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A NEW GENERALIZED LAPLACE TRANSFORM AND ITS APPLICATIONS TO FRACTIONAL BAGLEY-TORVIK AND FRACTIONAL HARMONIC VIBRATION PROBLEMS

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Abstract. In this paper, a new generalized Laplace transform is defined and its certain properties are given. By using the new transform, the solutions of fractional Bagley-Torvik and fractional harmonic vibration problems are obtained, as application. Also transformations of some elementary functions and the relationships between the new transform with other generalized Laplace transforms are given in separate tables.

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1. Introduction

Integral transformations are very powerful tools used to solve differential and integral equations that arise in various fields of science. These transformations usually consist of integrating the equation with a weight function of two variables, which will result a simplification of the considered particular problem. Many problems in the field of oscillation theory, thermal conductivity, neutron diffusion, hydrodynamics, theory of elasticity and physical kinetics can be solved with the help of these transformations.

The one of the most popular integral transformation is the Laplace transform [4] which defined as

$$\mathfrak{L}\{\kappa(\xi)\} = \int_0^\infty \exp(-\xi\eta)\kappa(\xi)d\xi,$$

where $\Re(\eta) > 0$, $\xi \geq 0$ and $\kappa(\xi)$ is a piecewise continuous and α-exponential function. There are many generalizations of Laplace transform in the literature such as, Aboodh [1], *HY* [2], Novel [3], Elzaki [5], Gupta [7], Jafari [8], Kamal [9], Kashuri [10], *N* [12], *G* [13], Mahgoub [15], Shedu [16], α-Laplace [17], Mohand [18], Sawi [19], ARA [21], Sadik [22], $\mathbb{M}$ [23], Upadhyaya [24], Sumudu [25] and ZZ [26] transforms (see Table 1 for details).

In this paper, we defined a new generalization of Laplace transform, which has a more general form than the transforms mentioned above. We determined some of its properties such as, linearity, convolution and derivative formula. We also calculate generalized Laplace transforms of some elementary functions and Caputo fractional derivative. Finally, we used the generalized Laplace transform for obtaining the solutions of fractional Bagley-Torvik and fractional harmonic vibration problems.

2. GENERALIZED LAPLACE TRANSFORM AND ITS PROPERTIES

In this section, we give the definitions of generalized Laplace ($\mathfrak{L}_n$) and inverse Laplace ($\mathfrak{L}_n^{-1}$) transforms and some of their properties.

Definition 2.1. Let $n \in \mathbb{R} - \{0\}$, $\Re(s^n) > 0$, $t \geq 0$ and $f(t)$ is a piecewise continuous and α-exponential function. Then, the generalized Laplace and the inverse Laplace transforms are defined respectively

$$\mathfrak{L}_n \{f(t); s\} := \hat{f}_n(s) = s^{n-1} \int_0^\infty \exp(-s^n t) f(t) dt,$$

and

$$\mathfrak{L}_n^{-1} \left\{\hat{f}_n(s); t\right\} := f(t).$$

Remark 2.1. For the special value $n = 1$, the generalized Laplace transform converts to the classical Laplace transform. Also, the special cases of $\mathfrak{L}_n$ transform are listed in Table 2.

Theorem 2.1. *If a function $f(t)$ is continuous or piecewise continuous and α-exponential in every finite interval $(0,T)$, then the function $f(t)$ has a generalized Laplace transform for all s^n with $\Re(s^n) > \alpha$.*

Proof. Let rewrite the generalized Laplace transform as follows:

$$\begin{aligned} \mathfrak{L}_n \{f(t); s\} &= \left(s^{n-1} \int_0^T \exp(-s^n t) f(t) dt\right) + \left(s^{n-1} \int_T^\infty \exp(-s^n t) f(t) dt\right) \\ &= I_1 + I_2. \end{aligned}$$

The I_1 integral is convergent, since it can be written as the sum of integrals over intervals where $\exp(-s^n t) f(t)$ is continuous. Considering $|f(t)| \leq M \exp(\alpha t)$ for the positive constant M and the entire $t > T$ in the range $0 \leq t < \infty$, we have

$$\begin{aligned} |I_2| &\leq \left|s^{n-1}\right| \int_T^\infty \exp(-s^n t) \left|f(t)\right| dt \\ &\leq \left|s^{n-1}\right| M \int_0^\infty \exp\left(-t(s^n - \alpha)\right) dt \\ &= \frac{M\left|s^{n-1}\right|}{s^n - \alpha}, \quad \left(\Re(s^n) > \alpha\right). \end{aligned}$$

□

Throughout the paper, unless otherwise stated, we assumed that the functions $f(t)$ and $g(t)$ have $\mathfrak{L}_n$ transforms with $n \in \mathbb{R} - \{0\}$, $\Re(s^n) > 0$, $t \geq 0$.

