# Relative information entropy in cosmology: The problem of information entanglement

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**Abstract**

The necessary information to distinguish a local inhomogeneous mass density field from its spatial average on a compact domain of the universe can be measured by relative information entropy. The Kullback-Leibler (KL) formula arises very naturally in this context, however, it provides a very complicated way to compute the mutual information between spatially separated but causally connected regions of the universe in a realistic, inhomogeneous model. To circumvent this issue, by considering a parametric extension of the KL measure, we develop a simple model to describe the mutual information which is entangled via the gravitational field equations. We show that the Tsallis relative entropy can be a good approximation in the case of small inhomogeneities, and for measuring the independent relative information inside the domain, we propose the Rényi relative entropy formula.

*Keywords:* Cosmic inhomogeneities, Information entropy, Large scale structure

## 1. Introduction

Understanding the evolution of local and global inhomogeneities of the universe is a central problem in modern cosmology. The question has been actively investigated from several different points of view (e.g. perturbation theory [1], exact models [2], numerical simulations [3]), and a more recent approach attacks from the direction of information theory [4, 5]. In the work of Hosoya *et al.* it has been shown [4], that the Kullback-Leibler relative information entropy [6] – a standard notion in information theory – arises very naturally in this context. It can describe the "distinguishability" of an actual mass density field from its spatial average on a compact domain of the universe, and its time derivative can also provide the commutation relation between the time evolution and the volume average operation on the density field.

The KL entropy is therefore a very useful measure to study the evolution of cosmic inhomogeneities inside a given domain, however, since gravity is a long-range interaction, it is often the case that the mutual information between spatially separated but causally connected domains of the universe is also important to consider. In order to compute the entangled information via the gravitational field equations in a realistic, inhomogeneous model, one has to face a rather formidable problem. The exact way to do so would be to follow a dynamical volume partitioning method (see e.g. the work of Wiegand and Buchert [7]), where the matter distribution function would have to be known globally, by solving complicated system of the Einstein's field equations. From an observationally motivated point of view for example, this approach is particularly difficult.

To circumvent this issue, in this Letter, we propose a simple model where we consider a parametric extension of the KL measure to describe the evolution of cosmic inhomogeneities on a domain whose dynamics is informationally entangled to its causally connected surroundings. In order to extract the mutual information between separated domains, instead of computing the explicit relations between the distribution functions via the exact method above, we consider a different approach where the distribution functions are formally treated as independent, but the corresponding relative entropy functions are nonadditive. We show, that under some physically motivated assumptions, this approach leads quite generally to the Tsallis relative entropy formula [9], which can parametrically incorporate the long-range interaction property of the gravitational field, based on the causal structure of the problem. Although nonadditive (nonextensive) phenomena have been known in cosmology and gravitation theory for decades, and the standard Tsallis entropy [10] has also been investigated several times in cosmological applications [11], as far as we know, this is the first time that a similar approach is considered in connection with cosmic inhomogeneities and information entropy. For simplicity, in the present work we restrict our investigations to linearly perturbed, spatially flat, Friedmann-Lemaître-Robertson-Walker (FLRW) dust cosmologies, however we expect that our model can be extended to more general, inhomogeneous cosmological models as well. Throughout

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this Letter we use units such that $c = G = 1$.

## 2. Background

The KL relative information entropy for continuous probability distributions $p(x)$ and $\overline{p}(x)$ on a compact domain, $D$, is defined as

$$S_{KL}\{p|\overline{p}\} = \int_D p(x) \ln \frac{p(x)}{\overline{p}(x)} dx. \tag{1}$$

