The number of countable models via Algebraic logic

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Abstract

Vaught's Conjecture states that if T is a complete first order theory in a countable language that has more than $\aleph_0$ pairwise non isomorphic countable models, then T has $2^{\aleph_0}$ such models. Morley showed that if T has more than $\aleph_1$ pairwise non isomorphic countable models, then it has $2^{\aleph_0}$ such models.

In this paper, we first show how we can use algebraic logic, namely the representation theory of cylindric and quasi-polyadic algebras, to study Vaught's conjecture (count models), and we re-prove Morley's above mentioned theorem. Second, we show that Morley's theorem holds for the number of non isomorphic countable models omitting a countable family of types. We go further by giving examples showing that although this number can only take the values given by Morley's theorem, it can be different from the number of all non isomorphic countable models. Moreover, our examples show that the number of countable models omitting a family of types can also be either $\aleph_1$ or 2 and therefore different from the possible values provided by Vaught's conjecture and by his well known theorem; in the case of $\aleph_1$, however, the family is uncountable. Finally, we discuss an omitting types theorem of Shelah.

The algebraic proof of Morley's theorem has been independently examined, essentially the same way, by T.S. Ahmed and M. Assem in Cairo and by G. Sági and D. Sziráki in Budapest, (see [12]). The rest of the paper is the joint work of the authors.

1 Introduction

The famous conjecture of Vaught says that if T is a countable complete theory, then the number of pairwise non isomorphic models of T is either at most countable or $2^{\aleph_0}$. The conjecture, in this form, has not been settled yet, though it is proven in a number of particular nontrivial cases by many prominent logicians (like Bouscaren, Buechler, Burgess, Lascar, Steel and Shelah). The most general result obtained in this connection is Morley's theorem, which excluded all other possibilities except for $\aleph_1$.

Morley's proof uses essentially the logic $L_{\omega_1,\omega}$ which allows infinite conjunctions. The proof distinguishes basically between two cases. One is when the theory T is *not* scattered, meaning that for some infinitary fragment $F \subseteq L_{\omega_1,\omega}$ and $n \in \omega$, the set $S_n(T, F)$ of all maximal F-types is uncountable. Examples of such theories are $Th\mathbb{N}$ (true arithmetic) and the theory of real closed fields.

In the other case, the theory T *is* scattered, meaning that $S_n(T, F)$ is countable for every infinite fragment F and every n. Examples include the theories of dense linear orders, algebraically closed fields and random graphs. *This is the critical case.* In other words, Vaught's conjecture is actually equivalent to the fact that if T is a scattered theory then the number of non isomorphic models of T is at most countable. A violation of Vaught's conjecture would thus entail constructing a scattered theory which has exactly $\aleph_1$ many pairwise non isomorphic countable models.

There are other proofs of Morley's result existing in the literature. One such proof uses Burgess' famous theorem that addresses a much more general context, namely, group actions on Polish spaces. It is motivated by the topological version of Vaught's conjecture, which deals with general actions of various Polish groups on Polish spaces rather than the action of S_∞ on the Polish space of countable models of a given countable complete theory. (This set can be endowed with a very natural topology).

Our proof also uses Burgess' theorem, so why do we claim that it is new? At the technical level, we apply Burgess' theorem to a topological space completely different from the space Morley originally dealt with. This technical difference provides a uniform algebraic way that may be utilized to study Vaught's conjecture and related problems in several modifications (restrictions and generalizations) of first order logics, like certain logics with infinitary predicates.

In more detail, here are the novelties that we point out:

(1) our proof is algebraic; it uses the representation theory of cylindric and quasi-polyadic algebras which is implemented by counting what we call Henkin ultrafilters in the Stone space of the Boolean part of the algebra representing the theory in question. Such ultrafilters correspond exactly to countable models, and the natural topology on the set of countable models mentioned above corresponds just to the Stone topology on these ultrafilters.

    This approach is a novel one initiated by Sági. Indeed, it opens a new avenue between algebraic logic, model theory and descriptive set theory. Also it has the asset of obtaining model-theoretic results for first order logic with equality and without equality using essentially one algebraic argument. Other results on model theoretic spectrum functions (like the number of non isomorphic models of a given theory in a given cardinality) may be systematically investigated on this basis; in this connection we refer to [10].

(2) Our proof is also significantly distinct from *all* the known proofs because of the following. We apply Burgess' theorem to the Stone dual space of certain cylindric-like algebras. These seem to be much more comprehensive than other "usual" spaces originating from certain fragments of infinitary logics that can be associated to countable

models. Moreover, as we mentioned, our approach provides a uniform way to treat related problems in rather different settings.

(3) We assume that our theories are only consistent, we do not assume completeness; (this is not a major difference, though). Algebraically we deal with arbitrary locally finite algebras that are not necessarily simple.

(4) The method enables us to count the number of non isomorphic models that omit a given countable family of non principal (finitary) types; this is done by stipulating that the desired ultrafilters preserve additional sets of meets.

