

An extension of the direction problem

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Abstract

Let U be a point set in the n -dimensional affine space $\text{AG}(n, q)$ over the finite field of q elements and $0 \leq k \leq n - 2$. In this paper we extend the definition of *directions determined by U* : a k -dimensional subspace S_k at infinity is *determined* by U if there is an affine $(k + 1)$ -dimensional subspace T_{k+1} through S_k such that $U \cap T_{k+1}$ spans T_{k+1} . We examine the extremal case $|U| = q^{n-1}$, and classify point sets *not* determining every k -subspace in certain cases.

1 Introduction

Let $\text{AG}(n, q)$ and $\text{PG}(n, q)$ denote the n -dimensional affine and projective geometries over the finite field of q elements. We will think about $\text{PG}(n, q)$ as $\text{AG}(n, q) \cup H_\infty$, where H_∞ is called the *hyperplane at infinity* or the *ideal hyperplane*.

Let U be a point set in the affine plane $\text{AG}(2, q) \subset \text{PG}(2, q)$. We say a direction d (i.e. a point at infinity) is *determined* by U if there is an affine line with the ideal point d containing at least two points of U . It is a well-studied question that how many directions are determined by a point set and how the “interesting” point sets U look like, see [1], [2]. It seems to be natural to extend this definition in the following way:

Definition 1 *Let $U \subset \text{AG}(n, q) \subset \text{PG}(n, q)$, and k be a fixed integer, $k \leq n - 2$. We say a subspace S_k of dimension k in H_∞ is determined by U if there is an affine subspace T_{k+1} of dimension $k + 1$, having S_k as its hyperplane at infinity, containing at least $k + 2$ affinely independent points of U (i.e. spanning T_{k+1}).*

The question is that for a fixed k how many subspaces of dimension k are determined by U , or how large the point set can be, if it does not determine all of the k -subspaces at infinity, and can we say something about the structure of U in that case. Consider a subspace S_k of dimension k at infinity. There are q^{n-k-1} (pairwise disjoint) affine subspaces of dimension $k + 1$ with the ideal hyperplane S_k . Suppose that S_k is not determined, and consider an affine $(k + 1)$ -subspace T_{k+1} through S_k . Then the points of the intersection of U and T_{k+1} are contained in some subspace of dimension k since S_k is undetermined and so T_{k+1} is not spanned by them. An affine subspace of dimension k consists of q^k points, so (any) T_{k+1} can contain at most q^k points of U . This implies $|U| \leq q^{n-1}$ if it does not determine all the k -subspaces at infinity.

2 The extremal case

We will examine point sets of the extremal cardinality, so let $|U| = q^{n-1}$. Our aim is to find out a (strong) structure of U , if there are relatively many undetermined subspaces at infinity.

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Let $|U| = q^{n-1}$. Suppose that there exists an undetermined k -subspace $S_k \subset H_\infty$. Then in each of the q^{n-k-1} affine $(k+1)$ -subspaces whose ideal hyperplane is S_k there lie exactly q^k points of U constituting one complete k -subspace.

We can construct point sets not determining many subspaces in the following way:

Let $U_0 \subset \text{AG}(m, q) \subset \text{PG}(m, q)$, $|U_0| = q^{m-1}$. Denote by N_l^0 the set of the l -subspaces in H_∞ which are not determined by U_0 . We can embed U_0 into $\text{AG}(n, q)$, where $n > m$. Consider a subspace V on the ideal hyperplane of $\text{AG}(n, q)$, $\dim V = v$, where $v = n - m - 1$, completely disjoint from the original m -dimensional space. We construct a cone (cylinder) with base U_0 and vertex V such that we take the union of the affine subspaces spanned by a point of U_0 and V . These subspaces are of dimension $n - m$, and in that way we get a point set U in $\text{AG}(n, q)$, $|U| = q^{n-1}$.

We will show that there are “many” subspaces at infinity which are not determined by U , and we characterize them.

Denote by N_r the set of subspaces of dimension r on the ideal hyperplane which are not determined by U .

Proposition 2 *Let $U_0 \subset \text{AG}(m, q) \subset \text{PG}(m, q)$, $|U_0| = q^{m-1}$ embedded into $\text{AG}(n, q)$, let $V \subseteq H_\infty$ be a subspace in $\text{PG}(n, q)$ completely disjoint from $\text{PG}(m, q)$, $\dim V = n - m - 1$, let $U \subset \text{AG}(n, q) \subset \text{PG}(n, q)$ be the cone (cylinder) constructed as above with base U_0 and vertex V . Let $W \subseteq H_\infty$ in $\text{PG}(n, q)$, $\dim W = r$. Then W is non-determined, i.e. $W \in N_r$ if and only if, after projecting W from V to the m -space, the projected image $W_0 = \text{PG}(m, q) \cap \langle V, W \rangle$ is non-determined by U_0 .*

Proof: For the proof let $s = \dim(W \cap V)$ and $r_0 = \dim W_0 = r - s - 1$. (Note that W and W_0 are contained in the hyperplane at infinity.) By projection we always mean projection from the center V (see below).

