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σ-CENTRALIZERS OF GENERALIZED MATRIX ALGEBRAS

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Abstract. In the present paper, we characterize Lie (Jordan) σ-centralizers of generalized matrix algebras. More precisely, we obtain some conditions under which every Lie σ-centralizer of a generalized matrix algebra can be expressed as the sum of a σ-centralizer and a center-valued mapping. Further, it is shown that under certain appropriate assumptions every Jordan σ-centralizer of a generalized matrix algebra is a σ-centralizer. Finally, the main results are applied to triangular algebras.

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1. INTRODUCTION

Let $\mathcal{A}$ be an algebra over a commutative unital ring $\mathcal{R}$ with center $\mathcal{Z}(\mathcal{A})$. For any $a,b \in \mathcal{A}$, $[a,b] = ab - ba$ (resp. $a \circ b = ab + ba$) denotes the Lie product (resp. Jordan product). Let σ be an automorphism of $\mathcal{A}$. An $\mathcal{R}$-linear mapping $L : \mathcal{A} \to \mathcal{A}$ is called a left σ-centralizer (resp. right σ-centralizer) if $L(ab) = L(a)\sigma(b)$ (resp. $L(ab) = \sigma(a)L(b)$) for all $a,b \in \mathcal{A}$. It is called a σ-*centralizer* if it is both a left σ-centralizer as well as a right σ-centralizer. An $\mathcal{R}$-linear mapping $L : \mathcal{A} \to \mathcal{A}$ is called a *Lie* σ-*centralizer* if $L([a,b]) = [L(a),\sigma(b)]$ (or $L([a,b]) = [\sigma(a),L(b)]$) for all $a,b \in \mathcal{A}$. An $\mathcal{R}$-linear mapping $L : \mathcal{A} \to \mathcal{A}$ is called a *Jordan* σ-*centralizer* if $L(a \circ b) = L(a) \circ \sigma(b)$ (or $L(a \circ b) = \sigma(a) \circ L(b)$) for all $a,b \in \mathcal{A}$. One can easily see that the conditions $L([a,b]) = [L(a),\sigma(b)]$ (resp. $L(a \circ b) = L(a) \circ \sigma(b)$) and $L([a,b]) = [\sigma(a),L(b)]$ (resp. $L(a\circ b) = \sigma(a)\circ L(b)$) are equivalent. Obviously, every σ-centralizer is a Lie σ-centralizer as well as a Jordan σ-centralizer but the converse statements are not true in general. If $d : \mathcal{A} \to \mathcal{A}$ is a σ-centralizer and $\ell : \mathcal{A} \to \mathcal{Z}(\mathcal{A})$ is a linear mapping, then $d + \ell$ is a Lie σ-centralizer on $\mathcal{A}$ if and only if $\ell([a,b]) = 0$ for all $a,b \in \mathcal{A}$. A Lie σ-centralizer is called *proper* if it can be written as the sum of a σ-centralizer and a center-valued mapping.

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Let $\mathcal{R}$ be a commutative ring with identity. A Morita context consists of two $\mathcal{R}$-algebras $\mathcal{A}$ and $\mathcal{B}$, two bimodules ${}_{\mathcal{A}}\mathcal{M}_{\mathcal{B}}$ and ${}_{\mathcal{B}}\mathcal{N}_{\mathcal{A}}$, and two bimodule homomorphisms called the pairings $\zeta_{\mathcal{M}\mathcal{N}} : \mathcal{M} \bigotimes_{\mathcal{B}} \mathcal{N} \to \mathcal{A}$ and $\psi_{\mathcal{N}\mathcal{M}} : \mathcal{N} \bigotimes_{\mathcal{A}} \mathcal{M} \to \mathcal{B}$ satisfying the following commutative diagrams:

$$\begin{tikzcd} \mathcal{M} \underset{\mathcal{B}}{\otimes} \mathcal{N} \underset{\mathcal{A}}{\otimes} \mathcal{M} \arrow[r, "\zeta_{\mathcal{M}\mathcal{N}} \otimes I_{\mathcal{M}}"] \arrow[d, "I_{\mathcal{M}} \otimes \psi_{\mathcal{N}\mathcal{M}}"] & \mathcal{A} \underset{\mathcal{A}}{\otimes} \mathcal{M} \arrow[d, "\cong"] \\ \mathcal{M} \underset{\mathcal{B}}{\otimes} \mathcal{B} \arrow[r, "\cong"] & \mathcal{M} \end{tikzcd} \quad and \quad \begin{tikzcd} \mathcal{N} \underset{\mathcal{A}}{\otimes} \mathcal{M} \underset{\mathcal{B}}{\otimes} \mathcal{N} \arrow[r, "\psi_{\mathcal{N}\mathcal{M}} \otimes I_{\mathcal{N}}"] \arrow[d, "I_{\mathcal{N}} \otimes \zeta_{\mathcal{M}\mathcal{N}}"] & \mathcal{B} \underset{\mathcal{B}}{\otimes} \mathcal{N} \arrow[d, "\cong"] \\ \mathcal{N} \underset{\mathcal{A}}{\otimes} \mathcal{A} \arrow[r, "\cong"] & \mathcal{N}. \end{tikzcd}$$

We denote this Morita context by $(\mathcal{A}, \mathcal{B}, \mathcal{M}, \mathcal{N}, \zeta_{\mathcal{M}\mathcal{N}}, \psi_{\mathcal{N}\mathcal{M}})$. If $(\mathcal{A}, \mathcal{B}, \mathcal{M}, \mathcal{N}, \zeta_{\mathcal{M}\mathcal{N}}, \psi_{\mathcal{N}\mathcal{M}})$ is a Morita context, then the set

$$\begin{pmatrix} \mathcal{A} & \mathcal{M} \\ \mathcal{N} & \mathcal{B} \end{pmatrix} = \left\{ \begin{pmatrix} a & m \\ n & b \end{pmatrix} \middle| \, a \in \mathcal{A}, m \in \mathcal{M}, n \in \mathcal{N}, b \in \mathcal{B} \right\}$$

forms an $\mathcal{R}$-algebra under the usual matrix operations, where at least one of the two bimodules $\mathcal{M}$ and $\mathcal{N}$ is distinct from zero. Such an $\mathcal{R}$-algebra is called a *generalized matrix algebra* and it is denoted by $\mathcal{G} = G(\mathcal{A}, \mathcal{M}, \mathcal{N}, \mathcal{B})$.

Throughout the paper, we assume that $\mathcal{A}$ and $\mathcal{B}$ are unital algebras with identity elements $1_{\mathcal{A}}$ and $1_{\mathcal{B}}$, respectively and $\mathcal{M}$ is a faithful $(\mathcal{A}, \mathcal{B})$-bimodule, but no any restrictions on $\mathcal{N}$. In view of [19, Lemma 3.1], the center of $\mathcal{G}$ is given by

$$\mathcal{Z}(\mathcal{G}) = \left\{ a \oplus b \, \middle| \, a \in \mathcal{A}, \;\; b \in \mathcal{B}, \;\; am = mb, \;\; na = bn \;\; \text{for all} \;\; m \in \mathcal{M}, \;\; n \in \mathcal{N} \right\},$$

where $a \oplus b = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$. Define two natural projections $\pi_{\mathcal{A}} : \mathcal{G} \to \mathcal{A}$ and $\pi_{\mathcal{B}} : \mathcal{G} \to \mathcal{B}$ by $\pi_{\mathcal{A}} \begin{pmatrix} a & m \\ n & b \end{pmatrix} = a$ and $\pi_{\mathcal{B}} \begin{pmatrix} a & m \\ n & b \end{pmatrix} = b$. By [19, Lemmas 3.1 and 3.2], it is easy to see that $\pi_{\mathcal{A}}(\mathcal{Z}(\mathcal{G})) \subseteq \mathcal{Z}(\mathcal{A})$, $\pi_{\mathcal{B}}(\mathcal{Z}(\mathcal{G})) \subseteq \mathcal{Z}(\mathcal{B})$ and there exists a unique algebra isomorphism $\xi : \pi_{\mathcal{A}}(\mathcal{Z}(\mathcal{G})) \to \pi_{\mathcal{B}}(\mathcal{Z}(\mathcal{G}))$ such that $am = m\xi(a)$ and $na = \xi(a)n$ for all $m \in \mathcal{M}$, $n \in \mathcal{N}$.

The aim of this paper is to characterize Lie (Jordan) σ-centralizers of generalized matrix algebras. Over the past few decades, a lot of work concerning characterizations of Lie (Jordan) mappings on various rings and algebras have been done (see [2, 3, 5, 9, 10, 15, 16, 21] and references therein). A related problem in this area is to characterize centralizers on various rings and algebras. Centralizers on different rings and algebras have been broadly examined by many algebraists (see [6–8] and references therein). Zalar [20] introduced the notion of Jordan centralizers and proved that every Jordan centralizer on a 2-torsion free semiprime ring is a centralizer. Vukman and Kosi-Ulbl [17, 18] extensively studied centralizers mainly on semiprime rings. In the year 2019, Fošner and Jing [6] introduced the notion of Lie centralizers and investigated the additivity of Lie centralizers of triangular rings. Recently, in [14], Liu studied the structure of nonlinear Lie centralizers of generalized matrix algebras. Inspired by these results, in this paper we investigate Lie (Jordan) σ-centralizers of generalized matrix algebras. In fact, we prove that under certain restrictions every Lie σ-centralizer of a generalized matrix algebra is proper (Theorem 1), and every Jordan σ-centralizer of a generalized matrix algebra is a σ-centralizer (Theorem 2).

The study of group of automorphisms is an important key for understanding the underlying algebraic structure. Hence, the study of group of automorphisms of various kinds of algebraic structures have been extensively investigated in the literature (see [1, 4, 11–13] and references therein). However, it is not always possible to determine all the automorphisms of the given algebraic structure in the general case. For example, Boboc et al. in [4, Remark 4.8] explained that Morita context rings are the rings with non-trivial idempotents, and hence there is virtually no hope of finding all automorphisms of Morita context rings in the general case. Therefore, we shall use the following class of automorphisms of a generalized matrix algebra $\mathcal{G}$ which, under certain restrictions on $\mathcal{G}$, coincides with the group of all automorphisms of $\mathcal{G}$.

