Polyadic Algebras

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1 Polyadic Algebras

Polyadic algebras were introduced and intensively studied by Halmos, after having studied cylindric algebras in Tarski's seminar in Berkeley; we refer to Section 5.4 of [11], see also [9]. This class of algebras can be regarded as an alternative approach to algebraize first order logic. After a thorough reformulation of Henkin, Monk, and Tarski, polyadic algebras also can be regarded as certain generalizations of cylindric algebras. On one hand, polyadic algebras have nice representation properties, on the other, their languages are rather large (in the ω-dimensional case the cardinality of their set of operations is continuum), which makes their equational theory recursively undecidable for trivial reasons. This is undesirable from metalogical point of view, hence, during the last decades, certain countable (even finite) reducts of polyadic algebras have also been intensively studied.The goal of this research direction is to find a countable reduct of polyadic algebras which has nice representation properties, and, at the same time, their equational theory is recursively enumerable.

This subsection is closely related to, and is based on Section 5.4 of [11]. In more detail, in subsection 1.1 we recall the definition of certain classes of polyadic algebras. In subsection 1.2 we are dealing with the representation theory of polyadic algebras; in subsection 1.3 we are establishing some connections between polyadic and cylindric algebras. Finally, subsection 1.4 is devoted to study the recursion theoretic complexity of the equational theories of certain classes of (reducts of) polyadic algebras.

1.1 Basic Properties of Polyadic Algebras

We start by recalling the definitions of polyadic and polyadic equality algebras and their representable subclasses. These definitions are the same as Definitions 5.4.1 and 5.4.22 of [11].

Definition. *Let α be an ordinal. An algebra*

$$\mathcal{A} = \langle A; \cdot, \sim, 0, 1, c_{(\Gamma)}, s_\tau \rangle_{\Gamma \subseteq \alpha,\ \tau \in {}^\alpha\alpha}$$

is defined to be a polyadic algebra of dimension α iff the following equational stipulations hold for any $x, y \in A$, for any $\Gamma, \Delta \subseteq \alpha$ and for any $\sigma, \tau \in {}^\alpha\alpha$.

(P_0) $\langle A; \cdot, \sim, 0, 1 \rangle$ *is a Boolean Algebra;*

(P_1) $c_{(\Gamma)}0 = 0$;
(P_2) $x \leq c_{(\Gamma)}x$;
(P_3) $c_{(\Gamma)}(x \cdot c_{(\Gamma)}(y)) = c_{(\Gamma)}(x) \cdot c_{(\Gamma)}(y)$;
(P_4) $c_{(\emptyset)}x = x$;
(P_5) $c_{(\Gamma)}c_{(\Delta)}x = c_{(\Gamma\cup\Delta)}x$;
(P_6) $s_{id}x = x$;
(P_7) $s_\sigma s_\tau x = s_{\sigma\circ\tau}x$;
(P_8) $s_\sigma(x \cdot y) = s_\sigma(x) \cdot s_\sigma(y)$;
(P_9) $s_\sigma(\sim x) = \sim s_\sigma(x)$;
(P_{10}) *if* $\sigma|_{\alpha-\Gamma} = \tau|_{\alpha-\Gamma}$ *then* $s_\sigma c_{(\Gamma)}x = s_\tau c_{(\Gamma)}x$;
(P_{11}) *if* $\Delta = \tau^{-1}[\Gamma]$ *and* $\tau|_\Delta$ *is one-one then* $c_{(\Gamma)}s_\tau x = s_\tau c_{(\Delta)}x$.

The class of all α *dimensional polyadic algebras will be denoted by* PA_α.

Definition. *Let* α *be an ordinal. An algebra*

$$\mathcal{A} = \langle A; \cdot, \sim, 0, 1, c_{(\Gamma)}, s_\tau, d_{ij}\rangle_{\Gamma\subseteq\alpha,\ \tau\in{}^\alpha\alpha,\ i,j\in\alpha}$$

is defined to be a polyadic equality algebra of dimension α *iff its* PA_α*-type reduct is a polyadic algebra of dimension* α*, for all* $i, j \in \alpha$ *we have* $d_{ij} \in A$ *and, in addition, the following equational stipulations hold for any* $x \in A$*, for any* $i, j \in \alpha$ *and for any* $\tau \in {}^\alpha\alpha$.

(E_1) $d_{ii} = 1$;
(E_2) $x \cdot d_{ij} \leq s_{[i/j]}x$;
(E_3) $s_\tau d_{ij} = d_{\tau(i)\tau(j)}$.

The class of all α *dimensional polyadic equality algebras will be denoted by* PEA_α.

Next we recall the definition of representable polyadic algebras; this definition is the same as Definition 5.4.22 of [11].

Definition. *Let* α *be an ordinal,* U *a set and let* $W \subseteq {}^\alpha U$. *For* $\tau \in {}^\alpha\alpha, \Gamma \subseteq \alpha,\ i, j \in \alpha$ *and* $x \subseteq W$ *the operations of cylindrification* $C^W_{(\Gamma)}$*, substitution* S^W_τ *and diagonal elements* D^W_{ij} *are defined as follows.*

$$C^W_{(\Gamma)}(x) = \{z \in W :\ (\exists r \in x)(z|_{\alpha-\Gamma} = r|_{\alpha-\Gamma})\};$$
$$S_\tau(x) = \{z \in W :\ z \circ \tau \in x\} \text{ and}$$
$$D_{ij} = \{z \in W :\ z_i = z_j\}.$$

The structure

$$\mathcal{A} = \langle \mathcal{P}(W); \cap, \sim, \emptyset, W, C^W_{(\Gamma)}, S^W_\tau, D^W_{ij} \rangle_{\Gamma \subseteq \alpha, \tau \in {}^\alpha\alpha, i,j \in \alpha}$$

is called the full α-dimensional relativized polyadic set algebra of W. If $W = {}^\alpha U$ then $\mathcal{A}$ is called the α-dimensional full polyadic set algebra of U. In addition, according to Definition 5.4.22 of [11],

(i) *the class Pse_α of polyadic equality set algebras of dimension α consists of all subalgebras of full polyadic set algebras (of appropriate dimension);*

(ii) *the class $Rppe_\alpha$ of representable polyadic equality algebras of dimension α consists of all subdirect products of full polyadic set algebras (of appropriate dimension);*

(iii) *the class Gp_α of α dimensional generalized polyadic set algebras consists of all subalgebras of relativized polyadic set algebras of $W = \cup_{i \in I} {}^\alpha U_i$ (of appropriate dimension).*

Note, that according to the last item of the previous definition, generalized polyadic set algebras are relativied polyadic set algebras of some W, where W is the union of the α^{th} direct power of some sets U_i. We emphasize, that we do not require the different U_i to be disjoint from each other. The class of relativized subalgebras of the disjoint unions of ${}^\alpha U_i$ is called the class Gwp_α of generalized weak polyadic set algebras. It is easy to see, that $Rppe_\alpha = \mathbf{I}Gwp_\alpha$.

Remark 1.1 By a representation of a polyadic (equality) algebra $\mathcal{A}$ we mean an isomorphism between $\mathcal{A}$ and an *Rppe* (or *RPA*, respectively). Representability with relativized algebras has also deserved considerable attention - this research direction can be well motivated by the Resek-Thompson theorem in cylindric algebra theory. In this connection we refer to Remark 3.2.88 of [11].

