Miskolc Mathematical Notes
Vol. 24 (2023), No. 2, pp. 909–914

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2023.4045

ON THE COMPARISON OF THE MARCINKIEWICZ AND LUXEMBURG NORMS OF THE EXPONENTIAL SPACE

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Received 23 December, 2021

Abstract. The exponential space, which is a Banach function space, can be defined with two very differently looking, but equivalent norms. In this paper, we give estimates for the best constants of the ratio of these two norms. Our result answers a question of C. Bennett, and R. Sharpley.

2010 *Mathematics Subject Classification:* 46B20; 46E30

Keywords: $L_{\exp}$ space, Marcinkiewicz norm, Luxemburg norm, Lorentz spaces, Orlicz spaces, Marcinkiewicz spaces

1. Introduction

1.1. *The $L_{\exp}$ space*

In 1928, A. Zygmund and E. C. Titchmarsh independently introduced the $L\log L$ *logarithm space*, and the $L_{\exp}$ *exponential space* during their studies on Fourier analysis [5–7] (see also the historical remarks on p. 288 of [1]). Later, these spaces turned out to be important in the theory of interpolation of operators, too [1]. The exponential space can be defined by a Marcinkiewicz space norm, and via an Orlicz space norm. These norms, although are guaranteed to be equivalent, look very differently. The authors of [1] wrote that "the exact relationship" between the two norms "is not clear" (p. 271). Our aim in this paper is to clarify this relation by giving approximations of the best constants for the ratio of the two norms. The logarithm space was treated in our previous paper [3].

The $L_{\exp} = L_{\exp}[0,1]$ space is the complete vector space of measurable functions on $[0,1]$ with finite norm, where the norm is given by the expression

$$\|f\|_{L_{\exp}} = \sup_{0<t<1} \frac{f^{**}(t)}{1+\log\left(\frac{1}{t}\right)}. \tag{1.1}$$

Here

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\mathrm{d}s$$

is the *maximal function* of f, and $f^*(s)$ is the *decreasing rearrangement* of f. The reader can find more on these notions in the classical text [2] or in the newer [1, 4]. Norms of the type (1.1) are called Marcinkiewicz-type norms [4, Chapter 11].

The exponential space can be viewed as an Orlicz space, too. In this case $f \in L_{\exp}$ is equipped with the following norm, called *Luxemburg norm* [4, Notes on p. 234]:

$$\|f\|_\Phi = \inf\left\{\lambda > 0 : \int_0^1 \Phi\left(\frac{|f(t)|}{\lambda}\right) \mathrm{d}t \le 1\right\}, \tag{1.2}$$

where

$$\Phi(t) = \begin{cases} t, & 0 \le t \le 1; \\ e^{t-1}, & t > 1. \end{cases} \tag{1.3}$$

The fact that (1.1) and (1.2) are equivalent can be verified by showing that the two corresponding spaces contain the same set of functions (see [1, p. 271] for more details, and detailed explanation). It therefore follows that there are two positive, finite constants c_1 and c_2 such that $c_1 \le \frac{\|f\|_{L_{\exp}}}{\|f\|_\Phi} \le c_2$. In the following theorem we give estimates for these constants.

Theorem 1. *There exist constants c_1 and c_2, such that for all $f \in L_{\exp}$*

$$c_1 \le \frac{\|f\|_{L_{\exp}}}{\|f\|_\Phi} \le c_2 \tag{1.4}$$

with

$$0 < c_1 \le 1, \quad \textit{and} \quad 1 + \frac{1}{e} \le c_2 \le 2(1 + \log 2).$$

Before proving our theorem, a remark and a question are in order.

Question. The constants in our theorem unfortunately do not seem to be optimal, so we put the following question: what are the optimal constants in (1.4)?

2. The proof of Theorem 1

2.1. *The estimate of c_1*

We will consider step functions[1] f which are decreasing and right-continuous (so that, conveniently, $f^* = f$):

$$f(t) = \sum_{k=0}^{n} a_k \chi_{]i_k, i_{k+1}]}(t) \tag{2.1}$$

[1]Note that $L_{\exp}$ is a non-separable space. The theory of Marcinkiewicz spaces says that if the fundamental function ϕ of the space is so that $\phi(0+) = 0$, and $\phi'(0+) = \infty$, then the given Marcinkiewicz space is non-separable [4, p. 164]. The fundamental function of the exponential space is

$$\phi(t) = \frac{1}{1 + \log\left(\frac{1}{t}\right)},$$

which behaves at zero exactly as we said above. Therefore, the step functions do not form a dense set in $L_{\exp}$.

with $0 = i_0 < i_1 < \cdots i_{n+1} = 1$ and $a_0 > a_1 > \cdots > a_n \geq 0$. (Here χ_I is the characteristic function of the interval I.)

