{(\alpha+1)/\alpha} \sum_{k=1}^{\infty} \frac{r^{((1-\alpha)/\alpha)k}}{\gamma^{2/\alpha} + r^{2k/\alpha}} - \frac{p}{q} \gamma^{(\alpha-1)/\alpha} \sum_{k=0}^{\infty} \frac{1}{\gamma^{2/\alpha} r^{((3-\alpha)/\alpha)k} + r^{((1-\alpha)/\alpha)k}},$$

and Lévy function

$$R_\gamma(x) = -\gamma q^{\lfloor \log_r(\gamma x^\alpha) \rfloor} = -\frac{r^{\{\log_r(\gamma x^\alpha)\}}}{x^\alpha}, \quad x > 0. \quad (6)$$

From this form, it is clear that  $W_\gamma$  is a semistable random variable with characteristic exponent  $\alpha$ . For the precise rate of the convergence in (4) see Csörgő [6], where short merging asymptotic expansions are provided, and also additional historical background and references are given. Merging asymptotic expansions are proved by Pap [21], where the length of the expansion depends on the parameter  $\alpha$ : the closer  $\alpha$  is to 0, the longer expansion is possible. Pap [21] also shows non-uniform asymptotic expansions. The natural framework of the merging theorems is the class of semistable distributions, see Csörgő and Megyesi [8]. In Section 2.3, we briefly collect the definition and basic properties of semistable distributions.

## 1.2. The maximum

It turns out that the maximum  $X_n^*$  has similar asymptotic behavior as the sum. Let us consider the classical case again, that is,  $\alpha = 1, p = 1/2$ . For  $\gamma \in (1/2, 1]$ , introduce the distribution function

$$H_\gamma(x) = \begin{cases} 0, & \text{for } x \leq 0, \\ \exp(-\gamma 2^{-\lfloor \log_2(\gamma x) \rfloor}), & \text{for } x > 0. \end{cases}$$

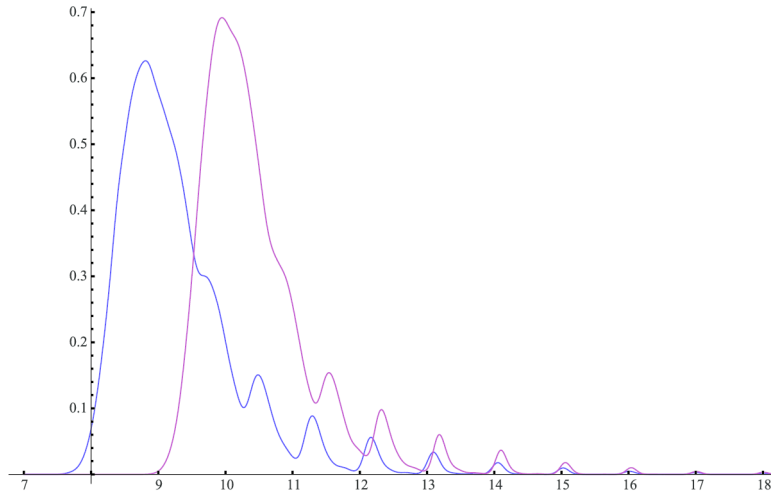
Berkes, Csáki and Csörgő [2] showed that although there is no limit theorem for the normed maximum through the whole sequence, the following merging theorem holds:

$$\sup_{x \in \mathbb{R}} \left| \mathbb{P} \left\{ \frac{X_n^*}{n} \leq x \right\} - H_{\gamma_n}(x) \right| = O(n^{-1}) \quad \text{as } n \rightarrow \infty, \quad (7)$$

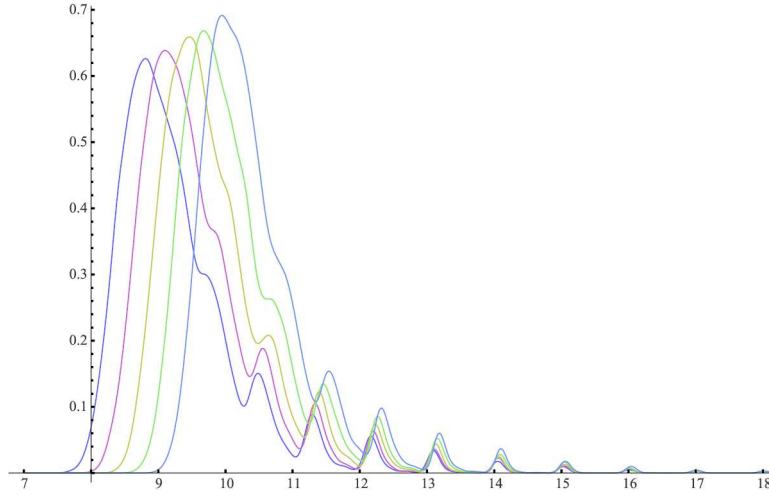
with the positional parameter  $\gamma_n$  defined in (2). Note that even though the “limiting” distribution function is not continuous, merging holds in uniform distance. A more general setup is treated by Megyesi [20], see in particular Theorem 4 in [20].

The merging theorems (4) and (7) immediately imply that in the classical case  $S_n/n - \log_2 n$  and  $X_n^*/n$  converges along the subsequence  $\{n_k\}$  if and only if  $\gamma_{n_k} \rightarrow \gamma$ , as  $k \rightarrow \infty$ , for some  $\gamma \in [1/2, 1]$ , or  $\{\gamma_{n_k}\}$  has exactly two limit points,  $1/2$  and  $1$ . The latter is called *circular convergence*, as it can be seen as convergence on the interval  $[1/2, 1]$ ,  $1/2$  and  $1$  identified. See [5] and [6]. Similar statement holds in the general case.

Having seen these similarities it is tempting to investigate the maximum and the sum together. In Figures 1 and 2 (all the figures correspond to the classical case), one can see that the histograms of  $\log_2 S_n$  are mixtures of unimodal densities such that the first



**Figure 1.** The histograms of  $\log_2 S_n$  for  $n = 2^6$  and for  $n = 2^7$ .

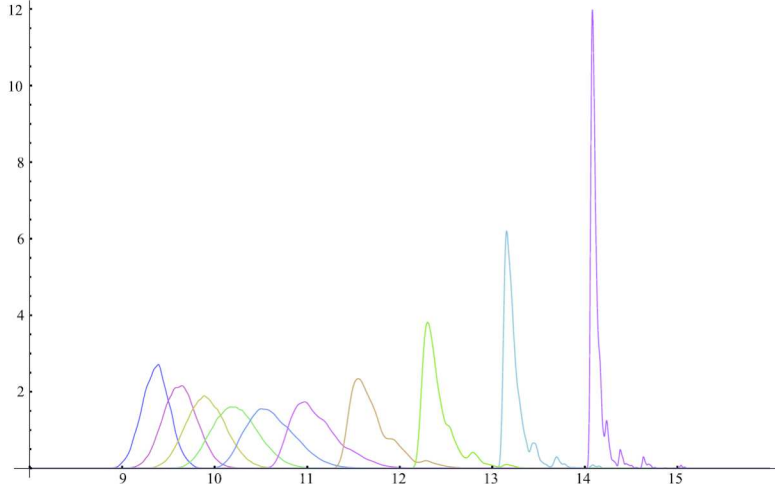


**Figure 2.** The histograms of  $\log_2 S_n$  for  $n = 2^{6+\eta}$ ,  $\eta = 0, 0.25, 0.5, 0.75, 1$ .

lobe is a mixture of overlapping densities, while the side lobes have disjoint support. For doubling  $n$ , in Figure 1 the pairs of corresponding side lobes are almost identical, which suggests an oscillating behavior governed by the parameter  $\gamma_n$  in (2). Figure 2 shows the histograms of  $\log_2 S_n$  for  $n = \lfloor 2^{6+\eta} \rfloor$ ,  $\eta = 0, 0.25, 0.5, 0.75, 1$ , that is for different values of  $\gamma_n$ .

We mention that investigating the joint behavior of the sum and the maximum goes back to Chow and Teugels [4]. Let  $Y, Y_1, Y_2, \dots$  be i.i.d. random variables,  $Z_n$  and  $Y_n^*$  their partial sum and partial maximum, respectively. In [4], Chow and Teugels show that for some deterministic sequences  $a_n > 0, c_n > 0, b_n, d_n$ ,  $(Z_n/a_n - b_n, Y_n^*/c_n - d_n)$  converges in distribution to  $(U, V)$ , where neither  $U$  nor  $V$  is degenerate, if and only if  $Y$  belongs to the domain of attraction of a stable law, and also belongs to the maximum domain of attraction of some extreme value distribution. Moreover, they also characterize when  $U$  and  $V$  are independent. The key technique in their proof is the “hybrid” function: characteristic function of the sum, and distribution function of the maximum. The same results using point process methods were proved by Kasahara [15] and by Resnick [22]. Arov and Bobrov [1] consider the maximum modulus term instead of the maximum. The joint convergence is also studied in case of non-independent random variables, we only mention a recent paper by Silvestrov and Teugels [24]. Without the proof, we mention that the method of Chow and Teugels can be used to obtain subsequential joint limit theorems for the sum and for the maximum in our setup.

In the present paper, we investigate together the maximum and the sum of the St. Petersburg random variables. In Section 2, we determine the asymptotic distribution of  $S_n$  conditioning on the maximum value, and we demonstrate how the limit depends on the maximum. Figure 3 shows the different blocks of the smoothed histogram of  $\log_2 S_n$ ,  $n = 2^7$ , such that in each block the maximum is the same, that is each lobe is the smoothed



**Figure 3.** The conditional histograms for  $\log_2 S_n$ ,  $n = 2^7$ .

conditional histogram for  $S_n$  given that  $X_n^* = 2^k$ , for  $k = 5, 6, \dots, 14$ . Comparing it with Figure 1 it is visible that the lobes are determined by the behavior of the maximum term. As (7) states, the typical value for  $k$  is  $\log_2 n$ . The first lobes correspond to smaller values of  $X_n^*$ , and so it is natural to expect a Gaussian limit; Proposition 3 deals with this case. The typical values of the maximum make the important contribution, and this is where the limiting semistable law appears. The middle lobes are the density functions of infinitely divisible distribution functions, each of these has finite expectation. This conditional limit theorem is stated in Proposition 6. Finally, as the maximum becomes larger and larger it dominates the whole sum  $S_n$ . The conditional limit for large maximum is contained in Proposition 7.

