A small minimal blocking set in PG(n, p^t), spanning a $(t-1)$-space, is linear

Peter Sziklai   Geertrui Van de Voorde

September 28, 2011

Abstract

In this paper, we show that a small minimal blocking set with exponent e in PG(n, p^t), p prime, spanning a $(t/e-1)$-dimensional space, is an $\mathbb{F}_{p^e}$-linear set, provided that $p > 5(t/e) - 11$. As a corollary, we get that all small minimal blocking sets in PG(n, p^t), p prime, $p > 5t - 11$, spanning a $(t-1)$-dimensional space, are $\mathbb{F}_p$-linear, hence confirming the linearity conjecture for blocking sets in this particular case.

Keywords: Blocking set, linearity conjecture, linear set

1 Introduction

In this section, we introduce the necessary background and notation. If V is a vectorspace, then we denote the corresponding projective space by PG(V). If V has dimension $n+1$ over the finite field $\mathbb{F}_q$, with q elements, $q = p^t$, p prime, then we also write V as V$(n+1, q)$ and PG(V) as PG(n, q).

A *blocking set* in PG(n, q) is a set B of points such that every hyperplane of PG(n, q) contains at least one point of B. Such a blocking set is sometimes called a 1-*blocking set*, or a *blocking set with respect to hyperplanes*. A blocking set B is called *small* if $|B| < 3(q+1)/2$ and *minimal* if no proper subset of B is a blocking set.

A point set S in PG(V), where V $=$ V$(n+1, p^t)$ is called $\mathbb{F}_{q_0}$-*linear* if there exists a subset U of V that forms an $\mathbb{F}_{q_0}$-vector space for some $\mathbb{F}_{q_0} \subset \mathbb{F}_{p^t}$, such that $S = \mathcal{B}(U)$, where

$$\mathcal{B}(U) := \{\langle u \rangle_{\mathbb{F}_{p^t}} \ : \ u \in U \setminus \{0\}\}.$$

We have a one-to-one correspondence between the points of $\mathrm{PG}(n, q_0^h)$ and the elements of a Desarguesian $(h-1)$-spread $\mathcal{D}$ of $\mathrm{PG}(h(n+1)-1, q_0)$. This gives us a different view on linear sets; namely, an $\mathbb{F}_{q_0}$-linear set is a set S of points of $\mathrm{PG}(n, q_0^h)$ for which there exists a subspace π in $\mathrm{PG}(h(n+1)-1, q_0)$ such that the points of S correspond to the elements of $\mathcal{D}$ that have a non-empty intersection with π. We identify the elements of $\mathcal{D}$ with the points of $\mathrm{PG}(n, q_0^h)$, so we can view $\mathcal{B}(\pi)$ as a subset of $\mathcal{D}$, i.e.

$$\mathcal{B}(\pi) = \{R \in \mathcal{D} | R \cap \pi \neq \emptyset\}.$$

For more information on this approach to linear sets, we refer to [5].

The *linearity conjecture* for blocking sets (see [11]) states that

(LC) All small minimal blocking sets in $\mathrm{PG}(n, q)$ are linear sets.

Up to our knowledge, this is the complete list of cases in which the linearity conjecture for blocking sets in $\mathrm{PG}(n, p^t)$, p prime, with respect to hyperplanes, has been proven.

- $t = 1$ (for $n = 2$, see [1]; for $n > 2$, see [10])
- $t = 2$ (for $n = 2$, see [9]; for $n > 2$, see [8])
- $t = 3$, (for $n = 2$, see [6]; for $n > 2$, see [8])
- B is of Rédei-type, i.e., there is a hyperplane meeting B in $|B| - p^t$ points (for $n = 2$, see [2]; for $n > 2$, see [7])
- $\langle B \rangle = t$ (see [10]).

In this paper, we show that if $\langle B \rangle = t - 1$, and the characteristic of the field is sufficiently large, B is a linear set, as a corollary of the main theorem.

Main Theorem. *A small minimal blocking set B in $\mathrm{PG}(n, q)$, with exponent e, $q = p^t$, p prime, $q_0 := p^e$, $q_0 \geq 7$, $t/e = h$, spanning an $(h-1)$-dimensional space is an $\mathbb{F}_{q_0}$-linear set.*

2 The intersection of a small minimal blocking set and a subspace

A subspace clearly meets an $\mathbb{F}_p$-linear set in 0 or 1 mod p points. The following theorem shows that for a small minimal blocking set, the same holds.

