Designing Decidable Logics of Epistemology

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Abstract. We investigate the following "epistemic" extensions of (fragments of) first order logics: if φ is a formula, then $\Box_i\varphi$ is also a formula, where I is a fixed finite set. The intended meaning of $\Box_i\varphi$ is "the i^{th} agent (i^{th} participant of the model) knows φ". The main result of the paper is Theorem 1: if L is such a fragment of first order logic whose consequence relation is weakly decidable, then the consequence relation of the epistemic extension of L remains weakly decidable, as well.

1 Introduction

Definition 1. *Let L be a fragment of first order logic and let I be any finite set. The set E_L of elementary epistemic formulas over L is defined to be the smallest set satisfying the following two stipulations:*
• *E_L contains all formulas of L and*
• *for any $i \in I$ and $\varphi \in E_L$ we have $\Box_i\varphi \in Form_{\mathcal{E},I}(L)$ (that is, E_L is closed for the operations $\Box_i$, for any $i \in I$).*
In addition, $Form_{\mathcal{E},I}(L)$ is defined to be the set of all Boolean combinations of E_L.

The intended meaning of $\Box_i\varphi$ is "the i^{th} agent (i^{th} participant of the model) knows φ", where φ is a formula that may also contain $\Box_j$ operations.

Logics of epistemology has been studied intensively, for related investigations we refer to [1], [2] and the references therein.

Our main aim is to provide semantics for the formulas $Form_{\mathcal{E},I}(L)$ in such a way, that the consequence relation of our semantics remains decidable, whenever the consequence relation of L is decidable. For a quite expressive fragment of first order logic with (weakly) decidable consequence relation, we refer to [3].

To achieve our goal, we need further preparations. In Section 2 we are summing up the preliminaries and definitions we need, in Section 3 we present the proofs.

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2 Technical Introduction

Our notation is standard, however, the following short list may help the reader. Throughout $\mathbf{N}$ denotes the set of natural numbers. Let L be a logic. Then $Form_L$ and $\models_L$ denote respectively, the set of formulas of L and the consequence relation of L (as usual, $\models_L$ also denotes the satisfaction relation of L). If $\mathcal{A}$ is a model for L then $Th(\mathcal{A})$ denotes the theory of $\mathcal{A}$ which is defined to be

$$Th(\mathcal{A}) = \{\varphi \in Form_L : \mathcal{A} \models_L \varphi\}.$$

Throughout, by Gödel numbering we mean an injective function

$$g : Form_{\mathcal{E},I}(L) \to \mathbf{N}$$

such that both g and g^{-1} is computable. It is well known, that such a g function exists (in fact, there exists a primitive recursive such g with g^{-1} primitive recursive, as well). We do not specify g further, because below we will use the fact only, that such a g exists (and we do not use the particular form, or further properties of such a g).

Definition 2. *Let $\varphi \in Form_{\mathcal{E},I}(L)$. Then the tautological skeleton $taut(\varphi)$ is defined inductively as follows.*

if φ is a formula of L, then $taut(\varphi) = \varphi$;
$taut(\neg\psi) = \neg taut(\psi)$;
$taut(\psi \wedge \varrho) = taut(\psi) \wedge taut(\varrho)$;
$taut(\Box_i\psi) = Z_n$ where Z_n is the n^{th} propositional variable and n is the Gödel-number of $\Box_i\psi$.

In addition, if $X \subseteq Form_{\mathcal{E},I}(L)$, then $taut(X)$ is defined to be

$$taut(X) = \{taut(\varphi) : \varphi \in X\}.$$

Remark 1. It is easy to see, that *taut* is a computable function, that is, there exists an algorithm computing $taut(\varphi)$ from φ. Moreover, φ is also computable from $taut(\varphi)$, because each propositional variable Z_n corresponds at most one formula $\psi \in Form_{\mathcal{E},I}(L)$, namely, Z_n corresponds to that ψ (if any) whose Gödel number is n.