For studies of cosmic inhomogeneities, the relevant distributions to consider are the matter density $\rho(t, x^i)$ and its spatial average, $\overline{\rho}(t)$, over an arbitrary domain of the universe. Hosoya *et al.* [4] have shown, that for a dust continuum cosmological model, described by the metric

$$ds^2 = -dt^2 + g_{ik} dx^i dx^k, \tag{2}$$

with a predefined time-orthogonal foliation, the KL relative entropy can be written as

$$\frac{S_{KL}\{\varrho|\overline{\varrho}\}}{V_D} = \overline{\varrho \ln \frac{\varrho}{\overline{\varrho}}}\ , \tag{3}$$

where overbar denotes the volume average operation:

$$\overline{\psi}(t) = \frac{1}{V_D} \int_D \psi(t, x^i) \sqrt{g}\, d^3x, \tag{4}$$

defined for any scalar field $\psi$ in the volume $V_D = \int_D \sqrt{g} d^3x$. In (4) $g$ is the determinant of the 3-metric $g_{ik}$, and $x^i$ are coordinates on a $t =$ const. hypersurface within a comoving gauge. By exploiting the identity $\dot{\overline{\psi}} - \overline{\dot{\psi}} = \overline{\psi\,\theta} - \overline{\psi}\,\overline{\theta}$ with $\theta = \dot{V}_D/V_D$ being the local expansion rate, and also applying the continuity equation $\dot{\overline{\varrho}} + \overline{\theta}\,\overline{\varrho} = \overline{\dot{\varrho} + \theta\varrho} = 0$, it can be shown [4] that the KL relative entropy is the generating functional of the commutation relation between the volume average and the time evolution of the density field, in the sense that

$$-\frac{\dot{S}_{KL}\{\varrho|\overline{\varrho}\}}{V_D} = \dot{\overline{\varrho}} - \overline{\dot{\varrho}}\ . \tag{5}$$

Hosoya *et al.* analyzed this relation, and argued that after long enough time, and on sufficiently large scales of averaging, $S_{KL}\{\varrho \,|\, \overline{\varrho}\}$ is an increasing function of time, and hence a reasonable entropy description for the evolution of local inhomogeneities.

## 3. The causal structure of the problem

The KL relative information entropy is additive for factorizing probabilities. More precisely, $S_{KL}\{\mathcal{A} \otimes \mathcal{B}\} = S_{KL}\{\mathcal{A}\} + S_{KL}\{\mathcal{B}\}$, if $\mathcal{A}$ and $\mathcal{B}$ are two *independent* systems in the sense that the probability distributions $p(x)$ and $\overline{p}(x)$ of $\mathcal{A} \otimes \mathcal{B}$ *factorizes* into those of $\mathcal{A}$ and of $\mathcal{B}$. As a consequence, the *mutual information*, defined usually as

$$I(\mathcal{A}, \mathcal{B}) \ = \ S\{\mathcal{A}\} + S\{\mathcal{B}\} - S\{\mathcal{A} \otimes \mathcal{B}\}, \tag{6}$$

is zero for composite independent systems. The assumption of independence, however, is obviously not valid for gravitationally coupled systems, like those in cosmology, where the evolution of a density field on an arbitrary compact domain of the universe is always influenced by its dynamics on its neighboring, causally connected regions via the gravitational field equations. On Fig. 1, we schematically plotted the causal structure of the evolution for an FLRW model. Here $J^-(\mathcal{A})$ is the causal past of $\mathcal{A}$ on

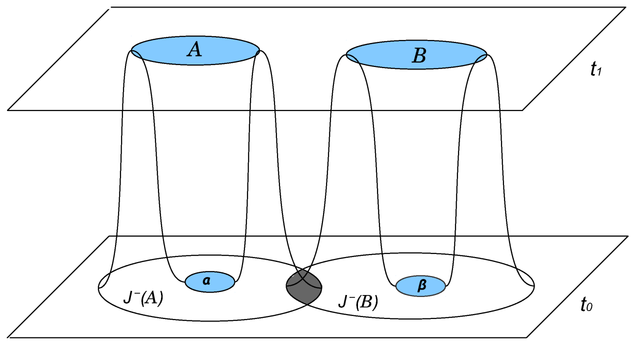


Figure 1: The causal structure of the evolution of two spatially separated domains in an FLRW universe. $\alpha$ and $\beta$ are the cosmological pasts of $\mathcal{A}$ and $\mathcal{B}$ respectively, while $J^-(\mathcal{A})$ and $J^-(\mathcal{B})$ are their intersecting causal pasts on $t_0$.