Theorem 1. [10, Theorem 2.7] *If B is a small minimal blocking set in $\mathrm{PG}(n,p^t)$, p prime, then B intersects every subspace of $\mathrm{PG}(n,p^t)$ in 1 mod p or zero points.*

From this theorem, we get that every small minimal blocking set B in $\mathrm{PG}(n,p^t)$, p prime, has an *exponent* $e \geq 1$, which is the largest integer for which every hyperplane intersects B in 1 mod p^e points.

2.1 The intersection with a line

The following theorem by Sziklai characterises the intersection of particular lines with a small minimal blocking set as linear sets.

Theorem 2. [11, Corollary 5.2] *Let B be a small minimal blocking set with exponent e in $\mathrm{PG}(n,q)$, $q = p^t$, p prime. If for a certain line L, $|L \cap B| = p^e + 1$, then $\mathbb{F}_{p^e}$ is a subfield of $\mathbb{F}_q$ and $L \cap B$ is $\mathbb{F}_{p^e}$-linear.*

Using the 1 mod p-result (Theorem 1), it is not too hard to derive an upper bound on the size of a small minimal blocking set in $\mathrm{PG}(n,q)$ as done in [12]. This bound is a weaker version of the bound in Corrolary 5.2 of [11].

Lemma 3. [12, Lemma 1] *The size of a small minimal blocking set B with exponent e in $\mathrm{PG}(n,q_0^h)$, $q_0 := p^e \geq 7$, p prime is at most $q_0^h + q_0^{h-1} + q_0^{h-2} + 3q_0^{h-3}$.*

In this paper, we will make use of the fact that we can find lower bounds on the number of secant lines to a small minimal blocking set. In the next lemma, one considers the number of (q_0+1)-secants to the blocking set B, which will give a linear intersection with the blocking set by Theorem 2.

Lemma 4. [12, Lemma 4] *A point of a small minimal blocking set B with exponent e in $\mathrm{PG}(n,q_0^h)$, $q_0 := p^e \geq 7$, p prime, lying on a (q_0+1)-secant, lies on at least $q_0^{h-1} - 4q_0^{h-2} + 1$ (q_0+1)-secants.*

For the proof of Lemma 7, we will make use of the concept of point exponents of a blocking set and the well-known fact that the projection of a small minimal blocking set is a small minimal blocking set.

Lemma 5. [10, Corollary 3.2] *Let $n \geq 3$. The projection of a small minimal blocking set in $\mathrm{PG}(n,q)$, from a point $Q \notin B$ onto a hyperplane skew to Q, is a small minimal blocking set in $\mathrm{PG}(n-1,q)$.*

The exponent e_P of a point P of a small minimal blocking set B is the largest number for which every line through P meets in 1 mod p^{e_P} or zero points. The following lemma is essentially due to Blokhuis.

Lemma 6. See [3, Lemma 2.4(1)] *If B is a small minimal blocking set in* $\mathrm{PG}(2, q)$*,* $q = p^t$*,* p *prime, with* $|B| = q + \kappa$*, and* P *is a point with exponent* e_P*, then the number of secants to* B *through* P*, is at least*

$$(q - \kappa + 1)/p^{e_P} + 1.$$

Lemma 7. *A point P with exponent $e_P = 2e$ of a small minimal blocking set B in* $\mathrm{PG}(n, q_0^h)$*,* $q_0 := p^e \geq 7$*,* p *prime, lies on at least* $q_0^{h-2} - q_0^{h-3} - q_0^{h-4} - 3q_0^{h-5} + 1$ *secant lines to* B*.*

Proof. If $n = 2$, Lemma 3, together with Lemma 6, shows that the number of secant lines to B is at least $(q_0^h - q_0^{h-1} - q_0^{h-2} - 3q_0^{h-3} + 1)/q_0^2 + 1 \geq q_0^{h-2} - q_0^{h-3} - q_0^{h-4} - 3q_0^{h-5} + 1$.