Lemma 1. *[4, Proposition 2.1] Let $(\gamma,\delta,u,v,m_0,n_0)$ be a 6-tuple such that $\gamma:\mathcal{A}\to\mathcal{A}$ and $\delta:\mathcal{B}\to\mathcal{B}$ are automorphisms, $u:\mathcal{M}\to\mathcal{M}$ is a (γ,δ)-bimodule isomorphism, $v:\mathcal{N}\to\mathcal{N}$ is a (δ,γ)-bimodule isomorphism, $m_0\in\mathcal{M}$ and $n_0\in\mathcal{N}$ are fixed elements, such that the following conditions are satisfied:*

(i) $m_0\mathcal{N}=0$ *and* $\mathcal{N}m_0=0$,

(ii) $\mathcal{M}n_0=0$ *and* $n_0\mathcal{M}=0$,

(iii) $u(m)v(n)=\gamma(mn)$ *and* $v(n)u(m)=\delta(nm)$ *for all* $m\in\mathcal{M}$ *and* $n\in\mathcal{N}$.

Then the mapping $\sigma:\mathcal{G}\to\mathcal{G}$ defined by

$$\sigma\begin{pmatrix} a & m \\ n & b \end{pmatrix}=\begin{pmatrix} \gamma(a) & \gamma(a)m_0-m_0\delta(b)+u(m) \\ n_0\gamma(a)-\delta(b)n_0+v(n) & \delta(b) \end{pmatrix}$$

is an automorphism.

Let us denote the set of all automorphisms of $\mathcal{G}$ defined in Lemma 1 by $Aut_0^0(\mathcal{G})$ and the group of all automorphisms of $\mathcal{G}$ by $Aut(\mathcal{G})$. It is easy to see that $Aut_0^0(\mathcal{G})$ is a subgroup $Aut(\mathcal{G})$ containing the identity automorphism. In [4, Theorem 4.7], Boboc et al. proved that if $\mathcal{G}=G(\mathcal{A},\mathcal{M},\mathcal{N},\mathcal{B})$ is a generalized matrix algebra such that $\mathcal{A}$ and $\mathcal{B}$ have only trivial idempotents and both the bilinear mappings $\zeta_{\mathcal{M}\mathcal{N}},\psi_{\mathcal{N}\mathcal{M}}$ are zero, then $Aut_0^0(\mathcal{G})\cup Aut_0^1(\mathcal{G})=Aut(\mathcal{G})$, where $Aut_0^1(\mathcal{G})$ is the set of all automorphisms of $\mathcal{G}$ as given in [4, Proposition 2.2]. Note that if $\mathcal{A}$ and $\mathcal{B}$ are not isomorphic, then $Aut_0^1(\mathcal{G}) = \varnothing$. Therefore, if $\mathcal{G} = G(\mathcal{A}, \mathcal{M}, \mathcal{N}, \mathcal{B})$ is a generalized matrix algebra such that $\mathcal{A}$ and $\mathcal{B}$ have only trivial idempotents, $\mathcal{A}$ and $\mathcal{B}$ are not isomorphic and both the bilinear mappings $\zeta_{\mathcal{M}\mathcal{N}}, \psi_{\mathcal{N}\mathcal{M}}$ are zero, then $Aut_0^0(\mathcal{G})$ coincides with $Aut(\mathcal{G})$.

2. σ-CENTRALIZERS OF GENERALIZED MATRIX ALGEBRAS

In this section, we give the structure of a σ-centralizer $\Delta : \mathcal{G} \to \mathcal{G}$ with associated automorphism σ of $\mathcal{G}$ as given in Lemma 1.

Proposition 1. *Let $\Delta : \mathcal{G} \to \mathcal{G}$ be a σ-centralizer with associated automorphism σ of $\mathcal{G}$ as given in Lemma 1. Then Δ is of the form*

$$\Delta\begin{pmatrix} a & m \\ n & b \end{pmatrix} = \begin{pmatrix} A_{11}(a) & A_{11}(a)m_0 - m_0B_{22}(b) + C_{12}(m) \\ n_0A_{11}(a) - B_{22}(b)n_0 + D_{21}(n) & B_{22}(b) \end{pmatrix},$$

where $A_{11} : \mathcal{A} \to \mathcal{A}$, $C_{12} : \mathcal{M} \to \mathcal{M}$, $D_{21} : \mathcal{N} \to \mathcal{N}$ and $B_{22} : \mathcal{B} \to \mathcal{B}$ are $\mathcal{R}$-linear mappings satisfying the following conditions:

(i) A_{11} *is a γ-centralizer of $\mathcal{A}$, B_{22} is a δ-centralizer of $\mathcal{B}$;*
(ii) $A_{11}(mn) = C_{12}(m)v(n) = u(m)D_{21}(n)$, $B_{22}(nm) = D_{21}(n)u(m) = v(n)C_{12}(m)$;
(iii) $C_{12}(am) = A_{11}(a)u(m) = \gamma(a)C_{12}(m)$, $C_{12}(mb) = C_{12}(m)\delta(b) = u(m)B_{22}(b)$;
(iv) $D_{21}(na) = v(n)A_{11}(a) = D_{21}(n)\gamma(a)$, $D_{21}(bn) = \delta(b)D_{21}(n) = B_{22}(b)v(n)$.

Proof. Suppose that the σ-centralizer Δ is of the form

$$\Delta\begin{pmatrix} a & m \\ n & b \end{pmatrix} = \begin{pmatrix} A_{11}(a)+B_{11}(b)+C_{11}(m)+D_{11}(n) & A_{12}(a)+B_{12}(b)+C_{12}(m)+D_{12}(n) \\ A_{21}(a)+B_{21}(b)+C_{21}(m)+D_{21}(n) & A_{22}(a)+B_{22}(b)+C_{22}(m)+D_{22}(n) \end{pmatrix},$$

where $A_{11}, B_{11}, C_{11}, D_{11}$ are $\mathcal{R}$-linear mappings from $\mathcal{A}, \mathcal{B}, \mathcal{M}, \mathcal{N}$ to $\mathcal{A}$, respectively; $A_{12}, B_{12}, C_{12}, D_{12}$ are $\mathcal{R}$-linear mappings from $\mathcal{A}, \mathcal{B}, \mathcal{M}, \mathcal{N}$ to $\mathcal{M}$, respectively; A_{21}, B_{21},
C_{21}, D_{21} are $\mathcal{R}$-linear mappings from $\mathcal{A}, \mathcal{B}, \mathcal{M}, \mathcal{N}$ to $\mathcal{N}$, respectively; and A_{22}, B_{22}, C_{22}, D_{22} are $\mathcal{R}$-linear mappings from $\mathcal{A}, \mathcal{B}, \mathcal{M}, \mathcal{N}$ to $\mathcal{B}$, respectively.

Since Δ is a σ-centralizer, we have

$$\Delta(xy) = \Delta(x)\sigma(y) = \sigma(x)\Delta(y) \text{ for all } x, y \in \mathcal{G}. \tag{2.1}$$

Let us choose $x = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ and $y = \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix}$ in (2.1). Then, we find that

$$\begin{pmatrix} C_{11}(am) & C_{12}(am) \\ C_{21}(am) & C_{22}(am) \end{pmatrix} = \begin{pmatrix} 0 & A_{11}(a)u(m) \\ 0 & A_{21}(a)u(m) \end{pmatrix}$$
$$= \begin{pmatrix} \gamma(a)C_{11}(m) & \gamma(a)C_{12}(m) + \gamma(a)m_0C_{22}(m) \\ n_0\gamma(a)C_{11}(m) & n_0\gamma(a)C_{12}(m) \end{pmatrix}.$$

Thus,

$$C_{11}(am) = 0 = \gamma(a)C_{11}(m),$$
$$C_{12}(am) = A_{11}(a)u(m) = \gamma(a)C_{12}(m) + \gamma(a)m_0C_{22}(m),$$
$$C_{21}(am) = 0 = n_0\gamma(a)C_{11}(m) \text{ and}$$

$$C_{22}(am) = A_{21}(a)u(m) = n_0\gamma(a)C_{12}(m).$$

Putting $a = 1_{\mathcal{A}}$, we get

$$C_{11}(m) = 0, m_0C_{22}(m) = 0, C_{21}(m) = 0 \text{ and } C_{22}(m) = n_0C_{12}(m) = 0.$$

Hence, $C_{12}(am) = A_{11}(a)u(m) = \gamma(a)C_{12}(m)$. Similarly, choose $x = \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix}$ and $y = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}$ in (2.1) to obtain $C_{12}(mb) = C_{12}(m)\delta(b) = u(m)B_{22}(b)$.

If we take $x = \begin{pmatrix} 0 & 0 \\ n & 0 \end{pmatrix}$ and $y = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ in (2.1), then we have

$$\begin{aligned}\begin{pmatrix} D_{11}(na) & D_{12}(na) \\ D_{21}(na) & D_{22}(na) \end{pmatrix} &= \begin{pmatrix} D_{11}(n)\gamma(a) & D_{11}(n)\gamma(a)m_0 \\ D_{21}(n)\gamma(a) + D_{22}(n)n_0\gamma(a) & D_{21}(n)\gamma(a)m_0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ v(n)A_{11}(a) & v(n)A_{12}(a) \end{pmatrix}.\end{aligned}$$

This implies that $D_{11}(na) = D_{11}(n)\gamma(a) = 0$, $D_{12}(na) = D_{11}(n)\gamma(a)m_0 = 0$, $D_{21}(na) = D_{21}(n)\gamma(a) + D_{22}(n)n_0\gamma(a) = v(n)A_{11}(a)$ and $D_{22}(na) = D_{21}(n)\gamma(a)m_0 = v(n)A_{12}(a)$. Putting $a = 1_{\mathcal{A}}$, we get $D_{11}(n) = 0$, $D_{12}(n) = 0$, $D_{22}(n)n_0 = 0$ and $D_{22}(n) = D_{21}(n)m_0 = 0$. Hence, $D_{21}(na) = D_{21}(n)\gamma(a) = v(n)A_{11}(a)$. Similarly, by taking $x = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}$ and $y = \begin{pmatrix} 0 & 0 \\ n & 0 \end{pmatrix}$ in (2.1), we can obtain $D_{21}(bn) = B_{22}(b)v(n) = \delta(b)D_{21}(n)$.