For the number of non isomorphic countable models omitting a countable family of types, we obtain the same possibilities as the ones that were provided by Morley's theorem for the number of all countable models. However, for certain particular theories and families of types, there are interesting discrepancies between the two actual values. For example, the set of ultrafilters (models) which do not omit the given types may have the power of the continuum. There are theories T (like the Peano arithmetic) and types satisfying the above property such that T has exactly one model that omits the given types. By the first property, these theories have $2^{\aleph_0}$ non isomorphic models, showing that the two values can be different.

Another example shows that, in contrast with a famous theorem of Vaught, there is a (countable and complete) theory T and a non principal type Γ of T such that the number of countable models omitting Γ is exactly 2.

Also, the number of countable models omitting types can be $\aleph_1$, and therefore different from the possible values provided by Vaught's conjecture. This is very easy to show using infinitary types, but we also give an example of a family of finitary types omitted by $\aleph_1$ countable models. The family in our example is of size $2^{\aleph_0}$, and if Vaught's conjecture for $L_{\omega_1,\omega}$ holds, then any such family has to be uncountable.

Notation

Our system of notation is mostly standard, but the following list may be useful. Throughout, ω denotes the set of natural numbers, and for every $n \in \omega$ we have $n = \{0, \dots, n-1\}$. Let A and B be sets. Then ${}^{A}B$ denotes the set of functions whose domain is A and whose range is a subset of B. In addition, $|A|$ denotes the cardinality of A and $\mathcal{P}(A)$ denotes the power set of A, that is, the set of all subsets of A. If $f : A \longrightarrow B$ is a function and $X \subseteq A$, then $f^*(X) = \{f(x) : x \in X\}$ and $f|_X$ is the restriction of f to X. Moreover, $f^{-1} : \mathcal{P}(B) \longrightarrow \mathcal{P}(A)$ acts between the power sets.

We denote structures by letters $\mathcal{M}, \mathcal{N}, \dots$ while their underlying sets are denoted by $M, N, \dots$.

2 Preliminaries from the Theory of Cylindric and Quasi-Polyadic Algebras

In this section, we recall the needed preliminaries from algebraic logic given in [10]. The section has survey character and is included to make the paper self contained and available to readers not so familiar with algebraic logic.

2.1 Cylindric Algebras

Cylindric algebras emerged in the first half of the twentieth century, due to the work of Alfred Tarski and his students. Their original intention with this theory was to provide an algebraic treatment of first order logic (with equality), just as Boolean algebras do for sentential calculus. Let α be an ordinal. Then an α-dimensional cylindric algebra is an algebraic structure of the form

$$\mathcal{A} = \langle A, \wedge, -, c_i, d_{ij} \rangle_{i,j<\alpha}$$

where the c_i are unary operations and d_{ij} are constants and $\mathcal{A}$ satisfies extra equational postulates, (see, e.g., Definition 1.1.1 of [4]). There are two main methods of constructing cylindric algebras, called "algebraizing syntax" and "algebraizing semantics". We start by describing **algebraizing syntax**. Suppose L is a first order language with equality, and let T be a theory in L. Two formulas ϕ and ψ are defined to be equivalent mod T, in symbols $\phi \equiv_T \psi$, iff $T \models \forall(\phi \leftrightarrow \psi)$, where $\forall(\phi \leftrightarrow \psi)$ is the universal closure of $\phi \leftrightarrow \psi$. Clearly, $\equiv_T$ is an equivalence relation on the set of L-formulas. Quotienting the formula algebra (the completely free algebra) with this equivalence relation, we obtain an ω-dimensional cylindric algebra which is denoted by $\mathsf{CA}(T)$ and sometimes called the *Lindenbaum algebra* of T.

Algebraizing semantics may be described as follows. Let $\mathcal{M}$ be a model of T, and for a formula ϕ, let $\|\phi\|^{\mathcal{M}} = \{s \in {}^{\omega}M : \mathcal{M} \models \phi[s]\}$; (we write $\|\phi\|$ when $\mathcal{M}$ is understood). The set $\{\|\phi\| : \phi \text{ is an } L\text{-formula}\}$, together with the operations $c_i\|\phi\| = \|\exists v_i \phi\|$ and $d_{ij} = \|v_i = v_j\|$ (and set theoretical intersection and complementation) forms an ω-dimensional cylindric algebra denoted by $\mathsf{Cs}(\mathcal{M})$.

2.2 Quasi-Polyadic Algebras

Suppose ϕ, ψ are L-formulas and ψ is obtained from ϕ by substituting the variable v_j for v_i, (both assumed free in ϕ). Let $s^i_j x$ be the cylindric term $c_i(x \wedge d_{ij})$. Then in $\mathsf{CA}(T)$ we have $\psi/\equiv_T = s^i_j(\phi/\equiv_T)$, and in $\mathsf{Cs}(\mathcal{M})$ we have $\|\psi\| = s^i_j\|\phi\|$. For $i, j \in \omega$, the cylindric algebraic terms s^i_j are called *substitutions*.

When one wants to apply the same process to theories in languages not containing the equality symbol, substitution operations are no longer term definable. To overcome this difficulty, the s^i_j are defined to be basic operations when algebraizing equality free logics.