In Section 3, we consider an application of this approach. As a consequence of Proposition 6, in Theorem 1 we show that

$$G_\gamma(x) = \sum_{j=-\infty}^{\infty} \tilde{G}_{j,\gamma}(x) p_{j,\gamma},$$

where  $G_\gamma$  is the merging distribution function appearing in (4). Here  $\tilde{G}_{j,\gamma_n}$  corresponds to the distribution function of the sum conditioned on  $X_n^* = r^{(\lceil \log_r n \rceil + j)/\alpha}$ , and  $p_{j,\gamma_n}$  is the approximate probability of this event. The decomposition shows that the merging property is caused by the asymptotic properties of the maximum.

Finally, we note that recently Gut and Martin-Löf [13] investigated the so-called max-trimmed St. Petersburg games in the classical case, where from the sum all the maximal observations are discarded. They obtained the asymptotic behavior of the trimmed sum along subsequences of the form  $(\lfloor \gamma 2^n \rfloor)_{n \in \mathbb{N}}$ .

## 2. Conditioning on the maximum

In this section, first we revisit the limit properties of  $X_n^*$ , and then conditioning on different values of the maximum, we determine the limit distribution of the sums.

### 2.1. Asymptotics of the maximum

For  $j \in \mathbb{Z}$  and  $\gamma \in [q, 1]$  introduce the notation

$$p_{j,\gamma} = e^{-\gamma q^j} (1 - e^{-\gamma(r-1)q^j}).$$

The following lemma is a reformulation of (7) in the general case. We give the short proof for completeness. Recall the definition of  $\gamma_n$  in (2).

**Lemma 1.** *For any  $\alpha > 0$  we have that*

$$\sup_{j \in \mathbb{Z}} |\mathbb{P}\{X_n^* = r^{(\lceil \log_r n \rceil + j)/\alpha}\} - p_{j,\gamma_n}| = O(n^{-1}). \quad (8)$$

*In particular for any  $j \in \mathbb{Z}$ , as  $n \rightarrow \infty$*

$$\mathbb{P}\{X_n^* = r^{(\lceil \log_r n \rceil + j)/\alpha}\} \sim e^{-\gamma_n q^j} (1 - e^{-\gamma_n(r-1)q^j}).$$

**Proof.** For any  $k = 1, 2, \dots$  we have  $\mathbb{P}\{X_n^* \leq r^{k/\alpha}\} = (1 - q^k)^n$ , and so

$$\begin{aligned} |\mathbb{P}\{X_n^* \leq r^{(\lceil \log_r n \rceil + j)/\alpha}\} - e^{-\gamma_n q^j}| &= |(1 - q^{\lceil \log_r n \rceil + j})^n - e^{-\gamma_n q^j}| \\ &= \left| \left(1 - \frac{\gamma_n q^j}{n}\right)^n - e^{-\gamma_n q^j} \right| \\ &= O(n^{-1}). \end{aligned}$$

Since the latter holds uniformly, that is,

$$\sup_{0 \leq y \leq nq} \left| \left(1 - \frac{y}{n}\right)^n - e^{-y} \right| = O(n^{-1}),$$

and

$$\mathbb{P}\{X_n^* = r^{k/\alpha}\} = \mathbb{P}\{X_n^* \leq r^{k/\alpha}\} - \mathbb{P}\{X_n^* \leq r^{(k-1)/\alpha}\},$$

the proof is complete.  $\square$

**Remark 1.** The random variables  $\alpha \log_r X_n^* - \lceil \log_r n \rceil$  have a limit distribution along subsequences  $\{n_k = \lfloor \gamma r^k \rfloor\}_{k \in \mathbb{N}}$ , with  $q < \gamma \leq 1$ , since using Lemma 1 above, as  $k \rightarrow \infty$

$$\mathbb{P}\{\alpha \log_r X_{n_k}^* - \lceil \log_r n_k \rceil = j\} \rightarrow e^{-\gamma q^j} (1 - e^{-\gamma(r-1)q^j}) = p_{j,\gamma}. \quad (9)$$

**Table 1.** Limit distribution of  $\log_2 X_{n_k}^* - \lfloor \log_2 n_k \rfloor$  in the classical case with  $\gamma = 1$ 

$j$	-2	-1	0	1	2	3	4	5
$p_{j,1}$	0.018	0.117	0.233	0.239	0.172	0.104	0.057	0.03

Table 1 contains the few largest values of  $p_{j,1}$ . This is the main part of the limit distribution, as  $\sum_{j=-2}^5 p_{j,1} \approx 0.943$ .

The asymptotic distribution (8) implies that  $\inf_n \text{Var}(\log_r X_n^*) > 0$ , while in the classical case Györfi and Kevei (Remark 2 in [14]) showed that  $\text{Var}(\log_2 S_n) = O(1/\log_2 n)$ .

**Remark 2.** Consider again the classical case. We note that the merging theorem (9) already appears in Földes [10]. Let  $\mu(n)$  be the longest tail-run after tossing a fair coin  $n$  times. Then Theorem 4 in [10] states that for any integer  $j$

$$\mathbb{P}\{\mu(n) - \lfloor \log_2 n \rfloor < j\} = e^{-2^{-(j+1-\lfloor \log_2 n \rfloor)}} + o(1).$$

Since each single St. Petersburg game lasts till to the first heads, in our setup we are tossing the coin until a random time, until heads appears  $n$  times. Thus, the number of tosses has a negative binomial distribution with parameter  $n$ . Moreover, the values  $(\log_2 X_k) - 1$ ,  $k = 1, 2, \dots, n$ , are the number of tails between two consecutive heads, therefore the quantity  $\log_2 X_n^* - 1$  can be thought as the longest tail-run in this coin tossing sequence.

We investigate the conditional distribution of  $S_n$  given that  $X_n^* = r^{k/\alpha}$ . The following lemma determines this conditional distribution. The statement for continuous random variables is much simpler, as in that case the maximum value is almost surely unique, and so  $M_n = 1$  a.s. (see the definition below). For the continuous version, see Lemma 2.1 in [9].

**Lemma 2.** Let  $Y, Y_1, \dots, Y_n$  be discrete i.i.d. random variables with possible values  $\{y_1, y_2, \dots\}$ ,  $y_1 < y_2 < \dots$ . Put

$$G_k(y) = \mathbb{P}\{Y \leq y | Y \leq y_k\}.$$

Put  $Z_n = Y_1 + \dots + Y_n$  for the partial sum,  $Y_n^* = \max\{Y_1, \dots, Y_n\}$  for the partial maximum, and  $M_n = |\{k: 1 \leq k \leq n, Y_k = Y_n^*\}|$  for the multiplicity of the maximum. Then given that  $Y_n^* = y_k$  and  $M_n = m$

$$Z_n \stackrel{\mathcal{D}}{=} my_k + Z_{n-m}^{(k-1)},$$

where  $Z_n^{(k-1)} = Y_1^{(k-1)} + \dots + Y_n^{(k-1)}$ , with  $Y_1^{(k-1)}, \dots, Y_n^{(k-1)}$  are i.i.d. with distribution function  $G_{k-1}$ .



**Proof.** We have

$$\begin{aligned}
& \mathbb{P}\{Z_n \leq y | Y_n^* = y_k, M_n = m\} \\
&= \frac{\mathbb{P}\{Z_n \leq y, Y_n^* = y_k, M_n = m\}}{\mathbb{P}\{Y_n^* = y_k, M_n = m\}} \\
&= \frac{1}{\mathbb{P}\{Y_n^* = y_k, M_n = m\}} \binom{n}{m} \\
&\quad \times \mathbb{P}\left\{Y_1 = \dots = Y_m = y_k, \sum_{j=m+1}^n Y_j \leq y - my_k, \max\{Y_{m+1}, \dots, Y_n\} < y_k\right\} \\
&= \frac{\binom{n}{m} \mathbb{P}\{Y = y_k\}^m \mathbb{P}\{Y \leq y_{k-1}\}^{n-m}}{\mathbb{P}\{Y_n^* = y_k, M_n = m\}} \\
&\quad \times \mathbb{P}\left\{\sum_{j=m+1}^n Y_j \leq y - my_k \mid \max\{Y_j, j = m+1, \dots, n\} \leq y_{k-1}\right\} \\
&= G_{k-1}^{*(n-m)}(y - my_k),
\end{aligned}$$

as stated.  $\square$

Put

$$N_n = |\{k: 1 \leq k \leq n, X_k = X_n^*\}|.$$

According to the previous lemma in order to analyze the conditional behavior of  $S_n$ , we first have to understand the behavior of  $N_n$ .

**Lemma 3.** *The conditional generating function of  $N_n$  given  $X_n^*$  is*

$$g_{k,n}(s) = \mathbb{E}[s^{N_n} | X_n^* = r^{k/\alpha}] = \frac{(1 - q^{k-1}(1 - ps))^n - (1 - q^{k-1})^n}{(1 - q^k)^n - (1 - q^{k-1})^n}, \quad (10)$$

and the generating function of  $N_n$  is

$$g_n(s) = \mathbb{E}[s^{N_n}] = \sum_{k=1}^{\infty} [(1 - q^{k-1}(1 - ps))^n - (1 - q^{k-1})^n].$$

**Proof.** Simply

$$\begin{aligned}
\mathbb{P}\{N_n = m | X_n^* = r^{k/\alpha}\} &= \frac{\mathbb{P}\{N_n = m, X_n^* = r^{k/\alpha}\}}{\mathbb{P}\{X_n^* = r^{k/\alpha}\}} \\
&= \frac{\binom{n}{m} (q^{k-1}p)^m (1 - q^{k-1})^{n-m}}{(1 - q^k)^n - (1 - q^{k-1})^n}.
\end{aligned} \quad (11)$$

Therefore, by the binomial theorem the conditional generating function is

$$\begin{aligned} g_{k,n}(s) &= \sum_{m=1}^n s^m \frac{\binom{n}{m} (q^{k-1}p)^m (1-q^{k-1})^{n-m}}{(1-q^k)^n - (1-q^{k-1})^n} \\ &= \frac{1}{(1-q^k)^n - (1-q^{k-1})^n} [(sq^{k-1}p + 1 - q^{k-1})^n - (1 - q^{k-1})^n]. \end{aligned}$$

The unconditional version follows from the law of total probability.  $\square$

The distribution of  $N_n$  in the classical case is calculated by Gut and Martin-Löf, in particular formula (11) is formula (4.1) in [13]. Moreover, in (4.3) in [13] they determine the asymptotic behavior of  $N_n$  conditioned on typical maximum along geometric subsequences. This is formula (13) in the next proposition in the general merging framework.