Definition 3. *Let $X \subseteq Form_{\mathcal{E},I}(L)$ and let $i \in I$ be fixed. Then $cl_i(X)$ is defined to be*

$$cl_i(X) = \{\varphi, \Box_i\varphi : taut(X) \models_L taut(\varphi)\}.$$

Definition 4. *By an $\langle \mathcal{E}, I\rangle$-structure we mean a pair $\langle \mathcal{A}, f\rangle$ where $\mathcal{A}$ is an L-structure and $f : I \to \mathcal{P}(Form_{\mathcal{E},I}(L))$ is a function, such that for any $i \in I$*

- *If $\varphi \in Form_L$ and $\mathcal{A} \models_L \varphi$ then $\varphi \in f(i)$;*
- *$cl_i(f(i)) = f(i)$.*

Definition 5. *Let $\langle \mathcal{A}, f \rangle$ be an $\langle \mathcal{E}, I \rangle$-structure and let k be an evaluation over $\mathcal{A}$. Then the satisfaction relation $\langle \mathcal{A}, f \rangle \models_{\mathcal{E},L} \varphi[k]$ is defined recursively on the complexity of $\varphi \in Form_{\mathcal{E},I}(L)$ as follows.*

- *for an atomic (first order) formula $\langle \mathcal{A}, f \rangle \models_{\mathcal{E},L} \varphi[k]$ iff $\mathcal{A} \models \varphi[k]$;*
- *$\langle \mathcal{A}, f \rangle \models_{\mathcal{E},L} \neg\varphi[k]$ iff $\langle \mathcal{A}, f \rangle \not\models_{\mathcal{E},L} \varphi[k]$;*
- *$\langle \mathcal{A}, f \rangle \models_{\mathcal{E},L} \varphi \wedge \psi[k]$ iff $\langle \mathcal{A}, f \rangle \models_{\mathcal{E},L} \varphi[k]$ and $\langle \mathcal{A}, f \rangle \models_{\mathcal{E},L} \psi[k]$;*
- *$\langle \mathcal{A}, f \rangle \models_{\mathcal{E},L} \exists v_n \varphi[k]$ iff there exists an evaluation k' such that $\langle \mathcal{A}, f \rangle \models_{\mathcal{E},L} \varphi[k']$ and for any $m \neq n$ we have $k(m) = k'(m)$;*
- *$\langle \mathcal{A}, f \rangle \models_{\mathcal{E},L} \Box_i \varphi[k]$ iff $\varphi \in f(i)$.*

Finally, $\langle \mathcal{A}, f \rangle \models_{\mathcal{E},L} \varphi$ iff $\langle \mathcal{A}, f \rangle \models_{\mathcal{E},L} \varphi[k]$ for any evaluation k over $\mathcal{A}$.

Using the notation of the previous definition, it is easy to see, that for a first order formula $\varphi \in Form(L)$ the assertion $\langle \mathcal{A}, f \rangle \models_{\mathcal{E},L} \varphi$ is equivalent with $\mathcal{A} \models \varphi$.

Definition 6. *Let $\Sigma \subseteq Form_{\mathcal{E},I}(L)$ and let $\varphi \in Form_{\mathcal{E},I}(L)$. Then $\Sigma \models_{\mathcal{E},L} \varphi$ iff for any $\langle \mathcal{E}, I \rangle$-structure $\langle \mathcal{A}, f \rangle$ the following holds:*

$$\text{if for all } \psi \in \Sigma \text{ we have } \langle \mathcal{A}, f \rangle \models_{\mathcal{E},L} \psi \text{ then } \langle \mathcal{A}, f \rangle \models_{\mathcal{E},L} \varphi.$$

We say, that the consequence relation of a logic $\mathcal{L}$ is weakly decidable iff there exists an algorithm π whose input is a finite set $\Sigma \subseteq Form(\mathcal{L})$ and a formula $\varphi \in Form(\mathcal{L})$ and π always stops after a finite number of steps and provides a correct answer for the question "$\Sigma \overset{?}{\models}_{\mathcal{L}} \varphi$".

Our main result is as follows.

Theorem 1. *Suppose the consequence relation $\models_L$ of L is weakly decidable. Then $\models_{\mathcal{E},L}$ is also weakly decidable.*

The rest of this paper is devoted to prove this theorem. To do so, we need further preparations.