an arbitrary reference time slice $t_0$ in the past, while $\alpha$ is the cosmological past of $\mathcal{A}$. $\mathcal{B}$ is a spatially separated domain from $\mathcal{A}$ on $t_1$, $\beta$ is its cosmological-, while $J^-(\mathcal{B})$ is its causal past on $t_0$. It is clear from the picture that due to the long-range interaction property of the gravitational field (manifested by the intersection of $J^-(\mathcal{A})$ and $J^-(\mathcal{B})$), the mutual information $I(\mathcal{A}, \mathcal{B})$ at $t_1$ is obviously not zero, however as we have pointed out in the introduction, it is complicated to compute its value in general via the gravitational field equations in a realistic, inhomogeneous universe model by using an exact dynamical volume partitioning method, like the one in [7]. Nevertheless, since the time evolution of a density field on a domain $D$ is clearly not independent from its evolution on causally connected regions to $D$, it would be nice to have a simple way to estimate the information entanglement between them.

In the following sections, by considering a universe model described by a linearly perturbed FLRW metric with a dust continuum matter filed, we develop a parametric approximation for the entanglement problem. In this model we will stay completely inside the framework defined by Hosoya et al., i.e. we have a global, time orthogonal foliation with an inhomogeneous spacelike metric $g_{ik}$ that is comoving with the matter perturbations in the considered linear order.

## 4. A parametric entropy extension

On large enough scales, as nicely confirmed by CMBR experiments [8], the universe can be well described in a thermodynamic equilibrium during its evolution. Unlike in standard thermodynamics however, as we have seen in the previous section, spatially separated domains do not evolve independently in general relativity, so it is a natural consequence that the entropy function of these regions are *not additive* for composition. When computing the joint entropy of separated domains in the model of Hosoya et al., the nonadditive part in the KL measure arises from the mutual dependence of the density and metric functions of the domains via the gravitational interaction. The practical computability of this entanglement is very difficult, and our approach in this Letter is to develop a toy model instead, where we formally treat the density functions as independent, but consider a parametric extension of the KL entropy function which is nonadditive even for independent distributions. The new entropy parameter will then be used to describe the mutual information between causally connected regions of the universe.

Based on the concept of composability alone, Abe showed [13], that the most general nonadditive entropy composition rule which is compatible with equilibrium requirements can be written in the form

$$H_\lambda(S_{\mathcal{A}\otimes\mathcal{B}}) = H_\lambda(S_\mathcal{A}) + H_\lambda(S_\mathcal{B}) + \lambda H_\lambda(S_\mathcal{A})H_\lambda(S_\mathcal{B}), \quad (7)$$

where $H_\lambda$ is some differentiable function of $S$, $\lambda \in \mathbb{R}$ is a parameter, and $S_\mathcal{A}$, $S_\mathcal{B}$ and $S_{\mathcal{A}\otimes\mathcal{B}}$ are the entropies of the subsystems and the joint system, respectively.

For the gravitationally coupled problem of cosmological inhomogeneities, the explicit form of $H_\lambda$ is virtually impossible to compute, not to mention the entropy problem of the gravitational field itself. On the other hand, the most recent results of large scale structure measurements indicate that the matter distribution in the physical (approximately FRLW) universe can be extremely well approximated by a homogeneous and isotropic dust model above the $\sim 115Mpc$ scale [14]. Local inhomogeneities start to grow below this scale and their distribution becomes increasingly structured on smaller and smaller scales. Above this scale, no further contribution to the relative information entropy can be expected from the statistically homogeneous matter distribution, and for scales not much smaller than this (i.e. around the linear regime in inhomogeneities) the entropy composition rule is not expected to deviate too much from additivity. These considerations imply that the $\lambda$-parameter in Abe's formula is not zero but reasonably small, i.e. $|\lambda| << 1$, and also that the deviation of $H_\lambda$ from the identity function may not be too strong either. For regions with this magnitude of inhomogeneities, the nonadditive entropy composition rule in (7), in the leading order of the $\lambda$-parameter, can be approximated by the formula

$$S_{\mathcal{A}\otimes\mathcal{B}} = S_\mathcal{A} + S_\mathcal{B} + \lambda S_\mathcal{A} S_\mathcal{B} + \ldots \, . \quad (8)$$

This approximate form is not immediate to derive from (7) under our conditions, and we intend to present a detailed proof in a forthcoming communication [15].