Now we can determine the asymptotic behavior of  $N_n$ .

**Proposition 1.** *Conditionally on  $X_n^* = r^{k_n/\alpha}$ , where  $\log_r n - k_n \rightarrow \infty$*

$$\frac{N_n - \mathbb{E}[N_n | X_n^* = r^{k_n/\alpha}]}{\sqrt{\text{Var}(N_n | X_n^* = r^{k_n/\alpha})}} \xrightarrow{\mathcal{D}} N(0, 1) \quad \text{as } n \rightarrow \infty. \quad (12)$$

*Conditionally on  $X_n^* = r^{(\lceil \log_r n \rceil + j)/\alpha}$ ,  $j \in \mathbb{Z}$ ,*

$$\lim_{n \rightarrow \infty} |g_{\lceil \log_r n \rceil + j, n}(s) - h_{j, \gamma_n}(s)| = 0, \quad s \in [0, 1], \quad (13)$$

where

$$h_{j, \gamma}(s) = \frac{e^{-(1-ps)\gamma q^{j-1}} - e^{-\gamma q^{j-1}}}{e^{-\gamma q^j} - e^{-\gamma q^{j-1}}}, \quad (14)$$

*is the generating function of a Poisson( $pq^{j-1}\gamma$ ) random variable conditioned on not being zero. While, if  $k_n - \log_r n \rightarrow \infty$  then conditionally on  $X_n^* = r^{k_n/\alpha}$*

$$N_n \xrightarrow{\mathbb{P}} 1 \quad \text{as } n \rightarrow \infty. \quad (15)$$

That is, we have three different regimes. In the typical range, there are several random variables equal to the maximal value and the number of these observations is distributed according to  $h_{j, \gamma_n}$ . When the maximum is smaller than it should be, then there are a lot of maximum values, while for too big values there is a single maximal observation.

**Proof of Proposition 1.** Differentiating  $g_{k,n}$  in (10), we obtain

$$\mathbb{E}[N_n | X_n^* = r^{k/\alpha}] = \frac{nq^{k-1}p(1-q^k)^{-1}}{1 - (1 - ((pq^{k-1})/(1-q^k)))^n}. \quad (16)$$

First, we consider the case  $\log_r n - k_n \rightarrow \infty$ . Then

$$\left(1 - \frac{pq^{k_n-1}}{1 - q^{k_n}}\right)^n \rightarrow 0, \quad (17)$$

therefore

$$\mathbb{E}[N_n | X_n^* = r^{k_n/\alpha}] \sim \frac{nq^{k_n-1}p}{1 - q^{k_n}} =: \mu_{n,k_n}. \quad (18)$$

(Note that we do not assume that  $k_n \rightarrow \infty$  only that  $\log_r n - k_n \rightarrow \infty$ .) Using the simple identity that  $\text{Var}(N_n | X_n^* = r^{k_n/\alpha}) = g''_{k,n}(1) + g'_{k,n}(1) - (g'_{k,n}(1))^2$ , similar computation gives

$$\text{Var}(N_n | X_n^* = r^{k_n/\alpha}) \sim \frac{npq^{k_n-1}}{1 - q^{k_n}} \left(1 - \frac{pq^{k_n-1}}{1 - q^{k_n}}\right) =: \sigma_{n,k_n}^2. \quad (19)$$

Substituting into formula (10), we have

$$\begin{aligned} & \mathbb{E}[e^{it(N_n - \mu_{n,k_n})/\sigma_{n,k_n}} | X_n^* = r^{k_n/\alpha}] \\ &= e^{-it\mu_{n,k_n}/\sigma_{n,k_n}} \frac{(1 - q^{k_n-1}(1 - pe^{it/\sigma_{n,k_n}}))^n - (1 - q^{k_n-1})^n}{(1 - q^{k_n})^n - (1 - q^{k_n-1})^n} \\ &= e^{-it\mu_{n,k_n}/\sigma_{n,k_n}} \frac{(1 - (pq^{k_n-1}(1 - e^{it/\sigma_{n,k_n}})/(1 - q^{k_n})))^n - (1 - (pq^{k_n-1}/(1 - q^{k_n})))^n}{1 - (1 - ((pq^{k_n-1})/(1 - q^{k_n})))^n}. \end{aligned}$$

By (17), we have to determine the limit of

$$e^{-it\mu_{n,k_n}/\sigma_{n,k_n}} \left(1 - \frac{pq^{k_n-1}(1 - e^{it/\sigma_{n,k_n}})}{1 - q^{k_n}}\right)^n. \quad (20)$$

Notice that

$$1 - \frac{pq^{k_n-1}(1 - e^{it})}{1 - q^{k_n}}$$

is the characteristic function of a  $0/1$  Bernoulli( $pq^{k_n-1}/(1 - q^{k_n})$ ) random variable, and from (18) and (19) we see that  $\mu_{n,k_n}$  and  $\sigma_{n,k_n}^2$  is the mean and the variance of the sum, and so (20) is exactly the characteristic function of a properly centered and normed sum of i.i.d. random variables. Since  $\sigma_{n,k_n} \rightarrow \infty$ , a simple application of the Lindeberg–Feller theorem shows that the limit is  $e^{-t^2/2}$ , the characteristic function of the standard normal distribution. This proves (12).

We turn to the case of typical maximum. For any  $j \in \mathbb{Z}$

$$(1 - q^{\lceil \log_r n \rceil + j - 1}(1 - ps))^n = \left(1 - \frac{\gamma_n q^{j-1}(1 - ps)}{n}\right)^n \sim e^{-(1-ps)\gamma_n q^{j-1}},$$

and (13) follows.

For (15), it is easy to check that the expectation in (16) tends to 1, whenever  $k_n - \log_r n \rightarrow \infty$ . Since  $N_n \geq 1$ , the statement follows.  $\square$

For  $j \in \mathbb{Z}$  and  $m \geq 1$ , let denote

$$r_{j,\gamma}(m) = \frac{(pq^{j-1}\gamma)^m}{m!} (e^{pq^{j-1}\gamma} - 1)^{-1}.$$

Then

$$h_{j,\gamma}(s) = \sum_{m=1}^{\infty} r_{j,\gamma}(m) s^m.$$

From (13), we obtain that

$$\lim_{n \rightarrow \infty} \max_{1 \leq m \leq n} |\mathbb{P}\{N_n = m | X_n^* = r^{(\lfloor \log_r n \rfloor + j)/\alpha}\} - r_{j,\gamma_n}(m)| = 0. \quad (21)$$

As a consequence of Proposition 1 and Lemma 1, we obtain the unconditional asymptotic behavior of  $N_n$ , which also can be described through a merging phenomenon.

**Corollary 1.** *Let us denote*

$$h_\gamma(s) = \sum_{j=-\infty}^{\infty} (e^{-(1-ps)\gamma q^{j-1}} - e^{-\gamma q^{j-1}}).$$

*Then for the generating function of  $N_n$  we have*

$$\lim_{n \rightarrow \infty} |g_n(s) - h_{\gamma_n}(s)| = 0, \quad s \in [0, 1].$$

Given that  $X \leq r^{k/\alpha}$  for  $i \leq k$  we have  $\mathbb{P}\{X = r^{i/\alpha} | X \leq r^{k/\alpha}\} = pq^{i-1}/(1-q^k)$ . Introduce the corresponding distribution function

$$F_k(x) = \mathbb{P}\{X \leq x | X \leq r^{k/\alpha}\} = \begin{cases} \frac{1}{1-q^k} \left[ 1 - \frac{r^{\{\alpha \log_r x\}}}{x^\alpha} \right], & \text{for } x \in [r^{1/\alpha}, r^{k/\alpha}], \\ 1, & \text{for } x > r^{k/\alpha}. \end{cases} \quad (22)$$

In the following  $X^{(k)}, X_1^{(k)}, \dots$ , are i.i.d. random variables with distribution function  $F_k$ , and

$$S_n^{(k)} = X_1^{(k)} + \dots + X_n^{(k)} \quad (23)$$

stands for their partial sums. By Lemma 2, conditioning on  $X_n^* = r^{k/\alpha}$ ,  $N_n = m$

$$S_n \stackrel{\mathcal{D}}{=} mr^{k/\alpha} + \sum_{i=1}^{n-m} X_i^{(k-1)} = mr^{k/\alpha} + S_{n-m}^{(k-1)}. \quad (24)$$

Calculating the moments we obtain

$$\mathbb{E}(X^{(k)})^\ell = \frac{1}{1-q^k} \sum_{i=1}^k r^{(i\ell)/\alpha} q^{i-1} p \begin{cases} \frac{pr^{\ell/\alpha}}{1-q^k} \frac{r^{(\ell/\alpha-1)k} - 1}{r^{\ell/\alpha-1} - 1}, & \text{for } \ell \neq \alpha, \\ \frac{pr}{1-q^k} k, & \text{for } \ell = \alpha. \end{cases} \quad (25)$$

Note that for  $\alpha > \ell$  the truncated  $\ell$ th moment converges to  $\mathbb{E}X^\ell$  as  $k \rightarrow \infty$ , while in other cases the series diverges.

According to Lemma 1, the typical values for  $X_n^* = r^{k_n/\alpha}$  are of the form  $r^{(\lceil \log_r n \rceil + j)/\alpha}$ , for some  $j \in \mathbb{Z}$ . Therefore, the case  $r^{k_n}/n \rightarrow 0$  corresponds to small maximum, and  $r^{k_n}/n \rightarrow \infty$  corresponds to large one. In what follows, we determine the asymptotic behavior of the sum conditioned on small, typical and large maximum.

