Quantum electrodynamics in finite volume and nonrelativistic effective field theories

Z. Fodor[1,2,3], C. Hoelbling[1], S. D. Katz[3,4], L. Lellouch[5], A. Portelli[6], K. K. Szabo[1,2], B. C. Toth[1]

[1] *Department of Physics, University of Wuppertal, D-42119 Wuppertal, Germany*
[2] *Jülich Supercomputing Centre, Forschungszentrum Jülich, D-52428 Jülich, Germany*
[3] *Institute for Theoretical Physics, Eötvös University, H-1117 Budapest, Hungary*
[4] *MTA-ELTE Lendület Lattice Gauge Theory Research Group, H-1117 Budapest, Hungary*
[5] *CNRS, Aix Marseille U., U. de Toulon, CPT, UMR 7332, F-13288, Marseille, France*
[6] *School of Physics & Astronomy, University of Southampton, SO17 1BJ, UK*

Abstract

Electromagnetic effects are increasingly being accounted for in lattice quantum chromodynamics computations. Because of their long-range nature, they lead to large finite-size effects over which it is important to gain analytical control. Nonrelativistic effective field theories provide an efficient tool to describe these effects. Here we argue that some care has to be taken when applying these methods to quantum electrodynamics in a finite volume.

State-of-the-art lattice quantum chromodynamics (QCD) computations have reached such a high level of accuracy (see e.g. [1] and references therein) that small electromagnetic corrections, and other isospin breaking effects, are becoming important. These effects will have to be accounted for more and more systematically if the results of lattice calculations are to continue to be used to test the standard model and to search for new physics in increasingly precise experiments. Moreover, electromagnetic effects in hadrons are important in themselves. For instance, they are critical for understanding Big Bang nucleosynthesis as well as many properties of atomic nuclei or for determining the up and down quark masses. As a result, increasing attention is being focussed on including quantum electrodynamics (QED) corrections in lattice QCD calculations. A number of results concerning hadron and quark masses have been obtained, in the electroquenched approximation [2–8] and in full QCD+QED [9–12]. In addition, a method for including QED corrections in the lattice calculation of hadronic matrix elements has been proposed in [13].

A very important issue in such calculations is finite-volume effects. [1] Indeed, the vanishing mass of the photon implies that the leading finite-volume corrections are proportional to

[1] Throughout this paper we will be concerned with situations in which the linear dimensions of the lattice are much larger than the Compton wavelengths and the internal-structure length-scales of the particles under consideration.

inverse powers of the spatial dimension, L, of the lattice, instead of being exponentially suppressed, as they are in pure QCD calculations. These power corrections represent a large fraction–tens of percent–of the computed electromagnetic effects for typical lattice sizes $L \sim 3\,\text{fm}$, as shown in [5, 8, 12], where controlled infinite-volume extrapolations were performed. These corrections must be appropriately subtracted to obtain accurate results. Thus, it is important to know their precise analytical form and to constrain them as much as possible. [2]

An elegant and efficient method to determine the functional form of these effects was proposed and worked out in [14]. It is based on nonrelativistic effective field theories (NREFTs) and allows to compute these corrections in a systematic expansion in powers of $1/L$ for any spin-0 or 1/2 massive particles with (or without) internal structure, such as hadrons or nuclei. In this approach, the corrections are determined by the mass and by well-defined electromagnetic properties of the particle, such as its charge, charge radius, magnetic moment, etc. However, as pointed out in [12] and explained in more detail below, the results obtained in [14] do not fully agree with those of point-particle calculations performed in [12]. Since many checks of the point-particle calculation were performed in [12] and both results cannot be correct, we have been led to investigate the possible source of the discrepancy. As explained below, this investigation has led us to uncover a subtlety in the application of NREFTs to finite-volume QED, which goes beyond the calculation discussed in the present paper. [3]

To expose the problem and its solution, we focus on the calculation of the finite-volume corrections to the pole mass of spin-1/2 particles at $O(\alpha)$. As in [12, 14], we work with periodic boundary conditions in the QED_L formulation of QED in a finite volume. In this formulation, momentum modes of the photon field with $\vec{k} = \vec{0}$, for all values of k_0, are eliminated from the theory. This approach was first proposed in [16] and, as shown in [12], has many theoretical and practical advantages. Here we work with an infinite time direction, as it simplifies analytical calculations and is equivalent to the finite time extent, T, formulation with periodic boundary conditions, up to corrections which are smaller than any inverse power in T [12].