If $n > 2$, then let L be a line through P, meeting B in $q_0^2 + 1$ points. By Theorem 1, a plane through L, containing a point of B, not on L, contains at least q_0^3 points of B, not on L. By Lemma 3, this implies that there is a plane Π through L with no points of B, outside L. Let Q be a point of $\Pi \setminus L$ and let $\tilde{B}$ is the projection of B from Q onto a hyperplane through L. By Lemma 5, $\tilde{B}$ is a small minimal blocking set in $\mathrm{PG}(n-1, q)$. It is clear that every line through P meets $\tilde{B}$ in 1 mod q_0^2 or 0 points, and that there is a line, namely L, meeting $\tilde{B}$ in $1 + q_0^2$ points, so $e_P = 2e$ in the blocking set $\tilde{B}$. It follows that the number of secant lines through a point P with exponent $2e$ to B is at least the number of secant lines through the point P with exponent $2e$ to $\tilde{B}$ in $\mathrm{PG}(n-1, q_0^h)$. Continuing this process, we see that this number is at least the number of secant lines through the point P with exponent $2e$ in a small minimal blocking set $\tilde{B}$ in $\mathrm{PG}(2, q_0^h)$, and the statement follows. □

2.2 The intersection with a plane

In the following lemma, we will distinguish planes acording to their intersection size with a small minimal blocking set. We will call a plane with $q_0^2 + q_0 + 1$ non-collinear points of B a *good* plane, while all other planes will be called *bad*. Note that also planes meeting B in only points on a line, or skew to B are called *bad*. The following lemma shows that good planes meet a small minimal blocking set in a linear set.

Lemma 8. *If* Π *is a plane of* $\mathrm{PG}(n, q)$ *containing at least* 3 *non-collinear points of a small minimal blocking set* B *in* $\mathrm{PG}(n, q)$*, with exponent* e*,* $q = p^t$*,* p *prime,* $q_0 := p^e$*, then*

(i) $q_0^2 + q_0 + 1 \leq |B \cap \Pi|$.

(ii) If $|B \cap \Pi| = q_0^2 + q_0 + 1$, then $B \cap \Pi$ is $\mathbb{F}_{q_0}$-linear.

(iii) If $|B \cap \Pi| > q_0^2 + q_0 + 1$, then $|B \cap \Pi| \geq 2q_0^2 + q_0 + 1$.

Proof. (i) By Lemma 1, every line meets B in 1 mod q_0 or 0 points. Since we find 3 non-collinear points, it is easy to see that $|B \cap \Pi| \geq q_0^2 + q_0 + 1$.

(ii) From the previous argument, we easily see that if $|B \cap \Pi| = q_0^2 + q_0 + 1$, then every line in Π contains 0, 1 or $q_0 + 1$ points of B. Suppose that there exist two $(q_0 + 1)$-secants that meet in a point, not in B, then the number of points in $\Pi \cap B$ is at least $q_0^2 + q_0 + 1 + q_0$. Hence, every two $(q_0 + 1)$-secants meet in a point of B. Moreover, through two points of $B \cap \Pi$, there is a unique $(q_0 + 1)$-secant, so B meets Π in an $\mathbb{F}_{q_0}$-subplane.

(iii) By Theorem 1, if there is a line L of Π containing more than $(q_0 + 1)$ points of B, then $|L \cap B| \geq 2q_0 + 1$, and $|\Pi \cap B| \geq 2q_0^2 + q_0 + 1$. So from now on, we may assume that every line meets B in 0, 1 or $q_0 + 1$ points. If there is an $\mathbb{F}_{q_0}$-subplane strictly contained in $\Pi \cap B$, then clearly $|B \cap \Pi| \geq q_0^3 + q_0^2 + q_0 + 1$, so we may assume that there is no $\mathbb{F}_{q_0}$-subplane contained in $\Pi \cap B$.

Let L be a $(q_0 + 1)$-secant in Π, let P be a point of $B \cap L$, let Q be a point of $B \setminus L$ and let M be the line PQ. From Theorem 2, we know that $L \cap B$ and $M \cap B$ are sublines over $\mathbb{F}_{q_0}$. These sublines define a unique $\mathbb{F}_{q_0}$-subplane Π_0. Let N_1 be a line, not through P, through a point of $L \cap B$, say R_1 and $M \cap B$, say R_2. Let N_2 be another line, not through R_1 or R_2, meeting L in a point R_3 of B and M in a point R_4 of B. If T is the intersection point of N_1 and N_2, then T belongs to the subplane Π_0.