## 2.2. Conditioning on small maximum

From (24), we see that conditioning on the maximum value  $S_n$  is a sum of random number of i.i.d. random variables. Moreover, (12) says that conditioning on a small maximum  $N_n$  is asymptotically normal. To obtain limit distribution for random number of i.i.d. random variables, first we have to determine the behavior of the sum of  $n$  i.i.d. random variables.

The following proposition is the conditional counterpart of Theorem 4 in [14] (there only the classical case is treated), which states that for the sum of truncated variables at  $c_n$  the central limit theorem holds if and only if  $c_n/n \rightarrow 0$ . The proof is also similar, therefore we only sketch it.

If we condition on  $X_n^* = r^{1/\alpha}$ , then all the variables are degenerate, so we exclude this case in the following statement. Recall definitions (22), (23) and the notation after it.

**Proposition 2.** For  $\alpha \in (0, 2)$ ,  $k_n \geq 2$

$$\frac{S_n^{(k_n)} - \mathbb{E}S_n^{(k_n)}}{\sqrt{\text{Var}(S_n^{(k_n)})}} \xrightarrow{\mathcal{D}} N(0, 1) \quad (26)$$

if and only if  $\log_r n - k_n \rightarrow \infty$ .

**Proof.** We may assume that  $k_n \rightarrow \infty$ . From equation (25), we have that for any  $\alpha \in (0, 2)$

$$(\mathbb{E}X^{(k)})^2 = o(\mathbb{E}(X^{(k)})^2) \quad \text{as } k \rightarrow \infty, \quad (27)$$

therefore

$$\text{Var } X^{(k_n)} \sim \frac{pr^{2/\alpha}}{r^{2/\alpha-1} - 1} r^{(2/\alpha-1)k_n}.$$

Thus for the variance of the sum

$$s_n^2 = \text{Var } S_n^{(k_n)} = n \text{Var } X^{(k_n)} \sim n \frac{pr^{2/\alpha}}{r^{2/\alpha-1} - 1} r^{(2/\alpha-1)k_n}. \quad (28)$$

By the Lindeberg–Feller central limit theorem

$$\frac{S_n^{(k_n)} - \mathbb{E}S_n^{(k_n)}}{s_n} \xrightarrow{\mathcal{D}} N(0, 1)$$

holds if and only if for every  $\varepsilon > 0$

$$L_n(\varepsilon) = \frac{n}{s_n^2} \int_{\{|X^{(k_n)} - \mathbb{E}X^{(k_n)}| > \varepsilon s_n\}} (X^{(k_n)} - \mathbb{E}X^{(k_n)})^2 d\mathbb{P} \rightarrow 0.$$

By (27), it is easy to show that

$$L_n(\varepsilon) \sim \frac{n}{s_n^2} \int_{\{X^{(k_n)} > \varepsilon s_n\}} (X^{(k_n)})^2 d\mathbb{P}.$$

If  $r^{k_n}/n \rightarrow 0$ , then by (28) the domain of integration in  $L_n(\varepsilon)$  is empty for large  $n$ , therefore Lindeberg's condition holds.

While if  $r^{k_n}/n > \varepsilon$  for some  $\varepsilon > 0$  and  $n$ , then by (28) we have  $r^{k_n/\alpha} - \mathbb{E}X^{(k_n)} > \varepsilon' s_n$  for some  $\varepsilon'$ , thus the last jump of  $X^{(k_n)}$  belongs to the domain of integration. Therefore,

$$L_n(\varepsilon') \geq \frac{n}{s_n^2} r^{2k_n/\alpha} q^{k_n-1} p > \frac{1}{2} \frac{r^{2/\alpha-1} - 1}{r^{2/\alpha-1}}.$$

The proof is complete.  $\square$

Therefore CLT holds for the random index  $N_n$  (see (12)) and also for the corresponding deterministic term sums (previous proposition). Combining these two results the general theory for random sums (Theorem 4.1.1 in Gnedenko and Korolev [12]) implies the following.

**Proposition 3.** *Let  $\alpha \in (0, 2)$ . Given that  $X_n^* = r^{k_n/\alpha}$ ,  $k_n \geq 2$ , such that  $\log_r n - k_n \rightarrow \infty$*

$$\frac{S_n - \mathbb{E}[S_n | X_n^* = r^{k_n/\alpha}]}{\sqrt{\text{Var}(S_n | X_n^* = r^{k_n/\alpha})}} \xrightarrow{\mathcal{D}} N(0, 1). \quad (29)$$

**Proof.** By (24) given that  $X_n^* = r^{k/\alpha}$  we may write

$$S_n \stackrel{\mathcal{D}}{=} N_n r^{k/\alpha} + S_{n-N_n}^{(k-1)} = n r^{k/\alpha} + \sum_{i=1}^{n-N_n} (X_i^{(k-1)} - r^{k/\alpha}).$$

We apply Theorem 4.1.1 in [12] to the triangular array

$$\left\{ \frac{X_1^{(k_n-1)} - r^{k_n/\alpha}}{\sqrt{\text{Var } S_n^{(k_n-1)}}}, \dots, \frac{X_n^{(k_n-1)} - r^{k_n/\alpha}}{\sqrt{\text{Var } S_n^{(k_n-1)}}} \right\}_{n \geq 1}.$$

By Proposition 2

$$\sum_{i=1}^n \frac{X_i^{(k_n-1)} - r^{k_n/\alpha}}{\sqrt{\text{Var } S_n^{(k_n-1)}}} - \frac{n(\mathbb{E}X^{(k_n-1)} - r^{k_n/\alpha})}{\sqrt{\text{Var } S_n^{(k_n-1)}}} \xrightarrow{\mathcal{D}} N(0, 1),$$

that is condition (1.1) on page 93 in [12] holds. First assume that either  $k_n \rightarrow k$  for some  $k \in \mathbb{N}$ , or  $k_n \rightarrow \infty$ . Put  $u = 1 - \lim_{n \rightarrow \infty} \frac{q^{k_n-1}p}{1-q^{k_n}}$ . Using (19)

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{r^{k_n/\alpha} - \mathbb{E}X^{(k_n-1)}}{\sqrt{\text{Var } S_n^{(k_n-1)}}} \sqrt{\text{Var}(N_n | X_n^* = r^{k_n/\alpha})} \\ &= \lim_{n \rightarrow \infty} \frac{r^{k_n/\alpha} - \mathbb{E}X^{(k_n-1)}}{\sqrt{\text{Var } X^{(k_n-1)}}} \sqrt{\frac{pq^{k_n-1}}{1-q^{k_n}} \left(1 - \frac{pq^{k_n-1}}{1-q^{k_n}}\right)} =: v, \end{aligned} \tag{30}$$

and the latter limit exists both for  $k_n \equiv k$  and for  $k_n \rightarrow \infty$ . Using (12)

$$\left( \frac{n - N_n}{n}, \frac{n(\mathbb{E}X^{(k_n-1)} - r^{k_n/\alpha})}{\sqrt{\text{Var } S_n^{(k_n-1)}}} \frac{n - N_n}{n} - c_n \right) \xrightarrow{\mathcal{D}} (u, vZ),$$

where  $Z$  is a standard normal random variable and

$$c_n = -(n - \mathbb{E}[N_n | X_n^* = r^{k_n/\alpha}]) \frac{r^{k_n/\alpha} - \mathbb{E}X^{(k_n-1)}}{\sqrt{\text{Var } S_n^{(k_n-1)}}}.$$

That is, condition (1.9) on page 96 in [12] holds, so Theorem 4.1.1 applies, and we obtain that given  $X_n^* = r^{k_n/\alpha}$

$$\frac{\sum_{i=1}^{n-N_n} (X_i^{(k_n-1)} - r^{k_n/\alpha})}{\sqrt{\text{Var } S_n^{(k_n-1)}}} - c_n \xrightarrow{\mathcal{D}} N(0, v^2 + u).$$

Using (24), standard calculation gives that

$$\mathbb{E}[S_n | X_n^* = r^{k/\alpha}] = n\mathbb{E}X^{(k-1)} + \mathbb{E}[N_n | X_n^* = r^{k/\alpha}](r^{k/\alpha} - \mathbb{E}X^{(k-1)}),$$

and

$$\begin{aligned} \text{Var}(S_n | X_n^* = r^{k/\alpha}) &= \text{Var}(N_n | X_n^* = r^{k/\alpha})(r^{k/\alpha} - \mathbb{E}X^{(k-1)})^2 \\ &\quad + (n - \mathbb{E}[N_n | X_n^* = r^{k/\alpha}]) \text{Var } X^{(k-1)}. \end{aligned}$$

Substituting back the asymptotics (18) and using (30) we get that

$$\lim_{n \rightarrow \infty} \frac{\text{Var } S_n^{(k_n-1)}}{\text{Var}(S_n | X_n^* = r^{k_n/\alpha})} = \frac{1}{v^2 + u}.$$

Summarizing, we obtain (29).

Now let  $k_n$  be an arbitrary sequence. From any subsequence  $\{n'\}$ , one can choose a further subsequence  $\{n''\}$ , such that either  $k_{n''} \rightarrow k \in \mathbb{N}$  or  $k_{n''} \rightarrow \infty$  holds, and so on this subsequence the convergence takes place. This is equivalent to (29).  $\square$

**Remark 3.** Without proof, we note that convergence of moments also hold both in (29) and in (26). In view of the distributional convergence, it is enough (in fact equivalent) to show the uniform integrability of arbitrary powers of the corresponding random variables.

Using Chernoff's bounding technique, one can prove exponential bounds for the tail probabilities

$$\mathbb{P}\{S_n^{(k)} - \mathbb{E}S_n^{(k)} > n^{1/\alpha}x\},$$

from which uniform integrability follows. These bounds and a detailed proof of the statement will be published elsewhere, as a continuation of the present paper.

For  $\alpha > 2$ , the variance is finite thus usual central limit theorem holds without conditioning. As it was pointed out in the introduction, for  $\alpha = 2$  the generalized St. Petersburg(2,  $p$ ) distribution has infinite variance, but it is still in the domain of attraction of the normal law. However, the normalizing sequence is  $\sqrt{prn \log_r n}$ , therefore it is meaningful to ask what is the necessary and sufficient condition for (26).