Theorem 2.2. *Let $f(t)$ and $g(t)$ be functions whose $\mathfrak{L}_n$ transforms exist, and let k_1 and k_2 be constants. Then*

$$\mathfrak{L}_n\{k_1 f(t) + k_2 g(t); s\} = k_1 \mathfrak{L}_n\{f(t); s\} + k_2 \mathfrak{L}_n\{g(t); s\}.$$

Proof. Applying the $\mathfrak{L}_n$ transform, we have

$$\mathfrak{L}_n\{f(t); s\} = s^{n-1} \int_0^\infty \exp(-s^n t) f(t) dt, \quad \big(\Re(s^n) > \alpha_1\big),$$
$$\mathfrak{L}_n\{g(t); s\} = s^{n-1} \int_0^\infty \exp(-s^n t) g(t) dt, \quad \big(\Re(s^n) > \alpha_2\big).$$

With $\Re(s^n) > \max\{\alpha_1, \alpha_2\}$, we have

$$\begin{aligned} \mathfrak{L}_n\{k_1 f(t) + k_2 g(t); s\} &= s^{n-1} \int_0^\infty \exp(-s^n t)\big(k_1 f(t) + k_2 g(t)\big) dt \\ &= k_1 s^{n-1} \int_0^\infty \exp(-s^n t) f(t) dt + k_2 s^{n-1} \int_0^\infty \exp(-s^n t) g(t) dt \\ &= k_1 \mathfrak{L}_n\{f(t); s\} + k_2 \mathfrak{L}_n\{g(t); s\}. \end{aligned}$$

□

Example 2.1. Consider the function $f(t) = \exp(at)$ for $t > 0$. If we apply the $\mathfrak{L}_n$ transform, we get

$$\begin{aligned} \mathfrak{L}_n\{\exp(at); s\} &= s^{n-1} \int_0^\infty \exp(-s^n t) \exp(at) dt \\ &= s^{n-1} \left(\lim_{A \to \infty} \int_0^A \exp\big(-t(s^n - a)\big) dt \right) \\ &= \frac{s^{n-1}}{s^n - a}, \quad \text{for all } \Re(s^n) > a. \end{aligned}$$

Example 2.2. If we apply $\mathfrak{L}_n$ transform to the function $f(t) = t^m$, where m is a positive integer, then we get

$$\mathfrak{L}_n\{t^m; s\} = \frac{m!}{s^{mn+1}}.$$

To see this, we use induction. For $m = 1$, we have

$$\begin{aligned} \mathfrak{L}_n\{t; s\} &= s^{n-1} \int_0^\infty \exp(-s^n t) t dt \\ &= s^{n-1} \lim_{A \to \infty} \left(\frac{t \exp(-s^n t)}{-s^n} \Bigg|_0^A + \frac{1}{s^n} \int_0^A \exp(-s^n t) dt \right) \\ &= \frac{1}{s^{n+1}}. \end{aligned}$$

For $m=k$, let the following equation be true:

$$\mathfrak{L}_n\left\{t^k;s\right\}=\frac{k!}{s^{kn+1}}. \tag{2.1}$$

For $m=k+1$, considering equation (2.1), we have

$$\begin{aligned}\mathfrak{L}_n\left\{t^{k+1};s\right\}&=s^{n-1}\int_0^\infty \exp(-s^n t)t^{k+1}dt\\&=s^{n-1}\lim_{A\to\infty}\left(\frac{t^{k+1}\exp(-s^n t)}{-s^n}\bigg|_0^A+\frac{(k+1)}{s^n}\int_0^A \exp(-s^n t)t^k dt\right)\\&=\frac{(k+1)}{s^n}\frac{k!}{s^{nk+1}}\\&=\frac{(k+1)!}{s^{n(k+1)+1}}.\end{aligned}$$

Definition 2.2. Let $f(t)$ and $g(t)$ be two functions that are piecewise continuous on every finite closed interval $0\le t\le b$ and of exponential order. The function denoted by $f*g$ which defined by

$$f(t)*g(t)=s^{n-1}\int_0^t f(t-\tau)g(\tau)d\tau$$

is called the convolution of the functions $f(t)$ and $g(t)$.