Now suppose that T is a point of B, then N_1 meets B in a subline, containing 3 points of the subline $\Pi_0 \cap N_1$, hence, the subline $N_1 \cap B$ is completely contained in B. The same holds for the subline $N_2 \cap B$, and repeating the same argument, for every subline through T meeting L and M in points, different from P. Again repeating the same argument, for a point $T' \neq T$ on N_1, not on L or M, yields that Π_0 is contained in B, a contradiction. This implies that the $q_0 - 1$ points of B on the line N_1, not on L or M are different from the $q_0 - 1$ points of B on the line N_2, not on L or M. Varying N_1 and N_2 over all lines meeting L and M in points of B, we get that there are at least $q_0^2(q_0 - 1) + 2q_0 + 1$ points in $B \cap \Pi$. □

To avoid abundant notation, we continue with the following hypothesis on B.

> B is a small minimal blocking set in $\mathrm{PG}(n, q)$, with exponent e, $q = p^t$, p prime, $q_0 := p^e$, $t/e = h$, **spanning an $(h-1)$-dimensional space**.

Lemma 9. *A plane of $\mathrm{PG}(n, q)$ contains at most $q_0^3 + q_0^2 + q_0 + 1$ points of B.*

Proof. Suppose there exists a plane Π with more than $q_0^3+q_0^2+q_0+1$ points of B, then, by Theorem 1, $|\Pi\cap B|\geq q_0^3+q_0^2+2q_0+1$. We prove by induction that, for all $2\leq k\leq h-1$ there is a k-space, containing at least $(q_0^{k+2}-1)/(q_0-1)+q_0^{k-1}$ points of B. The case $k=2$ is already settled, so suppose there is a j-space Π_j, $j<h-1$, containing at least $(q_0^{j+2}-1)/(q_0-1)+q_0^{j-1}$ points of B. Since B spans an $(h-1)$-space and $j<h-1$, there is a point Q in B, not in Π_j. Because a line containing two points of B contains at least q_0+1 points of B, this implies that $|\langle Q,\Pi_j\rangle\cap B|\geq (q_0^{j+3}-1)/(q_0-1)+q_0^j$. By induction, we obtain that B contains at least $(q_0^{h+1}-1)/(q_0-1)+q_0^{h-2}$ points, a contradiction, since $|B|\leq q_0^h+q_0^{h-1}+q_0^{h-2}+3q_0^{h-3}$. □

Lemma 10. *Let L be a (q_0+1)-secant to B. Then either L lies on at least $q_0^{h-2}-4q_0^{h-3}+1$ good planes, or L lies on bad planes only. In the latter case, all planes with points of B, outside L contain at least $q_0^3+q_0+1$ points of B.*

Proof. Let Q be a point on L, not on B. We project B from Q onto a hyperplane H, not through Q, and denote the image of this projection by $\tilde{B}$. Let P be the point $L\cap H$. It follows from Lemma 5, that $\tilde{B}$ is a small minimal blocking set. Since every subspace meets B in 1 mod q_0 or 0 points, every subspace meets $\tilde{B}$ in 1 mod q_0 or 0 points. Suppose that P has exponent $e_P=1$, then it follows from Lemma 4 that P lies on at least $q_0^{h-1}-4q_0^{h-2}+1$ (q_0+1)-secants. This means that there are at least $q_0^{h-1}-4q_0^{h-2}+1$ planes through L containing at least $q_0^2+q_0+1$ points of B, which implies that $|B|\geq q_0^2(q_0^{h-1}-4q_0^{h-2}+1)$, a contradiction since $|B|\leq q_0^h+q_0^{h-1}+q_0^{h-2}+3q_0^{h-3}$ by Lemma 3.

If P has exponent e_P at least 4, we get that the planes through L which contain a point of B, not on L, contain at least $q_0^4+q_0+1$ points, which is impossible by Lemma 9. We conclude that P has exponent $e_P=2$ or $e_P=3$. If P has exponent $e_P=3$, then every plane through L that contains a point of B not on L, contains at least $q_0^3+q_0+1$ points, and hence, all planes through L are bad.