**Proposition 4.** *Let  $\alpha = 2$ . Then (26) holds if and only if*

$$\liminf_{n \rightarrow \infty} \frac{\log_r n}{k_n} \geq 1. \quad (31)$$

Note that the condition is much weaker than the condition for  $\alpha \in (0, 2)$ . In particular, it also covers the typical case  $k_n \sim \log_r n$ , and part of the large maximum case.

**Proof of Proposition 4.** The proof is exactly the same as in the  $\alpha < 2$  case, the only difference is the variance asymptotic.

We again assume that  $k_n \rightarrow \infty$ . From equation (25), we have for the variance of the sum

$$s_n^2 = \text{Var } S_n^{(k_n)} = n \text{Var } X^{(k_n)} \sim \frac{p}{q} n k_n. \quad (32)$$

By the Lindeberg–Feller theorem, CLT holds if and only if  $L_n(\varepsilon) \rightarrow 0$  for any  $\varepsilon > 0$ . We have

$$L_n(\varepsilon) \sim \frac{n}{s_n^2} \int_{\{X^{(k_n)} > \varepsilon s_n\}} (X^{(k_n)})^2 d\mathbb{P}$$



$$= \frac{1}{k_n} |\{k: r^{k/2} > \varepsilon s_n; k \leq k_n\}| = \frac{1}{k_n} \left( k_n - \left\lfloor \log_r \frac{\varepsilon^2 p n k_n}{q} \right\rfloor \right)_+,$$

and the latter goes to 0 if and only if

$$\liminf_{n \rightarrow \infty} \frac{\log_r(nk_n)}{k_n} \geq 1.$$

Since  $(\log_r k_n)/k_n \rightarrow 0$  this is equivalent to (31).  $\square$

### 2.3. Conditioning on typical maximum

According to Lemma 1, the typical value for  $X_n^*$  is  $r^{(\lceil \log_r n \rceil + j)/\alpha}$ ,  $j \in \mathbb{Z}$ . In the following, we investigate this case. Since semistability appears, first we briefly define the semistable distributions, and summarize their most important properties. For background, we refer to Meerschaert and Scheffler [18] and Megyesi [19] and the references therein.

Let  $Y$  be an infinitely divisible random variable with characteristic function  $\phi(t) = \mathbb{E}(e^{itY})$  in its Lévy form ([11], page 70), given for each  $t \in \mathbb{R}$  by

$$\phi(t) = \exp \left\{ it\theta - \frac{\sigma^2}{2} t^2 + \int_{-\infty}^0 \beta_t(x) dL(x) + \int_0^{\infty} \beta_t(x) dR(x) \right\},$$

where

$$\beta_t(x) = e^{itx} - 1 - \frac{itx}{1+x^2}.$$

We describe semistable laws in the present framework as follows: an infinitely divisible law is semistable if and only if either it is normal (as a semistable distribution of exponent 2), or there exist nonnegative bounded functions  $M_L(\cdot)$  on  $(-\infty, 0)$  and  $M_R(\cdot)$  on  $(0, \infty)$ , one of which has strictly positive infimum and the other one either has strictly positive infimum or is identically zero, such that  $L(x) = M_L(x)/|x|^\alpha$ ,  $x < 0$ , is left-continuous and non-decreasing on  $(-\infty, 0)$  and  $R(x) = -M_R(x)/x^\alpha$ ,  $x > 0$ , is right-continuous and non-decreasing on  $(0, \infty)$  and  $M_L(c^{1/\alpha}x) = M_L(x)$  for all  $x < 0$  and  $M_R(c^{1/\alpha}x) = M_R(x)$  for all  $x > 0$ , with the same period  $c > 1$ .

The following theorem of Kruglov [16] highlights the importance of semistability. Let  $Y_1, Y_2, \dots$  be independent and identically distributed random variables with the common distribution function  $G$ . If for some centering and norming constants  $c_{n_k} \in \mathbb{R}$  and  $a_{n_k} > 0$  the convergence in distribution

$$\frac{1}{a_{n_k}} \left( \sum_{j=1}^{n_k} Y_j - c_{n_k} \right) \xrightarrow{\mathcal{D}} W \quad (33)$$

holds along a subsequence  $\{n_k\}_{k=1}^\infty \subset \mathbb{N}$  satisfying

$$\lim_{k \rightarrow \infty} \frac{n_{k+1}}{n_k} = c \quad \text{for some } c \in [1, \infty), \quad (34)$$

then the non-degenerate limit  $W$  is necessarily semistable. When the exponent  $\alpha < 2$ , the  $c$  in the common multiplicative period of  $M_L(\cdot)$  and  $M_R(\cdot)$  is the  $c$  from the latter growth condition on  $\{n_k\}$ . Conversely, for an arbitrary semistable distribution there exists a distribution function  $G$  for which (33) holds along some  $\{n_k\} \subset \mathbb{N}$  satisfying (34).

Now we turn to the asymptotic behavior of  $S_n^{(\lceil \log_r n \rceil + j)}$  defined in (23). Recall the definition of  $\mu_n$  in (3).

**Proposition 5.** *Let  $\alpha \in (0, 2)$ ,  $j \in \mathbb{Z}$ . The centered and normed sum*

$$\frac{S_{n_k}^{(\lceil \log_r n \rceil + j)}}{n_k^{1/\alpha}} - \mu_{n_k}$$

*converges in distribution if and only if  $\gamma_{n_k} \rightarrow \gamma$ , for some  $\gamma \in [q, 1]$ . In this case the limit  $W_{j,\gamma}$  has characteristic function*

$$\varphi_{j,\gamma}(t) = \mathbb{E} e^{itW_{j,\gamma}} = \exp \left[ it u_{j,\gamma} + \int_0^\infty (e^{itx} - 1 - itx) dL_{j,\gamma}(x) \right], \quad (35)$$

*with*

$$L_{j,\gamma}(x) = \begin{cases} \gamma q^j - \frac{r^{\{\log_r(\gamma x^\alpha)\}}}{x^\alpha}, & \text{for } x < r^{j/\alpha} \gamma^{-1/\alpha}, \\ 0, & \text{for } x \geq r^{j/\alpha} \gamma^{-1/\alpha}, \end{cases} \quad (36)$$

*and*

$$u_{j,\gamma} = \begin{cases} \frac{pr^{1/\alpha}}{r^{1/\alpha-1} - 1} r^{j(\alpha^{-1}-1)} \gamma^{1-\alpha^{-1}}, & \alpha \neq 1, \\ pr \log_r \frac{r^j}{\gamma}, & \alpha = 1. \end{cases} \quad (37)$$

Note that the random variables  $W_{j,q}$  and  $W_{j+1,1}$  have the same distribution. This implies that when the set of limit points of the sequence  $\{\gamma_{n_k}\}_{k \in \mathbb{N}}$  is  $\{q, 1\}$  then convergence in distribution does not hold, contrary to the unconditional case described after (7).

**Proof of Proposition 5.** Recall the notation in (22). According to Theorem 25.1 in Gnedenko and Kolmogorov [11] the centered and normalized sum  $S_n^{(\lceil \log_r n \rceil + j)} / n^{1/\alpha} - A_n$  converges in distribution with some  $A_n$  along the subsequence  $\{n_k\}$  if and only if

$$n_k [1 - F_{\lceil \log_r n_k \rceil + j}(n_k^{1/\alpha} x)] \quad \text{converges} \quad (38)$$

and

$$n_k F_{\lceil \log_r n_k \rceil + j}(-n_k^{1/\alpha} x) \quad \text{converges,} \quad (39)$$

for any  $x > 0$ , which is a continuity point of the corresponding limit function, and

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \limsup_{k \rightarrow \infty} n_k \int_{|x| \leq \varepsilon} x^2 dF_{\lceil \log_r n_k \rceil + j}(n_k^{1/\alpha} x) \\ &= \lim_{\varepsilon \rightarrow 0} \liminf_{k \rightarrow \infty} n_k \int_{|x| \leq \varepsilon} x^2 dF_{\lceil \log_r n_k \rceil + j}(n_k^{1/\alpha} x) = \sigma^2. \end{aligned} \quad (40)$$

Condition (39) holds for any subsequence with 0 as the limit function. Using (22) for  $x < r^{j/\alpha} / \gamma_{n_k}^{1/\alpha}$

$$\begin{aligned} n_k [1 - F_{\lceil \log_r n_k \rceil + j}(n_k^{1/\alpha} x)] &= \frac{-n_k q^{\lceil \log_r n_k \rceil + j}}{1 - q^{\lceil \log_r n_k \rceil + j}} + \frac{r^{\{\log_r(n_k x^\alpha)\}} x^{-\alpha}}{1 - q^{\lceil \log_r n_k \rceil + j}} \\ &= \frac{-q^j \gamma_{n_k} + r^{\{\log_r(n_k x^\alpha)\}} x^{-\alpha}}{1 - q^{\lceil \log_r n_k \rceil + j}}, \end{aligned}$$

thus condition (38) reduces to the convergence of

$$-\frac{\gamma_{n_k}}{r^j} + \frac{r^{\{\log_r(n_k x^\alpha)\}}}{x^\alpha}$$

for  $x < r^{j/\alpha} / \gamma_{n_k}^{1/\alpha}$ , which is a continuity point of the limit. This holds if and only if  $\gamma_{n_k}$  converges to some  $\gamma \in [q, 1]$ , in which case the limit function is  $L_{j,\gamma}$  in (36), as stated.

Finally, for condition (40) assume that  $\varepsilon < r^{j/\alpha}$ . Then

$$\begin{aligned} n \int_{|x| \leq \varepsilon} x^2 dF_{\lceil \log_r n \rceil + j}(n^{1/\alpha} x) &= n^{1-2/\alpha} \int_{|y| \leq \varepsilon n^{1/\alpha}} y^2 dF_{\lceil \log_r n \rceil + j}(y) \\ &= n^{1-2/\alpha} \sum_{k: r^{k/\alpha} \leq \varepsilon n^{1/\alpha}} r^{2k/\alpha} \frac{pq^{k-1}}{1 - q^{\lceil \log_r n \rceil + j}} \\ &\leq \frac{\varepsilon^{2-\alpha}}{q - q^{2/\alpha}}, \end{aligned}$$

for  $n$  large enough, which shows that (40) holds along the whole sequence with  $\sigma^2 = 0$ .

