n-FINE RINGS

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Abstract. A ring $R$ is said to be $n$-fine if every nonzero element in $R$ can be written as a sum of a nilpotent and $n$ units in $R$. The class of these rings contains fine rings and $n$-good rings in which each element is a sum of $n$ units. Fundamental properties of such rings are obtained. One of the main results of this paper is that the $m \times m$ matrix ring $M_m(R)$ over any arbitrary ring $R$ is 2-fine. Furthermore, the $m \times m$ matrix ring $M_m(R)$ over a $n$-fine ring $R$ is $n$-fine.

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1. INTRODUCTION

All rings in this paper are assumed to be associative with identity. For a ring $R$, our terminology and notations are mainly in agreement with [4]. For instance, $U(R)$ is the multiplicative group of units of $R$, $\text{Nil}(R)$ is the set of all nilpotent elements of $R$ and $\text{Id}(R)$ is the set of all idempotents of $R$. If $R$ is commutative, then $\text{Nil}(R) = N(R)$ is the nil-radical of $R$. We denote by $M_m(R)$ the ring of $m \times m$ matrices over $R$ with the identity $I_m$.

In the last four decades, an additive theory has emerged in the study of these three interesting sets. A ring $R$ is called $n$-good if every element of $R$ is a sum of $n$ units. Many mathematicians for instance Vamos and Ashrafi studied 2-good rings extensively (see [1, 6, 7]). In 1977, Nicholson defined a ring element $a \in R$ to be clean if it can be written in the form $e + u$ where $e \in \text{Id}(R)$ and $u \in U(R)$ [5]. If every $a \in R$ is clean, $R$ is said to be a clean ring. The interest in the clean property of rings stems from its close connection to exchange rings, since clean is a concise property that implies exchange. Prompted by this, Xiao and Tong [9] called a ring $R$ $n$-clean if every element of $R$ is the sum of an idempotent and $n$ units (see also [8]). The class of these rings contains clean rings and $n$-good rings. Recently, G. Călugăreanu and T. Y. Lam defined a property in [2] related to 2-good in the following way: a nonzero element $a \in R$ is fine if $a = t + u$ where $t \in \text{Nil}(R)$ and $u \in U(R)$. The ring $R$ is called fine if every nonzero element of $R$ is fine. It is shown that fine rings form a proper class of simple ring [2, Theorem 2.3].
Guided by these definitions, we introduce in this work the following definition. Given a positive integer \( n \), we call a ring \( R \) \( n \)-fine if every nonzero element of \( R \) can be written as the sum of a nilpotent and \( n \) units in \( R \). It is clear that fine rings are \( 1 \)-fine. In section 2, some fundamental properties of \( n \)-fine rings are studied. We shall prove every \( n \)-fine ring is \((n + 1)\)-good. Furthermore, we prove that the class of \( n \)-fine rings is closed under factor rings and direct products. The main result of this section states that a ring \( R \) is \( n \)-fine if and only if every factor ring of \( R \) is \( n \)-fine if and only if every indecomposable factor ring of \( R \) is \( n \)-fine. Then in Section 3 we will look at matrix rings and, more generally, endomorphism rings of free modules of infinite rank. In fact, over any ring \( R \), we give an explicit \( 2 \)-fine decomposition for generic matrices of orders 2 and 3. In the main theorem, we prove that over any ring \( R \), the matrix ring \( M_m(R) \) is \( 2 \)-fine for each \( m \geq 2 \). As a consequence, we will revisit Henriksen's result that a proper matrix ring over any ring has unit sum number at most 3. An example shows that there exists a \( 2 \)-fine ring that is not fine. This shows that \( n \)-fine rings are a proper generalization of fine rings. We also show that if \( R \) is \( n \)-fine, then so is the matrix ring \( M_m(R) \) for any integer \( m \geq 1 \). Moreover, we prove that for any ring \( R \), the endomorphism ring of a free \( R \)-module of rank at least 2 is \( 2 \)-fine.