Theorem 2.3 (Convolution Theorem)**.** *Denoting $\mathfrak{L}_n$ transforms of $f(t)$ by $\hat{f}_n(s)$ and $g(t)$ by $\hat{g}_n(s)$, we have*

$$\mathfrak{L}_n\left\{f(t)*g(t);s\right\}=\hat{f}_n(s)\hat{g}_n(s).$$

Proof. By using the definitions of convolution and the $\mathfrak{L}_n$ transform, we have

$$\begin{aligned}\mathfrak{L}_n\left\{f(t)*g(t);s\right\}&=s^{n-1}\int_0^\infty \exp(-s^n t)s^{n-1}\int_0^t f(t-\tau)g(\tau)d\tau dt\\&=\left(s^{n-1}\int_0^\infty \exp(-s^n\tau)g(\tau)d\tau\right)\left(s^{n-1}\int_0^\infty \exp(-s^n x)f(x)dx\right)\\&=\hat{g}_n(s)\hat{f}_n(s).\end{aligned}$$ □

Theorem 2.4 (Translation Property)**.** *Suppose $f(t)$ is a function such that $\mathfrak{L}_n\left\{f(t)\right\}$ exist for $\Re(s^n)>\alpha$. For any constant a,*

$$\mathfrak{L}_n\left\{\exp(at)f(t);s\right\}=\frac{s^{n-1}}{(s^n-a)^{\frac{n-1}{n}}}\mathfrak{L}_n\left\{f(t);(s^n-a)^{\frac{1}{n}}\right\}$$

for $\Re(s^n)>\alpha+a$.

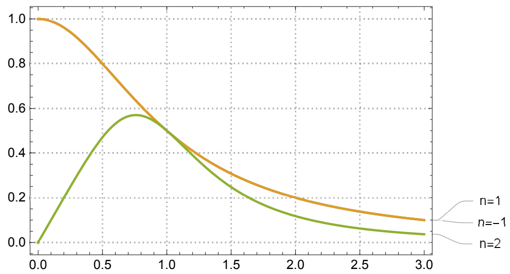
Proof. Using the $\mathfrak{L}_n$ transform, we have

$$\mathfrak{L}_n\left\{\exp(at)f(t);s\right\}=s^{n-1}\int_0^\infty \exp(-s^n t)\exp(at)f(t)dt$$

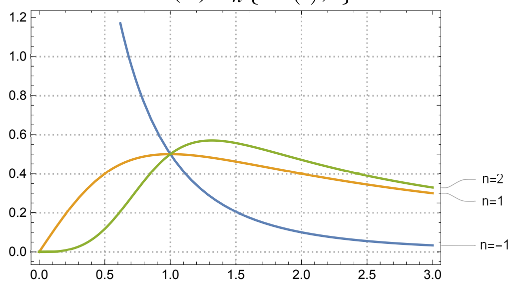
$$= \frac{s^{n-1}}{(s^n - a)^{\frac{n-1}{n}}} (s^n - a)^{\frac{n-1}{n}} \int_0^\infty \exp\left(-\left((s^n - a)^{\frac{1}{n}}\right)^n t\right) f(t)dt$$

$$= \frac{s^{n-1}}{(s^n - a)^{\frac{n-1}{n}}} \mathfrak{L}_n \left\{ f(t); (s^n - a)^{\frac{1}{n}} \right\}. \qquad \square$$

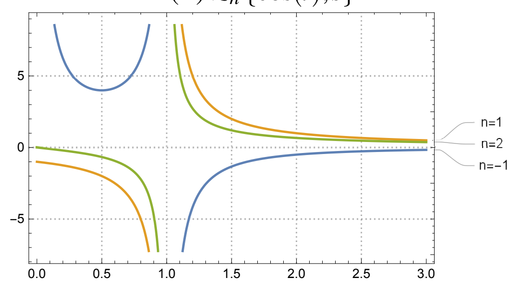
The new generalized Laplace transform of some functions, which can be obtained by direct calculations, are listed in Table 3. Also the graphics of the generalized Laplace transform of the functions $\sin(t)$, $\cos(t)$ and $\exp(t)$ can be found in Figure 1 for the values of $n = -1, n = 1$ and $n = 2$.



(A) $\mathfrak{L}_n \{\sin(t); s\}$



(B) $\mathfrak{L}_n \{\cos(t); s\}$



(C) $\mathfrak{L}_n \{\exp(t); s\}$

FIGURE 1. The graphics of the $\mathfrak{L}_n$ transforms of the functions $\sin(t), \cos(t)$ and $\exp(t)$ for $n = -1, n = 1$ and $n = 2$.