It turns out, that on one-step functions the norms (1.1) and (1.2) agree. To see this, let f be a one-step function. Then

$$f^{**}(t) = \frac{1}{t}\int_0^{i_1} a_0 \mathrm{d}s = \begin{cases} a_0, & \text{if } 0 < t \leq i_1; \\ \frac{a_0 i_1}{t}, & \text{if } t > i_1, \end{cases}$$

whence

$$\|f\|_{L_{\exp}} = \max\left\{ \sup_{0<t\leq i_1} \frac{a_0}{1+\log\left(\frac{1}{t}\right)}, \sup_{i_1<t<1} \frac{a_0 i_1/t}{1+\log\left(\frac{1}{t}\right)} \right\} = \frac{a_0}{1+\log\left(\frac{1}{i_1}\right)}. \tag{2.2}$$

On the other hand,

$$\|f\|_{\Psi} = \inf\left\{\lambda > 0 \,\middle|\, \int_0^{i_1} \Phi\left(\frac{a_0}{\lambda}\right) \mathrm{d}t \leq 1 \right\} = \inf\left\{\lambda > 0 \,\middle|\, i_1 \Phi\left(\frac{a_0}{\lambda}\right) \leq 1 \right\}.$$

By (1.3),

$$\Phi\left(\frac{a_0}{\lambda}\right) = \begin{cases} \frac{a_0}{\lambda}, & \lambda \geq a_0; \\ \exp(a_0/\lambda - 1), & 0 < \lambda < a_0. \end{cases}$$

It is seen, that $\Phi\left(\frac{a_0}{\lambda}\right) = 1$ has a unique solution:

$$\lambda^* = \frac{a_0}{1+\log\left(\frac{1}{i_1}\right)}. \tag{2.3}$$

And thus we have that

$$\|f\|_{\Phi} = \frac{a_0}{1+\log\left(\frac{1}{i_1}\right)}. \tag{2.4}$$

This indeed agrees with (2.2). It therefore follows that

$$c_1 = \inf_{f\in L_{\exp}} \frac{\|f\|_{L_{\exp}}}{\|f\|_{\Phi}} \leq \inf_{f\in S_1} \frac{\|f\|_{L_{\exp}}}{\|f\|_{\Phi}} = 1.$$

(That c_1 is positive is a trivial consequence of the equality of the norms.) Above S_1 is the set of one-step functions. The first part of our theorem is proved.

2.2. *The estimate of* c_2

To find lower estimate for

$$c_2 = \sup_{f\in L_{\exp}} \frac{\|f\|_{L_{\exp}}}{\|f\|_{\Phi}},$$

we consider step functions of special form. Let S be a class of step functions, for which the norm (1.2) is taken already on the first step, i.e., $\|f\|_{\Phi} \geq a_1$. In view of

(2.3), a step function f can belong to S only if

$$a_1 < \frac{a_0}{1+\log\left(\frac{1}{i_1}\right)} \tag{2.5}$$

(otherwise we would have that $\lambda < a_1$, and a_1 would therefore be needed to calculate $\|f\|_\Phi$).

Seeking for the supremum of $\frac{\|f\|_{L_{\exp}}}{\|f\|_\Phi}$, we can raise the steps of f and thus making the nominator bigger, while the denominator remains unchanged. We do this so that our function stays in S. Clearly, in limit, any rear steps in f become as high as a_1, and a_1 itself will be equal to $\frac{a_0}{1+\log\left(\frac{1}{i_1}\right)}$ (see (2.5)). For simplicity, let us denote i_1 by α. Thus, in the set S, the extremal ratio of the norms is attained for

$$f(t) = a_0\chi_{]0,\alpha}(t) + a_1\chi_{]\alpha,1]}(t) \tag{2.6}$$

with some $0 < \alpha < 1$, and $a_1 = \frac{a_0}{1+\log\left(\frac{1}{\alpha}\right)}$. For this f, (2.4) is still valid (with $i_1 = \alpha$), which gives us the Luxemburg norm of f. We still need to determine the Marcinkiewicz norm of f. This can be calculated as follows:

$$\begin{aligned}
\|f\|_{L_{\exp}} &= \max\left\{\sup_{0<t<\alpha}\frac{a_0}{1+\log\left(\frac{1}{t}\right)}, \sup_{\alpha\le t<1}\frac{a_0\alpha + a_1(t-\alpha)}{t\left(1+\log\left(\frac{1}{t}\right)\right)}\right\}\\
&= \max\left\{\frac{a_0}{1+\log\left(\frac{1}{\alpha}\right)}, \sup_{\alpha\le t<1}\frac{(a_0-a_1)\alpha + a_1 t}{t\left(1+\log\left(\frac{1}{t}\right)\right)}\right\}.
\end{aligned}$$