Finally, if P has exponent 2, we know from Lemma 7 that there are at least $s=q_0^{h-2}-q_0^{h-3}-q_0^{h-4}-3q_0^{h-5}+1$ secant lines through P, which implies that there are at least s planes through L containing a point of B outside L. Suppose t of the s planes are bad, than, using Lemma 8(iii), B contains at least $t(2q_0^2)+(s-t)(q_0^2)+q_0+1$ points. If we put $t=3q_0^{h-3}-q_0^{h-4}-3q_0^{h-5}+1$, we get a contradiction since $|B|\leq q_0^h+q_0^{h-1}+q_0^{h-2}+3q_0^{h-3}$ by Lemma 3. □

Lemma 11. *A point P of B lying on a (q_0+1)-secant, lies on at most one (q_0+1)-secant L that lies on only bad planes.*

Proof. Let P be a point of B, lying on a (q_0+1)-secant and let L be a line through P that only lies on bad planes. From Lemma 9 and Lemma 10, we get that $q_0^3+q_0+1 \le |\Pi \cap B| \le q_0^3+q_0^2+q_0+1$ for all planes Π through L, containing points of B outside L.

By Lemma 3, $|B| \le q_0^h + q_0^{h-1} + q_0^{h-2} + 3q_0^{h-3}$, so there are at most $q_0^{h-3}+2q_0^{h-4}$ planes through L containing points of B outside L. Since P lies on at least $q_0^{h-1}-4q_0^{h-2}+1$ (q_0+1)-secants, there are at least two planes Π_1 and Π_2 containing at least $q_0^2-6q_0+1$ (q_0+1)-secants through P. Suppose that L' is a (q_0+1)-secant through P, different from L, lying on only bad planes. At least one of the planes Π_1, Π_2, say Π_1, does not contain L'.

We will now show that for all $k \le h-2$, there exists a k-space through Π_1, not containing L', containing at least $q_0^k - 6q_0^{k-1}$ (q_0+1)-secants through P. For $k=2$, the statement is true, hence, suppose it holds for all $k \le j < h-2$. Let Π' be a j-space through Π_1, not containing L' and containing at least $q_0^j - 6q_0^{j-1}$ (q_0+1)-secants through P.

Let $|\Pi' \cap B| = A$, then a $(j+1)$-space Π'' through Π', containing a point of B, not in Π', contains at least $(q_0-1)A+1$ points of B, not in Π', and we see that the number of $(j+1)$-spaces containing a point of B, not in Π', is maximal if the number of points in Π' is minimal. Since $|B \cap \Pi_1| \ge q_0^3+q_0+1$, $|B \cap \Pi'| \ge (q_0^3+q_0+1)q_0^{j-3}+1$. This implies that the number of points of B in such a $(j+1)$-space, outside Π' is at least $q_0^{j+1}-p^j+q_0^{j-1}-q_0^{j-3}+p$. Since $|B| \le q_0^h+q_0^{h-1}+q_0^{h-2}+3q_0^{h-3}$, the number of such $(j+1)$-spaces is at most $q_0^{h-j-1}+2q_0^{h-j-2}+4q_0^{h-j-3}$. At most $(q_0^{j+1}-1)/(q_0-1)$ (q_0+1)-secants through P lie in Π'. Suppose that all $(j+1)$-spaces through Π', except possibly $\langle \Pi', L \rangle$ contain at most $q_0^j - 6q_0^{j-1}$ (q_0+1)-secants through P, not in Π', then the number of (q_0+1)-secants through P is at most

$$(q_0^{h-j-1}+2q_0^{h-j-2}+4q_0^{h-j-3}-1)(q_0^3-6q_0^2)+(q_0^{j+1}-1)/(q_0^{j-1}-1),$$

a contradiction if $j < h-2$, since there are at least $q_0^{h-1}-4q_0^{h-2}+1$ (q_0+1)-secants through P. We may conclude, by induction, that there exists an $(h-2)$-space Π'', not through L', that contains at least $q_0^{h-2}-6q_0^{h-3}$ (q_0+1)-secants through P. Since L' does not lie in Π'', this implies that there are at least $q_0^{h-2}-6q_0^{h-3}$ different planes through L' that each have at least q_0^3 points outside L, a contradiction since $|B| \le q_0^h+q_0^{h-1}+q_0^{h-2}+3q_0^{h-3}$. This implies that there is at most one line through P that lies on only bad planes. $\square$