Definition. *The classes of SPA_α and RPA_α consist of the diagonal-free reducts of elements of Pse_α and $Rppe_\alpha$ respectively.*

It is routine to check that $RPA_\alpha \subseteq PA_\alpha$ and $Rppe_\alpha \subseteq PEA_\alpha$. We will see in Subsection 1.2 below, that the aim of representation theory of polyadic algebras is establishing results related to the converse inclusions.

As we mentioned, if $\alpha \geq \omega$, then the cardinality of operations of PA_α is uncountable, making the equational theory undecidable for trivial reasons. Hence, finite and countably infinite reducts of polyadic algebras has also been intensively studied. Next, we recall the definitions of some countable reducts of polyadic algebras.

Definition 1.2 *Let α be given and let $G \subseteq {}^\alpha\alpha$ be a semigroup (under composition of functions). Then the class $G - PA_\alpha$ consists of all subreducts of elements of PA_α*

having the following set of operations:

- *the Boolean operations;*
- $\{c_{(\Gamma)} : \Gamma \subseteq \alpha$ *is finite* $\}$ *and*
- $\{s_\tau : \tau \in G\}$.

If $G = \{\tau \in {}^\alpha\alpha : \{i \in \alpha : \tau(i) \neq i\}$ *is finite* $\}$ *then the class* $G - PA_\alpha$ *is denoted by* QPA_α *and called the class of* α *dimensional quasi-polyadic algebras.*

$G - PEA_\alpha$ *and* $QPEA_\alpha$ *are defined similarly.*

Remark 1.3 Let α be an ordinal, and as usual, for $i, j \in \alpha$ let $[i/j] : \alpha \to \alpha$ be the function mapping i to j and leaving every other element fixed. Similarly, $[i, j]$ denotes the function that maps i onto j, maps j onto i and leaves every other element of α fixed. It is easy to check, that the semigroup $G = \{\tau \in {}^\alpha\alpha : \{i \in \alpha : \tau(i) \neq i\}$ is finite $\}$ can be generated by $G_0 = \{[i/j], [i, j] : i, j \in \alpha\}$. Hence, in QPA_α , for $\tau \in G$ the operation s_τ is term definable by the operations $\{s_\tau : \tau \in G_0\}$.

Suppose α is given. Then QPA_α can be regarded as the "minimalistic polyadic extension" of CA_α. Particularly, the language of QPA_ω contains countably many operation symbols only. However, there is an essential difference between the definition of QPA_α and CA_α.

For different α, the classes CA_α can be defined as a system of varieties. In more detail, this means the following. If α is given, and $\xi \in {}^\alpha\alpha$ is a function, then ξ acts on the equations of the language of CA_α in the natural way: if e is a CA_α-equation, then $\xi(e)$ can be obtained from e by replacing each occurrence of c_i by $c_{\xi(i)}$ and d_{ij} by $d_{\xi(i)\xi(j)}$, respectively. Then there is a finite set E of equations such that for any α, CA_α is the class of all models of $\{\xi[E] : \xi \in {}^\alpha\alpha$ is a permutation $\}$. This uniform definability may be useful, because it makes accessible some techniques of universal algebra.

The set of instances of the equations in the definition of QPA_α is not closed under all permutations of α; in addition, it is not obvious, if there exists an alternative definition of QPA_α containing schemas of equations whose set of instances is closed under permutations. To study the situation, Sain and Thompson in [24] introduced the classes FPA_α and $FPEA_\alpha$ of *Finitary Polyadic (Equality) Algebras* of dimension α.

Definition. *A finitary polyadic equality algebra of dimension of* α *is an algebra*

$$\mathcal{A} = \langle A; \cdot, \sim, 0, 1, c_i, s^i_j, p_{ij}, d_{ij} \rangle_{i,j \in \alpha}$$

where d_{ij} *are constants, and the following equational stipulations hold for any* $i, j, k \in \alpha$*:*

(F_0) $\langle A; \cdot, \sim, 0, 1\rangle$ *is a Boolean algebra,* $s^i_i = p_{ii} = d_{ii} = Id|_A$ *and* $p_{ij} = p_{ji}$

(F_1) $x \leq c_i x$*;*

(F_2) $c_i(x \vee y) = c_i(x) \vee c_i(y)$;
(F_3) $s^i_j c_i(x) = c_i(x)$;
(F_4) $c_i s^i_j(x) = s^i_j c_i(x)$ *if* $i \neq j$;
(F_5) $s^i_j c_k(x) = c_k s^i_j(x)$ *if* $k \notin \{i, j\}$;
(F_6) s^i_j *and* p_{ij} *are Boolean endomorphisms;*
(F_7) $p_{ij} p_{ij} x = x$;
(F_8) $p_{ij} p_{ik}(x) = p_{jk} p_{ij}(x)$ *if* i, j, k *are distinct;*
(F_9) $p_{ij} s^i_j(x) = s^j_i(x)$;
(F_{10}) $s^i_j d_{ij} = 1$;
(F_{11}) $x \cdot d_{ij} \leq s^i_j x$.

The class of finitary polyadic equality algebras of dimension α *is denoted by* $FPEA_\alpha$*; its diagonal free subreduct is denoted by* FPA_α*.*

It is easy to check, that the set of instances of defining equations of $FPEA_\alpha$ is closed under permutations of α.

Theorem *(Sain, Thompson). Let* $\alpha > 2$*.*
(i) The varieties $FPEA_\alpha$ *and* $QPEA_\alpha$ *are term definitionally equivalent.*
(ii) The varieties FPA_α *and* QPA_α *are term definitionally equivalent.*

The proof can be found in [24].

1.2 Representation Theory of Polyadic Algebras

We will cut this subsection into two parts: first we survey results from the representation theory of polyadic algebras (without diagonal elements) and next, we will deal with the case of polyadic equality algebras (containing diagonal elements). Both parts can be further divided to the finite dimensional and to the infinite dimensional case or to "positive" and "negative" results.

Representation theory, the diagonal-free case

The first theorem we should mention is the following celebrated result of Daigneault and Monk. The original proof can be found in [15]; see also Remark 5.4.41 of [11].

Theorem (Daigneault, Monk). For infinite α we have $PA_\alpha = RPA_\alpha$.

We also note, that independently, Keisler in [14] proved a completeness theorem for a version of first order logic with infinitary predicates; this completeness theorem may be considered as the logical version of the Daignault-Monk representation theorem.

On one hand, the Daigneault-Monk Theorem is elegant: it describes a finite schema of equations axiomatizing RPA_α (for infinite α). On the other hand, these equational schemas have continuum many instances for the smallest, $\alpha = \omega$ case. This is necessary, because the cardinality of the set of operations of RPA_ω is the continuum.

If α is finite, then the polyadic axiom schemas $P_0 - P_{11}$ have finitely many instances only. For finite α, the class RPA_α is a variety; however, as the next theorem indicates, its equational theory is rather complicated.

Theorem 1.4 *For finite $\alpha \geq 2$ the variety RPA_α is not finitely axiomatizable. In addition, $Rppe_\alpha$ "cannot be axiomatized by finitely many variables": if Σ is a set of polyadic equations such that $Mod(\Sigma) = RPA_\alpha$ then, for every $n \in \omega$ there is an equation $e_n \in \Sigma$ containing at least n distinct variables.*

For a proof and more details, see [1], [2] and [12].