Theorem 2.5 ($\mathfrak{L}_n$ Transforms of Derivatives)**.** *The $\mathfrak{L}_n$ transform of the function $f^{(k)}(t)$ is obtained as:*

$$\mathfrak{L}_n\left\{f^{(k)}(t);s\right\} = s^{kn}\hat{f}_n(s) - s^{n-1}\sum_{m=0}^{k-1}(s^n)^{k-1-m}f^{(m)}(0), \tag{2.2}$$

where $f^{(i)}(t)\exp(-s^n t) \to 0$, $(i = 0,1,\ldots,r-1)$ as $t \to \infty$.

Proof. We again use induction. For $k = 1$, we have

$$\begin{aligned}\mathfrak{L}_n\left\{f'(t);s\right\} &= s^{n-1}\int_0^\infty \exp(-s^n t)f'(t)dt \\ &= s^{n-1}\lim_{A\to\infty}\left(\exp(-s^n t)f(t)\Big|_0^A + s^n\int_0^A \exp(-s^n t)f(t)dt\right) \\ &= s^n\hat{f}_n(s) - s^{n-1}f(0),\end{aligned}$$

in which we assumed $f(t)\exp(-s^n t) \to 0$ as $t \to \infty$.
For $k = r$, let the following equation be true

$$\mathfrak{L}_n\left\{f^{(r)}(t);s\right\} = s^{rn}\hat{f}_n(s) - s^{n-1}\sum_{m=0}^{r-1}(s^n)^{r-1-m}f^{(m)}(0). \tag{2.3}$$

For $k = r+1$, considering equation (2.3), we have

$$\begin{aligned}\mathfrak{L}_n\left\{f^{(r+1)}(t);s\right\} &= s^{n-1}\int_0^\infty \exp(-s^n t)f^{(r+1)}(t)dt \\ &= s^{n-1}\left(-f^{(r)}(0) + s^n\int_0^\infty \exp(-s^n t)f^{(r)}(t)dt\right) \\ &= s^{(r+1)n}\hat{f}_n(s) - s^{n-1}\sum_{m=0}^{r}(s^n)^{r-m}f^{(m)}(0),\end{aligned}$$

in which we assumed $f^{(i)}(t)\exp(-s^n t) \to 0$, $(i = 0,1,\ldots,r)$ as $t \to \infty$. □

Theorem 2.6. *The $\mathfrak{L}_n$ transform of the Caputo fractional derivative of order α is obtained as:*

$$\mathfrak{L}_n\left\{{}_0^cD_t^\alpha f(t);s\right\} = s^{\alpha n}\hat{f}_n(s) - \sum_{k=0}^{m-1}s^{\alpha n-nk-1}f^{(k)}(0). \tag{2.4}$$

Proof. The Caputo fractional derivative [20] for $\Re(\alpha) > 0$, $m-1 < \Re(\alpha) < m$, $m \in \mathbb{N}$ is given by

$${}_0^cD_t^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\tau)^{m-\alpha-1}f^{(m)}(\tau)d\tau.$$

For $g(\tau) = f^{(m)}(\tau)$, we have

$$ {}_0^cD_t^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} g(\tau) d\tau. $$

Multiplied each sides by $\frac{s^{n-1}}{s^{n-1}}$, we have

$$ {}_0^cD_t^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)s^{n-1}} \left(s^{n-1} \int_0^t (t-\tau)^{m-\alpha-1} g(\tau) d\tau \right). $$

Considering the Convolution Theorem and applying the $\mathfrak{L}_n$ transform to both sides of the equation, we have

$$ \mathfrak{L}_n \left\{ {}_0^cD_t^\alpha f(t); s \right\} = s^{-(m-\alpha)n} \hat{g}_n(s). $$

Using equation (2.2), we have

$$ \mathfrak{L}_n \left\{ {}_0^cD_t^\alpha f(t); s \right\} = s^{\alpha n} \hat{f}_n(s) - \sum_{k=0}^{m-1} s^{\alpha n - nk - 1} f^{(k)}(0). \qquad \square $$

3. Applications of $\mathfrak{L}_n$ transform to differential problems

Problem 3.1. *Let the fractional Bagley-Torvik differential equation be given:*

$$ y''(x) + {}_0^cD_x^{\frac{3}{2}} y(x) + y(x) = x + 1, \quad y(0) = y'(0) = 1. $$