2. Basic Properties of \( n \)-Fine Rings

Definition 1. Let \( n \) be a positive integer. A nonzero element \( x \) of \( R \) is called \( n \)-fine if \( x = t + u_1 + \ldots + u_n \) where \( t \) is a nilpotent element of \( R \) and \( u_1, \ldots, u_n \) are units in \( R \). A ring \( R \) is called \( n \)-fine if every nonzero element of \( R \) is \( n \)-fine.

Proposition 1. Let \( R \) be a ring. Then the following statements hold:

1. If \( R \) is \( n \)-fine, then it is also \( m \)-fine for all \( m \geq n \).
2. If \( \text{Nil}(R) \) is additively closed (in particular, if \( R \) is a commutative ring), then the sum of \( n \)-fine and \( m \)-fine elements of \( R \) is \((n + m)\)-fine.
3. Every \( n \)-good ring is \( n \)-fine; if \( R \) is \( n \)-fine, then \( R \) is \((n + 1)\)-good.

Proof.

1. Let \( r \) be a nonzero element of \( R \) and let \( m > n \). Then, we can write \( r = (r - (m - n).1) + (m - n).1 \) and expressing \( r - (m - n).1 \) as a sum of a nilpotent element and \( n \) units of \( R \) gives a representation of \( r \) as a sum of a nilpotent and \( m \) units.
2. It suffices to notice that if \( \text{Nil}(R) \) is additively closed, then the sum of two nilpotent elements is a nilpotent element.
3. It is clear that every \( n \)-good ring is \( n \)-fine. For the second statement, let \( r \) be a nonzero element of \( R \). By hypothesis, \( r - 1 = t + u_1 + \ldots + u_n \) where \( t \in \text{Nil}(R) \) and \( u_1, \ldots, u_n \in U(R) \). Hence, \( r = (1 + t) + u_1 + \ldots + u_n \). Since, \((1 + t) \in U(R) \). Then, \( r \) is \((n + 1)\)-good.

\(\Box\)
In the next two lemmas we consider the effect of some ring operations on our invariants

**Lemma 1.** Let \( R \) be a ring, \( I \) an ideal of \( R \) and let \( J(R) \) denote the Jacobson radical of \( R \). If \( a \in R \) is \( n \)-fine, then so is \( \overline{a} \in R/I \). The converse also holds if \( I \subseteq J(R) \) and \( J(R) \) is nil.

**Proof.** The first part of the statement is clear since the image of a unit (resp., a nilpotent) is again a unit (resp., a nilpotent).
Assume now that \( I \subseteq J(R) \) and \( J(R) \) is nil. Let \( a \in R \) be such that \( \overline{a} \) is \( n \)-fine in \( R/I \).
Then there are unit elements \( \overline{u}_i \in R/I, 1 \leq i \leq n \) and a nilpotent element \( \overline{t} \in R/I \) such that \( \overline{a} = \overline{t} + \overline{u}_1 + \ldots + \overline{u}_n \).
Then \( u_i \in U(R) \) for all \( 1 \leq i \leq n \), and \( h = a - (t + u_1 + \ldots + u_n) \in J(R) \).
So, \( u_1 + h \in U(R) \). Also, \( \tau^m \in I \) for some integer \( m \geq 2 \). Hence, there is an integer \( m' \geq 2 \) such that \( \tau^{nm'} = 0 \) since \( I \subseteq J(R) \) is nil; that is \( t \) is a nilpotent element of \( R \).
It follows that \( a = t + (u_1 + h) + u_2 + \ldots + u_n \). This shows that \( a \) is \( n \)-fine. \( \square \)

**Lemma 2.** Let \( n \) be a positive integer. The following hold:

1. A homomorphic image of a \( n \)-fine ring is \( n \)-fine.
2. A direct product \( \prod R_\alpha \) of rings \( \{ R_\alpha \} \) is \( n \)-fine if and only if so is each \( \{ R_\alpha \} \).