3 The proof of the main theorem

Lemma 12. *Assume $h > 3$ and $q_0 > 5h-11$. Denote the (q_0+1)-secants, not lying on only bad planes, through a point P of B that lies on at least one (q_0+1)-secant, by $L_1, \ldots, L_s$. Let x be a point of the spread element* *corresponding to B in* $\mathrm{PG}(h(n+1)+1, q_0)$ *and let ℓ_i be the line through x such that $\mathcal{B}(\ell_i) = L_i \cap B$. Let $\mathcal{L} = \{\ell_1, \ldots, \ell_s\}$, then $\langle \mathcal{L} \rangle$ has dimension h.*

Proof. From Lemma 4 and Lemma 11 we get that s is at least $q_0^{h-1} - 4q_0^{h-2} + 1 - 1 = q_0^{h-1} - 4q_0^{h-2}$. From Lemmas 8(ii) and 10, we get that through every line L_i, $i = 1, \ldots, s$, there are at least $q_0^{h-2} - 4q_0^{h-3} + 1$ planes, say Π_{ij}, $j = 1, \ldots, t$, such that $B \cap \Pi_{ij} = \mathcal{B}(\pi_{ij})$, for a plane π_{ij} through ℓ_i. Denote the set of planes $\{\pi_{ij}, 1 \le i \le s, 1 \le j \le t\}$ by $\mathcal{V}$, and the set of lines $\{\ell_1, \ldots, \ell_s\}$ by $\mathcal{L}$.

A fixed plane π_{ij} of $\mathcal{V}$, say π_{11}, contains q_0+1 lines of $\mathcal{L}$, say $\ell_1, \ldots, \ell_{q_0+1}$. The lines $\ell_1, \ldots, \ell_{q_0+1}$ lie on a set of at least $(q_0+1)(q_0^{h-2} - 4q_0^{h-3} + 1) + 1$ different planes of $\mathcal{V}$. On these planes, there lie a set $\mathcal{P}$ of at least $(q_0 + 1)(q_0^{h-2} - 4q_0^{h-3} - 1)q_0^2$ different points $y_1, \ldots, y_u$, not in π_{11}, such that $\mathcal{B}(y_i) \subset B$.

We claim that $\mathcal{B}(y_r) = \mathcal{B}(y'_r)$ implies that $y_r = y'_r$ for y_r and y'_r in $\mathcal{P}$ $(*)$. We know that y_r lies on π_{ij} and y'_r lies on $\pi_{i'j'}$ for some i, i', j, j'. Since $\mathcal{B}(\pi_{ij}) = B \cap \Pi_{ij}$ and $\mathcal{B}(\pi_{i'j'}) = B \cap \Pi_{i'j'}$, the lines $\langle \mathcal{B}(xy_r) \rangle$ and $\langle \mathcal{B}(xy'_r) \rangle$ are (q_0+1)-secants to B. Since we assume that $\mathcal{B}(y_r) = \mathcal{B}(y'_r)$, these (q_0+1)-secants coincide. Moreover, $\mathcal{B}(xy_r) \subset B$ and $\mathcal{B}(xy'_r) \subset B$, so xy_r and xy'_r are transversal lines through the same regulus, which forces $y_r = y'_r$. This proves our claim, hence, different points of the pointset $\mathcal{P}$ give rise to different points of B.

We will prove that, for all $2 \le k \le h$ there exists an k-space through x with at least $q_0^{k-1} - (5k-11)q_0^{k-2}$ lines of $\mathcal{L}$. The existence of π_{11} proves this statement for $k = 2$. Assume, by induction, that there exists a j-space through x, say ν, where $j < h-1$, containing at least $q_0^{j-1} - (5j-11)q_0^{j-2}$ lines of $\mathcal{L}$.

We will now count the number of couples ($\ell \in \mathcal{L}$ contained in ν, r a point, not in ν with $\langle r, \ell \rangle \in \mathcal{V}$). The number of lines of $\mathcal{L}$ in ν is at least $q_0^{j-1} - (5j-11)q_0^{j-2}$, the number of points $r \notin \nu$ with $\langle r, \ell \rangle \in \mathcal{V}$ for some fixed ℓ, is at least $(q_0^{h-2} - 4q_0^{h-3})q_0^2 - (q_0^{j+1} - 1)/(q_0 - 1)$. The number of points r with $\langle r, \ell \rangle \in \mathcal{V}$, is by $(*)$ at most $|B|$, hence, the number of points $r \notin \nu$ with $\langle r, \ell \rangle \in \mathcal{V}$ is at most $|B| - (q_0^{j-1} - (5j-11)q_0^{j-2})q_0 - 1$.