Considering equation (2.4) for $m = 2$ and applying the $\mathfrak{L}_n$ transform to the Bagley-Torvik differential equation, we have

$$ \mathfrak{L}_n \left\{ y''(x) \right\} + \mathfrak{L}_n \left\{ {}_0^cD_x^{\frac{3}{2}} y(x) \right\} + \mathfrak{L}_n \left\{ y(x) \right\} = \mathfrak{L}_n \left\{ x \right\} + \mathfrak{L}_n \left\{ 1 \right\} $$

$$ s^{2n} \hat{y}_n(s) - s^{2n-1} - s^{n-1} + s^{\frac{3n}{2}} \hat{y}_n(s) - s^{\frac{3n}{2}-1} - s^{\frac{3n}{2}-n-1} + \hat{y}_n(s) = \frac{1}{s^{n+1}} + \frac{1}{s} $$

$$ \hat{y}_n(s) \left(s^{2n} + s^{\frac{3n}{2}} + 1 \right) - s^{2n-1} - s^{n-1} - s^{\frac{3n}{2}-1} - s^{\frac{3n}{2}-n-1} = \frac{1}{s^{n+1}} + \frac{1}{s}. $$

Then

$$ \hat{y}_n(s) = \frac{s^{-n-1} + s^{-1} + s^{2n-1} + s^{n-1} + s^{\frac{3n}{2}-1} + s^{\frac{3n}{2}-n-1}}{s^{2n} + s^{\frac{3n}{2}} + 1} = \frac{1}{s^{n+1}} + \frac{1}{s}. $$

By applying the $\mathfrak{L}_n^{-1}$ transform, we have

$$ \mathfrak{L}_n^{-1} \left\{ \hat{y}_n(s) \right\} = \mathfrak{L}_n^{-1} \left\{ \frac{1}{s^{n+1}} \right\} + \mathfrak{L}_n^{-1} \left\{ \frac{1}{s} \right\} $$

$$ y(x) = x + 1. $$

Problem 3.2. *Let* $1 < \Re(\alpha) < 2$ *and the fractional harmonic vibration differential equation be given:*

$$ {}_0^cD_x^\alpha y(x) + w^2 y(x) = 0, \quad y(0) = c_0, \; y'(0) = c_1. $$

Considering equation (2.4) for $m = 2$ and applying the $\mathfrak{L}_n$ transform to the harmonic vibration differential equation, we have

$$\mathfrak{L}_n\left\{{}_0^cD_x^\alpha y(x)\right\} + w^2\mathfrak{L}_n\left\{y(x)\right\} = 0$$
$$s^{\alpha n}\hat{y}_n(s) - s^{\alpha n-1}y(0) - s^{\alpha n-n-1}y'(0) + w^2\hat{y}_n(s) = 0$$
$$s^{\alpha n}\hat{y}_n(s) - c_0 s^{\alpha n-1} - c_1 s^{\alpha n-n-1} + w^2\hat{y}_n(s) = 0.$$

Then

$$\begin{aligned}\hat{y}_n(s) &= \frac{c_0 s^{\alpha n-1}}{s^{\alpha n}+w^2} + \frac{c_1 s^{\alpha n-n-1}}{s^{\alpha n}+w^2}\\ &= \frac{c_0 s^{-1}}{1+w^2 s^{-\alpha n}} + \frac{c_1 s^{-n-1}}{1+w^2 s^{-\alpha n}}\\ &= c_0\sum_{k=0}^{\infty}(-1)^k w^{2k} s^{-\alpha kn-1} + c_1\sum_{k=0}^{\infty}(-1)^k w^{2k} s^{-\alpha kn-n-1}.\end{aligned}$$

By applying the $\mathfrak{L}_n^{-1}$ transform, we obtain

$$\mathfrak{L}_n^{-1}\left\{\hat{y}_n(s)\right\} = c_0\sum_{k=0}^{\infty}(-1)^k w^{2k}\mathfrak{L}_n^{-1}\left\{s^{-\alpha kn-1}\right\} + c_1\sum_{k=0}^{\infty}(-1)^k w^{2k}\mathfrak{L}_n^{-1}\left\{s^{-\alpha kn-n-1}\right\}$$
$$\begin{aligned}y(x) &= c_0\sum_{k=0}^{\infty}\frac{\left(-w^2x^\alpha\right)^k}{\Gamma(\alpha k+1)} + c_1 x\sum_{k=0}^{\infty}\frac{\left(-w^2x^\alpha\right)^k}{\Gamma(\alpha k+2)}\\ &= c_0 E_{\alpha,1}\left(-w^2x^\alpha\right) + c_1 x E_{\alpha,2}\left(-w^2x^\alpha\right),\end{aligned}$$