Finite-volume, electromagnetic corrections to the mass of a particle can be obtained from the difference of its on-shell, electromagnetic self-energy at rest, in finite and infinite volumes. Physically, the corrections in inverse powers of L arise mainly from the particle under consideration exchanging a photon with itself around the periodic three-volume. In [14], Davoudi and Savage (DS) compute this effect for a generic spin-1/2 particle using NREFT to N^3LO, i.e. to order $1/L^4$. The use of NREFT is entirely justified here, because we are after an asymptotic expansion of particle properties in inverse powers of L. NREFT is a low-momentum effective theory of QED and in QED_L, it provides expansions of low-momentum particle properties in powers of the infrared momentum cutoff, $2\pi/L$.

To understand the NREFT approach, we have repeated their calculation. We stop at N^2LO, i.e. up to and including terms proportional to $1/L^3$, because this is the order at which the subtlety, described below, first arises. We use the same NREFT lagrangian as they do,

[2] The two leading finite-volume corrections, proportional to $1/L$ and $1/L^2$, are given in terms only of the particle's charge and infinite-volume mass [12, 14]. This important feature allows precise infinite-volume extrapolations of the QED contributions to particle masses in lattice calculations. In particular, the finite-volume corrections do not depend on spin nor on particle structure [12, 14]. As shown in [12], this *universality* follows, under very general hypotheses, from QED Ward-Takahashi identities and the work of Lüscher on the analyticity properties of propagators and vertex functions [15].

[3] It is important to note that this subtlety does not affect the two leading, *universal*, finite-volume corrections mentioned above, as it only enters at order $1/L^3$, as shown below.