Theorem 25.1 in [11] states that the centering sequence  $A_{n,j}$  can be chosen as

$$A_{n,j} = n \int_{|x| \leq \tau} x dF_{\lceil \log_r n \rceil + j}(n^{1/\alpha} x),$$

for arbitrary  $\tau > 0$ . Let us choose  $\tau > r^{(j+1)/\alpha}$ . Then by (25)

$$A_{n,j} = n^{1-\alpha^{-1}} \int_0^{\tau n^{1/\alpha}} x dF_{\lceil \log_r n \rceil + j}(x) = n^{1-\alpha^{-1}} \mathbb{E} X^{(\lceil \log_r n \rceil + j)}$$

$$= \begin{cases} \frac{pr^{1/\alpha}}{r^{1/\alpha-1}-1} r^{j(\alpha^{-1}-1)} \gamma_n^{1-\alpha^{-1}} + o(1), & \alpha < 1, \\ pr(\lceil \log_r n \rceil + j) + o(1), & \alpha = 1, \\ n^{1-\alpha^{-1}} \mathbb{E}X - \frac{pr^{1/\alpha}}{1-r^{1/\alpha-1}} r^{j(\alpha^{-1}-1)} \gamma_n^{1-\alpha^{-1}} + o(1), & \alpha > 1, \end{cases}$$

where  $o(1) \rightarrow 0$  as  $n \rightarrow \infty$ . We obtain that whenever  $\gamma_{n_k} \rightarrow \gamma$

$$\frac{S_{n_k}^{(\lceil \log_r n_k \rceil + j)}}{n_k^{1/\alpha}} - A_{n,j} \xrightarrow{\mathcal{D}} \widetilde{W}_{j,\gamma},$$

where

$$\mathbb{E}e^{it\widetilde{W}_{j,\gamma}} = \exp \left[ \int_0^\infty (e^{itx} - 1 - itx) dL_{j,\gamma}(x) \right].$$

Recall the definition of  $\mu_n$  in (3). We have

$$\mu_n - A_{n,j} = \begin{cases} -\frac{pr^{1/\alpha}}{r^{1/\alpha-1}-1} r^{j(\alpha^{-1}-1)} \gamma_n^{1-\alpha^{-1}} + o(1), & \alpha < 1, \\ -pr(j + \log_r \gamma_n^{-1}) + o(1), & \alpha = 1, \\ \frac{pr^{1/\alpha}}{1-r^{1/\alpha-1}} r^{j(\alpha^{-1}-1)} \gamma_n^{1-\alpha^{-1}} + o(1), & \alpha > 1. \end{cases}$$

Therefore,

$$\frac{S_{n_k}^{(\lceil \log_r n_k \rceil + j)}}{n_k^{1/\alpha}} - \mu_{n_k} \xrightarrow{\mathcal{D}} \widetilde{W}_{j,\gamma} + u_{j,\gamma},$$

with the constant  $u_{j,\gamma}$  in (37), as stated.  $\square$

The Lévy function  $L_{j,\gamma}$  is a pure jump function with jumps at  $r^{k/\alpha} \gamma^{-1/\alpha}$ ,  $k \leq j$ , such that  $L_{j,\gamma}(r^{k/\alpha} \gamma^{-1/\alpha}) - L_{j,\gamma}(r^{k/\alpha} \gamma^{-1/\alpha} -) = \gamma p q^{k-1}$ , for  $k \leq j$ . Introduce the notation

$$G_{j,\gamma}(x) = \mathbb{P}\{W_{j,\gamma} \leq x\}.$$

The form of the Lévy function  $L_{j,\gamma}$  in (36) implies that for any  $j \in \mathbb{Z}$ ,  $\gamma \in [q, 1]$ , the support of  $W_{j,\gamma}$  is  $\mathbb{R}$  for  $\alpha \geq 1$ , while for  $\alpha < 1$  the support of  $W_{j,\gamma}$  is  $[0, \infty)$ , since

$$u_{j,\gamma} - \int_0^\infty x dL_{j,\gamma}(x) = 0,$$

is the drift of the corresponding Lévy process. Moreover, the exponential moments  $\mathbb{E}e^{\lambda W_{j,\gamma}}$  are finite for any  $\lambda > 0$ ,  $\alpha \in (0, 2)$  and  $j \in \mathbb{Z}$ ,  $\gamma \in [q, 1]$ , see, for example, Sato [23], Chapter 5.

The logarithm of the characteristic function of  $W_{j,\gamma}$  can be written as

$$\log \varphi_{j,\gamma}(t) = it u_{j,\gamma} + \sum_{k=-\infty}^j \left( e^{it r^{k/\alpha} / \gamma^{1/\alpha}} - 1 - it \frac{r^{k/\alpha}}{\gamma^{1/\alpha}} \right) \gamma p q^{k-1}.$$

Thus,

$$\Re \log \varphi_{j,\gamma}(t) = \sum_{k=-\infty}^j \left( \cos \frac{t r^{k/\alpha}}{\gamma^{1/\alpha}} - 1 \right) \gamma p q^{k-1}.$$

Put

$$\kappa_\gamma(t) = \left\lfloor \alpha \log_r \frac{\gamma^{1/\alpha} \pi}{2|t|} \right\rfloor.$$

The same way as in the proof of Lemma 3 in [5] one has that

$$\begin{aligned} \Re \log \varphi_{j,\gamma}(t) &= - \sum_{k=-\infty}^j \left( 1 - \cos \frac{t r^{k/\alpha}}{\gamma^{1/\alpha}} \right) \gamma p q^{k-1} \\ &\leq - \frac{4pt^2}{q\pi^2} \gamma^{1-2/\alpha} \sum_{k=-\infty}^{j \wedge \kappa_\gamma(t)} r^{(2/\alpha-1)k} \\ &\leq - \frac{4p\gamma^{1-2/\alpha}}{q\pi^2(1-q^{2/\alpha-1})} t^2 r^{(2/\alpha-1)(j \wedge \kappa_\gamma(t))} \\ &\leq \begin{cases} -c_{\gamma;1}|t|^\alpha, & |t| > T_\gamma r^{-j/\alpha}, \\ -c_{\gamma;2} r^{j(2/\alpha-1)} t^2, & |t| \leq T_\gamma r^{-j/\alpha}, \end{cases} \end{aligned}$$

where

$$\begin{aligned} c_{\gamma;1} &= \frac{2^\alpha p}{q\pi^\alpha(r^{2/\alpha-1}-1)}, & c_{\gamma;2} &= \frac{4p\gamma^{1-2/\alpha}}{q\pi^2(1-q^{2/\alpha-1})}, \\ T_\gamma &= \frac{\gamma^{1/\alpha}\pi}{2}. \end{aligned}$$

By standard Fourier analysis, this implies that  $G_{j,\gamma}$  is infinitely many times differentiable. In particular, by the density inversion formula we obtain for  $g_{j,\gamma}(x) = (G_{j,\gamma}(x))'$

$$\begin{aligned} g_{j,\gamma}(x) &\leq \frac{1}{2\pi} \int_{-\infty}^{\infty} |\varphi_{j,\gamma}(t)| dt \\ &\leq \frac{1}{\pi} \left( \int_0^{T_\gamma q^{j/\alpha}} e^{-c_{\gamma;2} r^{j(2/\alpha-1)} t^2} dt + \int_{T_\gamma q^{j/\alpha}}^{\infty} e^{-c_{\gamma;1} t^\alpha} dt \right) \end{aligned}$$

$$\leq \frac{r^{-j(1/\alpha-1/2)}}{2\sqrt{\pi c_{\gamma;2}}} + \frac{\Gamma(\alpha^{-1})}{\alpha\pi(c_{\gamma;1})^{1/\alpha}}.$$

Differentiating the characteristic function, we can compute the first two moments of the variable  $W_{j,\gamma}$ . A little calculation gives that

$$\mathbb{E}W_{j,\gamma} = u_{j,\gamma} \quad \text{and} \quad \mathbb{E}(W_{j,\gamma})^2 = (\mathbb{E}W_{j,\gamma})^2 + \frac{p}{q - q^{2/\alpha}} \gamma^{1-2/\alpha} r^{(2/\alpha-1)j} \quad (41)$$

and so

$$\text{Var } W_{j,\gamma} = \frac{p}{q - q^{2/\alpha}} \gamma^{1-2/\alpha} r^{(2/\alpha-1)j}.$$

As a simple corollary we obtain the following merging theorem.

**Corollary 2.** *On the whole sequence of natural numbers*

$$\sup_{x \in \mathbb{R}} \left| \mathbb{P} \left\{ \frac{S_n^{(\lceil \log_r n \rceil + j)}}{n^{1/\alpha}} - \mu_n \leq x \right\} - G_{j,\gamma_n}(x) \right| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (42)$$

**Proof.** The simple proof relies upon the same compactness reasoning as the proof of Theorem 2 in [8]. We show that any subsequence  $\{n'\}$  contains a further subsequence on which (42) holds.

Let  $\{n'\}$  be an arbitrary subsequence. The Bolzano–Weierstrass theorem allows us to choose a further subsequence  $\{n''\}$  such that  $\gamma_{n''} \rightarrow \gamma$ , for some  $\gamma \in [q, 1]$ . As  $\varphi_{j,\gamma_{n''}}(t) \rightarrow \varphi_{j,\gamma}(t)$ , by the continuity of  $G_{j,\gamma}$  for any  $j$  and  $\gamma$  we have that  $G_{j,\gamma_{n''}}(x) \rightarrow G_{j,\gamma}(x)$  for any  $x$ . Using Proposition 5 the statement follows.  $\square$

Now we turn to the conditional limit theorem.