where $E_{\alpha,\beta}(x)$ is the Mittag-Leffler function with two parameters [6], which defined as

$$E_{\alpha,\beta}(x) = \sum_{k=0}^{\infty}\frac{x^k}{\Gamma(\alpha k+\beta)}.$$

Remark 3.1. For the graphics of the approximate solution of Problem 3.2, we define the function $y_p(x)$ with finite sums as

$$y_p(x) = c_0\sum_{k=0}^{p}\frac{\left(-w^2x^\alpha\right)^k}{\Gamma(\alpha k+1)} + c_1 x\sum_{k=0}^{p}\frac{\left(-w^2x^\alpha\right)^k}{\Gamma(\alpha k+2)}.$$

The graphics of the approximate solution $y_2(x)$ of the fractional harmonic vibration problem for the special values $c_0 = c_1 = w = 1$ with $\alpha = \frac{6}{5}, \frac{7}{5}, \frac{8}{5}, \frac{9}{5}$ can be found in Figure 2.

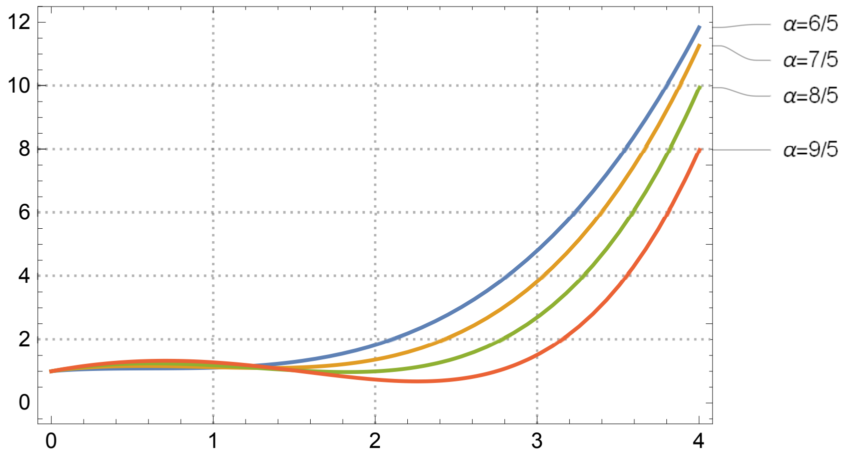


FIGURE 2. The behavior of approximate solution $y_2(x)$ of Problem 3.2 for different values of α, where $1 < \Re(\alpha) < 2$.

4. CONCLUSIONS

In this article, we introduced the generalized Laplace $\mathfrak{L}_n$ and inverse Laplace $\mathfrak{L}_n^{-1}$ transforms and examine their certain properties. Our motivation in doing this was to define an integral transform that has a more general form than many Laplace-like integral transforms found in the literature. We also presented a transformation table for $\mathfrak{L}_n$ transform (Table 1), and another table containing the relationships with other transforms (Table 2). Then, we use generalized Laplace and inverse Laplace transforms to arrive the analytical solutions of fractional Bagley-Torvik and harmonic vibration problems. Finally, we examined the behavior for different alpha values of the approximate solutions of the fractional harmonic vibration problem on a graph (Figure 2).

We should say that the solutions of the application problems we obtained with the generalized Laplace transform are in full agreement with the results obtained in the literature (see for example [11, 14]). As a result of these applications, we saw that the newly defined generalized Laplace transform is much more general and quite compatible with fractional problems. We conclude this research by stating that the generalized Laplace transform plays a very important role in finding analytical solutions of fractional problems, and therefore the results presented in this article are very important for application.

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TABLE 1. Generalized Laplace transforms