Suppose that W is non-determined. Let $L_0 \supset W_0$ be any of the affine $r_0 + 1$ -dimensional subspaces in $\text{PG}(m, q)$ through W_0 , we have to prove that $L_0 \cap U_0$ is a complete affine r_0 -space. Take any affine point P from the subspace $\langle V, L_0 \rangle$, then define $L = \langle P, W \rangle$. Now L_0 is the projected image of L , i.e. $L_0 = \text{PG}(m, q) \cap \langle V, L \rangle$. As $\dim L = r + 1$ and W is non-determined, $A = L \cap U$ must be a complete affine r -space. Hence $L_0 \cap U_0$ is the projected image of it, i.e. a complete affine r_0 -space. (We have used the fact that $\dim(\bar{A} \cap V) = s = \dim(W \cap V)$, where \bar{A} is the projective closure of A . Indeed, if $\dim(\bar{A} \cap V)$ were larger then $\dim(L \cap U)$ would be larger accordingly.) So W_0 is non-determined.

On the other hand, suppose that W_0 is non-determined. Then for any $r + 1$ -dimensional affine subspace L through W , and for its projected image L_0 , the intersection $L_0 \cap U_0$ is identical to the projected image of $L \cap U$. As $L_0 \cap U_0$ is a complete $r_0 + 1$ -dimensional affine subspace through W_0 , we have that $L \cap U$ must be $\langle V, L_0 \rangle \cap L$, i.e. a complete r -dimensional affine subspace through W . ■

For a given affine point set we can examine determined subspaces in H_∞ of different dimensions. We are trying to find out a hierarchy of the determined subspaces of different dimensions. In [3] the following theorem was proved in the classical case:

Result 3 *Let $U \subset \text{AG}(n, q) \subset \text{PG}(n, q)$, $|U| = q^{n-1}$ and let $D \subseteq H_\infty$ be the set of directions determined by U . Then D is the union of some complete $(n - 2)$ -dimensional subspaces of H_∞ . ■*

We find an analogous situation when determining higher dimensional subspaces:

Observation 4 *Let $U \subset \text{AG}(n, q) \subset \text{PG}(n, q)$, $|U| = q^{n-1}$ and k be a fixed integer, $k \leq n - 3$. If there is a subspace S_k of dimension k , $S_k \subset H_\infty$ determined by U , then there is a subspace S_{k+1} of dimension $k + 1$ in H_∞ , $S_k \subset S_{k+1}$, which is determined by U .*

Proof: Since S_k is determined by U , there exists an affine $(k + 1)$ -dimensional subspace A_{k+1} with the ideal hyperplane S_k which contains at least $k + 2$ linearly independent points from U . There will be at least one $(k + 2)$ -dimensional affine subspace A_{k+2} , $A_{k+1} \subset A_{k+2}$ containing at least one more point from U . So A_{k+2} contains at least $k + 3$ linearly independent points from U , so its ideal hyperplane will be determined. ■

Corollary 5 *Analogously to Result 3, through any determined k -subspace S_k , there exists a determined $(n - 2)$ -subspace.* ■

Proposition 6 *Let $U \subset \text{AG}(n, q) \subset \text{PG}(n, q)$, $|U| = q^{n-1}$ and k be a fixed integer, $k \leq n - 3$. If there is a subspace V_{n-2} of dimension $n - 2$, $V_{n-2} \subset H_\infty$ such that all of the k -dimensional subspaces of V_{n-2} are determined by U then V_{n-2} is determined by U as well.*