Formula (8), on the other hand, is a well known expression, called the Tsallis–Pareto composition rule, and the corresponding relative entropy measure has been developed by Tsallis in [9]. The general definition is given by

$$S_T\{p\,|\,\overline{p}\} = \frac{1}{q-1}\int_D p(x)\left[\left(\frac{p(x)}{\overline{p}(x)}\right)^{q-1} - 1\right]dx, \quad (9)$$

where $q = 1+\lambda$, and the $q \to 1$ ($\lambda \to 0$) limit recovers the KL relative entropy, as is expected by consistency requirements. Therefore, the Tsallis extension to the KL relative entropy formula arises very naturally from Abe's formula (7) as a consequence of the cosmological setup for small inhomogeneities in an FLRW universe, and it is easy to show that in our framework it takes the form

$$\frac{S_T\{\varrho\,|\,\overline{\varrho}\}}{V_D} \;=\; \frac{1}{V_D}\int_D \varrho\,\delta_\lambda\sqrt{g}\,d^3x \;\equiv\; \overline{\varrho\,\delta_\lambda}, \quad (10)$$

where we defined the $\lambda$-*deformed density contrast* function as

$$\delta_\lambda = \frac{1}{\lambda}\left[\left(\frac{\varrho}{\overline{\varrho}}\right)^\lambda - 1\right], \qquad \lim_{\lambda\to 0}\delta_\lambda = \ln(1+\delta), \quad (11)$$

with $\delta = (\varrho - \overline{\varrho})/\overline{\varrho}$ being the standard density contrast.

## 5. A geometric model for $\boldsymbol{\lambda}$

The $q$- or $\lambda$-parameter in the Tsallis formula is usually constant in different physical situations and its explicit value is a part of the problem to be solved. In our model however, we will consider a novel approach by allowing it to be time dependent in order to describe all possible causal relations between spatially separated domains during their evolution. In Abe's work [13], the $\lambda$-parameter appears originally as a separation constant in integrating the problem of the joint entropy function, hence, it seems to be a natural choice to connect it to some common but still independent property of the cosmological domains, i.e. their common causal past. Furthermore, we also expect that the mutual information between the causally connected domains should be proportional to the volume of their past causal intersection, so let us define

$$\lambda(t) \;:=\; -\lambda_0\frac{V_{J^-(\mathcal{A})\cap J^-(\mathcal{B})}}{V_{J^-(\mathcal{A})} + V_{J^-(\mathcal{B})} - V_{J^-(\mathcal{A})\cap J^-(\mathcal{B})}}, \quad (12)$$

where $V_{J^-(\mathcal{A})\cap J^-(\mathcal{B})}$ is the volume of the intersection of the causal pasts of $\mathcal{A}$ and $\mathcal{B}$ on $t_0$ (see Fig. 1). For our interest, $t_0$ can be e.g. the time of decoupling, ever since the dust model is considered to be a reasonable cosmological approximation.

It is easy to see from the definition that $\lambda(t)$ is zero for causally disconnected regions and $\lim_{t\to\infty}\lambda(t) = -\lambda_0$, so $\lambda(t) \in (-\lambda_0, 0]$ for all $t$. The $\lambda_0$ constant is expected to be different for different regions, and its explicit value may be constrained by observational data. On the other hand, based on our linear approximation in the model, we require $\lambda_0 \ll 1$ just below the $115Mpc$ scale. On gradually decreasing scales, $\lambda_0$ is expected to grow until it reaches the limit where our perturbative approach will eventually fail to be satisfied. Between the two extrema of $\lambda(t)$, there is an initial period for most spatially separated domains from $\mathcal{A}$, during which their causal past is disjoint from $J^-(\mathcal{A})$. After this, the explicit form of $\lambda(t)$ depends on the actual geometry of the domains in question. As an example, on Fig. 2, we have plotted the time dependence of $\lambda(t)$ (and its time derivative) for the case of two, initially spherical domains.