Hence, there is a point r, lying on (say) X different planes $\langle r, \ell \rangle$ of $\mathcal{V}$ with

$$X \ge \frac{(q_0^{j-1} - (5j-11)q_0^{j-2})(q_0^h - 4q_0^{h-1} - (q_0^{j+1} - 1)/(q_0 - 1))}{q_0^h + q_0^{h-1} + q_0^{h-2} + 3q_0^{h-3} - q_0^j + (5j-11)q_0^{j-1} - 1}.$$

This last expression is larger than $q_0^{j-1} - (5(j+1) - 11)q_0^{j-2}$, if $h > 3$, for all $j \le h - 1$.

This implies that the $j+1$-space $\langle r, \nu \rangle$, contains at least $(q_0^{j-1} - (5(j+1) -$

$11)q_0^{j-2})q_0+1$ lines of $\mathcal{L}$, hence, by induction, we find an h-dimensional-space through x containing at least $q_0^{h-1}-(5h-11)q_0^{h-2}$ lines of $\mathcal{L}$.

Suppose now that there is a line of the ℓ_i, say ℓ_s, not in this h-space ξ. Since by Lemma 10, there are at least $q_0^{h-2}-4q_0^{h-3}$ planes through ℓ_s, giving rise to $(q_0^{h-2}-4q_0^{h-3})(q_0^2-q_0)$ points z, which are not contained in ξ, such that $\mathcal{B}(z)\subset B$. By $(*)$, and the fact that there are at least $(q_0^{h-1}-(5h-11)q_0^{h-2})q_0+1$ points y in ξ such that $\mathcal{B}(y)\subset B$, we get that $|B|\geq q_0^h+q_0^{h-1}+q_0^{h-2}+3q_0^{h-3}$, a contradiction.

This shows that the dimension of $\langle\mathcal{L}\rangle$ is h. $\square$

We now use the following theorem, which is an extension of [9, Remark 3.3].

Theorem 13. [4, Corollary 1] *A blocking set of size smaller than* $2q$ *in* $\mathrm{PG}(n,q)$ *is uniquely reducible to a minimal blocking set.*

Main Theorem. *A small minimal blocking set* B *in* $\mathrm{PG}(n,q)$*, with exponent* e*,* $q=p^t$*,* p *prime,* $q_0:=p^e$*,* $q_0\geq 7$*,* $t/e=h$*, spanning an* $(h-1)$*-dimensional space is an* $\mathbb{F}_{q_0}$*-linear set.*

Proof. As seen in Lemma 12, there exists an h-dimensional space ξ in $\mathrm{PG}((n+1)h-1,q_0)$, such that $|\mathcal{B}(\xi)\cap B|\geq q_0^h-4q_0^{h-1}+1$. Define $\tilde{B}$ to be the union of $\mathcal{B}(\xi)$ and B and recall that $\mathcal{B}(\xi)$ is a small minimal $\mathbb{F}_{q_0}$-linear blocking set in $\mathrm{PG}(n,q)$. Clearly, $\tilde{B}$ is a blocking set, and its size is equal to $|B|+|\mathcal{B}(\xi)|-|B\cap B(\xi)|$. Hence, $|\tilde{B}|$ is at most $(q_0^{h+1}-1)/(q_0-1)+q_0^h+q_0^{h-1}+q_0^{h-2}+3q_0^{h-3}-(q_0^h-4q_0^{h-1}+1)<2q_0^h$. Theorem 13 shows that $B=\mathcal{B}(\xi)$, so we may conclude that B is an $\mathbb{F}_{q_0}$-linear set. $\square$

By the fact that the exponent of a small minimal blocking set in $\mathrm{PG}(n,q)$ is at least one (see Theorem 1), we get the following corollary.

Corollary 14. *All small minimal blocking sets in* $\mathrm{PG}(n,p^t)$*,* p *prime,* $p>5t-11$ *spanning a* $(t-1)$*-space, are* $\mathbb{F}_p$*-linear.*

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