More concretely, an α-dimensional *quasi-polyadic algebra* is an algebraic structure of the form

$$\mathcal{A} = \langle A, \wedge, -, c_i, s^i_j \rangle_{i,j<\alpha}$$

satisfying certain equational postulates. Here, the c_i correspond to existential quantifiers, and the s^i_j correspond to substitution operations. Algebraizing syntax and semantics may be defined similarly to the cylindric case; the resulting algebras are denoted by $\mathsf{QPA}(T)$ and by $\mathsf{Qs}(\mathcal{M})$, respectively.

2.3 Dimension Sets

Let $\mathcal{A}$ be an α-dimensional cylindric or quasi-polyadic algebra, and let $x \in A$. The dimension set Δx of x is defined to be $\Delta x = \{i \in \alpha : c_i x \neq x\}$. Intuitively, it is the set of free variables of x (as a formula). $\mathcal{A}$ is said to be *locally finite* if every element of it has a finite dimension set. Algebras obtained by the algebraization processes described above are locally finite, because each formula contains only a finite number of variables. We let Lf_α and LfQPA_α denote the classes of α-dimensional locally finite cylindric and quasi-polyadic algebras, respectively.

Simultaneous substitution in formulas can be described with a function $\tau \in {}^\omega\omega$: for every $i \in \omega$, let us substitute the variables $v_{\tau(i)}$ for the variables v_i simultaneously. As explained in Definitions 1.11.9 and 1.11.13 of [4], an operation corresponding to this kind of generalized substitution can be introduced in every locally finite algebra; it is denoted by s^+_τ.

Given an α dimensional cylindric algebra $\mathcal{A}$ and an ordinal $\beta < \alpha$, the *neat β-reduct* $\mathfrak{Nr}_\beta\mathcal{A}$ of $\mathcal{A}$ is defined to be the cylindric algebra whose universe is

$$\mathrm{Nr}_\beta\mathcal{A} = \{x \in A : c_i x = x \text{ for all } i \in \alpha - \beta\}$$

and whose operations are those of the β-dimensional reduct of $\mathcal{A}$, (i.e., the Boolean operations and c_i and d_{ij} for $i, j < \beta$,) restricted to $\mathrm{Nr}_\beta\mathcal{A}$; (it is easy to check that Nr_β is indeed closed under these operations and therefore $\mathfrak{Nr}_\beta\mathcal{A}$ is a β-dimensional cylindric algebra). Neat reducts in the quasi-polyadic case are defined similarly. If $\mathcal{A}$ is obtained by one of the above algebraization processes of first order logic and $n < \omega$, then $\mathfrak{Nr}_n\mathcal{A}$ corresponds to restricting our attention to those formulas in which only the free variables $v_0, v_1, \ldots, v_{n-1}$ are allowed.

2.4 Set Algebras

Let U be a set. Then *the full cylindric set algebra* on U of dimension α is the structure

$$\langle \mathcal{P}({}^\alpha U), \cap, -, C_i, D_{ij} \rangle_{i,j\in\alpha},$$

where $\cap$ is set theoretical intersection, $-$ is complementation (w.r.t. ${}^\alpha U$) and for any $X \subseteq {}^\alpha U$ and $i, j \in \alpha$,

$$C_i X = \{s \in {}^\alpha U : (\exists z \in X)(s|_{\alpha-\{i\}} = z|_{\alpha-\{i\}})\} \text{ and } D_{ij} = \{s \in {}^\alpha U : s(i) = s(j)\}.$$

The class Cs_α of α-dimensional cylindric set algebras is defined to be the class of all subalgebras of α-dimensional full cylindric set algebras. Similarly, *the full quasi-polyadic set algebra* on U of dimension α is the structure $\langle \mathcal{P}({}^\alpha U), \cap, -, C_i, S^i_j\rangle_{i,j\in\alpha}$, where for any $X \subseteq {}^\alpha U$ and $i, j \in \alpha$,

$$S^i_j X = \{s \in {}^\alpha U : s \circ [i/j] \in X\};$$

($[i/j]$ denotes the transformation that maps i to j and is the identity function on $\alpha - \{i\}$). The class Qs_α of α-dimensional quasi-polyadic set algebras is defined to be the class of all subalgebras of α-dimensional full quasi-polyadic set algebras.
An element X of an α-dimensional cylindric or quasi-polyadic set algebra is said to be *regular* iff for every $s, z \in {}^\alpha U$ we have

$$s \in X \text{ and } s|_{\Delta X} = z|_{\Delta X} \text{ imply } z \in X,$$

where $\Delta X = \{i \in \alpha : C_i X \neq X\}$. A cylindric or quasi-polyadic set algebra is regular if all its elements are regular. The classes of α-dimensional regular cylindric and quasi-polyadic set algebras are denoted by Cs^{reg}_α and Qs^{reg}_α, respectively. If $\mathcal{M}$ is an L-structure, then $\mathsf{Cs}(\mathcal{M})$ and $\mathsf{Qs}(\mathcal{M})$ are regular cylindric and quasi-polyadic set algebras, respectively.