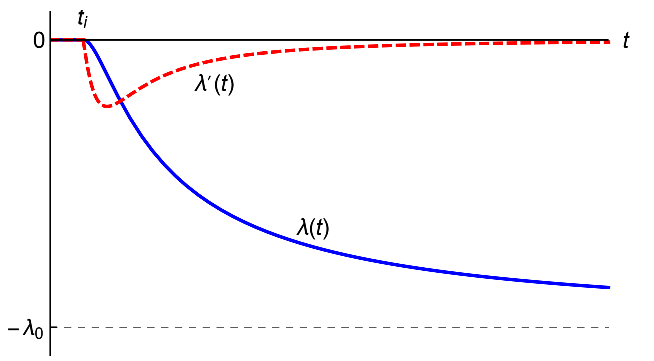


Figure 2: The time dependence of the $\lambda(t)$-parameter (blue, continuous curve) and its time derivative (red, dashed curve) for the case of two, initially spherical domains $\mathcal{A}$ and $\mathcal{B}$. $t_i \in [t_0, t_1]$ denotes the moment when the causal pasts $J^-(\mathcal{A})$ and $J^-(\mathcal{B})$ starts to intersect on $t_0$. The effects of inhomogeneities in the volume ratio are neglected on the plot.

The interpretation of $\lambda(t)$ is fairly straightforward, although it is interesting to note that within this model, the meaning of relative information entropy becomes a *relativistic* notion. In other words, it makes sense to talk about the relative information entropy of a domain only when we also specify a *reference domain* with respect to which we measure it. The $\lambda(t)$-parameter keeps track of the causal connection between the domains, and via the Tsallis formula we can also compute the mutual information between them. By substituting (8) into (6) we get $I_T(\mathcal{A},\mathcal{B}) = -\lambda S_T\{\mathcal{A}\} S_T\{\mathcal{B}\}$, which is always positive for causally connected regions, and zero otherwise. It can also be checked that the Tsallis relative entropy is a well defined measure for all $\lambda(t)$, i.e. it satisfies the necessary conditions: $S_T\{\varrho \,|\, \overline{\varrho}\} \geq 0$ for $\varrho > 0$, and $S_T\{\varrho \,|\, \overline{\varrho}\} = 0$ for $\varrho = \overline{\varrho}$, i.e. for homogeneous distributions the relative information entropy is zero [9].

Unless one is specifically interested in estimating the mutual information between, say, two disjoint superclusters, the most natural choice for a reference region to any domain is its entire causally connected surroundings. According to a result of Stewart and Walker [12], the relative information entropy functionals in our linear approximation are gauge invariant since their values are identically zero on the homogeneous background. This would allow us, in principle, to consider more general inhomogeneous cosmologies as well, not just our linearly perturbed FLRW case. Nevertheless, in this Letter we consider the simplest FLRW universe in the comoving and time orthogonal framework, which satisfies the condition that all domains of the scenario plotted on Fig. 3, can be described simultaneously within the comoving gauge condition during their evolution.

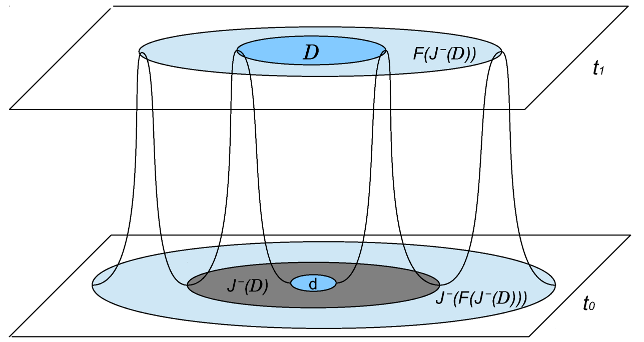


Figure 3: The causal structure of the evolution of a cosmological domain and its surroundings.