Consider $x = \begin{pmatrix} a_1 & 0 \\ 0 & 0 \end{pmatrix}$, $y = \begin{pmatrix} a_2 & 0 \\ 0 & 0 \end{pmatrix}$ in (2.1). Then

$$\begin{aligned}\begin{pmatrix} A_{11}(a_1a_2) & A_{12}(a_1a_2) \\ A_{21}(a_1a_2) & A_{22}(a_1a_2) \end{pmatrix} &= \begin{pmatrix} A_{11}(a_1)\gamma(a_2) & A_{11}(a_1)\gamma(a_2)m_0 \\ A_{21}(a_1)\gamma(a_2) + A_{22}(a_1)n_0\gamma(a_2) & A_{21}(a_1)\gamma(a_2)m_0 \end{pmatrix} \\ &= \begin{pmatrix} \gamma(a_1)A_{11}(a_2) & \gamma(a_1)A_{12}(a_2) + \gamma(a_1)m_0A_{22}(a_2) \\ n_0\gamma(a_1)A_{11}(a_2) & n_0\gamma(a_1)A_{12}(a_2) \end{pmatrix}.\end{aligned}$$

From the above expression, we see that

$$\begin{aligned} &A_{11}(a_1a_2) = A_{11}(a_1)\gamma(a_2) = \gamma(a_1)A_{11}(a_2); \\ &A_{12}(a_1a_2) = A_{11}(a_1)\gamma(a_2)m_0 = \gamma(a_1)A_{12}(a_2) + \gamma(a_1)m_0A_{22}(a_2); \\ &A_{21}(a_1a_2) = A_{21}(a_1)\gamma(a_2) + A_{22}(a_1)n_0\gamma(a_2) = n_0\gamma(a_1)A_{11}(a_2); \\ &A_{22}(a_1a_2) = A_{21}(a_1)\gamma(a_2)m_0 = n_0\gamma(a_1)A_{12}(a_2). \end{aligned}$$

Putting $a_2 = 1_{\mathcal{A}}$, we get $A_{12}(a_1) = A_{11}(a_1)m_0$ and $A_{22}(a_1) = A_{21}(a_1)m_0 = 0$. Taking $a_1 = 1_{\mathcal{A}}$, we have $A_{21}(a_2) = n_0A_{11}(a_2)$, $A_{22}(a_2) = n_0A_{12}(a_2) = 0$. Since $a_1, a_2 \in \mathcal{A}$ are arbitrary, we conclude that A_{11} is a γ-centralizer of $\mathcal{A}$, $A_{12}(a) = A_{11}(a)m_0$, $A_{21}(a) = n_0A_{11}(a)$ and $A_{22}(a) = 0$ for all $a \in \mathcal{A}$. Repeating the same computational process and choosing $x = \begin{pmatrix} 0 & 0 \\ 0 & b_1 \end{pmatrix}$ and $y = \begin{pmatrix} 0 & 0 \\ 0 & b_2 \end{pmatrix}$ in (2.1), we obtain $B_{22}(b_1b_2) = B_{22}(b_1)\delta(b_2) = \delta(b_1)B_{22}(b_2)$, for all $b_1, b_2 \in \mathcal{B}$, i.e., B_{22} is a δ-centralizer of $\mathcal{B}$ and $B_{11}(b) = 0$, $B_{12}(b) = -m_0B_{22}(b)$, $B_{21}(b) = -B_{22}(b)n_0$ for all $b \in \mathcal{B}$.

Putting $x = \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix}$ and $y = \begin{pmatrix} 0 & 0 \\ n & 0 \end{pmatrix}$ in (2.1), we arrive at

$$\begin{pmatrix} A_{11}(mn) & A_{12}(mn) \\ A_{21}(mn) & A_{22}(mn) \end{pmatrix} = \begin{pmatrix} C_{12}(m)v(n) & 0 \\ C_{22}(m)v(n) & 0 \end{pmatrix} = \begin{pmatrix} u(m)D_{21}(n) & u(m)D_{22}(n) \\ 0 & 0 \end{pmatrix}.$$

This yields $A_{11}(mn) = C_{12}(m)v(n) = u(m)D_{21}(n)$. Similarly, choosing $x = \begin{pmatrix} 0 & 0 \\ n & 0 \end{pmatrix}$ and $y = \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix}$ in (2.1), we get $B_{22}(nm) = D_{21}(n)u(m) = v(n)C_{12}(m)$. □

If $\mathcal{M}$ is a faithful $(\mathcal{A}, \mathcal{B})$-bimodule, then condition (i) in Proposition 1 become redundant and we obtain the following result:

Corollary 1. *Let $\mathcal{G} = G(\mathcal{A}, \mathcal{M}, \mathcal{N}, \mathcal{B})$ be a generalized matrix algebra with faithful $\mathcal{M}$ and $\Delta : \mathcal{G} \to \mathcal{G}$ be a σ-centralizer with associated automorphism σ of $\mathcal{G}$ as given in Lemma 1. Then Δ is of the form*

$$\Delta \begin{pmatrix} a & m \\ n & b \end{pmatrix} = \begin{pmatrix} A_{11}(a) & A_{11}(a)m_0 - m_0B_{22}(b) + C_{12}(m) \\ n_0A_{11}(a) - B_{22}(b)n_0 + D_{21}(n) & B_{22}(b) \end{pmatrix},$$

where $A_{11} : \mathcal{A} \to \mathcal{A}$, $C_{12} : \mathcal{M} \to \mathcal{M}$, $D_{21} : \mathcal{N} \to \mathcal{N}$ and $B_{22} : \mathcal{B} \to \mathcal{B}$ are $\mathcal{R}$-linear mappings satisfying the following conditions:

(i) $A_{11}(mn) = C_{12}(m)v(n) = u(m)D_{21}(n)$, $B_{22}(nm) = D_{21}(n)u(m) = v(n)C_{12}(m)$;
(ii) $C_{12}(am) = A_{11}(a)u(m) = \gamma(a)C_{12}(m)$, $C_{12}(mb) = C_{12}(m)\delta(b) = u(m)B_{22}(b)$;
(iii) $D_{21}(na) = v(n)A_{11}(a) = D_{21}(n)\gamma(a)$, $D_{21}(bn) = \delta(b)D_{21}(n) = B_{22}(b)v(n)$.

Proof. In view of Proposition 1, it is sufficient to show that if $\mathcal{M}$ is a faithful $(\mathcal{A}, \mathcal{B})$-bimodule, then A_{11} is a γ-centralizer of $\mathcal{A}$ and B_{22} is a δ-centralizer of $\mathcal{B}$. For all $a_1, a_2 \in \mathcal{A}$ and $m \in \mathcal{M}$, we see that

$$A_{11}(a_1a_2)u(m) = C_{12}(a_1a_2m) = A_{11}(a_1)u(a_2m) = A_{11}(a_1)\gamma(a_2)u(m).$$

Thus, $\{A_{11}(a_1a_2) - A_{11}(a_1)\gamma(a_2)\}\mathcal{M} = \{0\}$. Since $\mathcal{M}$ is a faithful left $\mathcal{A}$-module, we have $A_{11}(a_1a_2) = A_{11}(a_1)\gamma(a_2)$. Further,

$$A_{11}(a_1a_2)u(m) = C_{12}(a_1a_2m) = \gamma(a_1)C_{12}(a_2m) = \gamma(a_1)A_{11}(a_2)u(m)$$

for all $a_1, a_2 \in \mathcal{A}$, $m \in \mathcal{M}$. Hence $\{A_{11}(a_1a_2) - \gamma(a_1)A_{11}(a_2)\}\mathcal{M} = \{0\}$ which implies that $A_{11}(a_1a_2) = \gamma(a_1)A_{11}(a_2)$. Therefore, A_{11} is a γ-centralizer of $\mathcal{A}$. In a similar manner, one can prove that B_{22} is a δ-centralizer of $\mathcal{B}$. □

3. Lie σ-centralizers of generalized matrix algebras

In this section, we prove that under certain conditions every Lie σ-centralizer of a generalized matrix algebra is proper. To prove this, we first characterize a Lie σ-centralizer $\mathcal{L}:\mathcal{G}\to\mathcal{G}$ with associated automorphism σ as given in Lemma 1.