Suppose T is a theory in a language L with equality. Then $h : \mathsf{CA}(T) \longrightarrow \mathsf{Cs}(\mathcal{M})$, $\phi/\equiv_T \mapsto \|\phi\|$ is a surjective homomorphism. Thus, $\mathsf{Cs}(\mathcal{M})$, which is an element of $\mathsf{Lf}_\omega \cap \mathsf{Cs}^{reg}_\omega$, is a homomorphic image of $\mathsf{CA}(T)$. Conversely, an element $\mathcal{A}$ of $\mathsf{Lf}_\omega \cap \mathsf{Cs}^{reg}_\omega$, together with a distinguished set of its generators, determines a structure for some language; (this fact is well known, see [10]). This structure is a model of T iff there is a homomorphism from $\mathsf{CA}(T)$ onto $\mathcal{A}$ mapping the distinguished set of generators of $\mathsf{CA}(T)$ onto that of $\mathcal{A}$.

Similarly, if T is a theory in a language without equality, there is also the above one-one correspondence between models of T and surjective homomorphisms from $\mathsf{QPA}(T)$ onto elements of $\mathsf{LfQPA}_\omega \cap \mathsf{Qs}^{reg}_\omega$. Consequently, the problem of finding all the possible countable models of T is equivalent to finding all homomorphisms from $\mathsf{CA}(T)$ or $\mathsf{QPA}(T)$ onto some locally finite dimensional and regular set algebra with a countable base.

3 Representations

Let $\mathcal{A}$ be any Boolean algebra. The set of ultrafilters of $\mathcal{A}$ is denoted by $\mathcal{U}(\mathcal{A})$. The Stone topology makes $\mathcal{U}(\mathcal{A})$ a compact Hausdorff space; we denote this space by $\mathcal{A}^*$. Recall that the Stone topology has as its basic open sets the sets $\{N_x : x \in A\}$, where

$$N_x = \{\mathcal{F} \in \mathcal{U}(\mathcal{A}) : x \in \mathcal{F}\}.$$

It is easy to see that if A is countable, then $\mathcal{A}^*$ is *Polish*, (i.e., separable and completely metrizable).

Now, suppose $\mathcal{A}$ is a locally finite cylindric or quasi-polyadic ω-dimensional algebra with a countable universe. Note that if T is a theory in a countable language with (without) equality, then $\mathsf{CA}(T)$ (respectively $\mathsf{QPA}(T)$) satisfies these requirements. Let

$$\mathcal{H}(\mathcal{A}) = \bigcap_{\substack{i<\omega \\ x\in A}} (N_{-c_i x} \cup \bigcup_{j<\omega} N_{s^i_j x})$$

and, in the cylindric algebraic case, let

$$\mathcal{H}'(\mathcal{A}) = \mathcal{H}(\mathcal{A}) \cap \bigcap_{i\neq j\in\omega} N_{-d_{ij}}.$$

Note, for later use, that $\mathcal{H}(\mathcal{A})$ and $\mathcal{H}'(\mathcal{A})$ are G_δ subsets of $\mathcal{A}^*$, and are nonempty, – this latter fact can be seen, for example, from Theorem 3.1 below – and are therefore Polish spaces; (see Theorem 3.11 of [7]). Assume $\mathcal{F} \in \mathcal{H}(\mathcal{A})$. For any $x \in A$, define the function $\mathrm{rep}_{\mathcal{F}}$ to be

$$\mathrm{rep}_{\mathcal{F}}(x) = \{\tau \in {}^\omega\omega : s^+_\tau x \in \mathcal{F}\}.$$

We have the following results due to G. Sági and D. Sziráki; (see [10]).

Theorem 3.1. *If $\mathcal{F} \in \mathcal{H}'(\mathcal{A})$ (respectively $\mathcal{H}(\mathcal{A})$), then $\mathrm{rep}_{\mathcal{F}}$ is a homomorphism from $\mathcal{A}$ onto an element of $\mathsf{Lf}_\omega \cap \mathsf{Cs}^{reg}_\omega$ (respectively $\mathsf{LfQPA}_\omega \cap \mathsf{Qs}^{reg}_\omega$) with base ω. Conversely, if h is a homomorphism from $\mathcal{A}$ onto an element of $\mathsf{Lf}_\omega \cap \mathsf{Cs}^{reg}_\omega$ (respectively $\mathsf{LfQPA}_\omega \cap \mathsf{Qs}^{reg}_\omega$) with base ω, then there is a unique $\mathcal{F} \in \mathcal{H}'(\mathcal{A})$ (respectively $\mathcal{H}(\mathcal{A})$) such that $h = \mathrm{rep}_{\mathcal{F}}$.*