Proof: Suppose that there is a subspace V_{n-2} of dimension $n - 2$, $V_{n-2} \subset H_\infty$ which is not determined by U . We will show that there is a subspace V_k , $\dim V_k = k$, $V_k \subset V_{n-2}$ which is not determined by U . Each of the q affine $(n - 1)$ -subspaces whose ideal hyperplane is V_{n-2} contains precisely a whole $(n - 2)$ -subspace from U . Such an $(n - 2)$ -subspace has an ideal hyperplane of dimension $n - 3$ contained in V_{n-2} , so these $(n - 3)$ -subspaces cannot cover all the points of V_{n-2} . Consider an uncovered point P , and take a subspace $V_k \subset V_{n-2}$, $\dim V_k = k$ containing P . V_{n-2} and each of the affine subspaces of dimension $k + 1$ with the ideal hyperplane V_k span a subspace of dimension $n - 1$. Such an $(n - 1)$ -subspace contains precisely a whole $(n - 2)$ -subspace from U . The intersection of the $(n - 2)$ -subspace from U and the $(k + 1)$ -subspace is a subspace of dimension k (since this $(n - 2)$ -subspace cannot contain the $(k + 1)$ -subspace because $P \in V_k$.) So each of the $(k + 1)$ -spaces with the ideal hyperplane V_k contains precisely a whole k -space from U which means that V_k is not determined by U . ■

3 The 3-dimensional case

The question is whether we can give other constructions for U with many undetermined subspaces at infinity, or can we say something about the point set if there are less undetermined subspaces. In certain dimensions we found that in case of a few undetermined subspaces the point set has a strong structure.

Let $U \subset \text{AG}(3, q) \subset \text{PG}(3, q)$. We say a line $\ell \subset H_\infty$ is determined by U if there is an affine plane with the ideal line ℓ containing at least three points of U which are not collinear. Let $|U| = q^2$. Suppose that there exists an undetermined line ℓ . Then each of the q affine planes whose ideal line is ℓ contains precisely a complete line of U .

Theorem 7 *Let $U \subset \text{AG}(3, q) \subset \text{PG}(3, q)$, $|U| = q^2$. Let L be the set of lines in H_∞ determined by U and put N the set of non-determined lines. Then one of the following holds:*

- a) $|N| = 0$, i.e. U determines all the lines of H_∞ ;
- b) $|N| = 1$ and then there is a parallel class of affine planes such that U contains one (arbitrary) complete line in each of its planes;
- c) $|N| = 2$ and then U together with the two undetermined lines in H_∞ form a hyperbolic quadric

or U contains q parallel lines (U is a cylinder);

d) $|N| \geq 3$ and then U contains q parallel lines (U is a cylinder).

Proof: If all the points of U are contained in a plane, then U determines only one line, the line in H_∞ of the plane containing U , so $|L| = 1$. It is a special case of **d)**.

So from now on suppose that U is not contained in a plane.

Denote by f_∞ the ideal point of an affine line f , and let $\bar{f} = f \cup f_\infty$.

Throughout this proof let $|N| \geq 2$, $\ell_1, \ell_2 \subset H_\infty$ be undetermined lines, M the intersection point of ℓ_1 and ℓ_2 .

Lemma 8 *Let f and g be affine lines contained in U , \bar{f} intersecting ℓ_1 , \bar{g} intersecting ℓ_2 . Then f and g meet each other or their ideal point is M .*

Proof: Suppose to the contrary that \bar{f} and \bar{g} are skew lines, and at least for one of them, e. g. \bar{f} , $M \notin \bar{f}$. Then \bar{f} intersects the plane s determined by \bar{g} and ℓ_2 at an affine point P not contained in g . Then P and g would determine ℓ_2 , contradiction. ■

Lemma 9 *If there is an affine line f contained in U , \bar{f} intersecting ℓ_1 at M , then the ideal point of each of the q affine lines contained in U whose ideal points are in ℓ_1 is M .*

Proof: Suppose that there exists a line g contained in U , \bar{g} intersecting ℓ_1 not at M . Since \bar{f} intersects ℓ_1 at M , it intersects ℓ_2 as well, so by **Lemma 8** f meets g at an affine point P . Then f and g would determine ℓ_1 , contradiction. ■

Lemma 10 *If there exist f and g parallel affine lines contained in U , \bar{f}, \bar{g} intersecting ℓ_1 , then their ideal point is M .*

Proof: Suppose to the contrary that \bar{f} and \bar{g} meet ℓ_1 not at M . Consider a point $P \in U$ not contained in the plane $\langle f, g \rangle$, and the plane s determined by P and ℓ_2 . (If $\nexists P \in U$, $P \notin \langle f, g \rangle$ then all the points of U would be contained in a plane.) s cannot be parallel to \bar{f} and \bar{g} as then they would intersect in the ideal hyperplane (i.e. in ℓ_2). So s intersects f and g at two different affine points. The two intersection points and P cannot be collinear as $P \notin \langle f, g \rangle$, so they span s determining ℓ_2 , contradiction. ■

Corollary 11 *If there exist f and g parallel affine lines contained in U , \bar{f}, \bar{g} intersecting ℓ_1 , then U consists of q parallel lines whose ideal point is M .*