3 Proofs

Lemma 1. *Assume, that the consequence relation $\models_L$ of L is decidable. Let $X \subseteq Form_{\mathcal{E},I}(L)$ be a decidable subset of $Form_{\mathcal{E},I}(L)$ and let $i \in I$ be fixed. Then $cl_i(X)$ is a decidable subset of $Form_{\mathcal{E},I}(L)$.*

Proof. Clearly, if X is decidable, then so is $taut(X)$ (because by Remark 1, $taut^{-1}$ is computable and X is assumed to be decidable). Combining this with the assumption, that the consequence relation $\models_L$ of L is decidable, the statement follows immediately.

Now we will define a relation Ded and show, that this relation is decidable. Finally, we show, that Ded and the consequence relation $\models_{\mathcal{E},L}$ coincide, thus the algorithm deciding Ded also witnesses, that the consequence relation $\models_{\mathcal{E},L}$ is weakly decidable.

Definition 7. *Let $X \subseteq Form_{\mathcal{E},I}(L)$ and let $\varphi \in Form_{\mathcal{E},I}(L)$. Then*

$$Ded_0(X) = \{\psi \in Form_L : X \cap Form_L \models_L \psi\}.$$

Now suppose, that Ded_n has already been defined for some $n \in \mathbf{N}$. Then $Ded_{n+1}(X)$ is defined by recursion as follows.

> *$Ded_n(X) \subseteq Ded_{n+1}(X)$;*
> *if $\varphi = \Box_i \psi$ then $\varphi \in Ded_{n+1}(X)$ iff $\psi \in cl_i(Ded_n(X))$;*
> *if $\varphi = \neg\psi$ then $\varphi \in Ded_{n+1}(X)$ iff $\psi \notin Ded_{n+1}(X)$;*
> *if $\varphi = \psi \wedge \varrho$ then $\varphi \in Ded_{n+1}(X)$ iff $\psi \in Ded_{n+1}(X)$ and $\varrho \in Ded_{n+1}(X)$.*

Finally, let

$$Ded(X) = \bigcup_{n \in \mathbf{N}} Ded_n(X).$$

Theorem 2. *Assume, that the consequence relation $\models_L$ of L is weakly decidable. Let $X \subseteq Form_{\mathcal{E},I}(L)$ be a finite subset of $Form_{\mathcal{E},I}(L)$. Then $Ded(X)$ is a decidable subset of $Form_{\mathcal{E},I}(L)$.*

Proof. A simple inspection of Definition 7 together with Lemma 1 shows, that $Ded_n(X)$ is decidable for all $n \in \mathbf{N}$, in addition, (the Gödel number of) an algorithm deciding $Ded_n(X)$ may be computed from n. Moreover, $\varphi \in Ded(X)$ iff $\varphi \in Ded_n(X)$, where n is the number of all occurrences of $\Box$-operations in φ. It follows, that $Ded(X)$ is decidable, as desired.

Now we are ready to prove Theorem 1. We will split the proof into two parts.

Theorem 3. *Assume, that the consequence relation $\models_L$ of L is weakly decidable. Let $X \subseteq Form_{\mathcal{E},I}(L)$ be a finite subset of $Form_{\mathcal{E},I}(L)$ and let $\varphi \in Form_{\mathcal{E},I}(L)$. Then*

$$\varphi \in Ded(X) \quad \text{implies} \quad X \models_{\mathcal{E},L} \varphi.$$

Proof. Suppose $\varphi \in Ded(X)$. Then there exists $n \in \mathbf{N}$ such that $\varphi \in Ded_n(X)$. So it is enough to show

$$(*) \qquad \varphi \in Ded_n(X) \ \text{ implies } \ X \models_{\mathcal{E},L} \varphi \quad \text{and}$$
$$\langle \mathcal{A}, f \rangle \models X \ \text{ impies } \ (\forall i \in I) Ded_n(X) \subseteq f(i).$$

We apply induction on n. If $n = 0$, then $(*)$ holds, obviously. Now assume, that $(*)$ holds for $0, ..., n$; we shall show, that it remains true for $n+1$. To do so, let $\varphi \in Ded_{n+1}(X)$.

If $\varphi \in Form_L$ then, in fact, $\varphi \in Ded_0(X)$, hence $(*)$ follows for φ from the $n = 0$ case.