As we mentioned, QPA_α may be considered as the "minimalistic polyadic extension" of cylindric algebras. Even, this minimalistic class cannot be finitely axiomatized, as the following theorem says.

Theorem 1.5 *(Sain, Thomson).*
For $\alpha > 2$, the class $RQPA_\alpha$ of representable quasi-polyadic algebras of dimension α cannot be axiomatized by finitely many equations.

The proof, and stronger related results can be found in [24]. In fact, in [24] it was shown, that for $\alpha > 2$, the class of representable FPA_α (or equivalently, QPA_α) cannot be defined by finitely many equational schemas closed under permutations of the dimension set (i.e. these classes cannot be defined by finitely many Monk-type equational schemas).

By Theorem 1.5, for finite α, RPA_α is a proper subclass of PA_α. If we take smaller reducts, some positive results may be obtained.

Definition 1.6 *For a set U, the structure*

$$\mathcal{A} = \langle \mathcal{P}(^\alpha U); \cap, \sim, 0, 1, S_{[i/j]} \rangle_{i,j\in\alpha}$$

is called the α-dimensional full substitution set algebra of U.

(i) The class $SetSA_\alpha$ of substitution set algebras of dimension α consists of subalgebras of full substitution set algebras (of appropriate dimension);

(ii) the class RSA_α of representable substitution algebras of dimension α is defined to be $RSA_\alpha = \mathbf{ISP} SetSA_\alpha$.

These classes first was studied by Pinter (see [11], page 267). In [20] and in [19] the following were proved for RSA_n.

Theorem 1.7 *(Sági)*
(i) For finite $n \geq 2$ the class RSA_n is a finitely axiomatizable quasi-variety, but

not a variety;
(ii) the generated variety is also finitely axiomatized and it consists of (isomorphic copies of) the appropriate reducts of Gp_α;
(iii) the (quasi-)equational theory of RSA_α is decidable for $\alpha \leq \omega$.

We emphasize, that in (*iii*) above, $\alpha = \omega$ is allowed, as well.

Proof. We give a sketch only. Let $n \geq 2$ be finite and fixed. Let V be the variety generated by $SetSA_n$. Suppose, that $\mathcal{A} \in SetSA_n$ with base set U and $W \subseteq U$. Let $\mathcal{B} \in SetSA_n$ be the full set algebra over W. Then it is easy to see, that the function $\varphi_W : \mathcal{A} \to \mathcal{B}$ satisfying $\varphi_W(x) = x \cap {}^nW$ is a $SetSA_n$-homomorphism. Now let $a \in A - \{0\}$ be arbitrary. Then there exists $s \in A$. Let $W = ran(s)$. Then φ_W is a homomorphism from $\mathcal{A}$ which maps a to a non-zero element and the base set of its image is of cardinality at most n. Consequently, every $\mathcal{A} \in SetSA_n$ can be embedded into a direct product $\Pi_{i\in I}\mathcal{B}_i$ such that the base set of each $\mathcal{B}_i$ is of cardinality at most n. It follows, that V can be generated by finitely many finite algebras. In addition, V has a Boolean reduct, hence it is congruence distributive. Thus, by Baker's theorem, V is finitely axiomatizable. This proves the first part of (ii) and (iii) for finite α and for the equational theory of RSA_n.

Next, we show that RSA_n is not a variety. Let σ be the quasi-equation

$$s_{[0/1]}(x) \cdot s_{[1/0]}(y) = 0 \Rightarrow s_{[1/0]}(x) \cdot s_{[0/1]}(y) = 0.$$

It is easy to see, that σ holds in every $SetSA_n$, hence in RSA_n. Let $\mathcal{A} \in SetSA_n$ be the full set algebra on base set $\{0, ..., n-1\}$. Then $\mathcal{A}$ has a homomorphic image, in which σ is not true; for the details, see Theorem 3 of [20]. This shows, that RSA_n is not a variety.

Next, we show, that $SetSA_n$ is closed under ultraproducts. Let $\mathcal{A}_i \in SetSA_n$; suppose, that the base set of $\mathcal{A}_i$ is U_i and let $\mathcal{F}$ is an ultrafilter over I. Let $U = \Pi_{i\in I}U_i/\mathcal{F}$ and let $\varphi : \Pi_{i\in I}\mathcal{A}_i/\mathcal{F} \to \mathcal{P}({}^nU)$ defined to be

$$\varphi(\langle x(i) : i \in I\rangle/\mathcal{F}) = \{\langle s_0, ..., s_{n-1}\rangle/\mathcal{F} : \{i \in I : \langle s_0(i), ..., s_{n-1}(i)\rangle \in x(i)\} \in \mathcal{F}\}.$$

It is easy to check that φ is an embedding of $\Pi_{i\in I}\mathcal{A}_i/\mathcal{F}$. It follows, that $SetSA_n$, is closed under ultraproducts. Consequently RSA_n is also closed under ultraproducts, hence it is a quasi-variety. This proves the last two parts of (i).

The proofs of the remaining parts of (i),(ii) and (iii) are much more longer, hence we omit them. They can be found in [20] and in [19]. ∎

Parts of Theorem 1.7 not proved above, are based on a semigroup-theoretic observation. For a given set U, $NP(U)$ denotes the semigroup of finite, non-bijective selfmaps (that is,

$$NP(U) = \{t \in {}^U U : \ \{x \in U : x \neq t(x)\} \text{ is finite and } t \text{ is not bijective } \}$$

and the operation of $NP(U)$ is composition of functions). It was shown in [27] and [17] that there is a finite equational axiom system Σ such that, $NP(U)$ is presented by (the set of all instances of) Σ. Thus, $NP(U)$ can be presented by a finite set of "presentation schemas". These schemas can be used to describe the action of the substitution operations - in the finite dimensional case they have finitely many instances, and this fact implies that RSA_α is finitely axiomatizable for finite α. For completeness, we note, that Jónsson in [13] provided a finite schema presentation for the semigroup of finite transformations $F(U) = \{t \in {}^U U : \ \{x \in U : x \neq t(x)\}$ is finite$\}$.

It seems, that cylindrifications are the responsible for the negative results.

Representation theory, the diagonal case, negative results

It is natural to ask, whether the analogue of the Daigneault-Monk Theorem for $Rppe_\omega$ remains true. It will turn out, that the situation is essentially different from the diagonal free case.

We start by showing, that $Rppe_\omega$ is not closed under ultraproducts. As described in Remark 5.4.41 of [11], this was first proved by Monk. Later Johnson showed that certain ultraproducts of polyadic equality set algebras are not in $Rppe_\omega$, hence $Rppe_\omega$ cannot be axiomatized by any set of first order formulas (particularly, it is not a variety).

Theorem. $Rppe_\omega$ is not closed under ultraproducts.