Proposition 2. *Let $\mathcal{L}:\mathcal{G}\to\mathcal{G}$ be a Lie σ-centralizer with associated automorphism σ of $\mathcal{G}$ as given in Lemma 1. Then $\mathcal{L}$ is of the form*

$$\mathcal{L}\begin{pmatrix} a & m \\ n & b \end{pmatrix}=\begin{pmatrix} R_{11}(a)+S_{11}(b) & (R_{11}(a)+S_{11}(b))m_0-m_0(R_{22}(a)+S_{22}(b))+T_{12}(m) \\ n_0(R_{11}(a)+S_{11}(b))-(R_{22}(a)+S_{22}(b))n_0+U_{21}(n) & R_{22}(a)+S_{22}(b) \end{pmatrix},$$

where $R_{11}:\mathcal{A}\to\mathcal{A}$, $S_{11}:\mathcal{B}\to\mathcal{Z}(\mathcal{A})$, $T_{12}:\mathcal{M}\to\mathcal{M}$, $U_{21}:\mathcal{N}\to\mathcal{N}$, $R_{22}:\mathcal{A}\to\mathcal{Z}(\mathcal{B})$ and $S_{22}:\mathcal{B}\to\mathcal{B}$ are $\mathcal{R}$-linear mappings satisfying the following conditions:

(i) *R_{11} is a Lie γ-centralizer of $\mathcal{A}$, $R_{11}(mn)-S_{11}(nm)=T_{12}(m)v(n)=u(m)U_{21}(n)$;*
(ii) *S_{22} is a Lie δ-centralizer of $\mathcal{B}$, $S_{22}(nm)-R_{22}(mn)=U_{21}(n)u(m)=v(n)T_{12}(m)$;*
(iii) *$T_{12}(am)=R_{11}(a)u(m)-u(m)R_{22}(a)=\gamma(a)T_{12}(m)$, $T_{12}(mb)=T_{12}(m)\delta(b)=u(m)S_{22}(b)-S_{11}(b)u(m)$;*
(iv) *$U_{21}(na)=v(n)R_{11}(a)-R_{22}(a)v(n)=U_{21}(n)\gamma(a)$, $U_{21}(bn)=\delta(b)U_{21}(n)=S_{22}(b)v(n)-v(n)S_{11}(b)$;*
(v) *$R_{22}([a_1,a_2])=0$ and $S_{11}([b_1,b_2])=0$.*

Proof. Suppose that the Lie σ-centralizer $\mathcal{L}$ is of the form

$$\mathcal{L}\begin{pmatrix} a & m \\ n & b \end{pmatrix}=\begin{pmatrix} R_{11}(a)+S_{11}(b)+T_{11}(m)+U_{11}(n) & R_{12}(a)+S_{12}(b)+T_{12}(m)+U_{12}(n) \\ R_{21}(a)+S_{21}(b)+T_{21}(m)+U_{21}(n) & R_{22}(a)+S_{22}(b)+T_{22}(m)+U_{22}(n) \end{pmatrix},$$

where $R_{11},S_{11},T_{11},U_{11}$ are $\mathcal{R}$-linear mappings from $\mathcal{A},\mathcal{B},\mathcal{M},\mathcal{N}$ to $\mathcal{A}$, respectively; $R_{12},S_{12},T_{12},U_{12}$ are $\mathcal{R}$-linear mappings from $\mathcal{A},\mathcal{B},\mathcal{M},\mathcal{N}$ to $\mathcal{M}$, respectively; R_{21},S_{21}, T_{21},U_{21} are $\mathcal{R}$-linear mappings from $\mathcal{A},\mathcal{B},\mathcal{M},\mathcal{N}$ to $\mathcal{N}$, respectively; $R_{22},S_{22},T_{22},U_{22}$ are $\mathcal{R}$-linear mappings from $\mathcal{A},\mathcal{B},\mathcal{M},\mathcal{N}$ to $\mathcal{B}$, respectively.

Since $\mathcal{L}$ is a Lie σ-centralizer, we have

$$\mathcal{L}([x,y])=[\mathcal{L}(x),\sigma(y)]=[\sigma(x),\mathcal{L}(y)] \text{ for all } x,y\in\mathcal{G}. \tag{3.1}$$

Choosing $x=\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ and $y=\begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix}$ into (3.1), we get

$$\begin{pmatrix} T_{11}(am) & T_{12}(am) \\ T_{21}(am) & T_{22}(am) \end{pmatrix}=\begin{pmatrix} u(m)R_{21}(a) & R_{11}(a)u(m)-u(m)R_{22}(a) \\ 0 & R_{21}(a)u(m) \end{pmatrix}$$
$$=\begin{pmatrix} [\gamma(a),T_{11}(m)] & \gamma(a)T_{12}(m)+\gamma(a)m_0T_{22}(m)-T_{11}(m)\gamma(a)m_0 \\ n_0\gamma(a)T_{11}(m)-T_{21}(m)\gamma(a)-T_{22}(m)n_0\gamma(a) & n_0\gamma(a)T_{12}(m)-T_{21}(m)\gamma(a)m_0 \end{pmatrix}.$$

Thus, $T_{11}(am)=u(m)R_{21}(a)=[\gamma(a),T_{11}(m)]$, $T_{12}(am)=R_{11}(a)u(m)-u(m)R_{22}(a)=\gamma(a)T_{12}(m)+\gamma(a)m_0T_{22}(m)-T_{11}(m)\gamma(a)m_0$, $T_{21}(am)=0=n_0\gamma(a)T_{11}(m)-T_{21}(m)\gamma(a)$

$-T_{22}(m)n_0\gamma(a)$ and $T_{22}(am) = R_{21}(a)u(m) = n_0\gamma(a)T_{12}(m) - T_{21}(m)\gamma(a)m_0$. Putting $a = 1_{\mathcal{A}}$, we get $T_{11}(m) = 0$, $T_{21}(m) = 0$, and $T_{22}(m) = 0$. Thus, $T_{12}(am) = R_{11}(a)u(m) - u(m)R_{22}(a) = \gamma(a)T_{12}(m)$. Similarly, choose $x = \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix}$ and $y = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}$ to obtain $T_{12}(mb) = u(m)S_{22}(b) - S_{11}(b)u(m) = T_{12}(m)\delta(b)$.

If we take $x = \begin{pmatrix} 0 & 0 \\ n & 0 \end{pmatrix}$ and $y = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ in (3.1), then we have

$$\begin{pmatrix} U_{11}(na) & U_{12}(na) \\ U_{21}(na) & U_{22}(na) \end{pmatrix}$$
$$= \begin{pmatrix} [U_{11}(n),\gamma(a)] & U_{11}(n)\gamma(a)m_0 - \gamma(a)U_{12}(n) - \gamma(a)m_0U_{22}(n) \\ U_{21}(n)\gamma(a) + U_{22}(n)n_0\gamma(a) - n_0\gamma(a)U_{11}(n) & U_{21}(n)\gamma(a)m_0 - n_0\gamma(a)U_{12}(n) \end{pmatrix}$$
$$= \begin{pmatrix} -R_{12}(a)v(n) & 0 \\ v(n)R_{11}(a) - R_{22}(a)v(n) & v(n)R_{12}(a) \end{pmatrix}.$$

Thus,

$$U_{11}(na) = [U_{11}(n),\gamma(a)] = -R_{12}(a)v(n)$$
$$U_{12}(na) = U_{11}(n)\gamma(a)m_0 - \gamma(a)U_{12}(n) - \gamma(a)m_0U_{22}(n) = 0$$
$$U_{21}(na) = U_{21}(n)\gamma(a) + U_{22}(n)n_0\gamma(a) - n_0\gamma(a)U_{11}(n) = v(n)R_{11}(a) - R_{22}(a)v(n)$$

and

$$\begin{aligned} U_{22}(na) &= U_{21}(n)\gamma(a)m_0 - n_0\gamma(a)U_{12}(n) = v(n)R_{12}(a) \\ &= U_{21}(n)\gamma(a)m_0 - n_0\gamma(a)U_{12}(n). \end{aligned}$$

Putting $a = 1_{\mathcal{A}}$, we get $U_{11}(n) = 0$, $U_{12}(n) = 0$ and $U_{22}(n) = 0$. Hence, $U_{21}(na) = U_{21}(n)\gamma(a) = v(n)R_{11}(a) - R_{22}(a)v(n)$. Similarly, take $x = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}$ and $y = \begin{pmatrix} 0 & 0 \\ n & 0 \end{pmatrix}$ to obtain $U_{21}(bn) = \delta(b)U_{21}(n) = S_{22}(b)v(n) - v(n)S_{11}(b)$.

Taking $x = \begin{pmatrix} a_1 & 0 \\ 0 & 0 \end{pmatrix}$ and $y = \begin{pmatrix} a_2 & 0 \\ 0 & 0 \end{pmatrix}$ in (3.1), we get

$$\begin{pmatrix} R_{11}([a_1,a_2]) & R_{12}([a_1,a_2]) \\ R_{21}([a_1,a_2]) & R_{22}([a_1,a_2]) \end{pmatrix}$$
$$= \begin{pmatrix} [R_{11}(a_1),\gamma(a_2)] & R_{11}(a_1)\gamma(a_2)m_0 - \gamma(a_2)R_{12}(a_1) - \gamma(a_2)m_0R_{22}(a_1) \\ R_{21}(a_1)\gamma(a_2) + R_{22}(a_1)n_0\gamma(a_2) - n_0\gamma(a_2)R_{11}(a_1) & 0 \end{pmatrix}$$
$$= \begin{pmatrix} [\gamma(a_1),R_{11}(a_2)] & \gamma(a_1)R_{12}(a_2) + \gamma(a_1)m_0R_{22}(a_2) - R_{11}(a_2)\gamma(a_1)m_0 \\ n_0\gamma(a_1)R_{11}(a_2) - R_{21}(a_2)\gamma(a_1) - R_{22}(a_2)n_0\gamma(a_1) & 0 \end{pmatrix}.$$

Thus, $R_{11}([a_1,a_2]) = [R_{11}(a_1),\gamma(a_2)] = [\gamma(a_1),R_{11}(a_2)]$, i.e., R_{11} is a Lie γ-centralizer of $\mathcal{A}$, $R_{12}([a_1,a_2]) = R_{11}(a_1)\gamma(a_2)m_0 - \gamma(a_2)R_{12}(a_1) - \gamma(a_2)m_0R_{22}(a_1)$, $R_{21}([a_1,a_2]) =$

$R_{21}(a_1)\gamma(a_2)+R_{22}(a_1)n_0\gamma(a_2)-n_0\gamma(a_2)R_{11}(a_1)$, and $R_{22}([a_1,a_2])=0$. Taking $a_1=a$ and $a_2=1_{\mathcal{A}}$, we get $R_{12}(a)=R_{11}(a)m_0-m_0R_{22}(a)$, $R_{21}(a)=n_0R_{11}(a)-R_{22}(a)n_0$. Similarly, choosing $x=\begin{pmatrix} 0 & 0 \\ 0 & b_1 \end{pmatrix}$ and $y=\begin{pmatrix} 0 & 0 \\ 0 & b_2 \end{pmatrix}$ in (3.1), we get

$S_{11}([b_1,b_2])=0$, $S_{12}(b_1)=S_{11}(b_1)m_0-m_0S_{22}(b_1)$, $S_{21}(b_1)=n_0S_{11}(b_1)-S_{22}(b_1)n_0$ and $S_{22}([b_1,b_2])=[S_{22}(b_1),\delta(b_2)]=[\delta(b_1),S_{22}(b_2)]$, i.e., S_{22} is a Lie δ-centralizer of $\mathcal{B}$.