Aboodh Transform	[1]: $\mathbf{A}\{\kappa(\eta)\}=\frac{1}{\xi}\int_0^\infty \exp(-\xi\eta)\kappa(\eta)d\eta$
HY Transform	[2]: $HY\{\kappa(\eta)\}=\xi\int_0^\infty \exp\left(-\xi^2\eta\right)\kappa(\eta)d\eta$
Novel Transform	[3]: $M_n\{\kappa(\eta)\}=\int_0^\infty \exp(-\xi\eta)\eta^n\kappa(\eta)d\eta$
Laplace Transform	[4]: $\mathfrak{L}\{\kappa(\eta)\}=\int_0^\infty \exp(-\xi\eta)\kappa(\eta)d\eta$
Elzaki Transform	[5]: $E\{\kappa(\eta)\}=\xi\int_0^\infty \exp\left(-\frac{\eta}{\xi}\right)\kappa(\eta)d\eta$
Gupta Transform	[7]: $\dot{R}\{\kappa(\eta)\}=\frac{1}{\xi^3}\int_0^\infty \exp(-\xi\eta)\kappa(\eta)d\eta$
Kamal Transform	[9]: $K\{\kappa(\eta)\}=\int_0^\infty \exp\left(-\frac{\eta}{\xi}\right)\kappa(\eta)d\eta$
Kashuri Transform	[10]: $\mathcal{K}\{\kappa(\eta)\}=\frac{1}{\xi}\int_0^\infty \exp\left(-\frac{\eta}{\xi^2}\right)\kappa(\eta)d\eta$
N-Transform	[12]: $N\{\kappa(\eta)\}=\int_0^\infty \exp(-\xi\eta)\kappa(u\eta)d\eta$
G-Transform	[13]: $G\{\kappa(\eta)\}=\xi^\alpha\int_0^\infty \exp\left(-\frac{\eta}{\xi}\right)\kappa(\eta)d\eta$
Mahgoub Transform	[15]: $M\{\kappa(\eta)\}=\xi\int_0^\infty \exp(-\xi\eta)\kappa(\eta)d\eta$
Shedu Transform	[16]: $\mathbb{S}\{\kappa(\eta)\}=\int_0^\infty \exp\left(-\frac{\xi\eta}{u}\right)\kappa(\eta)d\eta$
α-Laplace Transform	[17]: $\mathfrak{L}_\alpha\{\kappa(\eta)\}=\int_0^\infty \exp\left(-\xi^{\frac{1}{\alpha}}\eta\right)\kappa(\eta)d\eta$
Mohand Transform	[18]: $\mathbf{M}\{\kappa(\eta)\}=\xi^2\int_0^\infty \exp(-\xi\eta)\kappa(\eta)d\eta$
Sawi Transform	[19]: $\mathbf{S}\{\kappa(\eta)\}=\frac{1}{\xi^2}\int_0^\infty \exp\left(-\frac{\eta}{\xi}\right)\kappa(\eta)d\eta$
ARA Transform	[21]: $G_n\{\kappa(\eta)\}=\xi\int_0^\infty \exp(-\xi\eta)\eta^{n-1}\kappa(\eta)d\eta$
Sadik Transform	[22]: $S\{\kappa(\eta)\}=\frac{1}{\xi^\beta}\int_0^\infty \exp(-\xi^\alpha\eta)\kappa(\eta)d\eta$
$\mathbb{M}$-Transform	[23]: $\mathbb{M}_{\rho,m}\{\kappa(\eta)\}(u,v)=\int_0^\infty \frac{\exp(-u\eta)\kappa(v\eta)}{(\eta^m+v^m)^\rho}d\eta$
Upadhyaya Transform	[24]: $U\{\kappa(\eta)\}=\lambda_1\int_0^\infty \exp(-\lambda_2\eta)\kappa(\lambda_3\eta)d\eta$
Sumudu Transform	[25]: $\mathscr{S}\{\kappa(\eta)\}=\frac{1}{\xi}\int_0^\infty \exp\left(-\frac{\eta}{\xi}\right)\kappa(\eta)d\eta$
ZZ Transform	[26]: $H\{\kappa(\eta)\}=\xi\int_0^\infty \exp(-\xi\eta)\kappa(u\eta)d\eta$