On the figure, the causal past of $D$ on $t_0$ is $J^-(D)$, the future development of $J^-(D)$ on $t_1$ is $F(J^-(D))$ (i.e. it is the entire causally connected region to $D$ in the present), while $J^-(F(J^-(D)))$ is the causal past of $F(J^-(D))$ on $t_0$. Now, according to definition (12), the $\lambda(t)$-parameter can be computed as

$$\lambda(t) = -\lambda_0 \, \frac{V_{J^-(\mathcal{D})}}{V_{J^-(F(J^-(D)))}} \,. \tag{13}$$

The most interesting feature of this result is that $\lambda(t)$ above can be approximated by a constant with very high accuracy. Indeed, as time passes, the regions $J^-(D)$ and $J^-(F(J^-(D)))$ on $t_0$ expand with the same $\sim ct$ rate, and hence their volume ratio (13) remains approximately constant during the cosmic evolution because the effects of inhomogeneities on $t_0$ can be neglected in the volume ratio. As a consequence: *every domain in the universe has an (approximately) constant $\lambda$-parameter in its relative information entropy function which can account for the maximal amount of information that is entangled (or mutual) between the domain and its causal surroundings in the model.*

The volume ratio in (13) can be roughly estimated for any region below the size of the statistically homogeneous scale today. Considering that $t_1 - t_0$ is around 13 billion years since the time of decoupling, the radius of the cosmological past of any domain in this scale can be neglected compared to the radii $\sim c(t_1 - t_0)$ of $J^-(D)$ and $\sim 2c(t_1 - t_0)$ of $J^-(F(J^-(D)))$ on $t_0$. The volume ratio is proportional to the cubes of these radii so $V_{J^-(\mathcal{D})}/V_{J^-(F(J^-(D)))} \approx 1/8$, and thus $\lambda(t) \approx -\lambda_0/8$. The real problem is thus how to estimate $\lambda_0$ for a given domain.

## 6. Time evolution

According to our geometric model, the Tsallis extension of the KL entropy can elegantly measure the relative information in a cosmological domain while also describing the information entanglement with its causally connected surroundings. It is therefore also expected that similarly to the KL entropy, the time derivative of this measure can also provide information about the commutation relation between the time evolution and the volume average operation inside the domain. By differentiating (10) with respect to time and after doing some algebra one can show that

$$-\frac{\dot{S}_T\{\varrho|\overline{\varrho}\}}{V_D} = \dot{\overline{\varrho_\lambda}} - \overline{\dot{\varrho}_\lambda} + \frac{\dot{\lambda}}{\lambda}\left[\overline{\varrho\delta_\lambda} - \overline{\varrho_\lambda \ln\frac{\varrho}{\overline{\varrho}}}\right], \qquad (14)$$

where we defined the $\lambda$-*deformed density field* as $\varrho_\lambda = \varrho(1+\lambda\delta_\lambda)$. In the limit of $\lambda \to 0$ it obviously recovers (5), and for the most relevant $\lambda \approx$ *const.* case, it reduces to $\dot{\overline{\varrho_\lambda}} - \overline{\dot{\varrho}_\lambda}$. Hence, the Tsallis measure can also serve as the generating functional of the commutation relation between the time evolution and the volume average operation inside the domain, however it provides this relation on $\varrho_\lambda$, which explicitly depends on the maximal mutual information between the domain and its surroundings via a simple, parametric form.