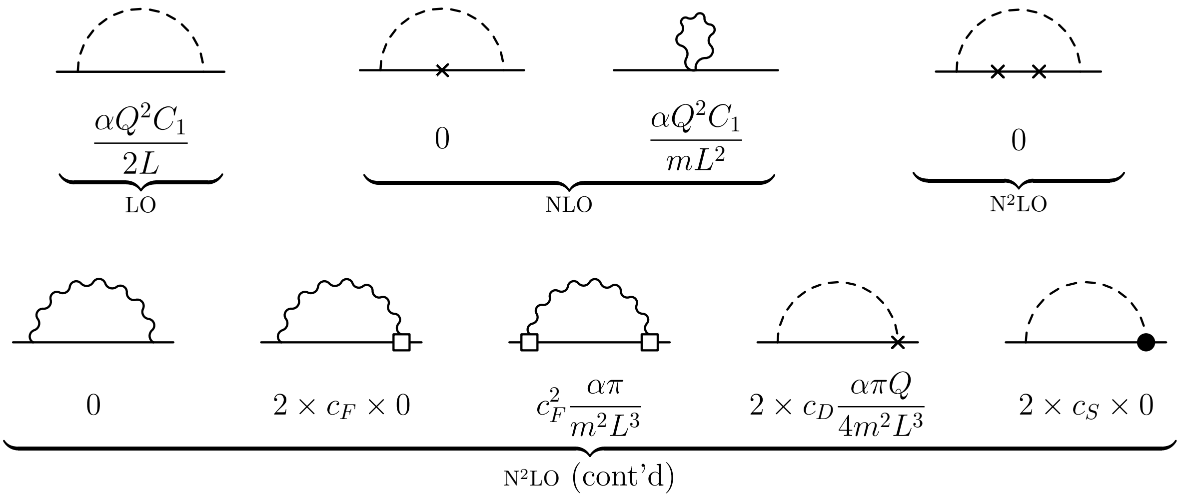

Figure 1: *NREFT self-energy diagrams which contribute to finite-volume effects on the mass of a spin-1/2 particle, up to order* $1/L^3$*. Their contributions were taken into account in [14]. The solid line corresponds to the fermion propagator; the dashed line to the temporal photon propagator; the wavy line to the transverse photon propagator; the cross to a kinetic term insertion; the simple vertices with these lines, to the fermion-temporal-photon interaction; to the fermion-transverse-photon interaction; to the fermion-to-two-transverse-photons interaction; the square vertex to the Fermi interaction; the crossed vertex to the Darwin interaction; the dot vertex to the spin-orbit interaction. Under each diagram we indicate its contribution to the finite-volume correction on the fermion mass. Calculations are performed in Coulomb gauge. An explicit factor of 2 appears in the single Fermi, Darwin and spin-orbit vertex terms. This indicates that to each of the three diagrams shown corresponds one in which the vertices are switched. These new diagrams give the same contributions as the original diagrams. The sum of all the contributions shown in the figure yields the result in Eq. (2).*

namely [17–24]:

$$\mathcal{L}_\psi = \psi^\dagger \Big[iD_0 + \frac{|\vec{D}|^2}{2m} + c_F \frac{e}{2m}\vec{\sigma}.\vec{B} + c_D \frac{e}{8m^2}\vec{\nabla}\cdot\vec{E} + ic_S \frac{e}{8m^2}\vec{\sigma}\cdot(\vec{D}\times\vec{E}-\vec{E}\times\vec{D}) + O(\vec{p}^4)\Big]\psi\ , \quad (1)$$

where ψ is a two-component spinor which annihilates a particle, $D_\mu = \partial_\mu + ieQA_\mu$, $c_F = Q+\kappa+O(\alpha)$, $c_D = Q + 4m^2\langle r^2\rangle/3 + O(\alpha)$, eQ is the charge of the particle, m its infinite-volume mass, $\langle r^2\rangle$ its mean-squared charge radius and κ its anomalous magnetic moment. The diagrams which contribute up to N^2LO are shown in Fig. 1. They yield the following finite-volume corrections:

$$\Delta_{\mathrm{DS}} m(L) \equiv m(L) - m \underset{L\to+\infty}{\sim} \alpha Q^2 \frac{C_1}{2L}\left[1+\frac{2}{mL}\right] + \frac{\pi\alpha}{m^2L^3}\left[c_F^2 + \frac{1}{2}c_D Q\right]\ , \quad (2)$$

where $m(L)$ is the value of the particle's mass in QED_L and $C_1 = -2.837297(1)$ [12, 25–27] is a known constant. This result fully agrees with DS's.

However, as noted above and in [12], if we reduce the corrections of Eq. (2) to the point particle case by setting $\langle r^2\rangle = 0$ and $\kappa = 0$, we find that it disagrees with the one that we computed directly in spinor QED. Indeed, in the reduced corrections, the term proportional to $(\pi\alpha Q^2/m^2L^3)$ has a coefficient of 3/2 instead of 3, as found in [12]. Moreover, in [12] we performed a precise, dedicated numerical study of finite-volume effects on the pole mass of a point-like fermion in QED_L. The study was conducted with fixed bare parameters and six lattice sizes ranging from $L/a = 24$ to $L/a = 128$, with a the lattice spacing. A fit of the measured, finite-volume mass, to a polynomial in a/L, highly favored our value for the $1/L^3$ coefficient over the one in [14]. To explain where this discrepancy comes from, we must now describe the main features of our relativistic calculation [12].