**Proof.**

1. The proof of (1) is clear.
2. Suppose that each \( \{ R_\alpha \} \) is \( n \)-fine. Let \( x = (x_\alpha) \in \prod R_\alpha \).
For each \( \alpha \), write

\[
x_\alpha = t_\alpha + u_\alpha^1 + \ldots + u_\alpha^n\]

where \( u_\alpha^i \in U(R_\alpha) \) for all \( 1 \leq i \leq n \) and \( t_\alpha \in \text{Nil}(R_\alpha) \).
Then, \( x = t + u^1 + \ldots + u^n \), where \( u_i = (u_\alpha^i) \in U(\prod R_\alpha) \) for all \( 1 \leq i \leq n \) and \( t = (t_\alpha) \in \text{Nil}(\prod R_\alpha) \). Hence, \( \prod R_\alpha \) is \( n \)-fine.

The converse immediately follows from Lemma 1. \( \square \)

We next determine when a polynomial ring or power series ring is a \( n \)-fine ring.

**Proposition 2.** Let \( R \) be a nonzero commutative ring and \( n \) be a positive integer.

1. \( R[X] \) is never a \( n \)-fine ring.
2. \( R[[x]] \) is \( n \)-fine if and only if so is \( R \).

**Proof.**

1. Note that \( \text{Nil}(R[X]) = \{ r_0 + r_1 X + \ldots + r_n X^n \mid r_i \in \sqrt{0} \ (i = 0, \ldots, n) \} \) and \( U(R[X]) = \{ r_0 + r_1 X + \ldots + r_n X^n \mid r_0 \in U(R), r_i \in \sqrt{0} \ (i = 1, \ldots, n) \} \).
If \( X \) is \( n \)-fine, we may let

\[
X = t + (u_1 + r_1 X + \ldots) + (u_2 + r_2 X + \ldots) + \ldots + (u_n + r_n X + \ldots),
\]

where \( t \in \text{Nil}(R), u_1, \ldots, u_n \in U(R) \) and \( r_1, \ldots, r_n \in \sqrt{0} \subseteq J(R) \) Jacobson radical of \( R \), for each \( 1 \leq i \leq n \). Then \( \sum_{i=1}^n r_i = 1 \in J(R) \), which is a contradiction. Thus \( R[X] \) is not a \( n \)-fine ring.
Theorem 1. Let $R$ be a nonzero ring. Then the following are equivalent:

1. $R$ is $n$-fine;
2. Every factor ring of $R$ is $n$-fine;
3. Every indecomposable factor ring of $R$ is $n$-fine.

Proof.

(1) $\Rightarrow$ (2) Follows from Lemma 1.

(2) $\Rightarrow$ (3) It is clear.

(3) $\Rightarrow$ (1) Suppose that (3) holds and $R$ is not $n$-fine. Then there is an element $a \in R$ which is not $n$-fine. Now let $S$ be the set of all proper ideals $I$ of $R$ such $\bar{a}$ is not $n$-fine in $R/I$. Clearly $0 \in S$, and the set $S$ is not empty. If $\{I_{\alpha}, \alpha \in A\}$ is a chain in $S$, let $I = \bigcup_{\alpha \in A} I_{\alpha}$. We prove that $\bar{a}$ is not $n$-fine in $R/I$. Suppose that $\bar{a}$ is $n$-fine in $R/I$. Then there are $\bar{v}_1, \ldots, \bar{v}_r \in U(R/I)$ (with inverses $v_1, \ldots, v_r$ respectively) and $\bar{t} \in \text{Nil}(R/I)$ such that $\bar{a} = \bar{t} + \bar{v}_1 + \ldots + \bar{v}_r$. Note that $t^m \in \bigcup_{\alpha \in A} I_{\alpha}$ for some positive integer $m \geq 2$ and $u_j v_i - 1 \in \bigcup_{\alpha \in A} I_{\alpha}$, hence $t^m \in I_{\bar{\alpha}_0}, u_j v_i \in I_{\bar{\alpha}_0}$ and $v_i u_j \in I_{\bar{\alpha}_0}$ for $\bar{\alpha}_0, \bar{\alpha}_i$ and $\alpha'_i \in A$. Now, we can use Zorn’s lemma to pick an ideal $I_0$ of $R$ maximal with respect to the property that $a$ is not $n$-fine in $R/I$. Then $R/I$ is decomposable as a ring by (3): $R/I_0 = R/I_1 \oplus R/I_2$, where both the ideals $I_1, I_2$ strictly contain $I_0$ and so by the choice of $I_0$, $\bar{a}$ is $n$-fine in $R/I_1$ and $R/I_2$. But then $\bar{a}$ is $n$-fine in $R/I_0$ by Lemma 2(2), a contradiction.