Proof. Let $pred : \omega \to \omega$ defined to be $pred(0) = 0$ and $pred(n+1) = n$ for each $n \in \omega$. Observe, that in each $\mathcal{A} \in Rppe_\omega$ the following "infinitary quasi-equation" is true for all $x \in A$:

$$(*) \qquad \left(\bigwedge_{i,j \in \omega} \ x \leq d_{ij} \right) \quad \Rightarrow \quad x = s_{pred}(x).$$

Now let $\mathcal{C} \in Cs_\omega$ be countable, whose base set U is infinite and let $\mathcal{A}$ be the full ω dimensional polyadic equality set algebra of U. Let $\mathcal{F}$ be a nonprincipal ultrafilter over ω, let $\mathcal{B} = {}^\omega \mathcal{A}/\mathcal{F}$ and finally let $b = \langle b_n : \ n \in \omega \rangle / \mathcal{F} \in B$ where

$$b_n = (\textstyle\prod_{i,j<n} d_{ij}) \cdot (\prod_{i<n} \sim d_{in}).$$

Then clearly, for every $i, j \in \omega$ we have $\mathcal{B} \models b \leq d_{ij}$ but $s^{\mathcal{B}}_{pred}(b) \neq b$. So $(*)$ does not hold in $\mathcal{B}$ and consequently, $\mathcal{B} \notin Rppe_\omega$. ∎

Remark 1.8 Now, by the previous theorem, there exists a nonrepresentable $\mathcal{B} \in PEA_\omega$. Using the notation of the previous proof, we can conclude, that the CA_ω-type reduct of $\mathcal{B}$ is in RCA_ω (i.e., it is representable). Indeed, $\mathcal{B}$ has been constructed as an ultrapower of a polyadic set-algebra $\mathcal{A}$ such that $\mathcal{B}$ is not representable. Since taking ultraproducts and forming reducts are commuting operations, it follows, that the CA_ω-type reduct of $\mathcal{B}$ coincides with the appropriate ultrapower of the CA_ω-type reduct of $\mathcal{A}$. Since this latter is a representable CA_ω, it follows from Theorem 3.1.109 of [11], that the CA_ω-type reduct of $\mathcal{B}$ is also in RCA_α, but according to the previous theorem, $\mathcal{B}$ itself is not in $Rppe_\omega$.

After the previous theorem, , the next problem is to axiomatize the variety generated by (or, equivalently, the equational theory of) $Rppe_\omega$. A finite axiomatization is impossible, because of the set of operations of $Rppe_\omega$ is of infinite cardinality. So a finite schema of equations would be desirable. A natural candidate for such a finite schema axiomatization is $P_0 - E_3$.

It turned out, that there is an equation valid in $\mathbf{HSP}Rppe_\omega$ but does not valid in PEA_ω. One could hope, that adding new, similar schemas to $P_0 - E_3$ might lead to an axiomatization of $Rppe_\omega$. We will see below, that this also cannot be done. We start by fixing the precise definition for "schemas similar to $P_0 - E_3$".

Having a look for $P_0 - E_3$, one can realize, that these schemas contain variables Γ, Δ ranging over subsets of ω and σ, τ ranging over ${}^\omega\omega$. Sometimes there is a condition between the sets and functions occurring in the names of operations in the schema, but such conditions always expressible in a certain first order language.

Consider P_7 as a typical example. It can be rephrased as follows: if $\varrho, \tau, \sigma \in {}^\omega\omega$ are such that $\varrho = \sigma \circ \tau$ then the equation $s_\varrho(x) = s_\sigma s_\tau(x)$ is an instance of P_7. Here ϱ, σ and τ can be treated as "variables" ranging over the "names" of certain polyadic operations; and, at the same time, they denote functions. So the names of polyadic operations have a structure, and one can use this structure to describe a general equational schema which applies for many polyadic operations. This motivates the next three definitions (originally introduced in Németi-Sági [16]; see also [19]).

Definition 1.9
(i) Let L be a first order language containing countable many unary function symbols $f_0, f_1, ...$, countably many unary relation symbols $r_0, r_1, ...$ and countably many constant symbols $n_0, n_1, ...$ (and nothing more).
(ii) Let L_{PT} be the similarity type (in the algebraic sense) of Boolean algebras endowed with unary operation symbols $s_{f_0}, s_{f_1}, ..., c_{r_0}, c_{r_1}, ...$ and constant symbols $d_{n_0 n_0}$, $d_{n_0 n_1}, ...$ Here the indices of the symbols c, s and d are the same as the corresponding symbols in L.
(iii) By a Halmos schema we mean a pair $\langle s, e\rangle$ where s is a first order sentence of L and e is an equation of L_{PT}.

In order to keep notation closer to intuition, we will write $s \Rightarrow e$ in place of $\langle s, e\rangle$.

Definition 1.10 *Let $s \Rightarrow e$ be a Halmos schema and let g be an equation in the language of PEA_α. Then g is defined to be an (α dimensional) instance of $s \Rightarrow e$ iff there are*

$$f_0^M, f_1^M, ... \in {}^\alpha\alpha, \quad r_0^M, r_1^M, ... \subseteq \alpha \quad \text{and} \quad n_0^M, n_1^M, ... \in \alpha \quad \text{such that}$$

$$\langle \alpha;\ r_0^M, r_1^M, ..., f_0^M, f_1^M, ...n_0^M, n_1^M, ...\rangle \models s$$

and g can be obtained from e by replacing r_i, f_i and n_i by r_i^M, f_i^M and n_i^M, respectively.

For example, the set of all instances of P_7 coincides with the set of all instances of the Halmos schema

$$(\forall v)(f_0(v) = f_1(f_2(v)) \quad \Rightarrow \quad s_{f_1}s_{f_2}(x) = s_{f_0}(x).$$

It is easy to see, that all elements of $P_0 - E_3$ can be expressed by a suitable Halmos schema in this sense.

The use of set theoretic structure of the names of polyadic operations makes the axiom system $P_0 - E_3$ so elegant (and, as we will see, also the structure of the names of operations makes the equational theory of $Rppe_\omega$ so complicated).

Definition 1.11 *Let $\mathcal{A} \in PEA_\alpha$. A Halmos schema is defined to be valid in $\mathcal{A}$ iff every α dimensional instance of it is valid in $\mathcal{A}$. A Halmos scehma is valid in a class of algebras iff it is valid in all elements of the class.*

Let PEA_α^+ be the class of all models of all α-dimensional instances of Halmos schemas valid in $Rppe_\alpha$. Note, that PEA_α^+ is the smallest variety containing $Rppe_\alpha$ and defined by Halmos schemas.

Now we are ready to state our non-axiomatizability result.

Theorem 1.12 *(Németi, Sági)*
$PEA_\omega^+ \neq \mathbf{HSP}Rppe_\omega$. That is, the equational theory of $Rppe_\omega$ is not axiomatizable by Halmos schemas.

The proof can be found in [16], see also [19]. ∎

In these papers it is shown, that there is an equation e_{CM} valid in $Rppe_\omega$ but not in PEA_ω^+. Although it is rather complicated, e_{CM} is explicitly given.

Next, one could try to axiomatize $Rppe_\omega$ by some kind of equation-schemas different from Halmos schemas. If $Rppe_\omega$ would be finitely axiomatizable by some kind of schemas Σ, then, as we will see in subsection 1.4, the equational consequences of Σ would form a Π_1^1-hard set (in the recursion theoretic sense). Since finite (schema) axiomatizability of a theory usually implies recursive enumerability, we can conclude, that $Rppe_\omega$ cannot be finitely axiomatized by any kind of "reasonable" schemas.

After these negative results it is natural to ask, what happens, if one takes small subreducts of PEA_ω. The analogue of Theorem 1.5 remains true for $RQPEA_\alpha$, $\alpha > 2$, as well.