Proof: Since f and g are parallel, \bar{f} and \bar{g} meet at M due to **Lemma 10**. Then, by **Lemma 9**, for all the lines h contained in U whose ideal points are in ℓ_1 , \bar{h} will intersect ℓ_1 at M , which means that they are all parallel. ■

Corollary 12 *If $|N| > 2$, $\ell_1 \subset H_\infty$ is an undetermined line, and there exist f, g parallel lines contained in U , \bar{f}, \bar{g} intersecting ℓ_1 , then all the undetermined lines in H_∞ intersect at the same point (and it is the ideal point of the lines of U).*

Proof: Let ℓ_2, ℓ_3 be undetermined lines, M the intersection point of ℓ_1 and ℓ_2 , K the intersection point of ℓ_1 and ℓ_3 . By **Lemma 10**, \bar{f} and \bar{g} meet ℓ_1 at M and also at K . ■

Corollary 13 *If there exist 2 lines contained in U whose ideal points are in ℓ_1 which are not parallel, then all the q lines whose ideal points are in ℓ_1 have q different ideal points (and none of them is M).*

Proof: By **Corollary 11**, if there exist two parallel lines, then all of them are parallel. So if they are not all parallel, then all of them are pairwise skew. It means that their ideal points in ℓ_1 are pairwise different. So there is a line $f \subset U$ such that \bar{f} intersects ℓ_1 not at M . Then by **Lemma 9** there cannot exist a line $g \subset U$ such that \bar{g} intersects ℓ_1 at M . ■

So we have two different cases: **(1)** all the q lines meeting $\ell_1 \in N$ are parallel and intersect the ideal hyperplane at the intersection point of the undetermined lines or **(2)** all of them are pairwise skew.

In the first case U consists of q (say) vertical lines (forming a cylinder), and the undetermined lines intersect each other at the ideal point of the vertical lines. Then every “horizontal” (affine) plane (intersecting the vertical lines) contains q points of U . Consider the directions not determined by these q points on a “horizontal” plane. The undetermined lines are exactly the lines connecting the ideal point of the vertical lines and an undetermined direction.

It is exactly the construction we saw in **Proposition 2**: the base of the cone is a point set of cardinality q contained in a horizontal plane, and the vertex of the cone is M .

There is a special case of this first case: the q vertical lines can be co-planar. Then U is contained in a plane. Then U determines only one line, so $|L| = 1$.

In the second case let ℓ_1, ℓ_2 be undetermined lines. By **Corollary 13** we know that the points of U form q lines whose ideal points are in ℓ_1 , and these ideal points are pairwise different, so the lines are skew. It also holds for ℓ_2 , so U forms q skew lines whose ideal points are in ℓ_2 , and these ideal points are pairwise different. Consider 3 of the skew lines whose ideal points are in ℓ_1 and denote them by f, g and h . By **Lemma 8** all the q lines whose ideal points are in ℓ_2 intersect f, g and h at one-one point. The intersection points are different as the q lines are pairwise skew. So the q lines whose ideal points are in ℓ_2 form a q -regulus of f, g and h , and with ℓ_1 it is a $(q + 1)$ -regulus of \bar{f}, \bar{g} and \bar{h} . The same holds for the lines f', g' and h' whose ideal points are in ℓ_2 : the q lines whose ideal points are in ℓ_1 together with ℓ_2 form a $(q + 1)$ -regulus of \bar{f}', \bar{g}' and \bar{h}' . It means that the q^2 points of U and the undetermined lines ℓ_1 and ℓ_2 form a hyperbolic quadric and **Theorem 7** is proved. ■

4 More quadrics

So we have the complete characterization in 3 dimensions. The question is whether we can have similar results in higher dimensions. In $\text{PG}(n, q)$ we will show that if we try to find a point set U , $|U| = q^{n-1}$ which does not determine all the subspaces of a certain dimension then the former examples will occur:

If the point set U corresponds to the affine part of a nonsingular quadric for which the ideal hyperplane is a tangent hyperplane, or U is a cone constructed as in **Proposition 2**, then there will be some undetermined subspaces. The open question is whether there are other point sets not

determining more than one subspaces or nonsingular quadrics and cones are the only examples.

Let $U \subset \text{AG}(n, q) \subset \text{PG}(n, q)$, $|U| = q^{n-1}$ be the affine part of a nonsingular quadric for which the ideal hyperplane is a tangent hyperplane (i.e. the intersection of the quadric and the ideal hyperplane is a cone based on an $(n-2)$ -dimensional quadric of the same character). Denote by g the projective index of the quadric, i.e. the dimension of the generators, the subspaces of maximum dimension contained in the quadric. Denote by w the character of the quadric, $w = 1, 2, 0$ for a parabolic, hyperbolic, elliptic quadric, respectively.