\[\square\]

3. $n$-Fineness of Matrix Rings

The purpose of this section is to investigate $n$-fine property of matrices over any arbitrary ring and of the endomorphism ring of a free $R$-module of rank at least 2. First we give the following interesting decomposition.

Theorem 2. Over any ring, the $2 \times 2$ and $3 \times 3$ matrices are 2-fine.
Proof. Let $R$ be a ring and let $M = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in M_2(R) \setminus \{0\}$. Put $N = \begin{pmatrix} a_{11} - 1 & 1 - a_{11} \\ a_{11} - 1 & 1 - a_{11} \end{pmatrix}$. It is checked easily that then $N^2 = 0$. Thus we have

$$M - N = \begin{pmatrix} 1 & a_{12} + a_{11} - 1 \\ a_{21} - a_{11} + 1 & a_{22} - a_{11} - 1 \end{pmatrix}.$$ 

Now there exist invertible matrices $P$ and $Q$ such that

$$P(M - N)Q = \begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ -1 & c \end{pmatrix}$$

for an appropriate $c$ and thus is a sum of two units. Hence $M$ is 2-fine. Now, let $M = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$ be $3 \times 3$ matrix over $R \setminus \{0\}$. We first construct an nilpotent in order to show 2-fineness of $M$. Set $N = \begin{pmatrix} b_{11} - 1 & b_{22} - 1 & 2 - b_{11} - b_{22} \\ b_{11} - 1 & b_{22} - 1 & 2 - b_{11} - b_{22} \\ b_{11} - 1 & b_{22} - 1 & 2 - b_{11} - b_{22} \end{pmatrix}$. It may be directly verified that $N^2 = 0$. Thus

$$M - N = \begin{pmatrix} 1 & b_{12} - b_{22} + 1 & b_{13} + b_{11} + b_{22} - 2 \\ b_{21} - b_{11} + 1 & 1 & b_{23} + b_{11} + b_{22} - 2 \\ b_{31} - b_{11} + 1 & b_{32} - b_{22} + 1 & b_{33} + b_{11} + b_{22} - 2 \end{pmatrix}.$$ 

We only need to show that $M - N$ is 2-good. Now there exist invertible matrices $P$ and $Q$ such that

$$P(M - N)Q = \begin{pmatrix} c_1 & 0 & c_2 \\ c_3 & 1 & 0 \\ 0 & c_4 & c_5 \end{pmatrix} \begin{pmatrix} 0 & 1 & c_2 \\ 0 & 0 & 1 \\ 1 & c_4 & c_5 \end{pmatrix} + \begin{pmatrix} c_1 & -1 & 0 \\ c_3 & 1 & -1 \\ -1 & 0 & 0 \end{pmatrix}$$

for an appropriate $c_i$ where $i \in \{1, \ldots, 5\}$ and thus is a sum of two units. Hence $M$ is 2-fine. This completes the proof. \hfill \qed 

Remark 1.

(1) For any ring $R$, $R$ can be embedded in the $2 \times 2$ matrix ring $M_2(R)$. That is, all rings can be embedded in a 2-fine ring by Theorem 2.

(2) It is known that fine rings are 2-fine rings. However, the converse is not true. For example, taking $R = M_2(\mathbb{Z})$, then $R$ is a 2-fine ring by Theorem 2. Let $M = \begin{pmatrix} 3 & 5 \\ 0 & 0 \end{pmatrix}$, then $M$ is not fine in $R$ by [2, Corollary 5.4], that is, $R$ is not a fine ring.
To facilitate the proof of Theorem 3, we isolate the following lemma, which is of some independent interest.