Theorem 1.13 *(Sain, Thomson)*
For $\alpha > 2$, the class $RQPEA_\alpha$ of representable quasi-polyadic equality algebras of dimension α cannot be axiomatized by finitely many Monk type scemas.

The proof can also be found in [24]. Of course, by Theorem 1.13, the class of representable quasi-polyadic equality algebras cannot be axiomatized by a finite set of equations.

Representation theory, the diagonal case, positive results

We start this subsection by a positive result due to I. Sain which is in a sharp contrast of the non-finite axiomatizability results presented so far and it has a definite knowledge theoretical significance.

Theorem 1.14 *(Sain)*
There is a finite reduct L of the language of PEA_ω such that
(i) all the CA_ω-operations are term-definable in L and
(ii) the class of L-subreduct of $Rppe_\omega$ is a finitely axiomatizable variety.

The proof can be found in [22] and it has also been based on semigroup theoretic investigations. At the technical level, the cornerstone was to find a finitely generated, finitely presented subsemigroup of ${}^\omega\omega$ with further nice properties. We emphasize, that in Theorem 1.14 the set Σ of equations axiomatizing the representable algebras is not only described by a finite set of equational schemas - Σ itself is a finite set of equations.

Next we give a sufficient condition which implies representability of a $QPEA_\omega$. Let $\mathcal{A} \in PA_\alpha$. As usual, the *dimension set* $\Delta(a)$ of $a \in A$ is defined to be $\Delta(a) = \{i \in \alpha : c_i(a) \neq a\}$. In addition, $\mathcal{A}$ is defined to be *locally finite-dimensional* iff every $a \in A$ has a finite-dimension set. A PEA_α is defined to be locally finite-dimensional iff its PA_α-reduct is locally finite-dimensional.

Our goal is to show, that every locally finite-dimensional $QPEA_\omega$ is representable. Although this is a classical result, for completeness we include here a proof, because other classical representation theorems for locally finite dimensional cylindric algebras can be quickly derived from this one. To present the proof, we need further preparations.

Remark 1.15 We recall a method of constructing homomorphisms from certain reducts of PEA_α into (relativized) set algebras. The idea comes from Andréka-Németi [3] (see also Remark 3.2.9 of [11]) where it was developed for locally finite-dimensional cylindric algebras. Below we adapt the method to certain elements of PEA_α.

Let α be any set, $\Gamma \subseteq \mathcal{P}(\alpha)$ and $\Lambda \subseteq {}^\alpha\alpha$. Let

$$\mathcal{A} = \langle A; \cdot, \sim, 0, 1, c_{(\gamma)}, d_{ij}, s_\tau \rangle_{\gamma \in \Gamma, i,j \in \alpha, \tau \in \Lambda}$$

be a reduct of a PEA_α. Let $\mathcal{F}$ be any ultrafilter on $\mathcal{A}$. Then the kernel $ker(\mathcal{F})$ of $\mathcal{F}$ is defined to be

$$ker(\mathcal{F}) = \{\langle i, j \rangle \in \omega \times \omega : d_{ij} \in \mathcal{F}\}.$$

It is easy to see, that $ker(\mathcal{F})$ is an equivalence relation. For any $\tau \in {}^\alpha\alpha$ we will denote by τ/E the function satisfying $\tau/E(i) = \tau(i)/E$ for every $i \in \alpha$. Finally, for each $a \in A$ let

$$rep_{\mathcal{F}}(a) = \{\tau/E : \tau \in \Lambda, s_\tau(a) \in \mathcal{F}\}.$$

Our aim is to show, that if $\mathcal{A}$ is locally finite dimensional, then $rep_{\mathcal{F}}$ is a $QPEA_\alpha$-homomorphism for some carefully chosen $\mathcal{F}$. To do so, we still need some further preparations.

Lemma 1.16 *Let $\mathcal{A}$ be a locally finite-dimenisonal $QPEA_\alpha$ and let $\mathcal{F}$ be an ultrafilter over I. Then*

(i) The set $A_{lf} := \{a \in {}^I A/\mathcal{F} : \Delta(a)$ is finite $\}$ is closed under the $QPEA_\alpha$-operations.

(ii) If $\mathcal{A}$ is an α-dimensional quasi-polyadic equality set algebra, then the $QPEA_\alpha$ generated by A_{lf} is isomorphic to an α-dimensional quasi-polyadic equality set algebra.

Proof. To see (i), let $a, b \in A_{lf}$ and let $c_{(\Gamma)}, s_\tau$ be $QPEA_\alpha$-operations. Then it is easy to see, that

$\Delta(a \cdot b) \subseteq \Delta(a) \cup \Delta(b)$;
$\Delta(d_{ij}) \subseteq \{i, j\}$ for every $i, j \in \alpha$;
$\Delta(c_{(\Gamma)}(a)) \subseteq \Delta(a)$ and
$\Delta(s_\tau(a)) \subseteq \tau^{-1}[\Delta(a)]$.

The right hand side is finite in all cases (for the last case we note, that $\tau^{-1}[\Delta(a)]$ is finite because $\{k \in \alpha : \tau(k) \neq k\}$ is finite).

The idea of the proof of (ii) is similar to that of Theorem 1.7 (i). Assume, that the base set of $\mathcal{A}$ is U. Define $\varphi : A_{lf} \to \mathcal{P}(^{\alpha}(^{I}U/\mathcal{F}))$ to be

$$\varphi(a) = \{\langle s_k/\mathcal{F} : k \in \alpha\rangle \in {}^{\alpha}(^{I}U/\mathcal{F}) : \{j \in I : \langle s_k(j) : k \in \alpha\rangle \in a_j\} \in \mathcal{F}\}$$

where $a = \langle a_j : j \in I\rangle/\mathcal{F}$. It is easy to check, that φ is an embedding from the $QPEA_\alpha$ generated by A_{lf} into the full α-dimensional quasi-polyadic equality set algebra on the base set ${}^{I}U/\mathcal{F}$. ∎

Definition 1.17 *Let $\mathcal{A}$ be a Boolean algebra (possibly with extra operations). The set of ultrafilters of $\mathcal{A}$ will be denoted by $\mathcal{U}(\mathcal{A})$. For any $a \in A$ we define N_a to be*

$$N_a = \{\mathcal{F} \in \mathcal{U}(\mathcal{A}) : a \in \mathcal{F}\}.$$

Remark 1.18 The following facts are well known: $\{N_a : a \in A\}$ is a basis of a topology on $\mathcal{U}(\mathcal{A})$; we will denote the generated topology by τ. $\mathcal{U}(\mathcal{A})$ endowed with τ is called the *Stone dual* space of $\mathcal{A}$ and is denoted by $\mathcal{A}^*$. It is a compact Hausdorff space.

Definition 1.19 *Suppose $X \subseteq A, \mathcal{F} \in \mathcal{U}(\mathcal{A})$ and $a \in A$ such that $a = sup(X)$. Then we say, that $\mathcal{F}$ preserves X iff*

$$a \in \mathcal{F} \quad \Rightarrow \quad (\exists b \in X)(b \in \mathcal{F}).$$

Note, that the converse implication always holds.

Lemma 1.20 *Suppose $X \subseteq A$ and $a \in A$ such that $a = sup(X)$. Then*

$$\mathcal{U}_X := \{\mathcal{F} \in \mathcal{U}(\mathcal{A}) : \mathcal{F} \text{ does not preserve } X\}$$

is nowhere dense in $\mathcal{A}^*$.