Theorem 3.2. *Let T be a consistent first order theory in a countable language with (without) equality. Let $\mathcal{M}_0$ and $\mathcal{M}_1$ be two models of T whose universe is ω. Suppose $\mathcal{F}_0, \mathcal{F}_1 \in \mathcal{H}'(\mathsf{CA}(T))$ (respectively $\mathcal{H}(\mathsf{QPA}(T))$) are such that $\mathrm{rep}_{\mathcal{F}_i}$ are homomorphisms from $\mathsf{CA}(T)$ (respectively $\mathsf{QPA}(T)$) onto $\mathsf{Cs}(\mathcal{M}_i)$ (respectively $\mathsf{Qs}(\mathcal{M}_i)$) for $i = 0, 1$. If $\rho : \omega \longrightarrow \omega$ is a bijection, then the following are equivalent:*

1. *$\rho : \mathcal{M}_0 \longrightarrow \mathcal{M}_1$ is an isomorphism.*
2. *$\mathcal{F}_1 = s^+_\rho \mathcal{F}_0 = \{s^+_\rho x : x \in \mathcal{F}_0\}$.*

These last two theorems allow us to study models and count them via corresponding ultrafilters.

4 Counting Models

4.1 A New Proof of Morley's Theorem

This section begins with our new proof of Morley's theorem. In this proof, we basically translate into algebraic logic some well known results from descriptive set theory, with the help of the above two theorems from the representation theory of cylindric and quasi-polyadic algebras.

Theorem 4.1.

1. (Morley) *Suppose T is a complete first order theory in a countable language with equality. If T has more than $\aleph_1$ pairwise non isomorphic countable models, then it has $2^{\aleph_0}$ such models.*

2. *The same statement holds for theories not necessarily complete, in countable languages with or without equality.*

Proof. Let T be a first order theory in a countable language with equality, and let $\mathcal{A} = \mathsf{CA}(T)$. We remark that in the case of equality free languages, all the coming work can be done with $\mathcal{A} = \mathsf{QPA}(T)$; we just replace $\mathcal{H}'$ by $\mathcal{H}$. Let $Sym(\omega)$ be the set of permutations on ω; (it is a Polish group w.r.t. composition of functions and the topology it inherits from the Baire space ${}^{\omega}\omega$). Consider the map $J : Sym(\omega) \times \mathcal{H}'(\mathcal{A}) \longrightarrow \mathcal{H}'(\mathcal{A})$ defined by $J(\rho, \mathcal{F}) = s^+_\rho \mathcal{F}$ for all $\rho \in Sym(\omega)$ and $\mathcal{F} \in \mathcal{H}'(\mathcal{A})$. It is easy to see that J is a well defined action of $Sym(\omega)$ on $\mathcal{H}'(\mathcal{A})$. We show that J is a continuous map from $Sym(\omega) \times \mathcal{H}'(\mathcal{A})$ (with the product topology) to $\mathcal{H}'(\mathcal{A})$. To do this, it is enough to show that for an arbitrary $a \in A$,

$$J^{-1}(N_a \cap \mathcal{H}'(\mathcal{A})) = \bigcup_{\tau \in Sym(\omega)} \left(\{\mu^{-1} : \mu \in Sym(\omega),\ \mu|_{\Delta a} = \tau|_{\Delta a}\} \times [N_{s^+_\tau a} \cap \mathcal{H}'(\mathcal{A})]\right).$$

This is enough, because, for fixed $\tau \in Sym(\omega)$, the set $\{\mu^{-1} : \mu \in Sym(\omega),\ \mu|_{\Delta a} = \tau|_{\Delta a}\}$ is an open subset of $Sym(\omega)$. To see this, let $f : Sym(\omega) \longrightarrow Sym(\omega)$ be the map given by $f(\tau) = \tau^{-1}$. As we know, f is continuous and open. Hence, $\{\mu^{-1} : \mu \in Sym(\omega),\ \mu|_{\Delta a} = \tau|_{\Delta a}\} = f^*(\{\mu \in Sym(\omega) : \mu|_{\Delta a} = \tau|_{\Delta a}\})$ is the image of an open set via an open map, and is therefore open.
Now, let $(\rho, \mathcal{F})$ be an arbitrary element in $Sym(\omega) \times \mathcal{H}'(\mathcal{A})$. We have the following:

$$\begin{aligned}
(\rho, \mathcal{F}) \in \bigcup_{\tau \in Sym(\omega)} & \left(\{\mu^{-1} : \mu \in Sym(\omega),\ \mu|_{\Delta a} = \tau|_{\Delta a}\} \times [N_{s^+_\tau a} \cap \mathcal{H}'(\mathcal{A})]\right) \\
& \Longleftrightarrow (\exists \tau \in Sym(\omega))[(\exists \mu \in Sym(\omega))(\mu|_{\Delta a} = \tau|_{\Delta a} \wedge \rho = \mu^{-1}) \wedge s^+_\tau a \in \mathcal{F}] \\
& \Longleftrightarrow (\exists \tau \in Sym(\omega))[(\exists \mu \in Sym(\omega))(\mu|_{\Delta a} = \tau|_{\Delta a} \wedge \rho^{-1} = \mu) \wedge s^+_\tau a \in \mathcal{F}] \\
& \Longleftrightarrow (\exists \tau \in Sym(\omega))[\rho^{-1}|_{\Delta a} = \tau|_{\Delta a} \wedge s^+_\tau a \in \mathcal{F}] \\
& \Longleftrightarrow (s^+_\rho)^{-1} a = s^+_{\rho^{-1}} a \in \mathcal{F} \\
& \Longleftrightarrow a \in s^+_\rho \mathcal{F} \\
& \Longleftrightarrow J(\rho, \mathcal{F}) = s^+_\rho \mathcal{F} \in N_a \cap \mathcal{H}'(\mathcal{A}) \\
& \Longleftrightarrow (\rho, \mathcal{F}) \in J^{-1}(N_a \cap \mathcal{H}'(\mathcal{A})).
\end{aligned}$$