If we consider $x=\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ and $y=\begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}$ in (3.1), then we have

$$\begin{aligned}
&\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\\
&=\begin{pmatrix} 0 & -R_{11}(a)m_0\delta(b)+R_{12}(a)\delta(b)+m_0\delta(b)R_{22}(b) \\ -R_{22}\delta(b)n_0+\delta(b)n_0R_{11}(a)-\delta(b)R_{21}(a) & [R_{22}(a),\delta(b)] \end{pmatrix}\\
&=\begin{pmatrix} [\gamma(a),S_{11}(b)] & \gamma(a)S_{12}(b)+\gamma(a)m_0S_{22}(b)-S_{11}(b)\gamma(a)m_0 \\ n_0\gamma(a)S_b-S_{21}(b)\gamma(a)-S_{22}(b)n_0\gamma(a) & 0 \end{pmatrix}.
\end{aligned}$$

This gives $[R_{22}(a),\delta(b)]=0$ and $[\gamma(a),S_{11}(b)]=0$ for all $a\in\mathcal{A}$, $b\in\mathcal{B}$. Since γ and δ are automorphisms of $\mathcal{A}$ and $\mathcal{B}$, respectively, we conclude $R_{22}(a)\in\mathcal{Z}(\mathcal{B})$ and $S_{11}(b)\in\mathcal{Z}(\mathcal{A})$.

Furthermore, if we take $x=\begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix}$ and $y=\begin{pmatrix} 0 & 0 \\ n & 0 \end{pmatrix}$ in (3.1), then we have

$$\begin{aligned}
\begin{pmatrix} R_{11}(mn)-S_{11}(nm) & R_{12}(mn)-S_{12}(nm) \\ R_{21}(mn)-S_{21}(nm) & R_{22}(mn)-S_{22}(nm) \end{pmatrix} &= \begin{pmatrix} T_{12}(m)v(n) & 0 \\ 0 & -v(n)T_{12}(n) \end{pmatrix}\\
&= \begin{pmatrix} u(m)U_{21}(n) & 0 \\ 0 & -U_{21}(n)u(m) \end{pmatrix}.
\end{aligned}$$

This implies that $R_{11}(mn)-S_{11}(nm)=T_{12}(m)v(n)=u(m)U_{21}(n)$ and $R_{22}(mn)-S_{22}(nm)=-v(n)T_{12}(n)=-U_{21}(n)u(m)$. □

If $\mathcal{M}$ is a faithful $(\mathcal{A},\mathcal{B})$-bimodule, then condition (v) in the above theorem can be obtained using condition (iii). Thus, we have

Corollary 2. *Let $\mathcal{G}=G(\mathcal{A},\mathcal{M},\mathcal{N},\mathcal{B})$ be a generalized matrix algebra with faithful $\mathcal{M}$ and $\mathcal{L}:\mathcal{G}\to\mathcal{G}$ be a Lie σ-centralizer with associated automorphism σ of $\mathcal{G}$ as given in Lemma 1. Then $\mathcal{L}$ is of the form*

$$\mathcal{L}\begin{pmatrix} a & m \\ n & b \end{pmatrix}=$$
$$\begin{pmatrix} R_{11}(a)+S_{11}(b) & n_0(R_{11}(a)+S_{11}(b))-(R_{22}(a)+S_{22}(b))n_0+U_{21}(n) \\ (R_{11}(a)+S_{11}(b))m_0-m_0(R_{22}(a)+S_{22}(b))+T_{12}(m) & R_{22}(a)+S_{22}(b) \end{pmatrix},$$

where $R_{11}:\mathcal{A}\to\mathcal{A}$, $S_{11}:\mathcal{B}\to\mathcal{Z}(\mathcal{A})$, $T_{12}:\mathcal{M}\to\mathcal{M}$, $U_{21}:\mathcal{N}\to\mathcal{N}$, $R_{22}:\mathcal{A}\to\mathcal{Z}(\mathcal{B})$ and $S_{22}:\mathcal{B}\to\mathcal{B}$ are $\mathcal{R}$-linear mappings satisfying the following conditions:

(i) *R_{11} is a Lie γ-centralizer of $\mathcal{A}$, $R_{11}(mn) - S_{11}(nm) = T_{12}(m)v(n) = u(m)U_{21}(n)$;*

(ii) *S_{22} is a Lie δ-centralizer of $\mathcal{B}$, $S_{22}(nm) - R_{22}(mn) = U_{21}(n)u(m) = v(n)T_{12}(m)$;*

(iii) *$T_{12}(am) = R_{11}(a)u(m) - u(m)R_{22}(a) = \gamma(a)T_{12}(m)$, $T_{12}(mb) = T_{12}(m)\delta(b) = u(m)S_{22}(b) - S_{11}(b)u(m)$;*

(iv) *$U_{21}(na) = v(n)R_{11}(a) - R_{22}(a)v(n) = U_{21}(n)\gamma(a)$, $U_{21}(bn) = \delta(b)U_{21}(n) = S_{22}(b)v(n) - v(n)S_{11}(b)$.*

The following proposition gives necessary and sufficient conditions for a Lie σ-centralizer with associated automorphism σ of $\mathcal{G}$ as given in Lemma 1 to be proper.

Proposition 3. *Let $\mathcal{G} = G(\mathcal{A}, \mathcal{M}, \mathcal{N}, \mathcal{B})$ be a generalized matrix algebra. A Lie σ-centralizer $\mathcal{L} : \mathcal{G} \to \mathcal{G}$ of the form presented in Proposition 2 is proper if and only if there exist linear mappings $\ell_{\mathcal{A}} : \mathcal{A} \to \mathcal{Z}(\mathcal{A})$ and $\ell_{\mathcal{B}} : \mathcal{B} \to \mathcal{Z}(\mathcal{B})$ satisfying the following conditions:*

(i) *$R_{11} - \ell_{\mathcal{A}}$ is a γ-centralizer of $\mathcal{A}$ and $S_{22} - \ell_{\mathcal{B}}$ is a δ-centralizer of $\mathcal{B}$;*

(ii) *$\ell_{\mathcal{A}}(a) \oplus R_{22}(a) \in \mathcal{Z}(\mathcal{G})$ and $S_{11}(b) \oplus \ell_{\mathcal{B}}(b) \in \mathcal{Z}(\mathcal{G})$ for all $a \in \mathcal{A}$, $b \in \mathcal{B}$;*

(iii) *$\ell_{\mathcal{A}}(mn) = S_{11}(nm)$ and $R_{22}(mn) = \ell_{\mathcal{B}}(nm)$ for all $m \in \mathcal{M}$, $n \in \mathcal{N}$.*

Proof. Assume that $\mathcal{L}$ is a Lie σ-centralizer of $\mathcal{G}$ of the form presented in Proposition 2 and there exist linear mappings $\ell_{\mathcal{A}} : \mathcal{A} \to \mathcal{Z}(\mathcal{A})$ and $\ell_{\mathcal{B}} : \mathcal{B} \to \mathcal{Z}(\mathcal{B})$ satisfying conditions $(i) - (iii)$. Define two mappings Δ and τ as follows:

$$\Delta\begin{pmatrix} a & m \\ n & b \end{pmatrix} = \begin{pmatrix} (R_{11} - \ell_{\mathcal{A}})(a) & (R_{11} - \ell_{\mathcal{A}})(a)m_0 - m_0(S_{22} - \ell_{\mathcal{B}})(b) + T_{12}(m) \\ n_0(R_{11} - \ell_{\mathcal{A}})(a) - (S_{22} - \ell_{\mathcal{B}})(b)n_0 + U_{21}(n) & (S_{22} - \ell_{\mathcal{B}})(b) \end{pmatrix}$$

and

$$\tau\begin{pmatrix} a & m \\ n & b \end{pmatrix} = \begin{pmatrix} \ell_{\mathcal{A}}(a) + S_{11}(b) & 0 \\ 0 & R_{22}(a) + \ell_{\mathcal{B}}(b) \end{pmatrix}.$$

It is easy to see that Δ and τ are $\mathcal{R}$-linear mappings and $\mathcal{L} = \Delta + \tau$. Moreover, it follows from Proposition 1 that Δ is a σ-centralizer of $\mathcal{G}$. It only remains to show that $\tau(\mathcal{G}) \subseteq \mathcal{Z}(\mathcal{G})$. Using assumption (ii), we have

$$(\ell_{\mathcal{A}}(a) + S_{11}(b))m = \ell_{\mathcal{A}}(a)m + S_{11}(b)m = mR_{22}(a) + m\ell_{\mathcal{B}}(b) = m(R_{22}(a) + \ell_{\mathcal{B}}(b))$$

for all $m \in \mathcal{M}$. Similarly, $n(\ell_{\mathcal{A}}(a) + S_{11}(b)) = (R_{22}(a) + \ell_{\mathcal{B}}(b))n$ for all $n \in \mathcal{N}$. Hence, it follows that $\tau(\mathcal{G}) \subseteq \mathcal{Z}(\mathcal{G})$.