Proof. Let G be a nonempty open set of $\mathcal{A}^*$. By shrinking it, if necessary, we may assume that G is basic open, that is, $G = N_b$ for some $0 \neq b \in A$. It is enough to show that there exists $0 < c < b$ such that $N_c \cap \mathcal{U}_X = \emptyset$. To do so, we will distinguish two cases.

Case 1: $b \cdot (\sim a) \neq 0$. in this case $c = b \cdot (\sim a)$ is suitable.

Case 2: $b \cdot (\sim a) = 0$. In this case $b \leq a$. Assume, seeking a contradiction, that for every $x \in X$ we have $b \cdot x = 0$. It follows, that $\sim b$ is an upper bound for X and hence $a \leq \sim b$. Consequently, $b \leq a \leq \sim b$, so $b = 0$, a contradiction.

By the previous paragraph, there exists $x \in X$ with $b \cdot x \neq 0$. Then, for $c = b \cdot x$ we have $N_c \cap \mathcal{U}_X = \emptyset$, as desired. ∎

Lemma 1.21 *Let $\mathcal{A} \in QPEA_\omega$ be countable and locally finite-dimensional. Let $a \in A - \{0\}$. For each $i \in \omega$ let $X_i \subseteq A$ and $b_i \in A$ be such that $b_i = sup(X_i)$. Then there exists $\mathcal{F} \in \mathcal{U}(\mathcal{A})$ such that $\mathcal{F}$ preserves all $X_i, i \in \omega$, moreover, $a \in \mathcal{F}$ and every equivalence class of $ker(\mathcal{F})$ is infinite.*

Proof. Let $\delta : \omega \to \omega$ be a function such that for every $k \in \omega$ the set $\{n \in \omega : \delta(n) = k\}$ is infinite. In addition, let $B_i := \{\mathcal{F} \in \mathcal{U}(\mathcal{A}) : \mathcal{F}$ does not preserve $X_i\}$. We will modify the standard proof of the Baire Category theorem. More concretely, by recursion we will define a sequence of elements $\langle a_n : n \in \omega\rangle$ of $\mathcal{A}$ such that the following hold for all $n, m \in \omega$:

(a) $a_0 = a$ and $a_n \neq 0$;
(b) if $n < m$ then $a_m \leq a_n$;
(c) $N_{a_{n+1}} \cap B_n = \emptyset$;
(d) $(\exists k \in \omega) n \leq k, a_{n+1} \leq d_{\delta(n)k}$.

Let $a_0 = a$; then (a)-(d) are clearly true. Next, suppose, that $n \in \omega$ and a_m has already been defined for all $m < n$. Then $\Delta(a_{n-1})$ is finite, hence exists $k \in \omega$ with $k \geq n - 1$ and $k \notin \Delta(a_{n-1})$. Then

$$\begin{aligned}
&c_k(a_{n-1} \cdot d_{\delta(n-1)k}) = \\
&c_k(c_k(a_{n-1}) \cdot d_{\delta(n-1)k}) = \\
&c_k(a_{n-1}) \cdot c_k(d_{\delta(n-1)k}) = \\
&a_{n-1} \cdot 1 = \\
&a_{n-1},
\end{aligned}$$

hence by (a), $a_{n-1} \cdot d_{\delta(n-1)k} \neq 0$. By Lemma 1.20, the set B_{n-1} is nowhere dense, hence there exists a nonzero $a_n \in A$ with $a_n \leq a_{n-1} \cdot d_{\delta(n-1)k}$ such that $N_{a_n} \cap B_{n-1} = \emptyset$. Clearly, (a)-(d) remains true. In this way, the sequence $\langle a_n, n \in \omega\rangle$ can be completely defined.

Combining (a) and (b), one obtains, that $\{N_{a_n} : n \in \omega\}$ has the finite intersection property. Since $\mathcal{A}^*$ is a compact space, it follows, that there exists $\mathcal{F} \in \cap_{n\in\omega} N_{a_n}$. Then by (c), $\mathcal{F}$ preserves X_i, for every $i \in \omega$. In addition, by (a), we have $a \in \mathcal{F}$. Finally, let $i, m \in \omega$ be arbitrary. We will show, that there exists $k \geq m$ such that $d_{ik} \in \mathcal{F}$. Let $n \in \omega$ be such that $n \geq m$ and $i = \delta(n)$. Then by (d), there exists $k \in \omega$ such that $m \leq n \leq k$ and $a_{n+1} \leq d_{\delta(n)k}$ Therefore $d_{\delta(n)k} = d_{ik} \in \mathcal{F}$, as desired. It follows, that $i/ker(\mathcal{F})$ is unbounded in ω. ∎

Theorem 1.22 *Let $\mathcal{A} \in QPEA_\omega$ be locally finite-dimensional. Then*

(i) For each $0 \neq a \in A$ there exist an α-dimensional quasi-polyadic equality set algebra $\mathcal{B}_a$ and a homomorphism $\varphi_a : \mathcal{A} \to \mathcal{B}_a$ such that $\varphi_a(a) \neq 0$.

(ii) $\mathcal{A}$ is representable.

Proof. Let $0 \neq a \in A$ be fixed. Let $\mathcal{A}_0$ be a countable elementary substructure of $\mathcal{A}$ containing a. For any $\tau \in {}^\omega\omega$ define $s_\tau : A \to A$ to be $s_\tau(x) = s_{\tau'}(x)$, where $\tau|_{\Delta(x)} = \tau'|_{\Delta(x)}$ and $\tau'|_{\omega-\Delta(x)}$ is the identity function. This is meaningful, since $s_{\tau'}$ is a quasi-polyadic operation. Throughout this proof, we assume that s_τ is a basic operation of $\mathcal{A}_0$ for every $\tau \in {}^\omega\omega$.

Now we turn to the proof of (i). For each $i \in \omega$ and $b \in A_0$ let $X_{i,b} = \{s_{[i/j]}(b) : j \in \omega - \Delta(b)\}$. Then, by item 1.11.6(i) of [10] we have $b = sup(X_{i,b})$. By Lemma 1.21 there exists an ultrafilter $\mathcal{F} \in N_a$ preserving every $X_{i,b}$ such that every equivalence class of $ker(\mathcal{F})$ is infinite. Clearly, $rep_{\mathcal{F}}(a) \neq \emptyset$.

We claim, that, in fact, $rep_{\mathcal{F}}$ is a homomorphism. Here is a sketch for a proof of this claim. One can verify by a straightforward computation, that $rep_{\mathcal{F}}$ preserves $\cdot$ and d_{ij} for any $i, j \in \omega$. Next, one can show, that if $\tau, \tau' \in \Lambda$ are such that, for any $i \in \omega$ we have $\langle \tau(i), \tau'(i) \rangle \in ker(\mathcal{F})$ then for any $x \in A_0$

$$(**) \qquad s_\tau(x) \in \mathcal{F} \quad \text{iff} \quad s_{\tau'}(x) \in \mathcal{F},$$

this may be established with an induction on $n := |\Delta(x) \cap \{i \in \omega : \tau(i) \neq \tau'(i)\}|$. Then $(**)$ implies, that $rep_{\mathcal{F}}$ preserves complementation and all the s_σ. Finally, combining $(**)$ with the fact, that $\mathcal{F}$ preserves each $X_{i,b}$ and using, that each equivalence class of $ker(\mathcal{F})$ is infinite, one obtains, that $rep_{\mathcal{F}}$ preserves c_i, for all $i \in \omega$.