And so, J is a continuous action of $Sym(\omega)$ on $\mathcal{H}'(\mathcal{A})$. It follows that the the orbit equivalence relation is analytic; (see 3.2 in [1]). Thus, by a result of Burgess, (see [2]), if there are more than $\aleph_1$ orbits, then there are $2^{\aleph_0}$ orbits. By Theorems 3.1 and 3.2, the number of orbits here is exactly the number of non isomorphic countably infinite models of T. This completes the proof. $\square$

4.2 Counting Models which Omit Types

The proof of Theorem 4.1 can be generalized to talk about special kinds of models. For example, suppose $(\Gamma_i : i < \omega)$ is a countable family of non principal types of T; (by a type, we mean a set of formulas which is finitary, i.e., is in finitely many variables and which is consistent with T, but not necessarily maximal, as in Definition 4.1.1 of [8] for example). Then we can take the space $\mathcal{H}_{\text{omit}}$, (and restrict the action to it), where $\mathcal{H}_{\text{omit}}$ corresponds to the set of countable models of T which omit the given family $(\Gamma_i : i < \omega)$ of non principal types of T. More precisely, we have the following theorem.

Theorem 4.2. *Let T be a theory in a countable language with or without equality. Let $(\Gamma_i : i < \omega)$ be a family of non principal types. Then the number of non isomorphic countable models of T omitting this family is either $\leq \aleph_1$ or $2^{\aleph_0}$.*

Proof. Set

$$\mathcal{H}_{\text{omit}} = \mathcal{H}'(\mathsf{CA}(T)) \cap \bigcap_{\substack{i\in\omega \\ \tau\in W}} \bigcup_{\phi\in\Gamma_i} N_{-s_\tau^+(\phi/\equiv_T)},$$

where $W = \{\tau \in {}^\omega\omega : |\{i : \tau(i) \neq i\}| < \omega\}$. It is easy to see that $\mathcal{H}_{\text{omit}}$ corresponds to the set of countable models of T which omit the family $(\Gamma_i : i < \omega)$. (Here, we have to use the fact that the Γ_i are types in finitely many variables, i.e., $\Delta = \bigcup\{\Delta(\phi/\equiv_T) : \phi \in \Gamma_i\}$ is finite. This is what allows us to replace ${}^\omega\omega$ by the countable W in the definition above.) Thus, $\mathcal{H}_{\text{omit}}$ is G_δ, and therefore Polish, see Theorem 3.11 of [7]. Hence, we can apply an argument similar to the one in the proof of Theorem 4.1. For languages without equality, the same argument works with $\mathcal{H}(\mathsf{QPA}(T))$. $\square$

Problem 4.3. It would be interesting to know whether Theorem 4.2 remains true if we allow families to be uncountable, in particular, for families of size $< \mathsf{cov}K$. This is the least cardinal for which the Baire category theorem fails, i.e., $\mathsf{cov}K$ is the least cardinal κ for which the real line, – or equivalently, any Polish space without isolated points – can be covered by κ many nowhere dense sets. Martin's axiom implies $\mathsf{cov}K = 2^{\aleph_0}$, but it is also consistent that $\mathsf{cov}K = \aleph_1 < 2^{\aleph_0}$, see [9].

The cardinal $\mathsf{cov}K$ is connected intimately to omitting types: if T is a countable theory, then any family of $< \mathsf{cov}K$ many non principal types can be omitted by a countable model of T, and for certain countable theories, (even complete ones,) one can find a non omissible family of $\mathsf{cov}K$ many types [9, 3]. With our method, it is easy to see the first statement: if T is a countable theory and $\mathcal{G} = (\Gamma_i : i < \lambda)$ is a family of non principal types for some $\lambda < \mathsf{cov}K$, then the sets

$$X_{i,\tau} \stackrel{\text{def}}{=} \bigcap_{\phi\in\Gamma_i} N_{s_\tau^+(\phi/\equiv_T)},$$

are nowhere dense for all $i < \lambda$ and $\tau \in W$, due to the fact that the types are non principal.

Thus, the set

$$\mathcal{H}_{\text{omit}} = \mathcal{H}'(\mathsf{CA}(T)) - \bigcup_{\substack{i\in\lambda \\ \tau\in W}} X_{i,\tau}$$

is nonempty, by the properties of $\mathsf{cov}K$, implying that there is a countable model of T omitting the types in $\mathcal{G}$.