The computation in spinor QED is a straightforward asymptotic expansion in large L, of the difference between the finite and infinite-volume, on-shell self-energies of the particle [12]. As mentioned above, we consider this difference at $O(\alpha)$, i.e. we consider the usual, one-loop sunset diagram in which a photon is emitted and re-absorbed by the particle. The asymptotic expansion is most straightforwardly performed using a Poisson summation formula. In that approach, the corrections in powers of $1/L$ result from the nonanalyticities in the integrand/summand, associated with intermediate states going on shell in the domain of integration. The obvious singularities that arise in the present case are the particle and the positive and negative energy photon poles. The antiparticle pole, which is $2m$ away in energy, is only expected to contribute terms which fall off exponentially with $2mL$. This is an illustration of the decoupling of antiparticle modes that leads to the NREFT for a single, massive particle, used in [14].

Upon closer inspection, however, we find that this expectation is incorrect. Analyzing the different contributions shows that the antiparticle pole contributes to the contentious term of order $1/(m^2L^3)$, even though its propagator cannot go on shell for the given kinematics. This surprising result is due to the fact that the contribution of the zero modes of the photon are omitted from the loop sum. At order $1/L^3$, the finite-volume corrections come not only from singularities of the summand/integrand in the domain of integration, but also from the explicit subtraction of the photon zero modes. Since these modes couple particles to antiparticles, the latter also play a role in the calculation of finite-volume effects.

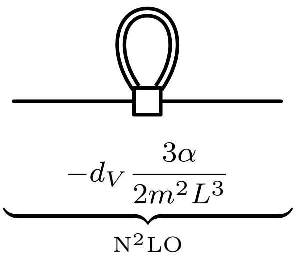

Figure 2: *Antiparticle contribution to the self-energy of a spin-1/2 particle which arises from the four-fermion lagrangian of Eq. (4). The double line corresponds to the antiparticle propagator. The vertex corresponds to the vector four-fermion coupling. Under the diagram we indicate its contribution to the finite-volume corrections on the fermion mass. The diagram contributes at order* $1/L^3$ *and provides the term which is missing in the calculation of [14].*

In the language of NREFT, we reach the same conclusions if we explicitly include antiparticle degrees of freedom, in addition to the usual contributions of antiparticles to higher-dimension particle operators. Thus, to the lagrangian density, $\mathcal{L}_\psi$, for particle fields given in Eq. (1) and used in [14], we add the one for antiparticle fields, χ, and include direct couplings of particles to antiparticles. Then, the lagrangian density becomes:

$$\mathcal{L}_{\mathrm{N^2LO}} = \mathcal{L}_\psi + \mathcal{L}_\chi + \mathcal{L}_{4f} + O(\vec{p}^{\,4}) \ , \tag{3}$$

where χ is a two-component spinor which annihilates antiparticles. In Eq. (3), $\mathcal{L}_\chi$ is obtained from $\mathcal{L}_\psi$ with the replacements $\psi \to \chi$ and $Q \to -Q$. The relevant four-fermion interaction, at $O(\alpha)$, originates from particle-antiparticle annihilation in the triplet channel and is given by [28]

$$\mathcal{L}_{4f} = d_V \frac{\alpha}{m^2} (\psi^\dagger \vec{\sigma} \sigma_2 \chi^*) \cdot (\chi^T \sigma_2 \vec{\sigma} \psi) + O(\alpha^2, \vec{p}^{\,4}) \ , \tag{4}$$

where the σ_i, $i = 1, 2, 3$, are the Pauli matrices. For point particles, $d_V = -\pi Q^2 + O(\alpha)$ [17,29].

From this four-fermion lagrangian, it is straightforward to compute the contribution of the antiparticle to the finite-volume effects in the particle's mass. It is given by the self-energy diagram of Fig. 2. We find this contribution to be

$$\Delta_{4f} m(L) = d_V \frac{3\alpha}{2m^2 L^3} \widehat{\sum_{\vec{q}\neq\vec{0}}} 1 \ , \tag{5}$$

where $\vec{q}$ is the momentum of the antiparticle in the loop and the sum, $\widehat{\sum}_{\vec{q}\neq\vec{0}}$, represents the difference between the sum over the finite-volume modes and the infinite-volume integral, i.e.

$$\frac{1}{L^3} \widehat{\sum_{\vec{q}\neq\vec{0}}} \equiv \frac{1}{L^3} \sum_{\vec{q}\neq\vec{0}} - \int \frac{d^3q}{(2\pi)^3} \ . \tag{6}$$