Proposition 14 *If $U \subset \text{AG}(n, q) \subset \text{PG}(n, q)$, $|U| = q^{n-1}$ is defined as above then the undetermined g -dimensional subspaces in H_∞ are exactly the generators contained in the intersection of the quadric and the ideal hyperplane, except the cases when $n = 2$ and q is even, or $n = 4$ and $q = 2$, or $n = 5$ and $q = 2$ and the quadric is elliptic.*

Proof: There are q^{n-1} affine points in the quadric. Denote by G a generator of the quadric contained in H_∞ . There are q^{n-g-1} affine $(g+1)$ -dimensional subspaces with the ideal hyperplane G . Each of these $(g+1)$ -subspaces contains at most q^g points from the quadric, otherwise the whole $(g+1)$ -subspace would be contained in the quadric, which is a contradiction. Since there are q^{n-1} affine points in the quadric, each $(g+1)$ -subspace contains exactly q^g points. The intersection of a $(g+1)$ -subspace and the quadric is a quadric of dimension $g+1$, containing G , a subspace of dimension g , which is a linear factor. So the rest of the intersection has to be a g -subspace as well. It means that each of the $(g+1)$ -dimensional affine subspaces with the ideal hyperplane G contains exactly an affine g -dimensional subspace from U , so G is undetermined.

On the other hand consider a g -dimensional subspace G' in H_∞ not contained in the quadric. Suppose it is undetermined. It means that each of the q^{n-g-1} $(g+1)$ -dimensional affine subspaces with the ideal hyperplane G' contains exactly a g -dimensional subspace from U (i.e. a generator). If the quadric is hyperbolic, such a $(g+1)$ -dimensional subspace should contain one more generator (from the other system), contradiction. If the quadric is elliptic or parabolic, the ideal hyperplanes of these generators are in G' and they are of dimension $g-1$. Since G' is not a generator, it can contain at most 2 $(g-1)$ -subspaces from the quadric. This implies that there is a $(g-1)$ -subspace in the intersection of G' and the quadric, which is the ideal hyperplane of at least $\frac{q^{n-g-1}}{2}$ generators. The number of generators through a fixed $(g-1)$ -subspace is the following:

$$\varrho(g-1, n, w) = (q^{2-w} + 1) \cdot (q^{2-w+1} + 1) \cdot \dots \cdot (q^{\frac{n-2g+1-w}{2}} + 1).$$

For an elliptic quadric $\varrho(\frac{n-5}{2}, n, 0) = q^2 + 1 < \frac{q^{\frac{n+1}{2}}}{2}$, if $n \geq 5$ and $q \geq 3$ or $n \geq 7$;

for a parabolic quadric $\varrho(\frac{n-4}{2}, n, 1) = q + 1 < \frac{q^{\frac{n+1}{2}}}{2}$, if $n \geq 4$ and $q \geq 3$ or $n \geq 6$, contradiction.

If $n = 3$, for an elliptic quadric it is clear that through a point in H_∞ not contained in the quadric there are q^2 affine lines, and $q+1$ of them are tangents to the quadric, all the others are secants or skew lines. Since there are q^2 affine points in the quadric, the point at infinity will be determined. Similarly, if $n = 2$, q odd, through a point in l_∞ not contained in the quadric, there is only one affine line which is tangent to the quadric, all the others are secants or skew lines, so the point will be determined.

There are exceptional cases. If $n = 2$ and q is even, the nucleus of the parabolic quadric is a not determined point in the ideal line, but it is not contained in the quadric. For a parabolic quadric in $\text{PG}(4, 2)$ and for an elliptic quadric in $\text{PG}(5, 2)$ there exist undetermined lines in H_∞ which are not generators. ■

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References

- [1] S. BALL, The number of directions determined by a function over a finite field, *J. Combin. Th. Ser. A*, **104** (2003), 341–350.
- [2] A. BLOKHUIS, S. BALL, A. BROUWER, L. STORME AND T. SZŐNYI, On the number of slopes determined by a function on a finite field, *J. Comb. Theory Ser. (A)* **86** (1999), 187–196.
- [3] L. STORME AND P. SZIKLAI, Linear point sets and Rédei type k -blocking sets in $PG(n, q)$, *J. Alg. Comb.* **14** (2001), 221–228.
- [4] J. W. P. HIRSCHFELD AND J. A. THAS, *General Galois geometries*, Clarendon Press, Oxford (1991)

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