TABLE 2. Relationships of $\mathfrak{L}_n$ transform with other transforms

Laplace Transform	$\mathfrak{L}_1\{\kappa(\eta)\} = \mathfrak{L}\{\kappa(\eta)\}$
Sumudu Transform	$\xi\mathfrak{L}_{-1}\{\kappa(\eta)\} = \mathscr{S}\{\kappa(\eta)\}$
N-Transform	$\mathfrak{L}_1\{\kappa(u\eta)\} = N\{\kappa(\eta)\}$
Elzaki Transform	$\xi^3\mathfrak{L}_{-1}\{\kappa(\eta)\} = E\{\kappa(\eta)\}$
Aboodh Transform	$\frac{1}{\xi}\mathfrak{L}_1\{\kappa(\eta)\} = \mathbf{A}\{\kappa(\eta)\}$
Kashuri Transform	$\xi^2\mathfrak{L}_{-2}\{\kappa(\eta)\} = \mathcal{K}\{\kappa(\eta)\}$
Novel Transform	$\mathfrak{L}_1\{\eta^n\kappa(\eta)\} = M_n\{\kappa(\eta)\}$
$\mathbb{M}$-Transform	$\mathfrak{L}_1\left\{\frac{\kappa(v\eta)}{(\eta^m+v^m)^\rho}\right\} = \mathbb{M}_{\rho,m}\{\kappa(\eta)\}$
α-Laplace Transform	$\xi^{\frac{\alpha-1}{\alpha}}\mathfrak{L}_{\frac{1}{\alpha}}\{\kappa(\eta)\} = \mathfrak{L}_\alpha\{\kappa(\eta)\}$
Kamal Transform	$\xi^2\mathfrak{L}_{-1}\{\kappa(\eta)\} = K\{\kappa(\eta)\}$
ZZ Transform	$\xi\mathfrak{L}_1\{\kappa(u\eta)\} = H\{\kappa(\eta)\}$
Mahgoub Transform	$\xi\mathfrak{L}_1\{\kappa(\eta)\} = M\{\kappa(\eta)\}$
G-Transform	$\xi^{\alpha+2}\mathfrak{L}_{-1}\{\kappa(\eta)\} = G\{\kappa(\eta)\}$
Mohand Transform	$\xi^2\mathfrak{L}_1\{\kappa(\eta)\} = \mathbf{M}\{\kappa(\eta)\}$
Sadik Transform	$\xi^{1-\alpha-\beta}\mathfrak{L}_\alpha\{\kappa(\eta)\} = S\{\kappa(\eta)\}$
HY Transform	$\mathfrak{L}_2\{\kappa(\eta)\} = HY\{\kappa(\eta)\}$
Sawi Transform	$\mathfrak{L}_{-1}\{\kappa(\eta)\} = \mathbf{S}\{\kappa(\eta)\}$
Shedu Transform	$\mathfrak{L}_1\left\{\kappa(\eta);\frac{\xi}{u}\right\} = \mathbb{S}\{\kappa(\eta)\}$
Upadhyaya Transform	$\lambda_1\mathfrak{L}_1\{\kappa(\lambda_3\eta);\lambda_2\} = U\{\kappa(\eta)\}$
ARA Transform	$\xi\mathfrak{L}_1\left\{\eta^{n-1}\kappa(\eta)\right\} = G_n\{\kappa(\eta)\}$
Gupta Transform	$\frac{1}{\xi^3}\mathfrak{L}_1\{\kappa(\eta)\} = \dot{R}\{\kappa(\eta)\}$

TABLE 3. $\mathfrak{L}_n$ transform table

$f(t)$	$\mathfrak{L}_n\{f(t);s\} = s^{n-1}\int_0^\infty \exp(-s^n t) f(t) dt$
1	$\frac{1}{s}$
$\exp(at)$	$\frac{s^{n-1}}{s^n - a}$
$\sin(at)$	$\frac{as^{n-1}}{s^{2n}+a^2}$
$\cos(at)$	$\frac{s^{2n-1}}{s^{2n}+a^2}$
$\sinh(at)$	$\frac{as^{n-1}}{s^{2n}-a^2}$
$\cosh(at)$	$\frac{s^{2n-1}}{s^{2n}-a^2}$
t	$\frac{1}{s^{n+1}}$
t^2	$\frac{2}{s^{2n+1}}$
t^m	$\frac{m!}{s^{mn+1}}$
$\exp(at)t^m$	$\frac{m!s^{n-1}}{(s^n-a)^{m+1}}$
$\exp(at)\sin(bt)$	$\frac{bs^{n-1}}{(s^n-a^2)+b^2}$
$\exp(at)\cos(bt)$	$\frac{s^{n-1}(s^n-a)}{(s^n-a)^2+b^2}$
$f'(t)$	$s^n \hat{f}_n(s) - s^{n-1} f(0)$
$f''(t)$	$s^{2n} \hat{f}_n(s) - s^{2n-1} f(0) - s^{n-1} f'(0)$
$f'''(t)$	$s^{3n} \hat{f}_n(s) - s^{3n-1} f(0) - s^{2n-1} f'(0) - s^{n-1} f''(0)$
$f^{(k)}(t)$	$s^{kn} \hat{f}_n(s) - s^{n-1} \sum_{m=0}^{k-1} (s^n)^{k-1-m} f^{(m)}(0)$
${}_0^c D_t^\alpha f(t)$	$s^{\alpha n} \hat{f}_n(s) - \sum_{m=0}^{k-1} s^{\alpha n - nm - 1} f^{(m)}(0)$

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