If $\varphi = \Box_i \psi$, then, according to Definition 7, we have $\psi \in cl_i(Ded_n(X))$. By induction, we have $X \models_{\mathcal{E},L} Ded_n(X)$. Assume $\langle \mathcal{A}, f \rangle \models_{\mathcal{E},L} X$. It follows, that

$\langle \mathcal{A}, f \rangle \models_{\mathcal{E},L} Ded_n(X)$. So, again by induction, we have $Ded_n(X) \subseteq f(i)$. Combining this with the second stipulation of Definition 4, we obtain $\langle \mathcal{A}, f \rangle \models_{\mathcal{E},L} \varphi$. This shows, that $(*)$ remains true for φ.

If $\varphi = \neg\psi$ or $\varphi = \psi \wedge \varrho$ then $(*)$ for φ may be derived from Definition 5 and Definition 7 in the usual way.

This completes the induction, and we are done.

Now we prove the converse of Theorem 3.

Theorem 4. *Assume, that the consequence relation $\models_L$ of L is weakly decidable. Let $X \subseteq Form_{\mathcal{E},I}(L)$ be a finite subset of $Form_{\mathcal{E},I}(L)$ and let $\varphi \in Form_{\mathcal{E},I}(L)$. Then*

$$X \models_{\mathcal{E},L} \varphi \quad \textit{implies} \quad \varphi \in Ded(X).$$

Proof. Assume, $\varphi \notin Ded(X)$; it is enough to show, that $X \not\models_{\mathcal{E},L} \varphi$. Do do so, we shall construct an $\langle \mathcal{E}, I \rangle$-structure $\langle \mathcal{A}, f \rangle$ such that $\langle \mathcal{A}, f \rangle \models_{\mathcal{E},L} X$ but $\langle \mathcal{A}, f \rangle \not\models_{\mathcal{E},L} \varphi$.

First we show, that $X \cap Form_L$ is consistent (in the sense of usual first order logic). Indeed, if $X \cap Form_L$ would be inconsistent, then it would follow, that $Ded_0 = Form_L$, consequently, we would have $Ded(X) = Form_{\mathcal{E},I}(L)$; particularly we would have $\varphi \in Ded(X)$. Thus, there exists a first order structure $\mathcal{A}$ such that $\mathcal{A} \models_L X \cap Form_L$.

Now, for any $i \in I$, let $f(i) = cl_i(Th_L(\mathcal{A}))$. Clearly, $\langle \mathcal{A}, f \rangle$ is an $\langle \mathcal{E}, I \rangle$-structure. Observe, that for any $\psi \in Form_{\mathcal{E},I}(L)$ we have $\langle \mathcal{A}, f \rangle \models_{\mathcal{E},L} \psi$ iff $\psi \in Ded(X)$. Particularly, $\langle \mathcal{A}, f \rangle \models_{\mathcal{E},L} X$ and $\langle \mathcal{A}, f \rangle \not\models_{\mathcal{E},L} \varphi$, as desired.

Now we are ready to prove the main result of the paper which is a more detailed version of Theorem 1.

Theorem 5. *Suppose the consequence relation $\models_L$ of L is weakly decidable. Let $X \subseteq Form_{\mathcal{E},I}(L)$ be a finite subset of $Form_{\mathcal{E},I}(L)$ and let $\varphi \in Form_{\mathcal{E},I}(L)$. Then we have*

(1) $X \models_{\mathcal{E},L} \varphi$ iff $\varphi \in Ded(X)$;
(2) $\models_{\mathcal{E},L}$ is weakly decidable, too.

Proof. Combining Theorems 3 and 4, (1) follows immediately. To prove (2) we note, that according to (1), for any finite $X \subseteq Form_{\mathcal{E},I}(L)$ and $\varphi \in Form_{\mathcal{E},I}(L)$ we have $X \models_{\mathcal{E},L} \varphi$ iff $\varphi \in Ded(X)$. But, by Theorem 2 $Ded(X)$ is decidable for any decidable X (in addition, an algorithm deciding $Ded(X)$ may be effectively constructed from an algorithm deciding X).

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