Unfortunately, the previous argument is not sufficient to determine the number of models omitting the family $\mathcal{G}$, i.e., the number of orbits of the action J on $\mathcal{H}_{\text{omit}}$ induced by isomorphism. In particular, while it does imply that $\mathcal{H}_{\text{omit}}$ is nonempty, it does not guarantee that it is G_δ. However, the above connection of $\mathsf{cov}K$ to omitting types makes Theorem 4.2 for models omitting families of size $< \mathsf{cov}K$ feasible.

As mentioned above, $\mathsf{cov}K$ is the least cardinal κ such that a family of κ many non principal types of a countable theory may not necessarily be omitted by a countable model. It is interesting to note that the situation is very different when we consider *maximal* types; in that case, even families of size $< 2^{\aleph_0}$ can always be omitted. This follows from a (deep as usual) result of Shelah, see Theorem 5.16 CH. IV [11]; (also see 7.2.4 and 7.2.5 in [6]). Here, the distinction between $\mathsf{cov}K$ and $2^{\aleph_0}$ is highly significant; for although $\mathsf{cov}K$ is an uncountable cardinal $\leq 2^{\aleph_0}$, one can show using iterated forcing that it is consistent that $\mathsf{cov}K < 2^{\aleph_0}$.

Suppose T is a countable theory, $\lambda < 2^{\aleph_0}$ and $(\Gamma_i : i < \lambda)$ are non principal maximal types of T. Defining the (nowhere dense) sets $X_{i,\tau} \stackrel{\text{def}}{=} \bigcap_{\phi\in\Gamma_i} N_{s_\tau^+(\phi/\equiv_T)}$ as above, Shelah's result tells us that

$$\mathcal{H}_{\text{omit}} = \mathcal{H}'(\mathsf{CA}(T)) - \bigcup_{\substack{i\in\lambda \\ \tau\in W}} X_{i,\tau} \neq \emptyset.$$

(On the other hand, if the types are not maximal, then we also know that the latter set can be empty; as mentioned above, there are known examples of countable theories, and even complete countable theories, that do not omit $\mathsf{cov}K$ many types.)

In light of the above, one can ask the following question: can the above result (that $\mathcal{H}_{\text{omit}} \neq \emptyset$ for families of $< 2^{\aleph_0}$ many maximal types) be obtained by a topological argument? If so, it can not rely solely on the Baire Category theorem, because this theorem simply does not apply in the uncountable case when we take $\geq \mathsf{cov}K$ many nowhere dense sets, and this is precisely the reason why the omitting types theorem can fail when we consider $\geq \mathsf{cov}K$ many non principal types that are not maximal. Perhaps there is a way to express the condition of maximality topologically that can be used in such an argument.

4.3 Some Examples

What we have proved in Theorem 4.2 is that for a countable theory T, the possible values of the number $|\mathcal{H}_{\text{omit}}|$ of non isomorphic countable models of T which omit a given countable family of types are exactly the values given by Morley's theorem for the number $|\mathcal{H}|$ of all non isomorphic countable models. But, as the examples below show, the values $|\mathcal{H}_{\rm omit}|$ and $|\mathcal{H}|$ may well be different for a theory T.

Example 4.4. Let T be $Th(\mathbb{N})$, true arithmetic. The standard model $\mathbb{N}$ of T is an atomic model, (see [8] p. 129). Thus, the neat n-reduct of $\mathcal{A} = \mathsf{CA}(T)$ is atomic for each $n < \omega$, (by an algebraic reformulation of a well known characterization of the existence of countable atomic models; see for example Theorem 4.2.10 of [8]). Now, let $\Gamma_n = \{-x : x \in \mathsf{At}\mathfrak{Nr}_n\mathcal{A}\}$, that is the set of the co-atoms of the neat reduct $\mathfrak{Nr}_n\mathcal{A}$. The Γ_n's are non principal types, and a model $\mathcal{M}$ omits them iff it is atomic; (it is not hard to show that any model that omits co-atoms actually omits any family of non principal types). Hence, any model that omits the Γ_n's is isomorphic to $\mathbb{N}$, by the unicity of countable atomic models. Thus, up to isomorphism, T has one countable model that omits the family $(\Gamma_n : n < \omega)$. On the other hand, it is known that T has $2^{\aleph_0}$ non isomorphic countable models, (see [8] p.155).

Example 4.5. Let T be the theory of algebraically closed fields of characteristic zero. Then T has countably many non isomorphic countable models; for each $\alpha \leq \omega$, there is one model of transcendence degree α over the rationals. Let $\mathcal{A} = \mathsf{CA}(T)$, and define the types Γ_n as above. Then the model of degree zero (the field of algebraic numbers) is atomic and is therefore the only countable model omitting the family $(\Gamma_n : n < \omega)$. However, as can be seen from Example 4.6 below, the above countable family can be replaced by just one type in this case.