Conversely, suppose that $\mathcal{L}$ is proper, that is, $\mathcal{L} = \Delta + \tau$, where Δ is a σ-centralizer and τ is a center-valued mapping. In view of the representations of $\mathcal{L}$ and Δ, the mapping $\tau = \mathcal{L} - \Delta$ has the following form:

$$\tau\begin{pmatrix} a & m \\ n & b \end{pmatrix} = \begin{pmatrix} (R_{11} - A_{11})(a) + S_{11}(b) & 0 \\ 0 & R_{22}(a) + (S_{22} - B_{22})(b) \end{pmatrix}.$$

Set $\ell_{\mathcal{A}} = R_{11} - A_{11}$ and $\ell_{\mathcal{B}} = S_{22} - B_{22}$. Then, it is straightforward to check that $\ell_{\mathcal{A}}$ and $\ell_{\mathcal{B}}$ are the desired mappings satisfying assumptions $(i) - (iii)$. $\square$

By Corollary 1, if $\mathcal{M}$ is a faithful $(\mathcal{A},\mathcal{B})$-bimodule, then condition (i) of the above proposition becomes superfluous. Thus, as a consequence of Proposition 3, we have the following corollary:

Corollary 3. *Let $\mathcal{G} = G(\mathcal{A},\mathcal{M},\mathcal{N},\mathcal{B})$ be a generalized matrix algebra and $\mathcal{L} : \mathcal{G} \to \mathcal{G}$ be a Lie σ-centralizer with associated automorphism σ of $\mathcal{G}$ as given in Lemma 1. If $\mathcal{L}$ is proper, then the following conditions hold:*

(i) *$R_{22}(\mathcal{A}) \subseteq \pi_{\mathcal{B}}(\mathcal{Z}(\mathcal{G}))$ and $S_{11}(\mathcal{B}) \subseteq \pi_{\mathcal{A}}(\mathcal{Z}(\mathcal{G}))$;*
(ii) *$S_{11}(nm) \oplus R_{22}(mn) \in \mathcal{Z}(\mathcal{G})$ for all $m \in \mathcal{M}$, $n \in \mathcal{N}$.*

The converse also holds provided $\mathcal{M}$ is faithful.

Proof. If $\mathcal{L}$ is a proper Lie σ-centralizer of $\mathcal{G}$, then the required conditions follow directly from Proposition 3. For the converse, suppose that $\mathcal{M}$ is a faithful $(\mathcal{A},\mathcal{B})$-bimodule and $\mathcal{L}$ is a Lie σ-centralizer of $\mathcal{G}$ of the form presented in Proposition 2 satisfying (i) and (ii). Since $\mathcal{M}$ is faithful, there exists a unique algebra isomorphism $\xi : \pi_{\mathcal{A}}(\mathcal{Z}(\mathcal{G})) \to \pi_{\mathcal{B}}(\mathcal{Z}(\mathcal{G}))$ such that $a \oplus \xi(a) \in \mathcal{Z}(\mathcal{G})$ for all $a \in \pi_{\mathcal{A}}(\mathcal{Z}(\mathcal{G}))$. Define $\ell_{\mathcal{A}} : \mathcal{A} \to \mathcal{Z}(\mathcal{A})$ and $\ell_{\mathcal{B}} : \mathcal{B} \to \mathcal{Z}(\mathcal{B})$ by $\ell_{\mathcal{A}} = \xi^{-1} \circ R_{22}$ and $\ell_{\mathcal{B}} = \xi \circ S_{11}$, respectively. It is easy to verify that $\ell_{\mathcal{A}}$ and $\ell_{\mathcal{B}}$ are linear mappings satisfying the hypotheses of Proposition 3. Therefore, $\mathcal{L}$ is proper. □

Now we are in a position to give a sufficient condition for a Lie σ-centralizer with associated automorphism σ of $\mathcal{G}$ as given in Lemma 1 to be proper.

Corollary 4. *Let $\mathcal{G} = G(\mathcal{A},\mathcal{M},\mathcal{N},\mathcal{B})$ be a generalized matrix algebra with faithful $\mathcal{M}$. A Lie σ-centralizer with associated automorphism σ of $\mathcal{G}$ as given in Lemma 1 is proper if the following conditions hold:*

(i) *$\pi_{\mathcal{A}}(\mathcal{Z}(\mathcal{G})) = \mathcal{Z}(\mathcal{A})$, $\pi_{\mathcal{B}}(\mathcal{Z}(\mathcal{G})) = \mathcal{Z}(\mathcal{B})$ and*
(ii) *either $\mathcal{A}$ or $\mathcal{B}$ does not contain nonzero central ideals.*

Proof. Suppose that $\mathcal{L}$ is a Lie σ-centralizer of $\mathcal{G}$ of the form presented in Proposition 2. We shall use Corollary 3 to prove that $\mathcal{L}$ is proper. Obviously, the condition $R_{22}(\mathcal{A}) \subseteq \pi_{\mathcal{B}}(\mathcal{Z}(\mathcal{G}))$ and $S_{11}(\mathcal{B}) \subseteq \pi_{\mathcal{A}}(\mathcal{Z}(\mathcal{G}))$ of Corollary 3 follows from the assumption (i). We only need to show that $S_{11}(nm) \oplus R_{22}(mn) \in \mathcal{Z}(\mathcal{G})$ for all $m \in \mathcal{M}$, $n \in \mathcal{N}$. Without loss of generality, assume that $\mathcal{A}$ does not contain nonzero central ideals. Define $\ell_{\mathcal{A}}(a) = \xi^{-1}(R_{22}(a))$. Then $\ell_{\mathcal{A}}(a) \oplus R_{22}(a) \in \mathcal{Z}(\mathcal{G})$ for all $a \in \mathcal{A}$. Set $A_{11} = R_{11} - \ell_{\mathcal{A}}$ and $\nu(a,b) = \ell_{\mathcal{A}}(a) + S_{11}(b)$. Using Proposition 2(iii), one can easily show that A_{11} is a γ-centralizer. Again, using Proposition 2, we have

$$\begin{aligned}\nu(mn,-nm) = \ell_{\mathcal{A}}(mn) - S_{11}(nm) &= R_{11}(mn) - A_{11}(mn) - S_{11}(nm)\\ &= T_{12}(m)v(n) - A_{11}(mn)\end{aligned}$$

for all $m \in \mathcal{M}$, $n \in \mathcal{N}$. Thus, using the fact that A_{11} is a γ-centralizer, we have

$$\begin{aligned}\nu(amn,-nam) &= T_{12}(am)v(n) - A_{11}(amn) = \gamma(a)T_{12}(m)v(n) - \gamma(a)A_{11}(mn)\\ &= \gamma(a)(T_{12}(m)v(n) - A_{11}(mn)) = \gamma(a)\nu(mn,-nm)\end{aligned}$$

for all $a \in \mathcal{A}$. Since γ is an automorphism of $\mathcal{A}$, the set $\mathcal{A}\nu(mn,-nm)$ is a central ideal of $\mathcal{A}$ for each $m \in \mathcal{M}$, $n \in \mathcal{N}$. Hence, $\ell_{\mathcal{A}}(mn) - S_{11}(nm) = \nu(mn,-nm) = 0$. Therefore, $S_{11}(nm) \oplus R_{22}(mn) = \ell_{\mathcal{A}}(mn) \oplus R_{22}(mn) \in \mathcal{Z}(\mathcal{G})$. $\square$

Recall that if $\mathcal{G} = G(\mathcal{A},\mathcal{M},\mathcal{N},\mathcal{B})$ is a generalized matrix algebra such that $\mathcal{A}$ and $\mathcal{B}$ have only trivial idempotents, $\mathcal{A}$ and $\mathcal{B}$ are not isomorphic and both the bilinear mappings $\zeta_{\mathcal{M}\mathcal{N}}, \psi_{\mathcal{N}\mathcal{M}}$ are zero, then the class of all automorphisms presented in Lemma 1 coincides with the group of all automorphisms of $\mathcal{G}$. Thus, in view of Corollary 4, we have the first main result of the paper which characterizes an arbitrary Lie σ-centralizer of a generalized matrix algebra.

Theorem 1. *Let $\mathcal{G} = G(\mathcal{A},\mathcal{M},\mathcal{N},\mathcal{B})$ be a generalized matrix algebra. Then every Lie σ-centralizer of $\mathcal{G}$ is proper if the following conditions hold:*

(i) *$\pi_{\mathcal{A}}(\mathcal{Z}(\mathcal{G})) = \mathcal{Z}(\mathcal{A})$ and $\mathcal{M}$ is a faithful left $\mathcal{A}$-module;*
(ii) *$\pi_{\mathcal{B}}(\mathcal{Z}(\mathcal{G})) = \mathcal{Z}(\mathcal{B})$ and $\mathcal{M}$ is a faithful right $\mathcal{B}$-module;*
(iii) *$\mathcal{A}$ and $\mathcal{B}$ have only trivial idempotents and $\mathcal{A}$ and $\mathcal{B}$ are not isomorphic;*
(iv) *either $\mathcal{A}$ or $\mathcal{B}$ does not contain nonzero central ideals;*
(v) *both the bilinear mappings $\zeta_{\mathcal{M}\mathcal{N}}, \psi_{\mathcal{N}\mathcal{M}}$ are zero.*

4. JORDAN σ-CENTRALIZER OF GENERALIZED MATRIX ALGEBRAS

In this section, we show that under certain restrictions every Jordan σ-centralizer of a generalized matrix algebra is a σ-centralizer. We begin this section with the following proposition which provides the structure of a Jordan σ-centralizer with associated automorphism σ as given in Lemma 1.