**Lemma 3.** Let $R$ be a ring, $n, m \geq 1$ and $k \geq 2$. If the matrix rings $M_n(R)$ and $M_m(R)$ are both $k$-fine, then so is the matrix ring $M_{n+m}(R)$.

**Proof.** Let $M \in M_{n+m}(R)$ which we will write in the block decomposition form

$$M = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where $A_{11} \in M_n(R)$, $A_{22} \in M_m(R)$ and $A_{12}, A_{21}$ are appropriately sized rectangular matrices. By hypothesis, there exist invertible $n \times n$, $m \times m$ matrices $U_1, U_2, \ldots, U_k$ and $V_1, V_2, \ldots, V_k$, and nilpotent matrices $N_1, N_2$ such that $A_{11} = N_1 + U_1 + U_2 + \ldots + U_k$ and $A_{22} = N_2 + V_1 + V_2 + \ldots + V_k$. Thus the decomposition

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} U_1 & A_{12} \\ 0 & V_1 \end{pmatrix} + \begin{pmatrix} U_2 & 0 \\ A_{21} & V_2 \end{pmatrix} + \ldots + \begin{pmatrix} U_k & 0 \\ 0 & V_k \end{pmatrix} + \begin{pmatrix} N_1 & 0 \\ 0 & N_2 \end{pmatrix}$$

shows that $M$ is $k$-fine. □

We now present the titular result.

**Theorem 3.** Over any ring $R$, the matrix ring $M_m(R)$ is 2-fine for any positive integer $m \geq 2$.

**Proof.** The result follows directly by combining Theorem 2 and Lemma 3. □

**Remark 2.** Commenting on Theorem 3, [2, Proposition 2.12] ensures that over any ring $S$, the ring $R = M_n(S)$ where $n \geq 2$ is 3-fine. Indeed, if $M$ is a matrix of $R$, then $M - I_n = T_1 + T_2$ where $T_1$ and $T_2$ are two fine matrices. Then, there are $N_1, N_2 \in \text{Nil}(R)$ and $U_1, U_2 \in U(R)$ such that $M - I_n = (N_1 + U_1) + (N_2 + U_2)$. Hence, $M = N_1 + (N_2 + I_n) + U_1 + U_2$ where $N_2 + I_n \in U(R)$, this implies that $M$ is 3-fine.

Again, combining Proposition 1 and Theorem 3 we recover a result due to Henriksen [3, Theorem 3].

**Corollary 1.** If $R$ is any nonzero ring with identity, and $m \geq 2$, then every matrix in $M_m(R)$ is the sum of three invertible matrices in $M_m(R)$.

**Theorem 4.** Let $n \geq 1$. If $R$ is a $n$-fine ring, then so is the matrix ring $M_m(R)$ for any positive integer $m$.

**Proof.** For $n = 1$, the result follows from [2, Theorem 3.1]. For $n \geq 2$, it is clear by induction and by Lemma 3. □

We will conclude this section by considering the 2-fineness of the endomorphism ring of a free $R$-module of rank at least 2.
**Proposition 3.** Let \( R \) be a ring and let the free \( R \)-module \( F \) be (isomorphic to) the direct sum of \( \alpha \geq 2 \) copies of \( R \) where \( \alpha \) is a cardinal number. Then the ring of endomorphisms \( E \) of \( F \) is 2-fine.

**Proof.** Assume first that \( \alpha \geq 2 \) is finite, so \( E \cong M_\alpha(R) \). Then \( E \) is 2-fine for \( \alpha = 2, 3 \) by Theorem 2 and the values \( \alpha < \omega \) for which \( E \) is 2-fine are closed under addition by Theorem 3. So \( E \) is 2-fine for all finite \( \alpha \).
Assume now that \( \alpha \) is infinite. Then from \( F \cong F \oplus F \), \( E \cong M_2(E) \), and so \( E \) is 2-fine by Theorem 2. \( \square \)

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**References**


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