The next example is motivated by a well known theorem of Vaught which says that a countable complete theory T cannot have exactly two non isomorphic countable models. In his proof, Vaught argued that if T has exactly two such models, he can always construct from these models a third one which is not isomorphic to either original model, a contradiction. If we require that the two models omit a given family of non principal types, then, in principle, the constructed third model might not omit this family of non principal types, so perhaps Vaught's argument fails in this new context. The example below shows that this is possible.

Example 4.6. Take the language $L = \{c_n : n \in \omega\}$, where the c_n are constant symbols. Let $T = \{c_n \neq c_m : n \neq m \in \omega\}$. Then a model $\mathcal{M}$ of T is determined up to isomorphism by how many "extra" elements it has, i.e., by $\big|\{b \in M : b \neq c_n^{\mathcal{M}} \text{ for all } n \in \omega\}\big|$. Thus, T is $\aleph_1$-categorical and so, since it has only infinite models, it is complete. Also, T has countably many non isomorphic countable models; these are the models $\mathcal{M}_\alpha$ with α many extra elements for $\alpha \leq \omega$.

Consider the following 2-type Γ of T:

$$\Gamma = \{v_0 \neq v_1\} \cup \{v_0 \neq c_n : n \in \omega\} \cup \{v_1 \neq c_n : n \in \omega\}.$$

Then Γ is a non principal type and is omitted by exactly two countable models of T; these are the models $\mathcal{M}_0$ and $\mathcal{M}_1$.

We note that this argument can be generalized for complete strongly minimal theories T which have (countable) models of dimension α for all $\alpha \leq \omega$; (see Section 6.1 of [8]). Suppose T is such a theory, and let Γ be the unique 2-type of two independent elements; (Γ is determined uniquely by Lemma 6.1.6 of [8]). Then the non principal type Γ is omitted by exactly two models of T, those of dimension 0 and 1. In particular, the theory in Example 4.5 above also has a non principal type that is omitted by exactly two models. Similarly, one can find, for all $0 < n < \omega$, an n-type omitted by exactly n countable models of T.

Another question arises naturally in the spirit of the last example and of Theorem 4.2: can we have a theory together with a family of (non principal) types such that the number of non isomorphic countable models omitting this family is exactly $\aleph_1$? An *infinitary* type that is omitted by exactly $\aleph_1$ such models is not hard to construct: let T be the theory of linear orders (in the language $L = \{<\}$) and let $\Gamma = \Gamma(v_0, v_1, \ldots) = \{v_n > v_{n+1} : n \in \omega\}$. Then Γ is omitted by a model $\mathcal{M}$ of T iff $\mathcal{M}$ is well ordered, implying that T has exactly $\aleph_1$ non isomorphic countable models which omit the type Γ.

A modification of the above idea answers the question affirmatively for finitary types as well; see Example 4.7 below. However, this example differs from those above in that the cardinality of our family of types is uncountable, namely $2^{\aleph_0}$. Managing to make the family of types countable is expected to be very hard. For if this could be done, then we would actually obtain a negative answer to Vaught's conjecture for $L_{\omega_1,\omega}$ sentences. In more detail, suppose that T is a countable theory and $(\Gamma_n : n \in \omega)$ is a family of finitary types such that the number of countable models of T omitting the Γ_n's is exactly $\aleph_1$. Consider now the $L_{\omega_1,\omega}$ sentence

$$\bigwedge T \wedge \bigwedge_{n\in\omega} \left(\neg(\exists \bar{v}_n) \bigwedge_{\phi\in\Gamma_n} \phi(\bar{v}_n) \right),$$

(where $\bar{v}_n$ are the free variables in Γ_n). Clearly, the models of this sentence are exactly those of T which omit the Γ_n's, and therefore this sentence violates the conjecture.

Example 4.7. Take the first order countable language $L = \{<, c_0, c_1, c_2, \ldots\}$ where $<$ is a binary relation symbol and $c_0, c_1, c_2, \ldots$ are constants, and let T be the theory of L which states that $<$ is a linear order and that $c_i \neq c_j$ for all $i \neq j \in \omega$. Define the following types of T: take the 1-type

$$\Gamma(v) = \{v \neq c_i : i \in \omega\}$$

and for every injective $f \in {}^{\omega}\omega$, let

$$\Gamma_f = \{c_{f(i)} > c_{f(i+1)} : i \in \omega\}.$$

Consider the family $\mathcal{G} = \{\Gamma(v)\} \cup \{\Gamma_f : f \in {}^{\omega}\omega \text{ is injective}\}$ of $2^{\aleph_0}$ many non principle types of T. Clearly, if a model omits $\mathcal{G}$, then it is a well order, (because of the following: if a model omits $\Gamma(v)$, then all its elements are presented by constants. If in addition it omits all the Γ_f's, then it has no infinite decreasing sequences, and is therefore a well order). This makes the number countable of models omitting $\mathcal{G}$ less than or equal to $\aleph_1$. Conversely, given any countable well order, it is easy to obtain a model $\mathcal{M}$ such that $(M, <)$ is isomorphic to that well order and $\mathcal{M}$ omits $\Gamma(v)$ and therefore $\mathcal{G}$. Thus, there are exactly $\aleph_1$ countable models of T which omit $\mathcal{G}$.

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