We now argue that the $\vec{q} = \vec{0}$ modes must be eliminated from the finite-volume sum, even though it is, initially, only photon zero modes which are removed. This is where the subtlety enters in NREFT. If the $\vec{q} = \vec{0}$ antiparticle modes were present, as one might guess, then the finite-volume correction of Eq. (5) would vanish and antiparticle degrees of freedom would not contribute. However, one must remember where internal particle or antiparticle lines in NREFT come from. In relativistic QED, an internal particle or antiparticle line in a diagram, such as the self-energy diagram under consideration, is produced at a vertex with a photon. But in QED_L that photon cannot have a vanishing three-momentum. Therefore in such diagrams, where all external particle lines have vanishing three-momenta, the internal antiparticle lines cannot have vanishing three-momenta. This justifies the omission of the $\vec{q}=\vec{0}$ antiparticle modes in the contribution of Eq. (5).

Now $\widehat{\sum}_{\vec{q}\neq\vec{0}} 1 = -1$, so the full NREFT expression for the finite-volume correction to the mass m of a spin-1/2 particle of charge Q is, to $O(1/L^3)$ in the presence of electromagnetism,

$$\Delta m(L) \underset{L\to+\infty}{\sim} \Delta_{\rm DS} m(L) - d_V \frac{3\alpha}{2m^2L^3} \ . \tag{7}$$

Using the value of d_V for a point particle given after Eq. (4), we find that this additional correction adds to the ones in $\Delta_{\rm DS}m(L)$ exactly the $(3\pi\alpha Q^2/2m^2L^3)$ term which is missing to reproduce the relativistic point-particle result of [12].

It is worth noting that the result of Eq. (7) can also be obtained by only reinstating the $\vec{q}=\vec{0}$ modes of the antiparticle field in the finite-volume NREFT. Then one uses the corresponding terms in the lagrangian to subtract these modes' contribution from the finite-volume self-energy. In that approach, the $\widehat{\sum}_{\vec{q}\neq\vec{0}} 1$ that appears in Eq. (5) would directly be replaced by -1. One need consider neither the spatial modes of the antiparticle field in finite volume, nor any antiparticle modes in infinite volume.

To conclude, we have considered the calculation of finite-volume corrections to the pole mass of a charged spin-1/2 particle, in the presence of electromagnetism. We have explained how the NREFT calculation of [14] can be reconciled with the relativistic QED result of [12]. In the process, we have shown that there are subtleties associated with applying NREFTs to the calculation of finite-volume effects in QED. In particular, we have argued that antiparticle degrees of freedom must be dealt with carefully, because photon zero modes are treated differently in finite and infinite volumes. Indeed, those modes can couple antiparticles to particles. Therefore, NREFT calculations of particle properties must account for the contribution of antiparticles to the removal of the photon zero modes in finite volume. However, once this contribution is suitably accounted for, the NREFT approach constitutes an elegant and efficient formalism to calculate finite-volume corrections.

Acknowledgments

This work was supported in part by the OCEVU Labex (ANR-11-LABX-0060) and the A*MIDEX project (ANR-11-IDEX-0001-02), funded by the "Investissements d'Avenir" French government program and managed by the ANR, by DFG grants FO 502/2 and SFB-TR 55 and by STFC grants ST/J000396/1 and ST/L000296/1.

References

[1] S. Aoki, Y. Aoki, C. Bernard, T. Blum, G. Colangelo, *et. al.*, *Eur.Phys.J.* **C74** (2014), no. 9 2890, [arXiv:1310.8555].

[2] A. Duncan, E. Eichten, and H. Thacker, *Phys.Rev.Lett.* **76** (1996) 3894–3897, [hep-lat/9602005].

[3] T. Blum, T. Doi, M. Hayakawa, T. Izubuchi, and N. Yamada, *Phys.Rev.* **D76** (2007) 114508, [arXiv:0708.0484].