Proposition 4. *Let $\mathcal{G} = G(\mathcal{A},\mathcal{M},\mathcal{N},\mathcal{B})$ be a 2-torsion free generalized matrix algebra. Then a Jordan σ-centralizer $J : \mathcal{G} \to \mathcal{G}$ with associated automorphism σ of $\mathcal{G}$ as given in Lemma 1 is of the form*

$$J\begin{pmatrix} a & m \\ n & b \end{pmatrix} = \begin{pmatrix} R_{11}(a) & R_{11}(a)m_0 - m_0S_{22}(b) + T_{12}(m) \\ n_0R_{11}(a) - S_{22}(b)n_0 + U_{21}(n) & S_{22}(b) \end{pmatrix},$$

where $R_{11} : \mathcal{A} \to \mathcal{A}$, $S_{11} : \mathcal{B} \to \mathcal{Z}(\mathcal{A})$, $T_{12} : \mathcal{M} \to \mathcal{M}$, $U_{21} : \mathcal{N} \to \mathcal{N}$, $S_{11} : \mathcal{A} \to \mathcal{Z}(\mathcal{B})$ and $S_{22} : \mathcal{B} \to \mathcal{B}$ are $\mathcal{R}$-linear mappings satisfying the following conditions:

(i) *R_{11} is a Jordan γ-centralizer of $\mathcal{A}$, S_{22} is a Jordan δ-centralizer of $\mathcal{B}$;*
(ii) *$R_{11}(mn) = T_{12}(m)v(n) = u(m)U_{21}(n)$, $S_{22}(nm) = U_{21}(n)u(m) = v(n)T_{12}(m)$;*
(iii) *$T_{12}(am) = R_{11}(a)u(m) = \gamma(a)T_{12}(m)$, $T_{12}(mb) = T_{12}(m)\delta(b) = u(m)S_{22}(b)$;*
(iv) *$U_{21}(na) = v(n)R_{11}(a) = U_{21}(n)\gamma(a)$, $U_{21}(bn) = \delta(b)U_{21}(n) = S_{22}(b)v(n)$.*

Proof. Suppose that the Jordan σ-centralizer J is of the form

$$J\begin{pmatrix} a & m \\ n & b \end{pmatrix} = \begin{pmatrix} R_{11}(a)+S_{11}(b)+T_{11}(m)+U_{11}(n) & R_{12}(a)+S_{12}(b)+T_{12}(m)+U_{12}(n) \\ R_{21}(a)+S_{21}(b)+T_{21}(m)+U_{21}(n) & R_{22}(a)+S_{22}(b)+T_{22}(m)+U_{22}(n) \end{pmatrix},$$

where $R_{11}, S_{11}, T_{11}, U_{11}$ are $\mathcal{R}$-linear mappings from $\mathcal{A}, \mathcal{B}, \mathcal{M}, \mathcal{N}$ to $\mathcal{A}$, respectively; $R_{12}, S_{12}, T_{12}, U_{12}$ are $\mathcal{R}$-linear mappings from $\mathcal{A}, \mathcal{B}, \mathcal{M}, \mathcal{N}$ to $\mathcal{M}$, respectively; R_{21}, S_{21},

T_{21}, U_{21} are $\mathcal{R}$-linear mappings from $\mathcal{A}, \mathcal{B}, \mathcal{M}, \mathcal{N}$ to $\mathcal{N}$, respectively; $R_{22}, S_{22}, T_{22}, U_{22}$ are $\mathcal{R}$-linear mappings from $\mathcal{A}, \mathcal{B}, \mathcal{M}, \mathcal{N}$ to $\mathcal{B}$, respectively.

Since J is a Jordan σ-centralizer, we have

$$J(x\circ y) = J(x)\circ\sigma(y) = \sigma(x)\circ J(y) \ \text{ for all } \ x, y \in \mathcal{G}. \tag{4.1}$$

Let us choose $x = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ and $y = \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix}$ in (4.1). Then

$$\begin{pmatrix} T_{11}(am) & T_{12}(am) \\ T_{21}(am) & T_{22}(am) \end{pmatrix} = \begin{pmatrix} u(m)R_{21}(a) & R_{11}(a)u(m) + u(m)R_{22}(a) \\ 0 & R_{21}(a)u(m) \end{pmatrix}$$
$$= \begin{pmatrix} \gamma(a)\circ T_{11}(m) & \gamma(a)T_{12}(m) + \gamma(a)m_0T_{22}(m) + T_{11}(m)\gamma(a)m_0 \\ n_0\gamma(a)T_{11}(m) + T_{21}(m)\gamma(a) + T_{22}(m)n_0\gamma(a) & n_0\gamma(a)T_{12}(m) + T_{21}(m)\gamma(a)m_0 \end{pmatrix}.$$

This yields $T_{11}(am) = u(m)R_{21}(a) = \gamma(a)\circ T_{11}(m)$, $T_{12}(am) = R_{11}(a)u(m) + u(m)R_{22}(a) = \gamma(a)T_{12}(m) + \gamma(a)m_0T_{22}(m) + T_{11}(m)\gamma(a)m_0$, $T_{21}(am) = 0 = n_0\gamma(a)T_{11}(m) + T_{21}(m)\gamma(a) + T_{22}(m)n_0\gamma(a)$ and $T_{22}(am) = R_{21}(a)u(m) = n_0\gamma(a)T_{12}(m) + T_{21}(m)\gamma(a)m_0$. Putting $a = 1_{\mathcal{A}}$, we get $T_{11}(m) = 0$, $T_{21}(m) = 0$, and $T_{22}(m) = 0$. Thus, $T_{12}(am) = R_{11}(a)u(m) + u(m)R_{22}(a) = \gamma(a)T_{12}(m)$. Similarly, choosing $x = \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix}$ and $y = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}$ to obtain $T_{12}(mb) = u(m)S_{22}(b) + S_{11}(b)u(m) = T_{12}(m)\delta(b)$.

Taking $x = \begin{pmatrix} 0 & 0 \\ n & 0 \end{pmatrix}$ and $y = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ in (4.1), we get

$$\begin{pmatrix} U_{11}(na) & U_{12}(na) \\ U_{21}(na) & U_{22}(na) \end{pmatrix}$$
$$= \begin{pmatrix} U_{11}(n)\circ\gamma(a) & U_{11}(n)\gamma(a)m_0 + \gamma(a)U_{12}(n) + \gamma(a)m_0U_{22}(n) \\ U_{21}(n)\gamma(a) + U_{22}(n)n_0\gamma(a) + n_0\gamma(a)U_{11}(n) & U_{21}(n)\gamma(a)m_0 + n_0\gamma(a)U_{12}(n) \end{pmatrix}$$
$$= \begin{pmatrix} R_{12}(a)v(n) & 0 \\ v(n)R_{11}(a) + R_{22}(a)v(n) & v(n)R_{12}(a) \end{pmatrix}.$$

Thus, $U_{11}(na) = U_{11}(n)\circ\gamma(a) = R_{12}(a)v(n)$, $U_{12}(na) = U_{11}(n)\gamma(a)m_0 + \gamma(a)U_{12}(n) + \gamma(a)m_0U_{22}(n) = 0$, $U_{21}(na) = U_{21}(n)\gamma(a) + U_{22}(n)n_0\gamma(a) + n_0\gamma(a)U_{11}(n) = v(n)R_{11}(a) + R_{22}(a)v(n)$ and $U_{22}(na) = U_{21}(n)\gamma(a)m_0 + n_0\gamma(a)U_{12}(n) = v(n)R_{12}(a) = U_{21}(n)\gamma(a)m_0 + n_0\gamma(a)U_{12}(n)$. Putting $a = 1_{\mathcal{A}}$, we get $U_{11}(n) = 0$, $U_{12}(n) = 0$ and $U_{22}(n) = 0$. Hence $U_{21}(na) = U_{21}(n)\gamma(a) = v(n)R_{11}(a) + R_{22}(a)v(n)$.

Similarly, by taking $x = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}$ and $y = \begin{pmatrix} 0 & 0 \\ n & 0 \end{pmatrix}$ in (4.1), one can obtain $U_{21}(bn) = \delta(b)U_{21}(n) = S_{22}(b)v(n) + v(n)S_{11}(b)$.

If we consider $x = \begin{pmatrix} a_1 & 0 \\ 0 & 0 \end{pmatrix}$ and $y = \begin{pmatrix} a_2 & 0 \\ 0 & 0 \end{pmatrix}$ in (4.1). Then

$$\begin{pmatrix} R_{11}(a_1\circ a_2) & R_{12}(a_1\circ a_2) \\ R_{21}(a_1\circ a_2) & R_{22}(a_1\circ a_2) \end{pmatrix}$$

$$= \begin{pmatrix} R_{11}(a_1) \circ \gamma(a_2) & R_{11}(a_1)\gamma(a_2)m_0 + \gamma(a_2)R_{12}(a_1) + \gamma(a_2)m_0R_{22}(a_1) \\ R_{21}(a_1)\gamma(a_2) + R_{22}(a_1)n_0\gamma(a_2) + n_0\gamma(a_2)R_{11}(a_1) & 0 \end{pmatrix}$$
$$= \begin{pmatrix} \gamma(a_1) \circ R_{11}(a_2) & \gamma(a_1)R_{12}(a_2) + \gamma(a_1)m_0R_{22}(a_2) + R_{11}(a_2)\gamma(a_1)m_0 \\ n_0\gamma(a_1)R_{11}(a_2) + R_{21}(a_2)\gamma(a_1) + R_{22}(a_2)n_0\gamma(a_1) & 0 \end{pmatrix}.$$

It follows from the above relation that $R_{11}(a_1 \circ a_2) = R_{11}(a_1) \circ \gamma(a_2) + \gamma(a_1) \circ R_{11}(a_2)$, i.e., R_{11} is a Jordan γ-centralizer on $\mathcal{A}$, $R_{12}(a_1) = R_{11}(a_1)m_0 + m_0R_{22}(a_1)$, $R_{21}(a_1) = n_0R_{11}(a_1) + R_{22}(a_1)n_0$, and $R_{22}(a_1 \circ a_2) = 0$. Symmetrically, consider $x = \begin{pmatrix} 0 & 0 \\ 0 & b_1 \end{pmatrix}$ and $y = \begin{pmatrix} 0 & 0 \\ 0 & b_2 \end{pmatrix}$ in (4.1) to obtain $S_{11}(b_1 \circ b_2) = 0$, $S_{12}(b_1) = S_{11}(b_1)m_0 + m_0S_{22}(b_1)$, $S_{21}(b_1) = n_0S_{11}(b_1) + S_{22}(b_1)n_0$ and $S_{22}(b_1 \circ b_2) = S_{22}(b_1) \circ \gamma(b_2) + \gamma(b_1) \circ S_{22}(b_2)$, i.e., S_{22} is a Jordan δ-centralizer on $\mathcal{B}$.