Thus, $rep_{\mathcal{F}}$ is a homomorphism from $\mathcal{A}_0$ into some $\mathcal{C}_a \in Pse_\omega$. Now let $\mathcal{U}$ be an $|A|$-regular ultrafilter[1] and let $\mathcal{D}_a = ({}^I\mathcal{A}_0/\mathcal{U})_{lf}$. Then $\mathcal{A}$ can be embedded into $\mathcal{D}_a$. Let $\mathcal{B}_a = ({}^I\mathcal{C}_a/\mathcal{U})_{lf}$. By Lemma 1.16, $\mathcal{B}_a \in Pse_\omega$; it is also a homomorphic image of $\mathcal{D}_a$. Hence, there exists a homomorphism φ_a from $\mathcal{A}$ into $\mathcal{B}_a$ mapping a to a nonzero element, as desired.

Now we turn to prove (ii). By (i), for each $0 \neq a \in A$ there exist an α-dimensional quasi-polyadic equality set algebra $\mathcal{B}_a$ and a homomorphism $\varphi_a : \mathcal{A} \to \mathcal{B}_a$ with $\varphi_a(a) \neq 0$. Define $\varphi : \mathcal{A} \to \Pi_{0 \neq a \in A}\mathcal{B}_a$ to be $\varphi(x) = \langle \varphi_a(x), a \in A - \{0\} \rangle$. Then φ is the desired embedding. ∎

1.3 Connections with Cylindric and Quasi-polyadic Algebeas

In this subsection we are comparing PEA_α with CA_α and $QPEA_\alpha$. Clearly, every PEA_α has a CA_α-type and a $QPEA_\alpha$-type reduct. In addition, the following facts are true:

(i) The Df_α-type reduct of a PA_α is a Df_α;
(ii) The CA_α-type reduct of a PEA_α is a CA_α. In addition, $s_{[i/j]}(x) = c_i(d_{ij} \cdot x)$;
(iii) If $\beta \geq \omega$, $\mathcal{A} \in PEA_\beta$ then the CA_β-type reduct of $\mathcal{A}$ is a representable CA_β.

[1] for the definition and basic properties of regular ultrafilters we refer to [5]

For the proofs, see Theorem 5.4.3 and Corollary 5.4.18 of [11].

Remark 1.23 We note, that the converse of the above (ii) is not true: there exists an $\mathcal{A} \in CA_\alpha$ which cannot be obtained as the cylindric reduct of a suitable PEA_α. Indeed, as was shown in Section 5.4 of [11], every $\mathcal{A} \in PEA_\omega$ satisfies the marry-go-round properties, but there exists a cylindric algebra $\mathcal{B}$, which does not satisfies these properties. The same argument shows, that there exists a CA_ω which cannot be embedded into the cylindric reduct of a $QPEA_\omega$ (because every $QPEA_\omega$ also satisfies the marry-go-round properties). This supports the view, that quasi-polyadic equality algebras are "between" cylindric and polyadic algebras.

After Remark 1.23, the next natural question is: if a cylindric algebra $\mathcal{A}$ satisfies the marry-go-round properties (i.e. $\mathcal{A} \in CA_\alpha^+$) then does it follow, that $\mathcal{A}$ is isomorphic to the CA_α-type reduct of a suitable $\mathcal{B} \in QPEA_\alpha$? The answer is negative:

Theorem 1.24 *(Sayed Ahmed)*
There exists $\mathcal{A} \in RCA_\omega$ such that $\mathcal{A}$ is not isomorphic to the CA_ω-type reduct of any $\mathcal{B} \in QPEA_\omega$. (Since $\mathcal{A}$ is representable, it obviously satisfies the marry-go-round properties).

For the proof and further details, see [25]. On the other hand, in [7], Ferenczi proved the following.

Theorem 1.25 *There is a weakening $QPEA_\alpha^-$ of the axioms of $QPEA_\alpha$ such that if $\mathcal{A} \in CA_\alpha^+$ (i.e. $\mathcal{A}$ is a CA_α satisfying the merry-go-round properties) then $\mathcal{A}$ is isomorphic to the CA_α-type reduct of a suitable $\mathcal{B} \in QPEA_\alpha^-$.*

It is also natural to search subclasses of CA_α (or, subclasses of CA_α^+) whose elements can be obtained as CA_α-type reducts of certain $QPEA_\alpha$. Of course, Lf_ω, Dc_ω and the class included in [11], Theorem 3.2.52 are such classes. The following result is a generalization of this latter theorem from CA_α to CA_α^+ due to Ferenczi (see, [8], Theorem 3.5).

An algebra $\mathcal{A}$ in CA_α *can be supplemented* to an algebra $\mathcal{A}$ in $FPEA_\alpha$ if supplementing $\mathcal{A}$ by the usual substitution operators s^i_j and by certain operators p_{ij}, $(i, j < \alpha)$ the algebra obtained is in $FPEA_\alpha$.

Theorem 1.26 *Suppose that $\mathcal{A} \in CA_\alpha^+$ and $\mathcal{A} = Nr_\alpha \mathcal{B}$ for some $\mathcal{B} \in CA_{\alpha+1}^+$ (where $\alpha \geq 4$). Then $\mathcal{A}$ can be supplemented to an algebra $\widetilde{\mathcal{A}} \in FPEA_\alpha$ such that $p_{ij}a = {}_k s(i,j)a$ for any $k \notin \Delta(a), k \leq \alpha$.*

Proof. As a consequence of the merry-go-round properties, $\mathcal{B} \in CA^+_{\alpha+1}$ implies, that the operation $p_{ij} = {}_ks(i,j)a$ satisfies axioms $(F_6), (F_7), (F_8)$ and (F_9), for any $k \notin \Delta(a), k \leq \alpha$. Further, ${}_ks(i,j)a \in A$ holds by definition, if $k < \alpha$ and by $\mathcal{A} = Nr_\alpha \mathcal{B}$, if $k = \alpha$. So $\mathcal{A}$ can be supplemented to an algebra $\widetilde{\mathcal{A}} \in FPEA_\alpha$.

We claim, that $\widetilde{\mathcal{A}}$ is $\overline{R}$-representable. By the Resek-Thompson theorem, $\mathcal{B}$ is representable by an algebra $\mathcal{B}^* \in Crs_{\alpha+1} \cap CA_{\alpha+1}$; let $\mathcal{A}$' be the image of $\mathcal{A}$ under this representation. Obviously, $\mathcal{A}'$ is an $\alpha-$dimensional algebra in $Crs_\alpha \cap CA_\alpha$. For any $a \in A$, we will denote by a' the element in A' corresponding to a. Combining the facts, that

- $\mathcal{A}' = Nr_\alpha \mathcal{B}^*$,
- $p_{ij}(a) = {}_ks(i,j)a,$ and
- ${}_ks(i,j)$ can be expressed by cylindrifications and diagonals included in $\mathcal{B}$,

one obtains, that $\mathcal{A}$' can also be supplemented to an algebra $\widetilde{\mathcal{A}'} \in FPEA_\alpha$ and, in addition, $P_{ij}(a') = {}_kS^V(i,j)a'$ for any $k \notin \Delta(a), k \leq \alpha$ where V is the unit of $\mathcal{A}'$ (so $\mathcal{A}'$ is closed under the operations ${}_kS^V(i,j)$, too). In addition, $\widetilde{\mathcal{A}} \simeq \widetilde{\mathcal{A}'}$.