[4] T. Blum, R. Zhou, T. Doi, M. Hayakawa, T. Izubuchi, *et. al.*, *Phys.Rev.* **D82** (2010) 094508, [arXiv:1006.1311].

[5] **Budapest-Marseille-Wuppertal** Collaboration, S. Borsanyi, S. Dürr, Z. Fodor, J. Frison, C. Hoelbling, *et. al.*, *Phys.Rev.Lett.* **111** (2013) 252001, [arXiv:1306.2287].

[6] **RM123** Collaboration, G. de Divitiis *et. al.*, *Phys.Rev.* **D87** (2013), no. 11 114505, [arXiv:1303.4896].

[7] S. Drury, T. Blum, M. Hayakawa, T. Izubuchi, C. Sachrajda, *et. al.*, *PoS* **LATTICE2013** (2014) 268, [arXiv:1312.0477].

[8] S. Basak *et. al.*, *PoS* **LATTICE2014** (Sept., 2014) [arXiv:1409.7139].

[9] S. Aoki, K. Ishikawa, N. Ishizuka, K. Kanaya, Y. Kuramashi, *et. al.*, *Phys.Rev.* **D86** (2012) 034507, [arXiv:1205.2961].

[10] T. Ishikawa, T. Blum, M. Hayakawa, T. Izubuchi, C. Jung, *et. al.*, *Phys.Rev.Lett.* **109** (2012) 072002, [arXiv:1202.6018].

[11] **QCDSF** Collaboration, R. Horsley *et. al.*, *PoS* **Lattice2013** (2014) 499, [arXiv:1311.4554].

[12] **Budapest-Marseille-Wuppertal** Collaboration, S. Borsanyi, S. Durr, Z. Fodor, C. Hoelbling, S. Katz, *et. al.*, arXiv:1406.4088.

[13] N. Carrasco, V. Lubicz, G. Martinelli, C. Sachrajda, N. Tantalo, *et. al.*, arXiv:1502.0025.

[14] Z. Davoudi and M. J. Savage, *Phys.Rev.* **D90** (2014), no. 5 054503, [arXiv:1402.6741].

[15] M. Lüscher, *Commun.Math.Phys.* **104** (1986) 177.

[16] M. Hayakawa and S. Uno, *Prog.Theor.Phys.* **120** (2008) 413–441, [arXiv:0804.2044].

[17] W. Caswell and G. Lepage, *Phys.Lett.* **B167** (1986) 437.

[18] B. Thacker and G. P. Lepage, *Phys.Rev.* **D43** (1991) 196–208.

[19] P. Labelle, hep-ph/9209266.

[20] A. V. Manohar, *Phys.Rev.* **D56** (1997) 230–237, [hep-ph/9701294].

[21] M. E. Luke and M. J. Savage, *Phys.Rev.* **D57** (1998) 413–423, [hep-ph/9707313].

[22] J.-W. Chen, G. Rupak, and M. J. Savage, *Nucl.Phys.* **A653** (1999) 386–412, [nucl-th/9902056].

[23] R. J. Hill, G. Lee, G. Paz, and M. P. Solon, *Phys.Rev.* **D87** (2013), no. 5 053017, [arXiv:1212.4508].

[24] J.-W. Lee and B. C. Tiburzi, *Phys.Rev.* **D90** (2014), no. 7 074036, [arXiv:1407.8159].

[25] M. Lüscher, *Commun.Math.Phys.* **105** (1986) 153–188.

[26] P. Hasenfratz and H. Leutwyler, *Nucl.Phys.* **B343** (1990) 241–284.

[27] M. Lüscher, *Nucl.Phys.* **B354** (1991) 531–578.

[28] P. Labelle, S. Zebarjad, and C. Burgess, *Phys.Rev.* **D56** (1997) 8053–8061, [hep-ph/9706449].

[29] T. Kinoshita and G. P. Lepage, in "Quantum Electrodynamics," edited by T. Kinoshita (World Scientific, Singapore, 1990), pp. 81-89.