If the maximum was the first value, we would be in the case of one step functions. Otherwise, we have that

$$\frac{\|f\|_{L_{\exp}}}{\|f\|_\Phi} = \frac{\sup_{\alpha\le t<1}\frac{(a_0-a_1)\alpha+a_1 t}{t\left(1+\log\left(\frac{1}{t}\right)\right)}}{a_1} = \sup_{\alpha\le t<1}\frac{\left(\frac{a_0}{a_1}-1\right)\frac{\alpha}{t}+1}{1+\log\left(\frac{1}{t}\right)}.$$

Since $a_0/a_1 = 1+\log\left(\frac{1}{\alpha}\right)$, it comes that

$$\frac{\|f\|_{L_{\exp}}}{\|f\|_\Phi} = \sup_{\alpha\le t<1}\frac{\log\left(\frac{1}{\alpha}\right)\frac{\alpha}{t}+1}{1+\log\left(\frac{1}{t}\right)} \tag{2.7}$$

for our particular f in (2.6). For any $\alpha \in]0,1[$, the function on the right-hand-side inside the supremum decreases, and then increases in t, so its extremal values in $t \in [\alpha, 1]$ are taken in the points $t = \alpha$ and $t = 1$. In $t = \alpha$ its value is one, and at $t = 1$ it is $\log\left(\frac{1}{\alpha}\right)\alpha + 1$. We choose α such that this expression is maximal. The maximum is reached when $\alpha = \frac{1}{e}$.

From these considerations and from (2.7), we altogether have that

$$\frac{\|f\|_{L_{\exp}}}{\|f\|_\Phi} = \frac{1}{e} + 1.$$

Since the set S is obviously smaller than the whole set $L_{\exp}$, we expect the supremum of the ratio on $L_{\exp}$ larger than it is on S. That is,

$$c_2 \geq \frac{1}{e}+1.$$

The only statement that still needs to be proven is that $c_2 \leq 2(1+\log 2)$. To prove this inequality, we recall that the finiteness of

$$K := \int_0^1 \exp(\lambda f^*(t))\mathrm{d}t \tag{2.8}$$

implies that

$$f^*(t) \leq c\left(1+\log\left(\frac{1}{t}\right)\right) \qquad (0<t<1). \tag{2.9}$$

Here $c = \frac{1}{\lambda}\max\{\log K, 1\}$, and λ is positive. See the argument on p. 244-245 of [1].

At this point we suppose that f belongs to the unit sphere of $(L_{\exp}, \|\cdot\|_\Phi)$ in the expression $\frac{\|f\|_{L_{\exp}}}{\|f\|_\Phi}$. This causes no lost of generality, because of the homogeneity of the norms. We therefore have that

$$\|f\|_\Phi = \int_A |f(t)|\mathrm{d}t + \int_B \exp(|f(t)|-1)\mathrm{d}t = 1. \tag{2.10}$$

Here

$$A = \{t \in [0,1] : |f(t)| \leq 1\}, \quad B = [0,1]\setminus A.$$

Next, we estimate the constant K in (2.8) (with $\lambda = 1$):

$$\int_0^1 \exp(f^*(t))\mathrm{d}t = \int_B \exp(|f(t)|)\mathrm{d}t + \int_A \exp(|f(t)|)\mathrm{d}t.$$

From (2.10) it comes that $\int_B \exp(|f(t)|)\mathrm{d}t \leq e$. Moreover, by the definition of the set A,

$$\int_A \exp(|f(t)|)\mathrm{d}t \leq \int_A \exp(1)\mathrm{d}t \leq e.$$

Hence

$$K = \int_0^1 \exp(f^*(t))\mathrm{d}t \leq 2e,$$

and this yields that in (2.9) the constant c is

$$c = \frac{1}{1}\max\{\log(2e), 1\} = 1+\log 2.$$

To estimate $\|f\|_{L_{\exp}}$, however, we need an estimate for $f^{**}(t)$. But this is easy after having (2.9) at hand:

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\mathrm{d}s \leq \frac{1+\log 2}{t}\int_0^t \left(1+\log\frac{1}{s}\right)\mathrm{d}s \leq 2(1+\log 2)\left(1+\log\frac{1}{t}\right).$$

Now, (1.1) immediately yields that, if f is on the unit sphere of $(L_{\exp}, \|\cdot\|_\Phi)$, then

$$\|f\|_{L_{\exp}} \leq 2(1+\log 2).$$

Our proof is done.

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