Considering $x = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ and $y = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}$ in (4.1), we have

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & -R_{11}(a)m_0\delta(b) + R_{12}(a)\delta(b) - m_0\delta(b)R_{22}(b) \\ -R_{22}\delta(b)n_0 - \delta(b)n_0R_{11}(a) + \delta(b)R_{21}(a) & R_{22}(a) \circ \delta(b) \end{pmatrix}$$
$$= \begin{pmatrix} \gamma(a) \circ S_{11}(b) & \gamma(a)S_{12}(b) + \gamma(a)m_0S_{22}(b) + S_{11}(b)\gamma(a)m_0 \\ n_0\gamma(a)S_b + S_{21}(b)\gamma(a) + S_{22}(b)n_0\gamma(a) & 0 \end{pmatrix}.$$

It follows from the above equation that $R_{22}(a) \circ \delta(b) = 0$ and $\gamma(a) \circ S_{11}(b) = 0$ for all $a \in \mathcal{A}$, $b \in \mathcal{B}$. Putting $b = 1_{\mathcal{B}}$ and $a = 1_{\mathcal{A}}$ in the above equations, respectively, we get $2R_{22}(a) = 0$ and $2S_{11}(b) = 0$. Since $\mathcal{G}$ is 2-torsion free, we obtain $R_{22}(a) = 0$ and $S_{11}(b) = 0$.

Furthermore, if we choose $x = \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix}$ and $y = \begin{pmatrix} 0 & 0 \\ n & 0 \end{pmatrix}$ in (4.1), then we have

$$\begin{pmatrix} R_{11}(mn) & R_{12}(mn) + S_{12}(nm) \\ R_{21}(mn) + S_{21}(nm) & S_{22}(nm) \end{pmatrix} = \begin{pmatrix} T_{12}(m)v(n) & 0 \\ 0 & v(n)T_{12}(n) \end{pmatrix}$$
$$= \begin{pmatrix} u(m)U_{21}(n) & 0 \\ 0 & U_{21}(n)u(m) \end{pmatrix}.$$

Thus, $R_{11}(mn) = T_{12}(m)v(n) = u(m)U_{21}(n)$ and $S_{22}(nm) = v(n)T_{12}(n) = U_{21}(n)u(m)$. □

If $\mathcal{M}$ is a faithful $(\mathcal{A}, \mathcal{B})$-bimodule, then condition (i) in Proposition 4 become superfluous.

Corollary 5. *Let $\mathcal{G} = G(\mathcal{A},\mathcal{M},\mathcal{N},\mathcal{B})$ be a 2-torsion free generalized matrix algebra with faithful $\mathcal{M}$. Then a Jordan σ-centralizer $J : \mathcal{G} \to \mathcal{G}$ with associated automorphism σ of $\mathcal{G}$ as given in Lemma 1 is of the form*

$$J\begin{pmatrix} a & m \\ n & b \end{pmatrix} = \begin{pmatrix} R_{11}(a) & R_{11}(a)m_0 - m_0S_{22}(b) + T_{12}(m) \\ n_0R_{11}(a) - S_{22}(b)n_0 + U_{21}(n) & S_{22}(b) \end{pmatrix},$$

where $R_{11} : \mathcal{A} \to \mathcal{A}$, $S_{11} : \mathcal{B} \to \mathcal{Z}(\mathcal{A})$, $T_{12} : \mathcal{M} \to \mathcal{M}$, $U_{21} : \mathcal{N} \to \mathcal{N}$, $S_{11} : \mathcal{A} \to \mathcal{Z}(\mathcal{B})$ and $S_{22} : \mathcal{B} \to \mathcal{B}$ are $\mathcal{R}$-linear mappings satisfying the following conditions:

(i) $R_{11}(mn) = T_{12}(m)v(n) = u(m)U_{21}(n)$, $S_{22}(nm) = U_{21}(n)u(m) = v(n)T_{12}(m)$;
(ii) $T_{12}(am) = R_{11}(a)u(m) = \gamma(a)T_{12}(m)$, $T_{12}(mb) = T_{12}(m)\delta(b) = u(m)S_{22}(b)$;
(iii) $U_{21}(na) = v(n)R_{11}(a) = U_{21}(n)\gamma(a)$, $U_{21}(bn) = \delta(b)U_{21}(n) = S_{22}(b)v(n)$.

Proof. In view of Proposition 4, it suffices to show that if $\mathcal{M}$ is a faithful $(\mathcal{A},\mathcal{B})$-bimodule, then R_{11} is a Jordan γ-centralizer of $\mathcal{A}$ and S_{22} is a Jordan δ-centralizer of $\mathcal{B}$. For any $a_1, a_2 \in \mathcal{A}$ and $m \in \mathcal{M}$, we have

$$\begin{aligned} R_{11}(a_1 \circ a_2)m &= T_{12}((a_1 \circ a_2)m) = T_{12}(a_1a_2m + a_2a_1m) \\ &= R_{11}(a_1)u(a_2m) + \gamma(a_2)T_{12}(a_1m) \\ &= R_{11}(a_1)\gamma(a_2)u(m) + \gamma(a_2)R_{11}(a_1)u(m) = (R_{11}(a_1) \circ \gamma(a_2))u(m). \end{aligned}$$

This implies that $\{R_{11}(a_1 \circ a_2) - R_{11}(a_1) \circ \gamma(a_2)\}\mathcal{M} = \{0\}$. Since $\mathcal{M}$ is faithful as a left $\mathcal{A}$-module, we conclude $R_{11}(a_1 \circ a_2) = R_{11}(a_1) \circ \gamma(a_2)$. Thus, R_{11} is a Jordan γ-centralizer of $\mathcal{A}$. Similarly, we can show that S_{22} is a Jordan δ-centralizer of $\mathcal{B}$. $\square$

Combining Corollaries 1 and 5, we get the following proposition:

Proposition 5. *Let $\mathcal{G} = G(\mathcal{A},\mathcal{M},\mathcal{N},\mathcal{B})$ be a 2-torsion free generalized matrix algebra with faithful $\mathcal{M}$. Then a Jordan σ-centralizer of $\mathcal{G}$ with associated automorphism σ of $\mathcal{G}$ as given in Lemma 1 is a σ-centralizer.*

In view of Proposition 5, we now obtain the second main result of the paper which characterizes an arbitrary Jordan σ-centralizer of a generalized matrix algebra.

Theorem 2. *Let $\mathcal{G} = G(\mathcal{A},\mathcal{M},\mathcal{N},\mathcal{B})$ be a 2-torsion free generalized matrix algebra with faithful $\mathcal{M}$. Then every Jordan σ-centralizer of $\mathcal{G}$ is a σ-centralizer if the following conditions hold:*

(i) *$\mathcal{A}$ and $\mathcal{B}$ have only trivial idempotents;*
(ii) *$\mathcal{A}$ and $\mathcal{B}$ are not isomorphic;*
(iii) *both the bilinear mappings $\zeta_{\mathcal{M}\mathcal{N}}, \psi_{\mathcal{N}\mathcal{M}}$ are zero.*

5. Applications

In this section, we apply Theorems 1 and 2 to triangular algebras. Recall that a triangular algebra is an algebra

$$\mathfrak{A} = \mathrm{Tri}(\mathcal{A},\mathcal{M},\mathcal{B}) = \left\{ \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \middle| \, a \in \mathcal{A}, m \in \mathcal{M}, b \in \mathcal{B} \right\}$$

with the usual matrix operations consisting of two unital algebras $\mathcal{A}$, $\mathcal{B}$ and an $(\mathcal{A},\mathcal{B})$-bimodule $\mathcal{M}$. Triangular algebras are classical example of generalized matrix algebras. Indeed, if we take $\mathcal{N}=\{0\}$ in the definition of generalized matrix algebra $\mathcal{G}=G(\mathcal{A},\mathcal{M},\mathcal{N},\mathcal{B})$, then $\mathcal{G}$ exactly degenerates to a triangular algebra $\mathfrak{A}=\mathrm{Tri}(\mathcal{A},\mathcal{M},\mathcal{B})$. It is easy to see that the conditions "either $\mathcal{A}$ or $\mathcal{B}$ does not contain nonzero central ideals" and "both the bilinear mappings $\zeta_{\mathcal{M}\mathcal{N}},\psi_{\mathcal{N}\mathcal{M}}$ are zero" are not required in case of triangular algebras. Further, in the case where $\mathfrak{A}=\mathrm{Tri}(\mathcal{A},\mathcal{M},\mathcal{B})$ is a triangular algebras such that $\mathcal{A}$ and $\mathcal{B}$ have only trivial idempotents, we have $Aut_0^0(\mathfrak{A})=Aut(\mathcal{G})$. Hence the condition "$\mathcal{A}$ and $\mathcal{B}$ are not isomorphic" in Theorems 1 and 2 also become redundant. Therefore, as an application of Theorems 1 and 2, we have the following results:

Corollary 6. *Let $\mathfrak{A}=\mathrm{Tri}(\mathcal{A},\mathcal{M},\mathcal{B})$ be a triangular algebra. Suppose that $\mathcal{A}$ and $\mathcal{B}$ have only trivial idempotents and $\mathcal{M}$ is a faithful $(\mathcal{A},\mathcal{B})$-bimodule. Then every Lie σ-centralizer of $\mathfrak{A}$ is proper if $\pi_{\mathcal{A}}(\mathcal{Z}(\mathcal{G}))=\mathcal{Z}(\mathcal{A})$ and $\pi_{\mathcal{B}}(\mathcal{Z}(\mathcal{G}))=\mathcal{Z}(\mathcal{B})$.*

Corollary 7. *Let $\mathfrak{A}=\mathrm{Tri}(\mathcal{A},\mathcal{M},\mathcal{B})$ be a 2-torsion free triangular algebra. Suppose that $\mathcal{A}$ and $\mathcal{B}$ have only trivial idempotents and $\mathcal{M}$ is a faithful $(\mathcal{A},\mathcal{B})$-bimodule. Then every Jordan σ-centralizer of $\mathfrak{A}$ is a σ-centralizer.*

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