We claim, that $\widetilde{\mathcal{A}'} \in Prs_\alpha$. Namely, $Rd_{ca}\widetilde{\mathcal{A}'} \in Crs_\alpha \cap CA_\alpha$ and we show that ${}_kS^V(i,j)$ coincides with the operation $s_{[i,j]}$ in $\widetilde{\mathcal{A}'}$. One can check that

$$s_{[i,\,j]}(a^*) \;=\; {}_kS^{\overline{V}}(i,j)\; a^* \tag{1}$$

where $\overline{V}$ is the unit of $\mathcal{B}^*$. Restricting (1) to $\mathcal{A}'$, we get $s_{[i,j]}(a') = {}_kS^V(i,j)\; a'$ because $\mathcal{A}' = Nr_\alpha \mathcal{B}^*$ (here V is the unit of $\mathcal{A}'$).

$s_{[i,j]}(V) = V$ follows from ${}_kS^V(i,j)V = V$ using that $Rd_{ca}\mathcal{A}' \in CA_\alpha$; $s_{[i,j]}$ satisfies the $FPEA$ axioms concerning the p'_{ij}s. So we have $\widetilde{\mathcal{A}'} \in Prs_\alpha \cap FPEA_\alpha$. Therefore $\widetilde{\mathcal{A}} \in \overline{R}FPEA_\alpha$ and $p_{ij}(a) = {}_ks(i,j)a$ for any $k \notin \Delta(a), k \leq \alpha$. ∎

Moreover, Andréka and Németi proved in [4] that for $4 \leq \alpha < \omega$ there exists a nonrepresentable $\mathcal{A} \in PEA_\alpha$ such that its CA_α-type reduct is representable (as a cylindric algebra).

We close this subsection by recalling the infinite dimensional analogs of this result. As we have seen in remark 1.8 above, there exists a nonrepresentable PEA_ω whose CA_ω-type reduct is representable. Moreover, by a recent result of T. Sayed Ahmed [26], there exists a nonrepresentable $QPEA_\omega$ with a representable CA_ω-type reduct. Related investigations can also be found in Sági [21].

1.4 Complexity of the Equational Theories of Certain Classes of Polyadic Algebras

In this subsection we study the recursion theoretic complexity of the equational theories of polyadic algebras of dimension α. We start by the finite dimensional case.

Theorem 1.27 *For $3 \leq \alpha < \omega$ the equational theory of $Rppe_\alpha$ is undecidable.*

This theorem may be derived from the analogous result for cylindric algebras, see e.g. [11].

As we mentioned, if α is infinite, then the langauge of PA_α contains continuum many operation symbols, hence, the equational theory of $Rppe_\alpha$ is not recursively enumerable for trivial reasons. We will see below, that the situation remains the same, if we study "rich enough" finite reducts. Again, the rest of this subsection is divided into two parts: first we will deal with polyadic algebras without diagonal elements and then with polyadic equality algebras.

Complexity of equational theories, the diagonal-free case

After the Daigneault-Monk Theorem one could think, that if L is any finite reduct of the language of PA_ω, then the set of equations written in L and valid in RPA_ω forms a recursively enumerable set. Indeed, usually, representation theorems imply completeness theorems, and completeness theorems usually imply recursive enumerability. It turned out, that this commonsense reasoning breaks down in the case of RPA_ω.

Theorem 1.28 *(Sági)*
There is a finite reduct L of the language of PA_ω such that the set of equational consequences of $P_0 - P_{11}$ written in L is not recursively enumerable.

The proof can be found in [18], see also [19].

There are some positive results, as well.

Theorem 1.29 *(Sain-Gyuris)*
There is a finite reduct L of the language of PA_ω such that all the CA_ω operations are term definable in L and the variety generated by the L-reducts of RPA_ω can be axiomatized by a recursive set Σ of equations. In addition, although Σ is infinite, it may be described by finitely many schemas.

We note, that the schemas occurring in Theorem 1.29 are essentially simpler than Halmos schemas in general. The proof of Theorem 1.29 can be found in [23].

Complexity of equational theories, the diagonal case

As we have seen in subsection 1.2, the equational theory of $Rppe_\omega$ is rather complex in the "axiomatic sense", that is, there is no way to axiomatize it by Halmos schemas. In this section we survey some results on the recursion theoretic complexity of the equational theory of $Rppe_\omega$.

We start by recalling some notions from recursion theory. Throughout $\mathcal{N}$ denotes the standard model of number theory. By a Π^1_1 formula we mean a second order formula in prenex form in which every second order variable is quantified universally. A set A of natural numbers is defined to be

- arithmetical iff A is definable in $\mathcal{N}$ by a first order formula of arithmetic;
- A is called a Π^1_1-set iff it is definable by a Π^1_1-formula of arithmetic.

As it is well known, recursively enumerable sets and their complements are arithmetical sets, and the family of arithmetical sets contains sets much more complicated than any recursively enumerable set. Clearly, arithmetical sets are Π^1_1, as well.

Theorem 1.30 *(Németi, Sági)*
There is a strictly finite reduct L of the language of $Rppe_\omega$ and a recursive function tr mapping Π^1_1 formulas of arithmetic to equations of L such that for any Π^1_1 sentence σ

$$\mathcal{N} \models \sigma \quad \textit{iff} \quad Rppe_\omega \models tr(\sigma).$$

The proof can be found in [16], see also [19].

Theorem 1.30 may be interpreted as follows: there is a finite reduct L of the language of $Rppe_\omega$ such that the set P of (Gödel numbers of) equations written in L and valid in $Rppe_\omega$ is at least as complicated as the set S of (Gödel numbers of) Π^1_1 formulas of arithmetic true in $\mathcal{N}$. By a (version of) Tarski's theorem of undefinability of truth, S is not Π^1_1. Hence P is not Π^1_1, as well. Consequently, $Eq(Rppe_\omega)$ cannot be axiomatized by any kind of finite equational schemas Σ whose set of consequences is recursively enumerable (or at least Π^1_1).

Although Theorems 1.28 and 1.30 have very strong consequences, the reducts L in them are rather artificial: the indices of the substitution operations are carefully chosen, tricky recursive functions on ω. Hence, it is natural to ask what can be said about "more natural" reducts of $Rppe_\omega$. The following theorem is due to R. McKenzie and it shows, that even, some "natural reducts" of $Rppe_\omega$ may have a complicated equational theory.

Recall, that $pred, suc : \omega \to \omega$ are the functions defined by

$$pred(0) = 0, \ pred(n+1) = n \ \text{ and } \ suc(n) = n+1$$

for every $n \in \omega$.

Theorem 1.31 *(McKenzie)*
Let L be any countable reduct of the language of $Rppe_\omega$ containing the set of operations $\{\cdot, \sim, c_{(\omega)}, s_{suc}, s_{pred}, s_{[0,1]}, s_{[i/j]}, d_{ij} : \ i, j \in \omega\}$. Then the set of equations valid in $Rppe_\omega$ and written in L is not recursively enumerable.

For the details, see Chapter 11 of [6].

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