**Proposition 6.** *For  $\alpha \in (0, 2)$ ,  $j \in \mathbb{Z}$  we have*

$$\sup_{x \in \mathbb{R}} \left| \mathbb{P} \left\{ \frac{S_n}{n^{1/\alpha}} - \mu_n \leq x \mid X_n^* = r^{(\lceil \log_r n \rceil + j)/\alpha} \right\} - \tilde{G}_{j,\gamma_n}(x) \right| \rightarrow 0, \quad (43)$$

where

$$\tilde{G}_{j,\gamma}(x) = \sum_{m=1}^{\infty} G_{j-1,\gamma} \left( x - m \frac{r^{j/\alpha}}{\gamma^{1/\alpha}} \right) r_{j,\gamma}(m). \quad (44)$$

**Remark 4.** For any  $j \in \mathbb{Z}$  let  $(W_{j-1,\gamma})_{\gamma \in [q,1]}$  be random variables with characteristic function  $\varphi_{j-1,\gamma}$  defined in (35), and independently let  $(M_{j,\gamma})_{\gamma \in [q,1]}$  be positive integer valued random variables with generating function  $h_{j,\gamma}$  in (14). Then conditioning on  $X_n^* = r^{(\lceil \log_r n \rceil + j)/\alpha}$  the sum  $\frac{S_n}{n^{1/\alpha}} - \mu_n$  is close in distribution to

$$U_{j,\gamma} = W_{j-1,\gamma} + M_{j,\gamma} r^{j/\alpha} / \gamma^{1/\alpha}. \quad (45)$$

In fact

$$\mathbb{P}\{W_{j-1,\gamma} + M_{j,\gamma} r^{j/\alpha} / \gamma^{1/\alpha} \leq x\} = \mathbb{P}\{U_{j,\gamma} \leq x\} = \tilde{G}_{j,\gamma}(x).$$

By (14)  $M_{j,\gamma}$  is a Poisson random variable conditioned on being nonzero, thus it has finite exponential moments for any  $j \in \mathbb{Z}$  and  $\gamma \in [q, 1]$ . Moreover,  $W_{j-1,\gamma}$  and  $M_{j,\gamma}$  are independent,  $W_{j-1,\gamma}$  has finite exponential moments, therefore  $U_{j,\gamma}$  also has finite exponential moments. We can easily determine the moments of  $U_{j,\gamma}$ . We have

$$\mathbb{E}M_{j,\gamma} = \sum_{m=1}^{\infty} m r_{j,\gamma}(m) = \frac{pq^{j-1}\gamma e^{pq^{j-1}\gamma}}{e^{pq^{j-1}\gamma} - 1},$$

and

$$\text{Var } M_{j,\gamma} = \mathbb{E}M_{j,\gamma}^2 - (\mathbb{E}M_{j,\gamma})^2 = \frac{pq^{j-1}\gamma e^{pq^{j-1}\gamma}}{e^{pq^{j-1}\gamma} - 1} - \frac{(pq^{j-1}\gamma)^2 e^{pq^{j-1}\gamma}}{(e^{pq^{j-1}\gamma} - 1)^2}.$$

Therefore, by (45)

$$\begin{aligned} \mathbb{E}U_{j,\gamma} &= \mathbb{E}W_{j-1,\gamma} + \mathbb{E}M_{j,\gamma} \frac{r^{j/\alpha}}{\gamma^{1/\alpha}}, \\ \text{Var } U_{j,\gamma} &= \text{Var } W_{j-1,\gamma} + \frac{r^{2j/\alpha}}{\gamma^{2/\alpha}} \text{Var } M_{j,\gamma}. \end{aligned} \tag{46}$$

**Proof of Proposition 6.** According to (24) conditioning on  $X_n^* = r^{(\lceil \log_r n \rceil + j)/\alpha}$ ,  $N_n = m$

$$S_n \stackrel{\mathcal{D}}{=} m r^{(\lceil \log_r n \rceil + j)/\alpha} + S_{n-m}^{(\lceil \log_r n \rceil + j - 1)},$$

and by Corollary 2 we know the behavior of the latter sum, as for each fixed  $m \geq 1$

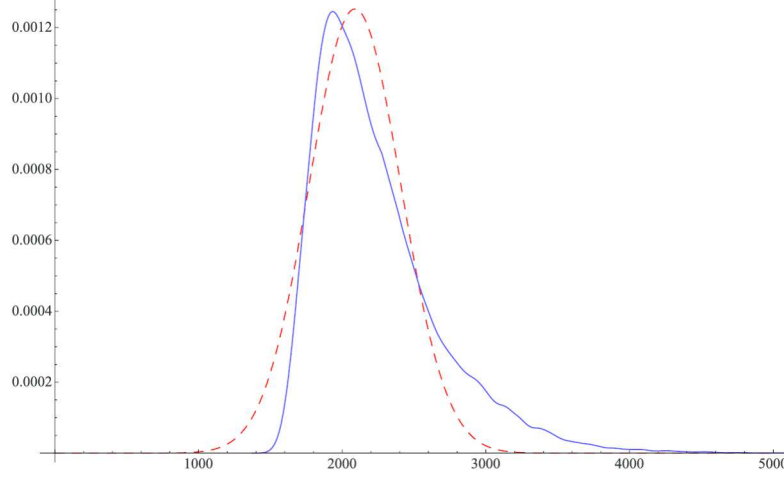
$$\sup_{x \in \mathbb{R}} \left| \mathbb{P} \left\{ \frac{S_{n-m}^{(\lceil \log_r n \rceil + j - 1)}}{n^{1/\alpha}} - \mu_n \leq x \right\} - G_{j,\gamma}(x) \right| \rightarrow 0.$$

By the law of total probability

$$\begin{aligned} &\mathbb{P} \left\{ \frac{S_n}{n^{1/\alpha}} - \mu_n \leq x \mid X_n^* = r^{(\lceil \log_r n \rceil + j)/\alpha} \right\} \\ &= \sum_{m=1}^n \mathbb{P} \left\{ \frac{S_{n-m}^{(\lceil \log_r n \rceil + j - 1)}}{n^{1/\alpha}} - \mu_n + \frac{m r^{j/\alpha}}{\gamma_n^{1/\alpha}} \leq x \right\} \mathbb{P}\{N_n = m \mid X_n^* = r^{(\lceil \log_r n \rceil + j)/\alpha}\}. \end{aligned} \tag{47}$$

Combining (21) with (47) it is routine to obtain (43).  $\square$

Figure 4 illustrates the histogram of  $S_n$  for  $n = 2^7$  ( $\alpha = 1$ ,  $p = 1/2$ ) conditioned on  $X_n^* = 2^{10}$  and a fitted Gaussian density. The histogram has the property of positive



**Figure 4.** The histogram of  $S_n$  for  $n = 2^7$  ( $\alpha = 1$ ,  $p = 1/2$ ) conditioned on  $X_n^* = 2^{10}$  (solid) and a fitted Gaussian density (dashed).

skewness, which means that the right-hand side tail is larger than the left-hand side one. The scaled and translated version of the histogram corresponds to the density function of  $U_{3,1}$ .

## 2.4. Conditioning on large maximum

As we mentioned in the Introduction, the side lobes in Figures 1 and 2 correspond to the conditional histograms of  $\log_2 S_n$  conditioning the large values of  $X_n^*$ , such that they have disjoint support contained in an interval of length 1. It means that  $\log_2 X_n^* < \log_2 S_n < \log_2 X_n^* + 1$ , or equivalently  $X_n^* < S_n < 2X_n^*$ , if  $X_n^*$  is large enough. In the next proposition, we make this observation precise.

In the following we investigate the case when  $X_n^* = r^{k_n/\alpha}$  is large, that is, what happens for  $k_n > \log_r n$ . We restrict ourselves to the  $\alpha \in (0, 2)$  case, since for  $\alpha \geq 2$  CLT holds, and thus the corresponding statements are not interesting.

**Proposition 7.** *Let  $\alpha \in (0, 2)$ . Assume that  $k_n - \log_r n \rightarrow \infty$ . Given that  $X_n^* = r^{k_n/\alpha}$*

$$\frac{S_n}{X_n^*} - A_n \xrightarrow{\mathbb{P}} 1,$$

where

$$A_n = \begin{cases} 0, & \alpha < 1, \\ \frac{p}{q} \frac{nk_n}{r^{k_n}}, & \alpha = 1, \\ \frac{n}{r^{k_n/\alpha}} \frac{p}{q^{1/\alpha} - q}, & \alpha > 1. \end{cases}$$



**Proof.** As  $k_n - \log_r n \rightarrow \infty$  by Proposition 1 we have that  $\mathbb{P}\{N_n = 1 | X_n^* = r^{k_n/\alpha}\} \rightarrow 1$ . Therefore, we may condition on the event  $\{X_n^* = r^{k_n/\alpha}, N_n = 1\}$ , and given this event by Lemma 2

$$S_n \stackrel{\mathcal{D}}{=} r^{k_n/\alpha} + S_{n-1}^{(k_n-1)}.$$

Proceeding as in the proof of Proposition 5, one can see that in order to obtain a non-degenerate limit the normalization for  $S_{n-1}^{(k_n-1)}$  should be  $n^{1/\alpha}$ , but  $r^{k_n/\alpha}/n^{1/\alpha} \rightarrow \infty$ , so the maximum alone is too large. That is in this case there is no non-degenerate limit distribution.

We shall determine the limit behavior of the sum  $S_{n-1}^{(k_n-1)}/r^{k_n/\alpha} - A_n$ , with some centering  $A_n$ . Using Theorem 25.1 in [11], one can check as in the proof of Proposition 5 that condition (40) holds, and also (38) and (39) hold with constant 0 as the limit function. Choosing  $\tau > 2$ , we get the centering sequence

$$A_n = n \int_{|x| \leq \tau} x \, dF_{k_n-1}(r^{k_n/\alpha} x) \sim \frac{n}{r^{k_n/\alpha}} \mathbb{E}X^{(k_n-1)}.$$

For  $\alpha < 1$ , using formula (25) we see that  $A_n \rightarrow 0$ , while for  $\alpha = 1$ ,

$$A_n \sim \frac{p n k_n}{q r^{k_n}}.$$

Finally, for  $\alpha > 1$  the expectation  $\mathbb{E}X = p/(q^{1/\alpha} - q) < \infty$ , therefore

$$A_n \sim \frac{n}{r^{k_n/\alpha}} \frac{p}{q^{1/\alpha} - q}.$$

In all cases the limit distribution is degenerate at 0, so we obtain that

$$\frac{S_{n-1}^{(k_n-1)}}{r^{k_n/\alpha}} - A_n \rightarrow 0, \tag{48}$$

in distribution, and so in probability. Adding the maximum term we obtain the statement.  $\square$

**Remark 5.** Note that contrary to the case  $\alpha \geq 1$  for  $\alpha < 1$  there is no need for centering for any  $k_n$  which satisfies  $n/r^{k_n} \rightarrow \infty$ . That is, given that  $X_n^* = r^{k_n/\alpha}$

$$\frac{S_n}{X_n^*} \xrightarrow{\mathbb{P}} 1,$$

so the maximum term alone dominates the whole sum. This is not surprising given the results of Darling [9] and Breiman [3]. In Theorem 5.1 in [9] Darling shows that if  $Y, Y_1, Y_2, \dots$  are nonnegative i.i.d. random variables from the domain of attraction of an  $\alpha$ -stable law,  $\alpha \in (0, 1)$ , then

$$\frac{\max\{Y_1, \dots, Y_n\}}{\sum_{i=1}^n Y_i}$$

converges in distribution to a non-degenerate random variable. On the other hand, Breiman in Theorem 4 [3] proves that this property characterizes the domain of attraction. Intuitively, when the tail of the distribution function behaves as  $x^{-\alpha}$ ,  $\alpha \in (0, 1)$ , the maximum term is about the same order as the whole sum. In Proposition 7, we assume that the maximum is larger than it should be, so it is reasonable to expect that it dominates the whole sum.