## 7. Independent information

After obtaining the mutual information between causally connected regions in the model, one might also be interested in how to determine the independent information inside a given domain. In a recent paper [16], Biró and Ván suggested a formalism, called the "formal logarithm" method, on how to describe systems with nonadditive thermodynamic properties in order to be compatible with the standard laws of thermodynamics. They showed that all equilibrium compatible entropy function that follows Abe's nonadditive composition rule (7) can be mapped by the function

$$L(S) = \frac{1}{\lambda}\ln[1 + \lambda H_\lambda(S)], \qquad (15)$$

to an additive one, i.e. $L(S_{\mathcal{A}\otimes\mathcal{B}}) = L(S_\mathcal{A}) + L(S_\mathcal{B})$, which is also a well defined, meaningful entropy measure. The best example of this result is the standard Tsallis entropy function [10] (with the same composition rule (8)) whose formal logarithm turns out to be the Rényi entropy formula [17]. Based on the complete analogy of our problem to the standard Tsallis–Rényi correspondence, let us define the *Rényi relative entropy* function for our purposes (for more general definitions see e.g. [18] and references therein) as the formal logarithm of the Tsallis relative entropy, i.e.

$$S_R\{p|\overline{p}\} \;:=\; L(S_T\{p|\overline{p}\}) \;\equiv\; \frac{1}{\lambda}\ln\left[1 + \lambda S_T\{p|\overline{p}\}\right], \qquad (16)$$

and give its general form as

$$S_R\{p|\overline{p}\} = \frac{1}{q-1}\ln\left[\int_D p(x)\left(\frac{p(x)}{\overline{p}(x)}\right)^{q-1} dx\right], \qquad (17)$$

with $q = 1+\lambda$, as before. (17) clearly recovers the KL measure in the $\lambda \to 0$ limit and satisfies all the conditions that are required from a well defined relative entropy measure. In our cosmological setting it reads as

$$S_R\{\varrho|\overline{\varrho}\} = \frac{1}{\lambda}\ln\left[1 + \lambda V_D \overline{\varrho\,\delta_\lambda}\,\right], \qquad (18)$$

and since it is additive for composition, the corresponding mutual information, $I_R(\mathcal{A},\mathcal{B})$, between two domains in our model is always zero. Accordingly, formula (18) can measure the *independent information* inside the domain by also taking into account the causal relation between the domain and its surroundings via the $\lambda(t)$-parameter.

## 8. Summary and discussion

In this Letter, we have investigated the problem of information entanglement between causally connected regions of the universe, and shown that one can estimate the effect by a parametric relative information entropy measure using the Tsallis formula (9). We defined the parameter function of the Tsallis entropy in a geometric way (12) in order to describe the causal connection between the domain and its surroundings. For measuring the independent information inside the domain, we proposed the Rényi relative entropy (17). The present value of the $\lambda(t)$ parameter for every given domain may be estimated via observations, in particular, the $\lambda_0$ constant may be possible to obtain from large scale structure data.

As pointed out in [4], the commutation rule for averaging a scalar $\psi$ in cosmology is $\dot{\overline{\psi}} - \overline{\dot{\psi}} = \overline{\psi\delta\theta} = \overline{\delta\psi\theta}$, which upon substitution in (14) for the most relevant $\lambda \approx$ *const.* case, gives $-\dot{S}_T\{\varrho|\overline{\varrho}\}/V_D = \overline{\rho_\lambda\delta\theta} = \overline{\delta\rho_\lambda\theta}$. Therefore, the relative information entropy grows in time for overdense (contracting) or underdense (expanding) regions, corresponding to what is physically expected for large enough times in cosmology. In general, see e.g. [19], one does not expect time convexity in the growth of cosmological inhomogeneities. This seems to be the case for models of the present universe which include dust and a positive cosmological constant [20]. However, in those models, and whenever the cosmic no-hair conjecture [21] holds, one expects, for large enough times, that both $S_T$ and $S_R$ become increasing functions of time. Explicit examples of the evolution of those measures for perturbed FLRW cosmologies will be presented in [15].

Although in this work we derived our results in a linearly perturbed FLRW approximation, we expect that after appropriate generalizations, our model is also suitable to describe nonlinear effects as well as large, exact inhomogeneities. Beyond cosmology, our parametric approach may also be relevant in other, relativistic entropy problems, e.g. black hole formation, where the nonadditive property of the entropy function is also a longstanding problem [22, 23].

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