For  $\alpha = 1$ , let us consider the classical case. For  $k_n = \lfloor \log_2 n + \log_2 \log_2 n \rfloor + j$ ,  $j \in \mathbb{Z}$ , given that  $X_n^* = 2^{k_n}$  we again obtain a precise oscillatory behavior

$$\frac{S_n}{X_n^*} - 2^{-j} 2^{\{\log_2 n + \log_2 \log_2 n\}} \xrightarrow{\mathbb{P}} 1.$$

In fact, (48) states more. For  $k_n = \lfloor \log_2 n + a \log_2 \log_2 n \rfloor$ , with some  $a \in (0, 1)$ , given  $X_n^* = 2^{k_n}$

$$\frac{S_n}{X_n^*} - (\log_2 n)^{1-a} 2^{\{\log_2 n + a \log_2 \log_2 n\}} \xrightarrow{\mathbb{P}} 1.$$

Note the interesting phenomenon that although the maximum does not dominate the sum, it is large enough to cause a deterministic growth rate.

For  $\alpha > 1$  consider the case when  $k_n = \lfloor \beta \log_r n \rfloor$ , for some  $\beta > 1$ . For  $\beta > \alpha$  the centering goes to 0, and so conditioning on  $X_n^* = r^{k_n}$

$$\frac{S_n}{X_n^*} \xrightarrow{\mathbb{P}} 1,$$

thus the maximum dominates the whole sum. For  $\alpha = \beta$  we obtain again the oscillatory behavior, as

$$\frac{S_n}{X_n^*} - \frac{p}{q^{1/\alpha} - q} r^{\{\alpha \log_r n\}/\alpha} \xrightarrow{\mathbb{P}} 1,$$

while for  $1 < \beta < \alpha$  the ratio grows as  $n^{1-\beta/\alpha} p / (q^{1/\alpha} - q) r^{\{\beta \log_r n\}/\alpha}$ .

**Remark 6.** When  $A_n = o(1)$ , Proposition 7 says that  $S_n/X_n^* \xrightarrow{\mathbb{P}} 1$ , given  $X_n^* = r^{k_n/\alpha}$ . By Chebyshev's inequality one can get the following bound for the rate of convergence

$$\mathbb{P}\{S_n > (1 + \varepsilon)X_n^* | X_n^* = r^{k_n/\alpha}\} \leq \frac{4pr^{2/\alpha}}{\varepsilon^2(r^{2/\alpha} - 1)} \frac{n}{r^{k_n}}.$$

### 3. A series representation of the semistable limit

In this section,  $\alpha \in (0, 2)$ . The next theorem gives a representation of the semistable distribution function  $G_\gamma$  introduced in (4). Recall the notation  $\tilde{G}_{j,\gamma}$  in (44). The interesting feature of the statement is that the distribution functions  $\tilde{G}_{j,\gamma}$  in the representation are distribution functions of infinitely divisible random variables with finite exponential moments. The expectation and variance is calculated in (46).

**Theorem 1.** Let  $\alpha \in (0, 2)$ . For any  $\gamma \in [q, 1]$

$$G_\gamma(x) = \sum_{j=-\infty}^{\infty} \tilde{G}_{j,\gamma}(x) p_{j,\gamma}.$$

**Remark 7.** Before the proof we continue Remark 4. Let  $(W_{j,\gamma})_{j \in \mathbb{Z}, \gamma \in [q, 1]}$  be random variables with characteristic function  $\varphi_{j,\gamma}$  in (35), independently let  $(M_{j,\gamma})_{j \in \mathbb{Z}, \gamma \in [q, 1]}$  be positive integer valued random variables with generating function  $h_{j,\gamma}$  in (14), and independently let  $(Y_\gamma)_{\gamma \in [q, 1]}$  be integer valued random variable with probability distribution  $p_{j,\gamma}$ . Then

$$\mathbb{P} \left\{ W_{Y_\gamma-1,\gamma} + M_{Y_\gamma,\gamma} \frac{r^{Y_\gamma/\alpha}}{\gamma^{1/\alpha}} \leq x \right\} = G_\gamma(x),$$

or equivalently the semistable random variable  $W_\gamma$  has the representation

$$W_\gamma \stackrel{\mathcal{D}}{=} W_{Y_\gamma-1,\gamma} + M_{Y_\gamma,\gamma} \frac{r^{Y_\gamma/\alpha}}{\gamma^{1/\alpha}}.$$

We note that this probabilistic representation in the classical case is basically given in Section 8 in [13].

**Proof of Theorem 1.** We show that for any fixed  $x$ , one has

$$\left| \mathbb{P} \left\{ \frac{S_n}{n^{1/\alpha}} - \mu_n \leq x \right\} - \sum_{j=-\infty}^{\infty} \tilde{G}_{j,\gamma_n}(x) p_{j,\gamma_n} \right| \rightarrow 0,$$

which together with formula (4) implies the statement.

To ease the notation, introduce

$$F_{n,j}(x) = \mathbb{P} \left\{ \frac{S_n}{n^{1/\alpha}} - \mu_n \leq x \mid X_n^* = r^{(\lceil \log_r n \rceil + j)/\alpha} \right\}$$

and

$$q_{n,j} = \mathbb{P} \{ X_n^* = r^{(\lceil \log_r n \rceil + j)/\alpha} \}. \quad (49)$$

By the law of total probability,

$$\begin{aligned} & \mathbb{P} \left\{ \frac{S_n}{n^{1/\alpha}} - \mu_n \leq x \right\} \\ &= \sum_{j=1-\lceil \log_r n \rceil}^{\infty} \mathbb{P} \left\{ \frac{S_n}{n^{1/\alpha}} - \mu_n \leq x \mid X_n^* = r^{(\lceil \log_r n \rceil + j)/\alpha} \right\} \mathbb{P} \{ X_n^* = r^{(\lceil \log_r n \rceil + j)/\alpha} \} \\ &= \sum_{j=1-\lceil \log_r n \rceil}^{\infty} F_{n,j}(x) q_{n,j}. \end{aligned}$$

For  $\varepsilon > 0$  choose  $j_{\min} < 0 < j_{\max}$  such that for all  $n \geq 1$

$$\sum_{j=-\lceil \log_r n \rceil + 1}^{j_{\min}} q_{n,j} < \varepsilon/4, \quad \sum_{j=-\infty}^{j_{\min}} p_{j,\gamma_n} < \varepsilon/4$$

and

$$\sum_{j=j_{\max}+1}^{\infty} q_{n,j} < \varepsilon/4, \quad \sum_{j=j_{\max}+1}^{\infty} p_{j,\gamma_n} < \varepsilon/4.$$

By (7) and Lemma 1, this is possible. Thus,

$$\begin{aligned} & \left| \mathbb{P} \left\{ \frac{S_n}{n^{1/\alpha}} - \mu_n \leq x \right\} - \sum_{j=-\infty}^{\infty} \tilde{G}_{j,\gamma_n}(x) p_{j,\gamma_n} \right| \\ & \leq \sum_{j=-\lceil \log_r n \rceil + 1}^{j_{\min}} q_{n,j} + \sum_{j=-\infty}^{j_{\min}} p_{j,\gamma_n} + \sum_{j=j_{\max}+1}^{\infty} q_{n,j} + \sum_{j=j_{\max}+1}^{\infty} p_{j,\gamma_n} \\ & \quad + \left| \sum_{j=j_{\min}+1}^{j_{\max}} F_{n,j}(x) q_{n,j} - \sum_{j=j_{\min}+1}^{j_{\max}} \tilde{G}_{j,\gamma_n}(x) p_{j,\gamma_n} \right| \\ & \leq \varepsilon + \sum_{j=j_{\min}+1}^{j_{\max}} |F_{n,j}(x) - \tilde{G}_{j,\gamma_n}(x)| + \sum_{j=j_{\min}+1}^{j_{\max}} |q_{n,j} - p_{j,\gamma_n}| \rightarrow \varepsilon, \end{aligned}$$

where in the last step we applied Lemma 1 and Proposition 6.  $\square$

As a consequence of Theorem 1, using simply Chebyshev's inequality combined with the asymptotics of the first and second moments of  $W_{j,\gamma}$  in (41) one can obtain sharp bounds on the tail of  $G_\gamma$ .

**Corollary 3.** *For any  $\gamma \in [q, 1]$  for large enough  $x$  we have*

$$1 - G_\gamma(x) \leq \text{const} \cdot x^{-\alpha}.$$

However, the exact asymptotic behavior of the semistable tail is known. It follows from a general recent result by Watanabe and Yamamuro [25]. Recall that  $R_\gamma$  is the Lévy function of the semistable limit  $W_\gamma$  defined in (6). In Theorem 3 in [25], they show that

$$\liminf_{x \rightarrow \infty} x^\alpha [1 - G_\gamma(x)] = \inf_{1 \leq x \leq r^{1/\alpha}} x^\alpha (-R_\gamma(x)) = 1$$

and

$$\limsup_{x \rightarrow \infty} x^\alpha [1 - G_\gamma(x)] = \sup_{1 \leq x \leq r^{1/\alpha}} x^\alpha (-R_\gamma(x-)) = r.$$

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